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THE MATHEMATICAL ASSOCIATION OF AMERICA
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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York 53, N. Y.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo 14, N. Y.

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THIS IS THE OFFICIAL JOURNAL OF THE

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AWARD FOR DISTINGUISHED SERVICE TO PROFESSOR EDWARD JAMES McSHANE

Citation of Professor McShane. Professor Edward James McShane has been named as the recipient of the "Award for Distinguished Service to Mathematics." This award is made for "outstanding service to mathematics, other than mathematical research." Consequently in honoring him today we pass over his brilliant record of mathematical research, first in the absolute minimum problems of the calculus of variations and later in the theory of integration in general spaces, in functional analysis and the general theory of limits. Today we honor the man who contains the mathematician.

It is possible, acceptable, and indeed admirable for a mathematician in his public life to be wholly committed to his research, to speak only through his original mathematical writing and to influence others only through this medium. Consequently we feel an even greater admiration for a man who, though he could rest upon his outstanding contributions to mathematical research, goes beyond them to express himself in a spirit of service to the profession, to the nation, to the community of mathematicians in the world, to *people*, whether important or not, whether friend or stranger. McShane is such a man.

We recall some examples of Professor McShane's service to mathematics beyond his research, starting with his concern for mathematical exposition. Perhaps it began with the translation of Courant's Differential and Integral Calculus which he did while he was in Göttingen as a young postdoctoral fellow. Later, in 1952, this concern for exposition led him to write the MONTHLY article on Partial Orderings and Moore-Smith Limits which was awarded the Chauvenet prize for that period. This work and still later his exposition of The Theory of Limits in the Association film by that name, reflect the influence of E. H. Moore, himself a mathematician with a great concern for teaching, with whom McShane as a graduate student at Chicago was closely associated. We may cite also the McShane-Botts *Real Analysis* (1959) written on a foundation of Professor McShane's teaching at the University of Virginia "to present, in a form accessible to the mature senior or beginning graduate student, some widely useful parts of real function theory, of general topology, and of functional analysis."

Turning to the field of national service we recall that early in World War II, Professor McShane was drafted into the war effort as Mathematical Ballistician, Aberdeen Proving Ground. After the war when this phase of Professor McShane's career was over, his national service in peace was expressed through the National Research Council, in which he ultimately became Chairman of the Division of Mathematics, and the National Science Foundation, where he is now a member of the Board. Meanwhile he had been recognized by the mathematicians as President of both the American Mathematical Society and the Mathematical Association of America. He has also served the Conference Board of Mathematical Sciences in many capacities and is a member of the National Academy.

EDWARD JAMES McSHANE

Professor McShane's universally recognized friendliness, together with his fabulous facility with languages, enabled him to make friends easily among mathematicians and teachers in many countries of the world, in Germany, Italy, The Netherlands, France and, more briefly, in Colombia and Japan, to mention some of them. So we are confident that in these countries there are today many friends of McShane, the man and the mathematician, who would wish to join us in honoring him.

He has served the Association in many different capacities since 1926. As President he conceived and appointed the Committee on the Undergraduate Program in Mathematics. He sounded the warning that the teacher is vital to the life of mathematics, often saying that an excessive financial support of research which withdraws all the good teachers from the classroom could bring extinction to mathematical research as we know it. In his retiring address, stressing the need for communication, he said: "Everyone of us is touched in some way or other by the problems of mathematical communication. Every one of us can make some contribution, great or small, within his own proper sphere of activity. And every contribution is needed if mathematics is to grow healthily and usefully and beautifully."

As President of the Society he found himself in a struggle to reduce the tensions between the pure and applied mathematicians. No better leader could have been found for such a crisis; for his own contributions to both fields gave him authority in each camp and his personal qualities made him absolutely trusted by the embattled proponent of either point of view. Indeed, if there were enough mathematicians having a McShane-like grasp of both fields and a McShane-like spirit of good will towards all colleagues, there would be no problem of reconciliation between these different types of mathematicians.

Distinguished as Professor McShane's contributions to mathematics through national organizations have been, however, each of these positions tends to carry its own recognition and its own honor. Moreover, each such position of honor and responsibility came to him for the same reason that today he receives the Award for Distinguished Service. This is the great esteem in which so many individual mathematicians hold him. In thousands of different incidents which have gone unrecorded he has been helpful to graduate students, sympathetic to the problems of teachers in the many little-known schools and colleges he has visited, generous in bringing his brilliant mathematical intellect to the aid of some young mathematician grappling with a problem, wise in his counsel, trusted in difficulty, and inspiring in his readily-sensed ideals of human behavior. To the many people who have been touched by Professor McShane in these ways he is affectionately known as "Jimmy McShane." So today we salute Jimmy McShane.

AWARD OF THE 1964 CHAUVENET PRIZE TO PROFESSOR LEON A. HENKIN

The Board of Governors of the Mathematical Association of America at its meeting on August 25, 1963, at the University of Colorado voted to award the 1964 Chauvenet Prize to Professor Leon A. Henkin of the University of California, Berkeley, for his paper "Are Logic and Mathematics Identical?," published in SCIENCE, 138 (1962) 788–794.

A certificate and monetary award in the amount of one hundred dollars was presented to Professor Henkin at the time of the Business Meeting of the Association on January 26, 1964, at the University of Miami.

The Chauvenet Prize is awarded for a noteworthy expository paper, published in English, such as will come within range of profitable reading for members of the Association. The purpose of the prize is to stimulate expository contributions to mathematical journals on the part of younger American scholars. The 1964 Prize, awarded for a paper published in 1962 by a member ot the Association, is the fourteenth award of the Chauvenet Prize since its institution by the MAA in 1925. For a list of the names of the previous winners, see this MONTHLY, 66 (1959) 446–447, 67 (1960) 118, and 70 (1963) 2–3.

Professor Henkin was born on April 19, 1921, in Brooklyn, New York. He received his A.B. from Columbia University in 1941 and his M.A. from Princeton University in 1942. After an interval of four years as a mathematician in industry, he returned to Princeton where he obtained his Ph.D. in 1947 and then served successively as Henry B. Fine Instructor and as Frank B. Jewett postdoctoral fellow.

In 1949, Professor Henkin was appointed Assistant Professor of Mathematics at the University of Southern California, and since 1953, he has been a member of the faculty of the University of California, Berkeley. During 1954–55 he was a Fulbright Research Scholar at the University of Amsterdam; during 1960–61, he served as Visiting Professor at Dartmouth College; and during 1961–62, he was a Guggenheim Fellow and member of the Institute for Advanced Study.

Professor Henkin has served as a Visiting Lecturer for the MAA, as Chairman of the MAA Committee on Production of Films, and is presently a member of the Committee on the Undergraduate Program in Mathematics. He has served as Editor of the *Journal of Symbolic Logic*, and in 1962 was elected to three-year terms as President of the Association for Symbolic Logic and as Member-at-large of the Council of the American Mathematical Society.

Professor Henkin's many significant contributions to logic and the foundations of mathematics are contained in his thirty-five publications. His interest in improving mathematical education at all levels is evidenced by his participation in numerous NSF summer institutes and his book "Retracing Elementary Mathematics," published in 1962 in cooperation with W. N. Smith, V. J. Varineau and N. J. Walsh by Macmillan Company.

NEW MATHEMATICAL METHODS IN THE LIFE SCIENCES

GEORGE B. DANTZIG, University of California, Berkeley

Haldane stated "The only way of real advance in biology lies in taking as our starting point, not the separated parts of an organism and its environment, but the whole organism in its actual relation to environment, and defining the parts and activities in this whole implying their existing relationships to the other parts and activities" [1].

Recognizing the dangers inherent in oversimplification, we might nevertheless say that in medicine we are dealing functionally with highly complex systems of chemical reactions and with control mechanisms that affect the varying rates of these reactions.

A living animal is perhaps the most complex chemical factory that will ever be devised. Its highly dynamic character has been brought into even sharper focus by the recent introduction of isotope techniques. It is clear that new approaches should be sought for viewing the human body as a whole and to make it possible to integrate and evaluate the information that has been accumulated but not necessarily interrelated about thousands of parameters developed in the study of living systems.

Indeed it is interesting to note that the most recent developments have evolved out of the analogy between the automation and control of complex industrial processes (and decision making in general) and the complex metabolic and control processes of the human body. We appear to be at that point in human understanding when the greatest progress will be made if many processes are considered in relation to each other. Mathematical models of biochemical processes should now be undertaken that view the human body as a whole, as well as consider a particular organ which may incorporate a number of such processes.

The idea of building mathematical models of biological systems is, of course, not new. Lotka, Rashevsky, Henderson, Michaelis, and others were early investigators who proposed mathematical models of biological systems or who pioneered in the quantification of parts of such systems. But it is only in recent years that the field has attracted a great deal of attention because now better tools are at hand to develop the basic ideas.

The emerging nuclear and space age presents an especially difficult challenge to biological science and medical art. The stresses that may be placed on organisms by the completely new and stringent environments associated with these technologies are largely unknown. To understand the effects of these environments on the whole organism, especially on the human system, will be difficult and slow, if not impossible, within the present state of the biological arts and sciences. The reason is partly our lack of knowledge of individual biological phenomena, and partly the present lack of a technique for integrating a very large number of environmental effects. We should not try to decide in advance which lack is more important. Mathematical techniques and computing facilities

for integrating more and more complicated sets of components are now becoming available. These techniques should be pushed to the biological or mathematical limit—whichever comes first—in the hope that mathematical models for increasingly large portions of the human system can be constructed.

An example of a model. Perhaps one of the simplest systems of chemical reactions that takes place simultaneously in the human body is the exchange of CO_2 and O_2 between blood and air in the lungs.

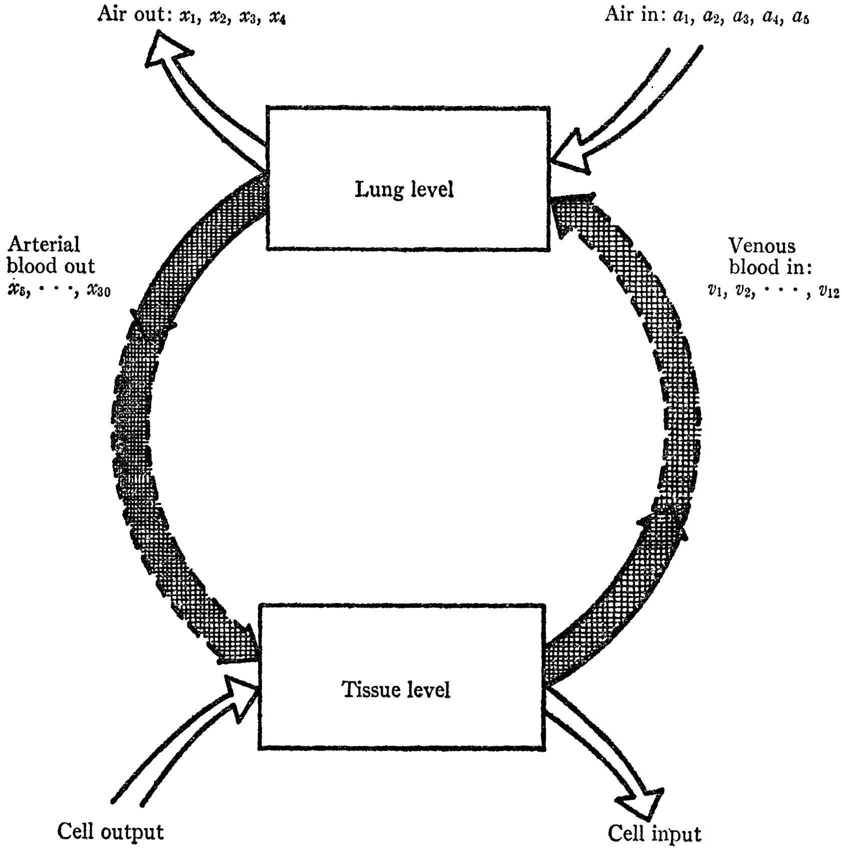


FIG. 1

In Fig. 1 the heavy arrows indicate the flow of blood from the tissue level to the lungs and the other arrows, the corresponding flow of air. The actual exchange takes place in tiny air sacs called alveoli shown in Fig. 2. There are close to 10^9 such sacs in the human lungs.

For a resting individual it can be assumed that the stale air that we breathe out and arterial blood are in equilibrium; that is, if we were to collect the stale air in the top of a big jar and expose it to the blood collected in the bottom, then

no further exchange would take place between them. For the purposes of building the chemical model we have essentially three *compartments* separated by membranes; the air compartment, the plasma (the fluid surrounding the cells), and the red cells.

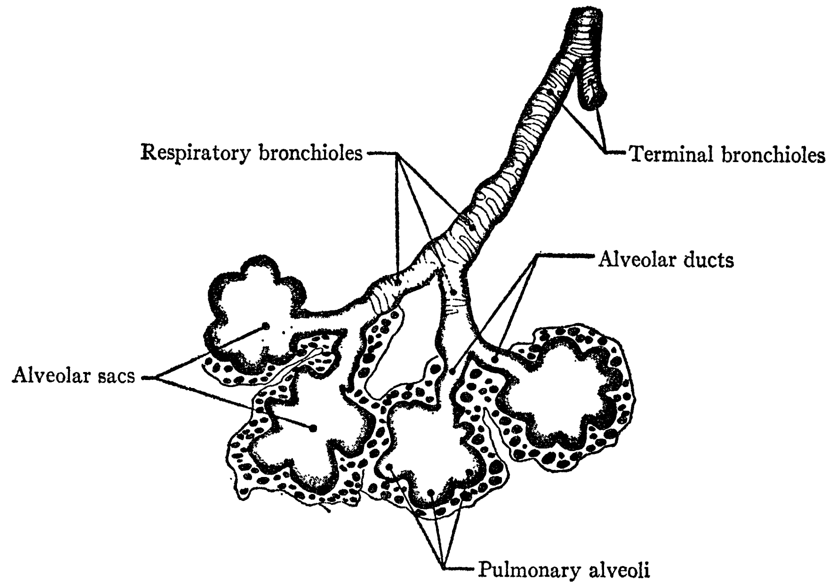


FIG. 2

To illustrate the principles for building a mathematical model for predicting the distribution of various type chemicals in the air and arterial blood leaving the lungs, let us extract a small piece of our respiratory model. For simplicity we have set aside most of the substances found in the blood plasma compartment except for carbon dioxide dissolved in water. This will result in the chemical species H_2O, H^+, OH^-, CO_2, HCO_3^-. We suppose that everything else is held at constant temperature and pressure and that sufficient time has lapsed for the mixture to settle down. Our problem is to predict the equilibrium distribution.

To build up the chemical equilibrium model we first distinguished the different molecule types by giving a formula by which they could be formed from elementary elements. A simple example of this is H_2O which indicates that water can be formed from two atoms of hydrogen and one of oxygen. Actually we could use any building block we please to build up the molecular types. Since organic molecules are very complex, we found it more convenient to use groups of atoms as our "building blocks." In our model we used H^+, OH^-, and CO_2. Thus, water is represented as a vector composed of one unit of H^+ and one unit of OH^- (indi-

TABLE I

Building Blocks (atoms)	Molecules in Mixture				
	H_2O	H^+	OH^-	CO_2	HCO_3^-
H^+	1	1			
OH^-	1		1		1
CO_2				1	1
Quantities in Mixture	x_{H_2O}	x_{H^+}	x_{OH^-}	x_{CO_2}	$x_{HCO_3^-}$

Mass Balance Relations

$$\begin{aligned}
\text{Input } H^+ &= x_{H_2O} + x_{H^+} \\
\text{Input } OH^- &= x_{H_2O} + x_{OH^-} + x_{HCO_3^-} \\
\text{Input } CO_2 &= x_{CO_2} + x_{HCO_3^-}
\end{aligned}$$

cated by the position of the unit entries in the H_2O column of Table I) while HCO_3^- is formed from one unit of OH^- and one unit of CO_2 (as shown by the position of the units in the HCO_3^- column).

To determine the unknown quantities of these molecules, x_j in the equilibrium mixture, we first express the chemical *law of mass balance.* In words, this law says that the total amount of each type of building block placed initially in the mixture is equal to the amounts used to form the various species in the mixture. The equations expressing each of these relations are shown in the bottom half of Table I and are formed from the top half by multiplying x_j by the corresponding entries in any row and summing.

For equilibrium, however, classical chemistry tells us that another law must also be satisfied, the *law of mass action.* There are various ways to express this law. One way is to state that the so-called free energy will "run down hill" until a minimum is reached. Now the function F which measures the free energy of the system has the simple form;

$$F(x) = x_{H_2O}\bar{F}_{H_2O} + x_{H^+}\bar{F}_{H^+} + \cdots + x_{HCO_3^-}\bar{F}_{HCO_3^-},$$

where $\bar{F}_{H_2O} = c_{H_2O} + \log\,(x_{H_2O}/\Sigma x_j)$. Mathematically our problem is:

Find $x_j \geqq 0$ and minimum F satisfying:

MASS BALANCE:

$$\sum_j a_{ij}x_j = b_i, \qquad \sum x_j = \bar{x} \qquad i = 1, 2, \cdots, m$$

$$F = \sum x_j(c_j + \log\,[x_j/\bar{x}]),$$

where F, called the Gibbs Free Energy Function, must be modified if there is more than one compartment.

This is like a linear program except that our function to be minimized is nonlinear. There are many methods for solving such systems in the physical-chemical literature.

The classical approach is to ignore the nonnegativity constraints—to assign Lagrange multipliers to the equations, and to set up the conditions that the partial derivatives must satisfy at equilibrium. In addition to the *Mass Balance* equations that we have already discussed, this gives rise to another set of conditions called the *Law of Mass Action.* If the Lagrange multipliers are eliminated in a systematic way this gives rise to a second set of equations each linear in the logs of the variables.

MASS ACTION LAW:

$$\log [x_{H_2O}] - \log [x_{H^+}] - \log [x_{OH^-}] = \log k_{H_2O},$$

where $[x_j]$ is a symbol for $x_j/\bar{x}_j$ the "concentration" of species j.

Mathematically, what the classical approach has done is to take a relatively simple linear inequality system with a homogeneous convex objective which is to be minimized and to replace it by a messy hybrid system consisting of linear equations and nonconvex conditions on the logs which is difficult if not impossible to analyze. It is perhaps for this reason that the physical-chemical literature is full of papers devoted to special cases.

The full respiratory model, which contains all the relations commonly described in a volume on the subject has the following compact appearance. (See Table II on the following page.) Notice that there are forty-four chemical species in this model. It includes a special fourth compartment called "Hemoglobin Structure." This part is based on a theory developed by Linus Pauling in 1935 which assumes that the four sites where oxygen attaches to the hemoglobin form a square. It also includes the theory of Roughton on the manner that CO_2 attaches to sites adjacent to the oxygen sites.

One direction of research has been to represent compactly all the different variants of a basic molecule. In the case of hemoglobin there are some 2^{150} different species of this molecule arising because there are about 150 sites where H^+ ions can attach or not attach to the long stem of the molecule. *Based on certain assumptions about the independence in the probability of attachments of ions to various sites*, it is possible to replace a large number of different columns by a few generic columns. This fourth compartment reflects such a reduction, [5].

The method of solution that has been proposed recently for solving such systems is to replace Gibbs Free Energy Function by a quadratic fit and to solve the resulting system (mathematically the same as a *quadratic program*) by a special technique. The solution is used to generate a new quadratic fit and the process is repeated. It takes about two minutes on a 7090 to solve such a system with great numerical accuracy (not to be confused with accuracy of input data). In short by going back to the original convex free energy proposed by Gibbs

TABLE II

Input "Element" (Moles): Type	Source: Air	Source: Venous Blood	Source: Total	I. Air Out: O_2 (1)	CO_2 (2)	N_2 (3)	H_2O (4)	II. Plasma: O_2 (5)	CO_2 (6)	N_2 (7)	H^+ (8)	OH^- (9)	Cl^- (10)	Na^+ (11)	H_2O (12)	HCO^- (13)	H_2CO_3 (14)	$CO^=$ (15)	$HP^=$ (16)	$P^=$ (17)	HHP_p (18)
1. O_2........	a_1	v_1	b_1	1				1													
2. CO_2........	a_2	v_2	b_2		1				1							1	1	1			
3. N_2........	a_3	v_3	b_3			1				1											
4. H^+........	a_4	v_4	b_4				1				1				1		1	−1		−1	1
5. OH^-.......	a_5	v_5	b_5				1					1			1	1	1	1			
6. Cl^-.......	......	v_6	b_6										1								
7. Na^+.......	......	v_7	b_7											1							
8. K^+........	......	v_8	b_8																		
9. HP_p^-......	......	v_9	b_9																1	1	1
10. HP_r^-......	......	v_{10}	b_{10}																		
11. Hb........	......	v_{11}	b_{11}																		
12. "A^-"*.....	....	....	0																		
13. "B"†......	......	....	0																		
14. "C^-"‡.....	......	....	0																		
15. Charge in II	......	....	0								−1	−1	−1	+1		−1		−2	−1	−2	
6. Free-energy values..........				c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}	c_{18}

Type	III. Red Cells: O_2 (19)	CO_2 (20)	N_2 (21)	H^+ (22)	OH^- (23)	Cl^- (24)	K^+ (25)	H_2O (26)	HCO_3^- (27)	H_2CO_3 (28)	$CO_3^=$ (29)	$HP_r^=$ (30)	$P_r^=$ (31)	HHP_r (32)	Hb (33)
1. O_2	1														
2. CO_2		1							1	1	1				
3. N_2			1												
4. H^+				1				1		1	−1		−1	1	
5. OH^-					1			1	1	1	1				
6. Cl^-						1									
7. Na^+															
8. K^+							1								
9. HP_p^-															
10. HP_r^-												1	1	1	
11. Hb															1
12. "A^-"*															$-n_1$
13. "B"†															−4
14. "C^-"‡															−8
15. Charge in II															
6. Free-energy values	c_{19}	c_{20}	c_{21}	c_{22}	c_{23}	c_{24}	c_{25}	c_{26}	c_{27}	c_{28}	c_{29}	c_{30}	c_{31}	c_{32}	0

Type	IV. Hb Structure: "A^-" (34)	H"A" (35)	"C^-" (36)	H"C" (37)	CO_2"C" (38)	"B" (39)	O_2"B" (40)	O_4"B_2"(a) (41)	O_4"B_2"(b) (42)	O_6"B_3" (43)	O_8"B_4" (44)
1. O_2							1	2	2	3	4
2. CO_2					1						
3. N_2											
4. H^+		1		1							
5. OH^-											
6. Cl^-											
7. Na^+											
8. K^+											
9. HP_p^-											
10. HP_r^-											
11. Hb											
12. "A^-"*	1	1			−1	−2					
13. "B"†					1	1	1	2	2	3	4
14. "C^-"‡			1	1	2	2					
15. Charge in II											
6. Free-energy values (← Hb free-energy relations →)	0	$\ln A$	0	$\ln A - \ln \beta$	c_{38}	$-\ln K'$	0	$\ln 4\alpha$	$\ln 2$	$\ln 4\alpha^2$	$\ln \alpha^4$

and minimizing it, instead of writing down all the conditions that hold at equilibrium, the mathematical structure is easily analyzed and solved on computers even though it may involve many chemical species competing with each other for the limited number of atoms in the mixture (cf. [2]).

An example of nonzero fluxes. In developing a model of transport of substances into and out of the body tissues, we assume that there is a *steady flux* of substances across the membranes that separate the various compartments.

Tissue Level

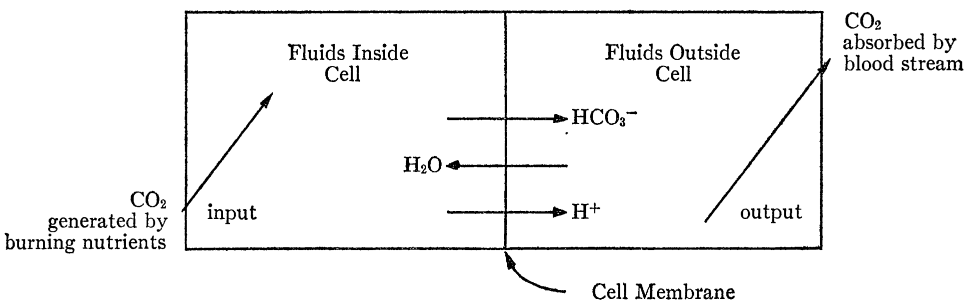


FIG. 3

The previous model no longer applies because there the net flux across membranes was assumed zero. One approach has been to *assume* each compartment to be in virtual equilibrium with respect to itself and to assume that concentration gradients appear only in the immediate neighborhood of the interface (like two lakes both nearly level except near the waterfall separating them).

Now the flow rates of dissolved O_2, say, depends on difference of their concentrations on one side and the other. For example, under certain assumptions based on Fick's Law:

$$\text{Flow } O_2 = k(\text{concentration dissolved oxygen outside} \\ -\text{concentration dissolved oxygen inside the cell}).$$

On the other hand, it can be shown mathematically that

$$\log(\text{concentration } O_2 \text{ inside}) = \text{Lagrange multipliers of the mass balance} \\ \text{oxygen-equation for inside the cell.}$$

This suggests relating the flow rates directly to Lagrange multipliers associated with equilibrium in each of the compartments.

This leads to the consideration of a new type of mathematical problem which is over-simplified here to indicate the approach. In the matrix equations below, the top set represents the mass balances in the inside of the cell (Compartment I) and the second, the mass balances outside the cell (Compartment II).

$$
\begin{array}{lcl|c}
 & & & \text{Lagrange Multipliers} \\
A_{\mathrm{I}}X_{\mathrm{I}} & & = b_{\mathrm{I}} \text{ (Inside)} & \lambda_{\mathrm{I}} \\
 & A_{\mathrm{II}}X_{\mathrm{II}} & = b_{\mathrm{II}} \text{ (Outside)} & \lambda_{\mathrm{II}} \\
\hline
F(X_{\mathrm{I}}) + F(X_{\mathrm{II}}) & & = \text{Min} &
\end{array}
$$

If we knew the vector of building blocks b_{I} or b_{II}, we could minimize $F(X_{\mathrm{I}})$ or $F(X_{\mathrm{II}})$ and find the equilibrium distributions X_{I} or X_{II}. We suppose, however, that we know only their sum $b_{\mathrm{I}}+b_{\mathrm{II}}=b$. We assume also that we know the fluxes and from them can deduce a relation at equilibrium between the Lagrange multipliers λ_{I} and λ_{II}. To simplify the discussion we assume that we can derive from the known fluxes that $\lambda_{\mathrm{I}}-\lambda_{\mathrm{II}}=\lambda$; i.e., λ is known. Then we can solve the above system by a simple device. What we do is add the first set of equations to the second as shown below. This induces a change in the Lagrange multipliers. In fact, those for the first set must be replaced by their difference λ (known) while the second remains unchanged. But mathematical theory tells us that in this case we can multiply the first set of equations by λ (transposed) and subtract the product from the function to be minimized, after that the first set can be ignored and the dotted system can be solved in its place.

$$
\begin{array}{lcl|l}
 & & & \text{Lagrange Multipliers} \\
A_{\mathrm{I}}X_{\mathrm{I}} & & = b_{I} & \lambda_{\mathrm{I}} - \lambda_{\mathrm{II}} = \lambda \\
\hdashline
A_{\mathrm{I}}X_{\mathrm{I}} & + A_{\mathrm{II}}X_{\mathrm{II}} & = b_{\mathrm{II}} + b_{\mathrm{I}} = b & \lambda_{\mathrm{II}} \\
\hline
F(X_{\mathrm{I}}) - \lambda^{T}A_{\mathrm{I}}X & + F(X_{\mathrm{II}}) & = \text{Min} & \\
\hdashline
\end{array}
$$

It is easy to see that the dotted system is mathematically of the same type as one obtained for a problem involving zero flux across the membrane.

Thus by this device a steady-state nonzero flux problem can be reduced to an ordinary zero flux case. Notice that in doing this we have applied our Lagrange multipliers only to part of the constraints and *not* to all of them. This is a modification of the usual Lagrange multiplier approach and has important applications in mathematical programming. In fact, it underlies the solution of nonlinear programming problems based on the *decomposition principle.*

Steady state models of this type have been coded up on computers by modifying the zero flux codes discussed earlier. They solve rapidly in about four minutes for very large scale systems. They have been used to investigate and predict known phenomena of transport across cell boundaries.

A mystery. There is, however, a complication. Living cells do not appear to believe in the laws of physics and appear to invent their own laws. One of the

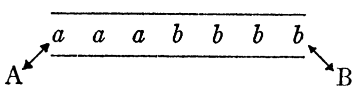
great mysteries* is this: The concentration of sodium ions within the cell is low relative to the outside; while for potassium ions the reverse is true. The relative amounts vary for different animals and the values selected below are illustrative only.

Inside Cell			Outside Cell		
Na^+	low	1	Na^+	high	9
K^+	high	19	K^+	low	1

There are various theories that explain this phenomenon of *active transport.* Some are complete with "trapdoors" which open and close at just the right moment so that Na^+ goes to one side and the K^+ to the other (Maxwell's Demon). The most popular explanation at present, however, is the so-called *carrier theory*, which is based on known phenomena.

Assuming the existence of a "carrier" substance (in the membrane that combines selectively with positive ions) we have been able to obtain some separation of Na^+ and K^+ and it might be possible that if we push this popular approach harder we will be able to explain the observed *quantitative* differences.

Recently, however, we have been excited by another type of mechanism, *statistical* in nature, for explaining the movement uphill of Na^+, say, against its concentration gradient [3].



We conjecture that there are passages interconnecting two compartments A and B each so narrow that n objects (molecules or ions) must line up in single

* One of the referees of this paper, a biologist, states: "The author's idea that these phenomena belong in the realm of black boxes is not valid. The phenomenon (which went unnamed) is called Donnan Equilibrium. The theory was worked out by Gibbs in 1876."

A second referee states: "I question whether it is justified to refer to this phenomenon as 'one of the great mysteries . . . .' As a biologist, I should object to his statement that 'Living cells do not believe in the laws of physics and appear to invent their own laws'."

Except for the adding of the words, "appear to" after "do not" above, I have left my remarks unchanged. I should say, however, a few words about what the model discussed up to now does include. It is, in fact, quite sophisticated: It includes Gibbs' theory, it includes volumetric effects as well as charge effects (Donnan Equilibrium), it includes effects of temperature, pressure, the laws of solubility, "salting in and out" phenomena, osmotic pressure, the passage of charged particles across a simple membrane with known distribution of potential, the exclusion of certain species from passage across the membrane and the varying of membrane permeability of other species. The author conducts a joint seminar with Nello Pace, the physiologist, in which this type of approach has been investigated to quantitatively predict phenomena of transport across cell walls. Theoretical considerations (as well as hundreds of trials on computers under a wide range of membrane characteristics) have convinced us that a more complicated theory of membrane structure is essential to explain quantitatively the phenomenon of active transport.

file in the passage. The objects originating in A will be denoted by a, those in B by b. In the figure $n=7$ and there are three objects from A and four from B. We will call the number of a objects in the passage the *state*. In the above case, the state is 3. The state of the passage can be changed, however, by the impact of an a object from A or by the impact of a b object from B. If impacts occur simultaneously at both ends, no change takes place. If not, we refer to the impact as an a impact-event if it is caused by an a object, and as a b impact-event in the other case. Only if the state is 0 can a b impact-event cause an object b to be driven out of the passage and become an object of A, or, if the state is n, can an a impact-event cause an object a to become an object of B. We are interested in the relative frequency of the events "an a object moves into the B compartment" and "a b object moves into the A compartment."

The matrix below gives the transition probabilities of going from state i to j after an event.

State Before an Event	State After an Event			
	0	1	2	3
0	q	p		
1	q	0	p	
2		q	0	p
3			q	p

Letting $\lambda=p/q=$ ratio of impacts and $p+q=1$, we can derive from the above two laws governing the flow rates:

1. THE DIFFERENCE IN FLOW RATES $(A\rightarrow B)-(B\rightarrow A)=$ CONSTANT (independent of n).

2. THE RATIO OF FLOW RATES IS PROPORTIONAL TO λ^{n+1}.

We now assume that A objects really are the totality of Na^+ and K^+ ions in the cell and that the B objects are the totality of these same ions outside the cell. In this case we have

	Inside	Outside
Na^+	1	9
K^+	19	1
Total	20 (in A)	10 (in B)

We next assume that the ratio of A impacts to B impacts is proportional to the total concentration of A objects to B objects or $\lambda = 20/10 = 2$. Suppose that the passage length is very small, say $n = 4$. Then

$$\text{Ratio of Total Flow Rates(IN/OUT)} = \lambda^5/1 = 2^5/1.$$

Notice that, because of the assumption of a narrow passage, if an A object happened to be a Na^+ (which will happen every 1/20 A impacts), then this Na^+ ion will be trapped in the passage if it is followed by another A object. Thus, out of every 2^5 of the A objects moving across 1/20 will be Na^+. Similarly, out of every k of the B objects moving in the reverse direction 9/10 will be Na^+. It follows that

$$\text{Ratio of } Na^+ \text{ Flow Rates IN/OUT} = (2^5 \times 1/20)/(1 \times 9/10) = 2^4/9 > 1.$$

Therefore

$$\text{Na Flow Rate (IN} \rightarrow \text{OUT)} > \text{Na Flow Rate (OUT} \rightarrow \text{IN)}.$$

Once we are able to explain any movement against the gradient it is not difficult to imagine another such mechanism for K^+ moving against its gradient in the reverse direction.

The narrow passage approach thus leads to a theory based on a Markov process to explain the mysterious natural life process of flows against the gradient. In its present state this is only a conjecture. It will be at least another year before experimental evidence can be analyzed to test this hypothesis.

Some concluding remarks. I have tried by way of examples to illustrate how mathematical, statistical and computer methods are currently being applied in the biological sciences. Those which I have selected, I believe, have some element of novelty in them, both from the view point of the mathematician and of the life scientist. The more classical methods based on simultaneous partial differential equations will, of course, remain the dominant tool; these will be coupled with the newer methods developed for solving large-scale inequality systems arising in the planning of large-scale enterprises.

Thus we see that mathematical, statistical, and computer methods are being applied with greater emphasis than ever in the biological sciences, and that a new field of "Bio-Mathematics" will undoubtedly emerge. The long term goals of such research will be the gradual development of mathematical models for the entire human system.

For the present, however, emphasis is placed on the representation of subsystems. By a model, as we have used it, is meant a representation of a current theory with its attendant hypotheses and assumptions. When such a theory fails to supply values of certain constants that represent, say, reaction rates or rates of physical flow, the model builders will undoubtedly insert "plausible" values and encourage laboratory experiments that will lead either to validation of the model or its replacement by another.

A partial list of mathematical sub-models that are under development is:

1. Control mechanisms of the central nervous and hormonal systems.
2. Neurological nets and learning processes.
3. The mechanisms of transport in the system as a whole.
4. The mechanisms of transport at the cell level.
5. Integration of metabolic processes at the cell level.
6. Chemical structures in the nucleus and other parts of the cell.
7. Stochastic theory of the dynamics of population genetics.
8. Cell multiplication, growth and aging processes.
9. System modification due to disease.

This paper is based on an invited address before the Mathematical Association of America in Berkeley, January 28, 1963. For further development of the discussion see the first two sections in [2].

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References

1. J. B. S. Haldane, Respiration, Yale University Press, New Haven, Conn., 1922.

2. G. B. Dantzig, J. C. DeHaven, I. Cooper, S. M. Johnson, E. C. DeLand, H. E. Kanter and C. F. Sams, A Mathematical Model of Human External Respiratory System, *Perspectives in Biology and Medicine*, Vol. IV, No. 3, Spring 1961.

3. G. B. Dantzig and Nello Pace, Molecular-Sized Channels and Flows Against the Gradient, Research Report 31, Operations Research Center, University of California, Berkeley, October 1962. Submitted for publication in the Proceedings of the National Academy of Sciences.

4. A. L. Hodgkin and R. D. Keynes, The Potassium Permeability of a Giant Nerve Fibre, J. Physiol., 128 (1955).

5. G. B. Dantzig and James C. DeHaven, On the Reduction of Certain Multiplicative Chemical Equilibrium Systems to Mathematically Equivalent Additive Systems, J. Chem. Phys., No. 10, 36 (1962) 2620–2627.

A PERIODIC OPTIMAL SEARCH

DAVID MATULA, University of California, Berkeley

1. Introduction. The search problem is a standard example in the application of dynamic programming methods. An object is in one of a finite set, I, of possible locations, with a priori probability p_i, ($\sum p_i = 1$, $i \in I$). Associated with each location i is a cost for searching that location c_i, and an overlook probability a_i, that if the object is in i and we search i, we do not find it. The problem is to find a program $\boldsymbol{\pi} = [\pi(1), \pi(2), \cdots]$, i.e., a sequence of locations to be searched such that the expected cost $V(\boldsymbol{\pi})$, of finding the object is minimal. Note that, in general, to be successful, each location must be searched infinitely often.

A program is called ultimately periodic if $\pi(j+\theta) = \pi(j)$ for all $j > T$, where T denotes the length of the transient phase and θ the length of the period. Our major result, Theorem 3, yields the conditions for the existence of an ultimately periodic optimal program and also determines the minimal period and the

minimal transient length. Whereas the general dynamic programming solution gives an optimal program recursively, our solution has the advantage of admitting a closed form expression and requires evaluation of only the first $T+\theta$ terms. A further virtue ensues from the observation that a periodic program yields for the expected cost a power series that is algebraically summable in closed form, which is, in general, not possible for arbitrary search procedures.

2. Optimal programs. To avoid trivial cases we will assume throughout this paper that we are dealing with a search problem over at least two locations satisfying the strict inequalities $0<a_i<1$, $0<c_i$, $0<p_i<1$, for $i\in I$.

For any program π, $M(i, N, \pi)$ is the number of searches of location i among the first N searches using program π. A program is called suitable if for $\pi(N)=j$,

$$a_j^{M(j,N-1,\pi)} b_j p_j/c_j = \max_{i\in I}\{a_i^{M(i,N-1,\pi)} b_i p_i/c_i\} \tag{1}$$

for all N. The following theorem of dynamic programming is stated without proof [1].

THEOREM 1. *A program is optimal if and only if it is suitable.*

Our first insight into the frequency of search of a particular location by an optimal program is

THEOREM 2. *Let π be an optimal program. Then*

$$\lim_{N\to\infty} M(i, N, \pi)/N = (1/\log a_i) \Big/ \sum_{j\in I} (1/\log a_j), \qquad i \in I.$$

Proof. Let

$$L_{ij}(N) = [a_i^{M(i,N,\pi)} b_i p_i/c_i]^{1/N}/[a_j^{M(j,N,\pi)} b_j p_j/c_j]^{1/N}. \tag{2}$$

We first shall show that the limit of this ratio is one.

For $i\neq j$ consider three cases.

(i) $\pi(N)=i$. Then $a_i^{M(i,N-1,\pi)}b_ip_i/c_i\geqq a_j^{M(j,N-1,\pi)}b_jp_j/c_j$, $L_{ij}(N-1)\geqq 1$, and $L_{ij}(N)=a_i^{1/N}[L_{ij}(N-1)]^{(N-1)/N}$.

(ii) $\pi(N)=j$. Then $a_i^{M(i,N-1,\pi)}b_ip_i/c_i\leqq a_j^{M(j,N-1,\pi)}b_jp_j/c_j$, $L_{ij}(N-1)\leqq 1$, and $L_{ij}(N)=(1/a_j)^{1/N}[L_{ij}(N-1)]^{(N-1)/N}$.

(iii) $\pi(N)\neq i$ or j. Then $L_{ij}(N)=[L_{ij}(N-1)]^{(N-1)/N}$.

Now we may choose N large enough so that both location i and j have been searched at least once. Then if $L_{ij}(N-1)>1$, $L_{ij}(N)$ is smaller but certainly no smaller than $a_i^{1/N}$; similarly if $L_{ij}(N-1)<1$ then $L_{ij}(N)$ is larger but no larger than $(1/a_j)^{1/N}$, and if $L_{ij}(N-1)=1$ then $L_{ij}(N)$ is one of the three values $(1/a_j)^{1/N}$, 1, $a_i^{1/N}$. In any case we have

$$(1/a_j)^{1/N} \geqq L_{ij}(N) \geqq a_i^{1/N}.$$

Since this is true for arbitrarily large N, $\lim_{N\to\infty} L_{ij}(N)=1$, or alternatively we have $\lim_{N\to\infty} \log L_{ij}(N)=0$.

Recalling (2), we see that

$$\lim_{N\to\infty} \left[(M(i, N, \pi)/N) \log a_i - (M(j, N, \pi)/N) \log a_j + (1/N) \log (p_i b_i c_j / p_j b_j c_i)\right] = 0,$$

$$\lim_{N\to\infty} \left[M(j, N, \pi)/N - (M(i, N, \pi)/N)(\log a_i/\log a_j)\right] = 0.$$

Summing this result over $j \in I$ yields

$$\lim_{N\to\infty} \left[1 - (M(i, N, \pi)/N) \sum_{j\in I} (\log a_i/\log a_j)\right] = 0,$$

$$\lim_{N\to\infty} M(i, N, \pi)/N = (1/\log a_i) \Big/ \sum_{j\in I} (1/\log a_j).$$

3. Periodic optimal programs. We wish to determine under what conditions a periodic optimal program exists, if indeed any exist at all. Assuming the existence, we may examine the properties of such a program, as in the

LEMMA. *If π is an ultimately periodic optimal program of transient length T and period $\theta = \sum_{i\in I} \sigma_i$, where σ_i is the number of searches of location i per period, then*

$$a_i^{\sigma_i} = a_j^{\sigma_j} \qquad \text{for } i, j \in I. \tag{3}$$

Proof. Let $\tau_i = M(i, T, \pi)$, so that $M(i, k\theta+T, \pi) = k\sigma_i + \tau_i$,

$$\lim_{K\to\infty} [M(i, K, \pi)/K] = \lim_{k\to\infty} [(k\sigma_i + \tau_i)/(T + k\theta)] = \sigma_i/\theta.$$

By Theorem 2, $\sigma_i/\theta = (1/\log a_i)/\sum_{j\in I} (1/\log a_j)$, or $\log a_i^{\sigma_i} = \theta/\sum_{j\in I} (1/\log a_j)$. Noting that the right hand side is independent of i, $a_i^{\sigma_i} = a_j^{\sigma_j}$ for $i, j \in I$.

This result is crucial and will motivate our remaining work. We may state this result in terms of the search problem parameters only as follows:

COROLLARY. *A necessary condition for the existence of an ultimately periodic optimal program is that the set of ratios $\{\log a_i/\log a_j\}$, $i, j \in I$, consist only of rational numbers.*

Having found that a periodic optimal program imposes necessary conditions on the overlook probabilities, we may ask if these conditions on the a_i alone are sufficient to insure the existence of an ultimately periodic optimal program. The answer is affirmative, and we now construct such a program. Furthermore we show that the method of construction used necessitates evaluation of a minimum number of terms of the program before the condition of periodicity can complete the solution.

THEOREM 3. *For the search problem where the ratios* $\{\log a_i/\log a_j\}$ *are rational numbers for* $i, j \in I$, *there exists a program* π^* *such that*

(a) π^* *is ultimately periodic of period* θ *and transient length* T, *where*

$$\theta = \min \left\{ \theta' \,\middle|\, \theta' \textit{ and } \theta' \middle/ \sum_{j \in I} (\log a_i/\log a_j) \textit{ are integers for } i \in I \right\} \tag{4}$$

$$T = \sum_{i \in I} \left[\min_{n=0,1,2,\cdots} \left\{ n \,\middle|\, a_i^n b_i p_i/c_i \leqq \min_{j \in I} \{ b_j p_j/(a_j c_j) \} \right\} \right], \tag{5}$$

(b) π^* *is optimal,*

(c) θ *is the minimal possible period,*

(d) T *is the minimal possible transient length.*

Proof. For any search problem there exists at least one suitable program π. For this π recall (1) and let

$$\phi(N) = \max_{i \in I} \{ a_i^{M(i,N-1,\pi)} b_i p_i/c_i \}, \qquad N = 1, 2, \cdots. \tag{6}$$

An inductive argument readily shows that $\phi(N)$ has the same value for any suitable program.

$\phi(N)$ is a monotonically nonincreasing function which approaches zero, hence we may define

$$Y = \min_{j \in I} \{ b_j p_j/(a_j c_j) \} \tag{7}$$

$$T = \min \{ N \mid \phi(N) \leqq Y \}. \tag{8}$$

Note that this definition of T agrees with (5).

Since the numbers $\{\log a_i/\log a_j\}$ are all rational, $1/\sum_{j \in I} (\log a_i/\log a_j)$ is rational for any $i \in I$, giving meaning to (4), and we define

$$\sigma_i = \theta \middle/ \sum_{j \in I} (\log a_i/\log a_j), \qquad i \in I \tag{9}$$

$$X = \exp \left\{ \theta \middle/ \sum_{j \in I} (1/\log a_j) \right\} \tag{10}$$

$$K = \min \{ K' \mid \phi(K') \leqq XY \}. \tag{11}$$

The following relations will be needed

$$\theta = \sum_{i \in I} \sigma_i, \tag{12}$$

$$X = a_i^{\sigma_i} \qquad \text{for all } i \in I. \tag{13}$$

Finally let

$$G = \{ i \mid b_i p_i/(a_i c_i) = Y \} \tag{14}$$

and list the elements of G, $i_0, i_1, i_2, \cdots, i_L$. Now we are in a position to define

π^* as follows:

$$\begin{aligned} \pi^*(j) &= \pi(j) && \text{for } 1 \leqq j \leqq K-1 \\ \pi^*(j+K) &= i_j && \text{for } 0 \leqq j \leqq L,\ i_j \in G \\ \pi^*(j+\theta) &= \pi^*(j) && \text{for } j > K+L-\theta \end{aligned}$$

(a) *π^* is ultimately periodic of period θ and transient length T.*

First we will show that $T = K+L+1-\theta$. Consider two cases for i.

Case 1: $i \in G$. We have from (6) and (11) $a_i^{M(i,K-1,\pi^*)} b_i p_i / c_i = XY$. Using (13), (14) and observing that $M(i, T-1, \pi^*) = 0$ for $i \in G$

$$\sigma_i = M(i, K-1, \pi^*) - M(i, T-1, \pi^*) + 1.$$

Case 2: $i \in I$, $i \notin G$. We have from (6) and (8)

$$a_i^{M(i,T-1,\pi^*)} b_i p_i / c_i \leqq T < a_i^{M(i,T-1,\pi^*)-1} b_i p_i / c_i. \tag{15}$$

Using (6), (11) similarly and then (13)

$$a_i^{M(i,K-1,\pi^*)-\sigma_i} b_i p_i / c_i \leqq T < a_i^{M(i,K-1,\pi^*)-\sigma_i-1} b_i p_i / c_i. \tag{16}$$

Combining the left side of (15) and the right side of (16) yields

$$a_i^{M(i,T-1,\pi^*)} < a_i^{M(i,K-1,\pi^*)-\sigma_i-1}$$

$$\sigma_i \geqq M(i, K-1, \pi^*) - M(i, T-1, \pi^*).$$

Similarly the right side of (15) and the left side of (16) give

$$\sigma_i \leqq M(i, K-1, \pi^*) - M(i, T-1, \pi^*),$$

so finally $\sigma_i = M(i, K-1, \pi^*) - M(i, T-1, \pi^*)$. Using the results of Cases 1 and 2 and noting that there are $L+1$ elements of G,

$$\theta = \sum_{i \in I} \sigma_i = K - T + L + 1$$

$$T = K + L + 1 - \theta.$$

This shows that π^* has transient length T.

Now we observe that σ_i is the number of searches of location i per period θ. Our choice of θ assures us that g.c.d. $\{\sigma_i\} = 1$. Therefore π^* is ultimately periodic of period θ.

(b) *π^* is optimal.*

It will be proved by induction on k ($k = 0, 1, 2, \cdots$), that π^* is suitable through stage $K+L+k$.

For $k=0$ it is clear from the definition of π^* and the fact that π is suitable that π^* is suitable through stage $K+L$.

Assume π^* is suitable through stage $K+L+k$, and that $\pi^*(K+L+k+1) = j$, hence also $\pi^*(K+L+k+1-\theta) = j$.

$$\begin{aligned} a_j^{M(j,K+L+k,\pi^*)} b_j p_j/c_j &= a_j^{\sigma_j+M(j,K+L+k-\theta,\pi^*)} b_j p_j/c_j \\ &= X a_j^{M(j,K+L+k-\theta,\pi^*)} b_j p_j/c_j = X \max_{i\in I} \{a_i^{M(i,K+L+k-\theta,\pi^*)} b_i p_i/c_i\} \\ &= \max_{i\in I} \{a_i^{\sigma_i+M(i,K+L+k-\theta,\pi^*)} b_i p_i/c_i\} = \max_{i\in I} \{a_i^{M(i,K+L+k,\pi^*)} b_i p_i/c_i\}. \end{aligned}$$

This shows that π^* is suitable through stage $K+L+k+1$, and by the induction axiom suitable at all stages. Applying Theorem 1, we see that π^* is optimal.

(c) *Minimality of period.*

Let π' be any ultimately periodic optimal program of period θ' with $\theta' = \sum_{i\in I} s_i$, where s_i is the number of searches of location i per period θ'. By the lemma $a_i^{s_i} = a_j^{s_j}$ for all $i, j\in I$. Now recall that $a_i^{\sigma_i} = a_j^{\sigma_j}$ for all $i, j\in I$. Letting $q = s_1/\sigma_1$, we have $s_i = q\sigma_i$, $i\in I$. Since g.c.d. $\{\sigma_i\} = 1$, q is an integer and $\theta' = q\theta$. Therefore any other possible period is a multiple of θ.

(d) *Minimality of transient length.*

Assume that π' is an ultimately periodic program of transient length T' and period θ' (where $\theta' = q\theta$ by part *c*) with $T' < T$. For some $i\in G$ choose

$$J_i = \max \{J \mid \pi'(J) = i, J \leqq T' + \theta'\}.$$

Since i occurs $q\sigma_i$ times per period, we must have

$$\begin{aligned} a_i^{q\sigma_i-1} b_i p_i/c_i \geqq \phi(J_i) \geqq \phi(T'+\theta') &= X^q \phi(T') > X^q Y = X^q p_i b_i/(a_i c_i) \\ &= a_i^{q\sigma_i-1} b_i p_i/c_i. \end{aligned}$$

But this is a contradiction. So we have shown that T is the minimal transient length, thus completing the proof of the theorem.

Combining the corollary and Theorem 3 we may state the

PERIODIC SEARCH THEOREM. *A necessary and sufficient condition for the existence of an ultimately periodic optimal program is that the ratios* $\{\log a_i/\log a_j\}$, $i, j\in I$, *all be rational.*

The following numerical example illustrates the transient and periodic behavior of an ultimately periodic optimal program.

$p_1 = .28$	$a_1 = .50$	$b_1 = .50$	$c_1 = 2$
$p_2 = .64$	$a_2 = .25$	$b_2 = .75$	$c_2 = 2$
$p_3 = .08$	$a_3 = .25$	$b_3 = .75$	$c_3 = 4$

$Y = \min\{b_j p_j/(a_j c_j)\} = .06$. From equations (4) and (5)

$$\theta = 4, \qquad T = 2.$$

This implies that it is sufficient to find the first six terms of the program. From equation (6)

$$\phi(1) = \max\{b_i p_i / c_i\} = .24.$$

The maximum occurs for $i=2$, therefore $\pi^*(1)=2$. Using ϕ recursively to determine π^*

N	1	2	3	4	5	6
$\phi(N)$	.24	.07	.06	.035	.0175	.015
$\pi^*(N)$	2	1	2	1	1	3

We choose $\pi^*(6)=3$ rather than $\pi^*(6)=2$ because $3 \in G$, (see (14) and the definition of π^*). We see that π^* has the transient sequence (2, 1) followed indefinitely by the sequence (2, 1, 1, 3).

$$\pi^* = [2, 1, 2, 1, 1, 3, 2, 1, 1, 3, \cdots].$$

4. Conclusion. It is an interesting observation from Theorem 2 that the limiting frequency of search of a location for any optimal program depends only on the overlook probabilities, not on the initial probability distribution or even, surprisingly, the relative costs. For the ultimately periodic optimal program of Theorem 3, we see how the initial probability distribution and costs affects the transient phase (5). The order of search within the period also is affected by all parameters, but the length of the period is determined only by the overlook probabilities (4).

In practical applications it is generally desirable to find the minimal expected cost as well as an optimal program. The use of ultimately periodic programs presents an efficient method for estimating this cost when an optimal program itself is not periodic.

We first evaluate the first T terms (5), of a program we shall call π'. Then we approximate the ratios $\{\log a_i / \log a_j\}$ by rational numbers and find a period θ (4). The next θ terms of the program are then evaluated. π' is completely defined by repeating that period indefinitely. The expected cost $V(\pi')$ may then be calculated in closed form and a bound on its difference from the minimum should be easily determined.

Obviously the ratios $\{\log a_i / \log a_j\}$ may be approximated arbitrarily closely by rational numbers, so the expected cost may be found to any accuracy desired. The practical success of this method depends on one's ability to pick good approximations that at the same time keep θ manageably small. With electronic computers to aid us, this method should handle all search problems.

I wish to thank Dr. David Blackwell for his helpful suggestions during the preparation of this paper. This work was supported by the Computer Center at the University of California, Berkeley, California.

Reference

1. D. Blackwell, Notes on Dynamic Programming (unpublished), Dept. of Stat., University of California, 1962.

SIMPLE EXTENSIONS OF TOPOLOGIES

NORMAN LEVINE, Ohio State University

1. Introduction. Let $(X, \mathfrak{T})$ be a topological space and $\mathfrak{T}\subset\mathfrak{T}^*$. Then $\mathfrak{T}^*$ will be termed a simple extension of $\mathfrak{T}$ iff there exists an $A \notin \mathfrak{T}$ such that $\mathfrak{T}^* \equiv \{O\cup(O'\cap A): O, O'\in\mathfrak{T}\}$. In this case we write $\mathfrak{T}^*=\mathfrak{T}(A)$.

In this note we attempt to answer this general question: If $(X, \mathfrak{T})$ has property Q, under what conditions will $(X, \mathfrak{T}(A))$ also have property Q, where Q is some topological property?

2. Preliminaries.

LEMMA 1. *Let $(X, \mathfrak{T})$ be a topological space and $\mathfrak{T}^*=\mathfrak{T}(A)$ a simple extension. If $B\subset X$, then* $\mathrm{Int}^*\, B=\mathrm{Int}\, B\cup\mathrm{Int}_A(B\cap A)$, *where* Int^*, Int *and* Int_A *denote the interior operators relative to $\mathfrak{T}^*$, $\mathfrak{T}$ and $\mathfrak{T}\cap A$ respectively.*

We omit the easy proof.

LEMMA 2. *Let $(X, \mathfrak{T})$ be a topological space and $\mathfrak{T}^*=\mathfrak{T}(A)$ a simple extension. If $B\subset X$ then $c^*B=cB\cap\{\mathcal{C}A\cup(A\cap c(B\cap A))\}$, where $\mathcal{C}$ denotes the complement operator and c^* and c denote the closure operators relative to $\mathfrak{T}^*$ and $\mathfrak{T}$ respectively.*

Proof. $c^*B = \mathcal{C}\ \mathrm{Int}^*\ \mathcal{C}B = \mathcal{C}\{\mathrm{Int}\ \mathcal{C}B \cup \mathrm{Int}_A\ \mathcal{C}B \cap A\}$ (by Lemma 1) $=\mathcal{C}\ \mathrm{Int}\ \mathcal{C}B\cap\mathcal{C}\ \mathrm{Int}_A\ \mathcal{C}_A(B\cap A)=cB\cap\{\mathcal{C}A\cup\mathcal{C}_A\ \mathrm{Int}_A\ \mathcal{C}_A(B\cap A)\}=cB\cap\{\mathcal{C}A\cup c_A(B\cap A)=cB\cap\{\mathcal{C}A\cup(A\cap c(B\cap A))\}$.

LEMMA 3. *Let $(X, \mathfrak{T})$ be a topological space and $\mathfrak{T}^*=\mathfrak{T}(A)$ a simple extension. Then $(A, \mathfrak{T}\cap A)=(A, \mathfrak{T}^*\cap A)$ and $(\mathcal{C}A, \mathfrak{T}\cap\mathcal{C}A)=(\mathcal{C}A, \mathfrak{T}^*\cap\mathcal{C}A)$.*

The reader may easily supply the proof.

LEMMA 4. *Let $(X, \mathfrak{T})$ be a topological space and $\mathfrak{T}^*=\mathfrak{T}(A)$ a simple extension. If $B\subset X$, then $c^*(B\cap A)=c(B\cap A)$.*

Proof. $c^*(B\cap A)=c(B\cap A)\cap\{\mathcal{C}A\cup(A\cap c(B\cap A))$ (by Lemma 2) $=c(B\cap A)$.

COROLLARY 1. *Let $(X, \mathfrak{T})$ be a topological space and $\mathfrak{T}^*=\mathfrak{T}(A)$ a simple extension. Then A is closed in $(X, \mathfrak{T})$ iff A is closed in $(X, \mathfrak{T}^*)$ (A is always open in $(X, \mathfrak{T}^*)$).*

Proof. By Lemma 4, $c^*A=cA$ and thus $A=c^*A$ iff $A=cA$.

LEMMA 5. *Let $(X, \mathfrak{T})$ be a topological space and $\mathfrak{T}^*=\mathfrak{T}(A)$ a simple extension. If F is closed in $(X, \mathfrak{T}^*)$ or $(X, \mathfrak{T})$, then $F\cap A$ is closed in $(A, \mathfrak{T}^*\cap A)=(A, \mathfrak{T}\cap A)$ and $F\cap\mathcal{C}A$, is closed in $(\mathcal{C}A, \mathfrak{T}^*\cap\mathcal{C}A)=(\mathcal{C}A, \mathfrak{T}\cap\mathcal{C}A)$.*

The proof follows from Lemma 3.

LEMMA 6. *Let $(X, \mathfrak{T})$ be a topological space and D a dense subset of X. If $O\in\mathfrak{T}$, then $cO=c(O\cap D)$.*

This follows from [1] page 57.

3. Some examples.

Example 1. Let X: a, b and $\mathfrak{I}$: $\varnothing$, X. Then $(X, \mathfrak{I})$ is regular and completely regular, but $(X, \mathfrak{I}(\{a\}))$ is neither (see Theorems 2, 3 and 4).

Example 2. Let X: a, b, c and $\mathfrak{I}$: $\varnothing$, $\{a\}$, $\{a, b\}$, X. Then $(X, \mathfrak{I})$ is normal, but $(X, \mathfrak{I}(\{a, c\}))$ is not (see Theorem 5).

Example 3. Let X: a, b, c and $\mathfrak{I}$: $\varnothing$, $\{a\}$, X. The simple extension $\mathfrak{I}(\{b\})$ of $\mathfrak{I}$ is not a minimal extension for $\mathfrak{I}\subset\mathfrak{I}(\{a, b\})\subset\mathfrak{I}(\{b\})$, the inclusions being proper.

Example 4. Let $(X, \mathfrak{I})$ be the closed unit interval with the usual topology. Then $(X, \mathfrak{I})$ is compact, but $(X, \mathfrak{I}(\{\frac{1}{2}\}))$ is not. If $(X, \mathfrak{I}(\{\frac{1}{2}\}))$ were compact, then since $\mathcal{C}\{\frac{1}{2}\}$ is closed in $(X, \mathfrak{I}(\{\frac{1}{2}\}))$, $(\mathcal{C}\{\frac{1}{2}\}, \mathfrak{I}(\{\frac{1}{2}\})\cap\mathcal{C}\{\frac{1}{2}\})$ would be compact and by Lemma 3, $(\mathcal{C}\{\frac{1}{2}\}, \mathfrak{I}\cap\mathcal{C}\{\frac{1}{2}\})$ would be compact, a contradiction (see Theorem 6).

4. Continuation.

THEOREM 1. *Let $(X, \mathfrak{I})$ be a topological space which is T_0, T_1 or T_2 and $A\notin\mathfrak{I}$. Then $(X, \mathfrak{I}(A))$ is T_0, T_1 or T_2.*

This follows from the fact that $\mathfrak{I}\subset\mathfrak{I}(A)$.

THEOREM 2. *Let $(X, \mathfrak{I})$ be regular and $A\notin\mathfrak{I}$, $\mathcal{C}A\in\mathfrak{I}$. Then $(X, \mathfrak{I}(A))$ is regular.*

Proof. Let $x\in O\cup(O'\cap A)\in\mathfrak{I}(A)$. *Case* 1: $x\in O$. There exists then a $U\in\mathfrak{I}\subset\mathfrak{I}(A)$ such that $x\in U\subset c^*U\subset cU\subset O\subset O\cup(O'\cap A)$ since $(X, \mathfrak{I})$ is regular. *Case* 2: $x\notin O$. Then $x\in O'$ and there exists a $U'\in\mathfrak{I}\subset\mathfrak{I}(A)$ such that $x\in U'\subset cU'\subset O'$. Then $x\in U'\cap A\subset c^*(U'\cap A)=c(U'\cap A)$ (by Lemma 4) $\subset cU'\cap cA\subset O'\cap A\subset O\cup(O'\cap A)$.

THEOREM 3. *Let $(X, \mathfrak{I})$ be a topological space and $A\notin\mathfrak{I}$. If A is dense in $(X, \mathfrak{I})$, then $(X, \mathfrak{I}(A))$ is not regular.*

Proof. Suppose $(X, \mathfrak{I}(A))$ is regular. Let $x\in A-\operatorname{Int} A$. Then $x\in A\in\mathfrak{I}(A)$ and there exist sets O, $O'\in\mathfrak{I}$ such that $x\in O\cup(O'\cap A)\subset c^*(O\cup(O'\cap A))=c^*(O)\cup c^*(O'\cap A)\subset A$. Now $x\notin O$ lest $x\in\operatorname{Int} A$, a contradiction. Hence $x\in O'\cap A\subset c^*(O'\cap A)\subset A$. By Lemma 4, $c^*(O'\cap A)=c(O'\cap A)$ and by Lemma 6, $c(O'\cap A)=cO'$. Thus $x\in O'\subset cO'\subset A$ which implies that $x\in\operatorname{Int} A$, a contradiction.

THEOREM 4. *Let $(X, \mathfrak{I})$ be completely regular and $A\notin\mathfrak{I}$, $\mathcal{C}A\in\mathfrak{I}$. Then $(X, \mathfrak{I}(A))$ is completely regular.*

Proof. Let $x\in O\cup(O'\cap A)$. *Case* 1: $x\in O$. Then there exists an f: $X\to[0, 1]$ continuous relative to $\mathfrak{I}$ (and hence to $\mathfrak{I}(A)$) such that $f(x)=0$ and $f[\mathcal{C}O]=1$. Then $f[\mathcal{C}\{O\cup(O'\cap A)\}]\subset f[\mathcal{C}O]=1$. *Case* 2: $x\notin O$. Then $x\in O'\cap A\in\mathfrak{I}\cap A$. But $(A, \mathfrak{I}\cap A)$ is completely regular (a subspace of a completely regular space is completely regular) and there exists then an f: $A\to[0, 1]$ continuous relative

to $\mathfrak{I}\cap A$ and hence $\mathfrak{I}(A)\cap A$ by Lemma 3 such that $f(x)=0$ and $f[\mathcal{C}_A(O'\cap A)]=1$. Let $f^*\colon X\to[0,1]$ as follows: $f^*(a)\equiv f(a)$ for all $a\in A$ and $f^*(y)\equiv 1$ for all $y\in \mathcal{C}A$. Now A is open and closed in $\mathfrak{I}(A)$ and it follows that $f^*\colon X\to[0,1]$ is continuous relative to $\mathfrak{I}(A)$. Clearly $f^*(x)=0$ and $f^*[\mathcal{C}\{O\cup(O'\cap A)\}]\subset f^*[\mathcal{C}(O'\cap A)]=f^*[\mathcal{C}A\cup\mathcal{C}_A(O'\cap A)]=f^*[\mathcal{C}A]\cup f[\mathcal{C}_A(O'\cap A)]=1$.

COROLLARY 2. *Let $(X,\mathfrak{I})$ be a Tychonoff space (completely regular and T_1). If $A\notin\mathfrak{I}$, $\mathcal{C}A\in\mathfrak{I}$, then $(X,\mathfrak{I}(A))$ is Tychonoff.*

The proof follows from Theorems 1 and 4.

THEOREM 5. *Let $(X,\mathfrak{I})$ be normal and $A\notin\mathfrak{I}$, $\mathcal{C}A\in\mathfrak{I}$. Then $(X,\mathfrak{I}(A))$ is normal iff $(\mathcal{C}A,\mathfrak{I}\cap\mathcal{C}A)$ is normal.*

Proof. Necessity. If $(X,\mathfrak{I}(A))$ is normal, then $(\mathcal{C}A,\mathfrak{I}(A)\cap\mathcal{C}A)$ is normal since $\mathcal{C}A$ is closed in $(X,\mathfrak{I}(A))$. But $(\mathcal{C}A,\mathfrak{I}\cap\mathcal{C}A)=(\mathcal{C}A,\mathfrak{I}(A)\cap\mathcal{C}A)$ by Lemma 3.

Sufficiency. Let F and G be closed and disjoint in $(X,\mathfrak{I}(A))$. Then $F\cap A$ and $G\cap A$ are closed in $(A,\mathfrak{I}(A)\cap A)$ and hence in $(A,\mathfrak{I}\cap A)$ by Lemma 3. Since A is closed in $(X,\mathfrak{I})$, $F\cap A$ and $G\cap A$ are closed in $(X,\mathfrak{I})$. But $(X,\mathfrak{I})$ is normal and thus there exist disjoint sets U and V in $\mathfrak{I}$ (and in $\mathfrak{I}(A)$) such that $F\cap A\subset U$ and $G\cap A\subset V$. Also $F\cap\mathcal{C}A$ and $G\cap\mathcal{C}A$ are disjoint and closed in $(\mathcal{C}A,\mathfrak{I}(A)\cap\mathcal{C}A)=(\mathcal{C}A,\mathfrak{I}\cap\mathcal{C}A)$ (by Lemma 3) which is presumed to be normal. There exist then disjoint sets U^* and V^* open in $(\mathcal{C}A,\mathfrak{I}\cap\mathcal{C}A)=(\mathcal{C}A,\mathfrak{I}(A)\cap\mathcal{C}A)$ and thus open in $(X,\mathfrak{I}(A))$ (since $\mathcal{C}A\in\mathfrak{I}(A)$) such that $F\cap\mathcal{C}A\subset U^*$ and $G\cap\mathcal{C}A\subset V^*$. Then $F=(F\cap A)\cup(F\cap\mathcal{C}A)\subset(U\cap A)\cup U^*\equiv U''$ and $G=(G\cap A)\cup(G\cap\mathcal{C}A)\subset(V\cap A)\cup V^*\equiv V''$. U'' and V'' are clearly disjoint and open in $\mathfrak{I}(A)$.

THEOREM 6. *Let $(X,\mathfrak{I})$ be countably compact (compact or Lindelöf) and $A\notin\mathfrak{I}$. Then $(X,\mathfrak{I}(A))$ is countably compact (compact or Lindelöf) iff $\mathcal{C}A$ is countably compact (compact or Lindelöf) in $(X,\mathfrak{I})$.*

Proof. We prove the theorem only for the countably compact case.

Necessity. Suppose $(X,\mathfrak{I}(A))$ is countably compact. Now $\mathcal{C}A$ is closed in $(X,\mathfrak{I}(A))$ and hence is countably compact in $(X,\mathfrak{I}(A))$. Then $\mathcal{C}A$ is countably compact in $(X,\mathfrak{I})$ since $\mathfrak{I}\subset\mathfrak{I}(A)$.

Sufficiency. Let $X=\bigcup_1^\infty(O_i\cup(O_i'\cap A))$. Then $\mathcal{C}A\subset\bigcup_1^\infty O_i$ and since $\mathcal{C}A$ is countably compact, $\mathcal{C}A\subset\bigcup_1^N O_i$. But $X=\bigcup_1^\infty O_i\cup O_i'$ and thus $X=\bigcup_1^M O_i\cup O_i'$. Then $A\subset\bigcup_1^M O_i\cup(O_i'\cap A)$ and it follows then that $X=A\cup\mathcal{C}A=\bigcup_1^{M+N}O_i\cup(O_i'\cap A)$.

THEOREM 7. *Let $(X,\mathfrak{I})$ be a second axiom space and $A\notin\mathfrak{I}$. Then $(X,\mathfrak{I}(A))$ is a second axiom space.*

Proof. Let $\{O_i\}$ be a countable open base for $\mathfrak{I}$. Then $\{O_i\cup(O_j\cap A)\}$ is a countable open base for $\mathfrak{I}(A)$. We may assume that O_1 is the empty set.

THEOREM 8. *Let* $(X, \mathfrak{J})$ *be separable and* $A \notin \mathfrak{J}$. *Then* $(X, \mathfrak{J}(A))$ *is separable iff* $(A, \mathfrak{J} \cap A)$ *is separable.*

Proof. Necessity. If $(X, \mathfrak{J}(A))$ is separable, there exists an $H \subset X$ denumerable such that $c^*H = X$. Now $A \in \mathfrak{J}(A)$ and thus $c^*(H \cap A) = c^*A$ by Lemma 6. Then $A \subset c^*A = c^*(H \cap A) = c(H \cap A)$ by Lemma 4, or $A = A \cap c(H \cap A)$. Since $A \cap H$ is countable, $(A, \mathfrak{J} \cap A)$ is separable.

Sufficiency. Let H be countable and dense in $(X, \mathfrak{J})$ and J countable and dense in $(A, \mathfrak{J} \cap A)$. Then $J \cup H$ is countable and dense in $(X, \mathfrak{J}(A))$. For let $O \cup (O' \cap A) \neq \emptyset$. *Case* 1. $O \neq \emptyset$. Then $O \cap H \neq \emptyset$ and $\{O \cup (O' \cap A)\} \cap \{H \cup J\} \neq \emptyset$. *Case* 2. $O = \emptyset$. Then $O' \cap A \neq \emptyset$ and $J \cap (O' \cap A) \neq \emptyset$. Thus $\{O \cup (O' \cap A)\} \cap \{H \cup J\} \neq \emptyset$.

THEOREM 9. *Let* $(X, \mathfrak{J})$ *be a topological space and* $A \notin \mathfrak{J}$, $(A, \mathfrak{J} \cap A)$ *connected and* A *dense in* $(X, \mathfrak{J})$. *Then* $(X, \mathfrak{J}(A))$ *is connected.*

Proof. Deny. Let $O \cup (O' \cap A)$ and $G \cup (G' \cap A)$ constitute a separation in $(X, \mathfrak{J}(A))$. Then $O \cup O' \neq \emptyset$ and $G \cup G' \neq \emptyset$ and since A is dense, $(O \cup O') \cap A \neq \emptyset$ and $(G \cup G') \cap A \neq \emptyset$. Now $X = (O \cup O') \cup (G \cup G')$ and hence $A = \{(O \cup O') \cap A\} \cup \{(G \cup G') \cap A\}$ contrary to $(A, \mathfrak{J} \cap A)$ being connected.

Example 5. Let $(X, \mathfrak{J})$ be the plane with the usual topology and $A = \{(x, y): x = 0$ or y rational or both$\}$. Then A is dense in $(X, \mathfrak{J})$, connected and not open. Thus $(X, \mathfrak{J}(A))$ is connected by Theorem 9 and the topology for the plane is not maximal relative to connectedness.

Reference

1. J. L. Kelley, General Topology, Van Nostrand, New York, 1955.

SPLITTING CONSECUTIVE INTEGERS INTO CLASSES WITH EQUAL POWER SUMS

J. B. ROBERTS, Reed College, and Birkbeck College, University of London

In 1851 E. Prouhet [2] observed that if $n = b^k$ then the first n nonnegative integers can be split into b equinumerous classes with equal tth power sums for all t, satisfying $0 \leqq t < k$. This was first proved, in a more general form, by D. H. Lehmer [1] in 1947. Later proofs and generalizations have been given in [3, 4, 5, 6, 7]. In particular, the following result is an immediate consequence of a theorem proved in [4].

If q *is a factorization of* n *whose factors have least common multiple* L_q *then the first* n *nonnegative integers can be split into* L_q *classes with equal t-th power sums for all* t *satisfying*

$$0 \leqq t < q^* - \max_{0 < s < L_q} \nu_s, \tag{1}$$

where q^* *is the number of factors in* q *and* ν_s *is the number of them that divide* s,

We call (1) the *t-range* of the factorization q. In the present paper we discuss the following three problems.

I. What is the maximum t-range possible for factorizations q?

II. Given q, how can one find splittings with t-range (1)?

III. Given q, how many splittings exist with t-range (1)?

We give the answer to I and give an algorithmic solution to II that yields a whole family of splittings. The answer to III is not known but we do give some results concerning the problem.

Included also is an application to the theory of linear congruences.

1. The maximum t-range. For convenience we denote the right term in (1) by T. Further, we put Q_n for the set of all factorizations of n and, as above, write q^* and L_q for the number and least common multiple, respectively, of the factors in q when $q \in Q_n$. A number L is said to be *permissible* for n if $L = L_q$ for some $q \in Q_n$. In this section we determine the maximum value of T taken over all $q \in Q_n$ and also the maximum value of T taken over just those q for which L_q is equal to an arbitrary fixed permissible L.

First of all we note that a necessary and sufficient condition for a number L ($\leqq n$) to be permissible for n is that L contain every prime factor contained in n. The necessity is obvious and we prove the sufficiency as follows. Let

$$n = p_1^{\gamma_1} \cdots p_u^{\gamma_u}, \ L = p_1^{\alpha_1} \cdots p_u^{\alpha_u}; \qquad 1 \leqq \alpha_i \leqq \gamma_i \text{ for } 1 \leqq i \leqq u. \tag{2}$$

Then we may write

$$\gamma_i = b_i \alpha_i + r_i; \qquad 0 \leqq r_i < \alpha_i, \tag{3}$$

and the factorization

$$q = \{\underbrace{p_1^{\alpha_1}, \cdots, p_1^{\alpha_1}}_{b_1}, p_1^{r_1}, \cdots, \underbrace{p_u^{\alpha_u}, \cdots, p_u^{\alpha_u}}_{b_u}, p_u^{r_u}\} \tag{4}$$

is in Q_n with $L_q = L$.

Suppose now that n and L are as in (2) and $q \in Q_n$ with $L_q = L$. (This q may, but need not be, the factorization in (4).) Let $\beta_j(q)$ designate the number of factors in q that are divisible by $p_j^{\alpha_j}$. Then $\nu_{L/p_j} = q^* - \beta_j(q)$ and, since

$$\max_{0<s<L} \nu_s = \max_{1 \leqq j \leqq u} \nu_{L/p_j},$$

we have

$$\max_{0<s<L} \nu_s = \max_{1 \leqq j \leqq u} (q^* - \beta_j(q)) = q^* - \min_{1 \leqq j \leqq u} \beta_j(q),$$

and therefore,

$$T = q^* - \max_{0<s<L} \nu_s = \min_{1 \leqq j \leqq u} \beta_j(q). \tag{5}$$

As already noted, the factorization q in (4) is one for which (5) is true. But for this q we know $\beta_j(q)=b_j$. On the other hand, if q' is any other factorization of n with $L_{q'}=L$ we cannot have $\beta_j(q')>b_j$ for any j, and therefore

$$\min_{1\leqq j\leqq u}\beta_j(q')\leqq\min_{1\leqq j\leqq u}b_j=\min_{1\leqq j\leqq u}\beta_j(q).$$

Noting from (3) that $b_j=[\gamma_j/\alpha_j]$ we have a complete proof of the following theorem.

THEOREM 1. *For n and L as in* (2),

$$\max_{q\in Q_n,L_q=L}\left(q^*-\max_{0<s<L}\nu_s\right)=\min_{1\leqq j\leqq u}[\gamma_j/\alpha_j]. \tag{6}$$

COROLLARY. *Let L be a permissible divisor of n and write $n=L^kh$, where $1\leqq h<L$. Then if $h=h_1\cdots h_r$, where each h_i divides L, the factorization $\{L,\cdots,L,h_1,\cdots,h_r\}$, where there are k factors of L, gives the maximum value of T for a factorization with $L_q=L$. This maximum value is $T=k$.*

It is now very easy to determine the maximum value of T over all permissible L. We need only maximize the right side of (6) over all u-tuples $(\alpha_1,\cdots,\alpha_u)$, where the α_i are subject to the restrictions set out in (2). The maximum value may be taken on for many different values of L, but, since $[\gamma_j/\alpha_j]\leqq\gamma_j$ always, one value of L yielding the maximum is $L=p_1\cdots p_u$. This gives

$$\max_{q\in Q_n}\left(q^*-\max_{0<s<L}\nu_s\right)=\min_{1\leqq j\leqq u}\gamma_j$$

and hence proves the next theorem.

THEOREM 2. *The maximum value of T is the smallest exponent occurring in the canonical prime factorization of n.*

As an example let us consider the case where $n=b^k$, $b=108$. Prouhet's theorem tells us of the existence of a splitting with 108 classes and t-range $0\leqq t<k$. Since $n=2^{2k}\cdot 3^{3k}$, Theorem 2 tells us of the existence of a splitting with t-range $0\leqq t<2k$. Making use of (5), one readily sees that if q consists of $2k$ factors of 6 and k factors of 3 then $T=2k$. Thus, since $L_q=6$, there is a splitting with 6 classes and t-range $0\leqq t<2k$.

In this example we have increased the t-range while decreasing the number of classes and thereby increasing the size of each class.

Generally speaking, for arbitrary n, each $q\in Q_n$ gives rise to a family of splittings with t-range given by (1). In the next section we give a constructive procedure for the determination of this family.

2. Construction of splittings. Each string S of n symbols of L varieties can be used to separate the first n nonnegative integers into L disjoint sets. Denoting

the $w+1$st term of S by $w(S)$ we need only put the numbers r and s, $0 \leqq r$, $s \leqq n-1$, into the same set if and only if $r(S)=s(S)$.

We shall take the integers $0, \cdots, L-1$ as our varieties of symbols.

Our aim in this section is to show, for each $q \in Q_n$, how to construct a family of strings S, consisting of integers taken from $0, \cdots, L_q-1$, such that for each string the separate classes constitute a splitting having t-range given by (1).

The construction will be based on a theorem, proved in [4], which we quote below. Before this we introduce some terminology.

Let $q=\{n_1, \cdots, n_m\} \in Q_n$ and for each $j=1, 2, \cdots, m$ let h_{j-1} be a mapping, of period n_j, taking all integers *onto* the integers $0, 1, \cdots, n_j-1$. We shall call h_{j-1} a mod n_j *projection map*, and, if it happens to be the identity when restricted to $0, 1, \cdots, n_j-1$ we shall call it the *canonical* mod n_j projection map.

Now, for a fixed s, $0<s<L$, define the functions $f_1, \cdots, f_m$ by:

$$f_j(n) = \exp\left(2\pi s i h_{j-1}(n)/n_j\right), \qquad j = 1, \cdots, m.$$

It is clear that f_j has period n_j and that

$$\begin{aligned}\sum_{n=0}^{n_j-1} f_j(n) &= \sum_{n=0}^{n_j-1} (\exp(2\pi s i/n_j))^{h_{j-1}(n)} \\ &= \sum_{n=0}^{n_j-1} (\exp(2\pi s i/n_j))^{n}.\end{aligned}$$

This sum is zero if n_j does not divide s and is not zero otherwise. We write

$$\sum_{n=0}^{n_j-1} f_j(n) n^t = 0 \qquad \text{for } 0 \leqq t \leqq \alpha_j,$$

where $\alpha_j=-1$ or 0 depending on whether n_j divides s or not. (I.e., when n_j divides s we symbolize that the sum is not zero by writing the impossible side condition $0 \leqq t \leqq -1$.)

We are now ready to apply the following theorem from [4].

THEOREM. *If $n_1, \cdots, n_m$ are integers $\geqq 2$; $\alpha_1, \cdots, \alpha_m$ are integers $\geqq -1$; $f_1, \cdots, f_m$ are functions such that f_j is periodic of period n_j and $\sum_{n=0}^{n_j-1} f_j(n) n^t = 0$ for $0 \leqq t \leqq \alpha_j$; $p_0=1$, $p_j=n_1 \cdots n_j$ for $1 \leqq j \leqq m$, then*

$$\sum_{n=0}^{p_m-1} f_1(n) f_2([n/p_1]) \cdots f_m([n/p_{m-1}]) n^t = 0 \quad \text{for } 0 \leqq t < \alpha_1 + \cdots + \alpha_m + m.$$

The f_j defined above satisfy the hypotheses of this theorem and therefore we find

$$\sum_{n=0}^{p_m-1} \exp\left(2\pi s i (h_0(n)/n_1 + \cdots + h_{m-1}([n/p_{m-1}])/n_m)\right) n^t = 0$$

for $0 \leqq t < m - \nu_s$, where ν_s is the number of $n_1, \cdots, n_m$ that divide s.

We now put $L=L_q$ and define C_r, $0\leqq r<L$, to be the set of all n, $0\leqq n<p_m$, satisfying

$$Lh_0(n)/n_1+\cdots+Lh_{m-1}([n/p_{m-1}])/n_m\equiv r \pmod{L}. \tag{7}$$

With this notation the above equation becomes

$$\sum_{r=0}^{L-1}\left(\sum_{n\in C_r} n^t\right)(\exp(2\pi si/L))^r=0 \tag{8}$$

for $0\leqq t<m-\nu_s$. Since this is true for all s, $0<s<L$, the coefficients must all be equal and we have proved the next theorem.

THEOREM 3. *Let $0\leqq r<u<L$ and $\bar{\nu}=\max_{0<s<L}\nu_s$. Then*

$$\sum_{n\in C_r} n^t=\sum_{n\in C_u} n^t \qquad \textit{for } 0\leqq t<m-\bar{\nu}.$$

Observe now that the maps $h_0', \cdots, h_{m-1}'$ defined by $h_j'(v)=h_j(v)-h_j(0)$ are modular projection maps (with the same respective moduli as the h_j) and in fact give rise to the same classes C_r arising from the h_j (perhaps rearranged). Further, each h_j' maps 0 onto 0. Therefore we confine ourselves, in all that follows, without loss of generality, to using modular projection maps that take 0 onto 0.

Now Theorem 3 shows that for each $q\in Q_n$ and choice of modular projection maps $h_0, \cdots, h_{q^*-1}$ there is a splitting of the first n nonnegative integers into L_q classes with equal tth power sums for all t satisfying $0\leqq t<T$.

We are now ready to construct the strings promised at the beginning of this section. We suppose $q=\{n_1, \cdots, n_m\}\in Q_n$, $L_q=L$ and that $h_0, \cdots, h_{m-1}$ have been chosen. We put $H_j=Lh_j/n_{j+1}$ for $j=0, 1, \cdots, m-1$ and define S_m by:

$$\begin{cases} S_0=0 \\ S_{j+1}=S_j^{H_j(0)}\cdots S_j^{H_j(n_{j+1})}, \qquad j=0,\cdots,m-1, \end{cases} \tag{9}$$

where S_j^a denotes the string obtained from S_j by adding a modulo L, to each term of S_j, and $S_j^aS_j^b$ denotes the string obtained by juxtaposing the strings S_j^a and S_j^b.

We prove next that for each w, $0\leqq w<p_m$, the number $w(S_m)$ is congruent, modulo L, to the left side of (7) after the n there has been replaced by w.

THEOREM 4. *$w(S_m)=r$ if and only if $w\in C_r$.*

Indeed, let $w=a_0+a_1p_1+\cdots+a_{m-1}p_{m-1}$, where $0\leqq a_j<n_{j+1}$. Now put $w_1=w-a_{m-1}p_{m-1}$, $w_2=w_1-a_{m-2}p_{m-2}$, $\cdots$, $w_{m-1}=a_0$. Then

$$\begin{aligned} w(S_m) &\equiv w_1(S_{m-1})+H_{m-1}(a_{m-1}) \\ &\equiv w_2(S_{m-2})+H_{m-2}(a_{m-2})+H_{m-1}(a_{m-1}) \\ &\quad\cdots \\ &\equiv H_0(a_0)+\cdots+H_{m-1}(a_{m-1}) \\ &\equiv Lh_0(a_0)/n_1+\cdots+Lh_{m-1}(a_{m-1})/n_m \pmod{L}. \end{aligned}$$

Eliminating the a_i we have

$$(10)\qquad \begin{aligned} w(S_m) &\equiv Lh_0(w)/n_1 + \cdots + Lh_{m-1}([w/p_{m-1}])/n_m \\ &\equiv \sum_{i=1}^{m} Lh_{i-1}([w/n_1 \cdots n_{i-1}])/n_i \pmod{L}. \end{aligned}$$

We call S_m a *splitting sequence* (for q and the h_i, or alternatively of the classes C_r obtained in this way).

To illustrate the use of (9) in constructing splitting sequences let us consider $n=36$, $q=\{3, 3, 2, 2\}$ and compute the classes C_r, $0 \leqq r < 6$, relative to the maps h_0, h_1, h_2, h_3, where $h_0(1)=2$, $h_0(2)=1$ and the others are canonical. Then we have the table:

a	$H_0(a)$	$H_1(a)$	$H_2(a)$	$H_3(a)$
0	0	0	0	0
1	4	2	3	3
2	2	4		

From (9) we find:

$S_0 = 0$

$S_1 = 042$

$S_2 = 042\ 204\ 420$

$S_3 = 042\ 204\ 420\ 315\ 531\ 153$

$S_4 = 042\ 204\ 420\ 315\ 531\ 153\ 315\ 531\ 153\ 042\ 204\ 420.$

This splitting sequence S_4 yields:

$C_0 = \{0, 4, 8, 27, 31, 35\}$ $\qquad C_3 = \{9, 13, 17, 18, 22, 26\}$

$C_1 = \{10, 14, 15, 19, 23, 24\}$ $\qquad C_4 = \{1, 5, 6, 28, 32, 33\}$

$C_2 = \{2, 3, 7, 29, 30, 34\}$ $\qquad C_5 = \{11, 12, 16, 20, 21, 25\}.$

3. The number of splittings. In [4] we stated (but omitted the proof) that the number of distinct splitting sequences for n of the form b^k, where $L=b$, that had t-range $0 \leqq t < k$, was not less than $((b-1)!)^{k-1}$. This number of splitting sequences may be obtained by suitable choices of the mod b projection maps $h_0, \cdots, h_{k-1}$. All these splittings are associated with the factorization of b^k that has k factors of b. Call this factorization q_1. As we shall see later, other factorizations may lead to different splitting sequences with the same t-range and this proves that in general the above number of splitting sequences is too small. First, however, we give a very simple proof that the stated number is a lower bound.

We begin with the observation that if $h_0, \cdots, h_{k-1}$ and $h_0', \cdots, h_{k-1}'$ are two sequences of mod b projection maps associated with q_1 and if for some s and

t we have $h_s = h_s'$ and $h_t \neq h_t'$ then the splitting sequences S_k and S_k' corresponding to the maps are distinct. Indeed, choose a and a' such that

$$h_t(a) \neq h_t'(a), \qquad h_s(a') = h_t(a),$$

and put $n_1 = ab^t$, $n_2 = a'b^s$. Then, with $r = h_t(a)$,

$$n_1(S_k) = h_t(a) = r \neq h_t'(a) = n_1(S_k'),$$
$$n_2(S_k) = h_s(a') = h_s'(a') = r = n_2(S_k'),$$

so that $n_1 \in C_r - C_r'$ while $n_2 \in C_r \cap C_r'$. Thus C_r, C_r' are neither disjoint nor equal and this means S_k and S_k' are distinct. If one now fixes h_0 one sees that the number of distinct ways of choosing $h_1, \cdots, h_{k-1}$ is just $((b-1)!)^{k-1}$ and our assertion is proved.

Returning to the more general situation where n need not be of the form b^k we give next a general theorem concerned with the equivalence of splitting sequences obtained from distinct factorizations of a certain type.

THEOREM 5. *Let $q = \{a_1, \cdots, a_m\}$ be an ordered factorization of n and put $i_0 = 0$, $i_k = m$ and define*

$$b_j = a_{i_{j-1}+1} \cdots a_{i_j} \qquad \text{for } 1 \leqq j \leqq k, \qquad i_{j-1} < i_j. \tag{11}$$

Then $b_1 \cdots b_k = n$ and therefore $q_1 = \{b_1, \cdots, b_k\}$ is also an ordered factorization of n. Now, the following three propositions are equivalent.

I. *There exists a splitting sequence of q that is a splitting sequence of q_1.*

II. *Every splitting sequence of q is a splitting sequence of q_1.*

III. *For each j, the factors on the right of* (11) *are relatively prime in pairs.*

Before proving this theorem we shall illustrate it by an example. Suppose $n = b^k$, $b = p_1^{\alpha_1} \cdots p_u^{\alpha_u}$, where the p_i are distinct prime numbers. Then, by the results in Sections 1 and 2 every ordering of the factorization

$$q = \{p_1^{\alpha_1}, \cdots, p_1^{\alpha_1}, \cdots, p_u^{\alpha_u}, \cdots, p_u^{\alpha_u}\},$$

where each $p_j^{\alpha_j}$ occurs k times, gives $T = k$ for all choices of modular projection maps. This is the same t-range afforded by the factorization $q_1 = \{b, \cdots, b\}$, where there are k factors of b.

By Theorem 5 only those orderings of q which cannot be separated into blocks of length u, each block being a permutation of $p_1^{\alpha_1} \cdots, p_u^{\alpha_u}$, can lead to splitting sequences not contained among the splitting sequences of q_1.

For $n = 6^2$ we have $b = 6$, $k = 2$, $p_1 = 2$, $p_2 = 3$ and $q = \{2, 2, 3, 3\}$. The possible orderings of q are:

2, 2, 3, 3   2, 3, 2, 3   3, 2, 2, 3   2, 3, 3, 2   3, 2, 3, 2   3, 3, 2, 2.

By Theorem 5 only the first and last of these can lead to splitting sequences not contained among the splitting sequences arising from the factorization $\{6, 6\}$. We analyse these two separately.

A. For $\{2, 2, 3, 3\}$ the maps h_0, h_1 must be canonical but h_2 and h_3 may be either of the two maps:

$$\text{(a)}\quad h(0) = 0,\ h(1) = 1,\ h(2) = 2,$$

$$\text{(b)}\quad h(0) = 0,\ h(1) = 2,\ h(2) = 1.$$

Choosing them both (a) gives a splitting sequence equivalent to that obtained when both are taken (b). Similarly, choosing h_3 to be (a) and h_4 to be (b) gives an equivalent splitting sequence to that obtained when these are reversed. When they are both (a) the splitting sequence is different from that obtained when one is (a) and the other (b). The two new splitting sequences are:

024240 402351 513315 351513 135024 240402

042240 420315 531153 315531 153042 204420.

B. A similar analysis applied to $\{3, 3, 2, 2\}$ yields the new splitting sequences:

033025 524114 255241 140330 411403 302552

033025 524114 411403 302552 255241 140330.

By Theorem 5 all four of these are distinct from the 120 splitting sequences obtainable from the factorization $\{6, 6\}$.

4. Proof of Theorem 5. In the proof we shall make use of the following:

LEMMA 1. *For each* $j=0, 1, \cdots, m-1$ *let* h_j *be a* mod a_{j+1} *projection map and let* S_m *be the splitting sequence associated with these maps and the ordered factorization* $q=\{a_1, \cdots, a_m\}$ *of* n. *Further, suppose*

$$0 \leqq v < a_1 \cdots a_s, \quad 0 \leqq f < a_{s+t}, \quad \text{where} \quad s \geqq 1 \quad \text{and} \quad t \geqq 1.$$

Then if

$$w = v + fa_1 \cdots a_{s+t-1}$$

we must have (*putting* L *for* L_q)

$$w(S_m) \equiv v(S_m) + Lh_{s+t-1}(f)/a_{s+t} \pmod{L}.$$

Indeed, since

$$w < a_1 \cdots a_s + a_1 \cdots a_{s+t} - a_1 \cdots a_{s+t-1} < a_1 \cdots a_{s+t},$$

we have

$$[w/a_1 \cdots a_{i-1}] = \begin{cases} [v/a_1 \cdots a_{i-1}] + fa_s \cdots a_{s+t-1} & \text{for } 1 \leqq i < s+t, \\ f & \text{for } i = s+t, \\ 0 & \text{for } s+t < i. \end{cases}$$

Therefore

$$\begin{aligned} w(S_m) &\equiv L\sum_{i=1}^{m} h_{i-1}([w/a_1\cdots a_{i-1}])/a_i \\ &\equiv L\sum_{i=1}^{s+t-1} h_{i-1}([v/a_1\cdots a_{i-1}])/a_i + Lh_{s+t-1}(f)/a_{s+t} \\ &\equiv L\sum_{i=1}^{m} h_{i-1}([v/a_1\cdots a_{i-1}])/a_i + Lh_{s+t-1}(f)/a_{s+t} \\ &\equiv v(S_m) + Lh_{s+t-1}(f)/a_{s+t} \pmod{L}. \end{aligned}$$

We now proceed to the main proof. Since II implies I in an obvious fashion all will be proved if we succeed in showing that III implies II and that not-III implies not-I.

(i) III *implies* II. Let S_m be the splitting sequence of q with modular projection maps $h_0, \cdots, h_{m-1}$ and let $L=L_q$. Now, for $0\leqq j<k$, define

$$h_j'(m) = \sum_{i=i_j+1}^{i_{j+1}} b_{j+1}h_{i-1}([m/a_{i_j+1}\cdots a_{i-1}])/a_i.$$

We show that h_j' is a mod b_{j+1} projection map and that the splitting sequence S_k' associated with q_1 and these maps is equivalent to S_m. This will prove (i).

Since each $h_j(0)=0$ so also does $h_j'(0)=0$ for each j. Also, since h_j is periodic with period a_{j+1}, we see that $h_j'(m+b_{j+1})=h_j'(m)$. Finally, suppose $0\leqq s_1\leqq s_2 \leqq b_{j+1}-1$ and $h_j'(s_1)=h_j'(s_2)$. Then

$$\sum_{i=i_j+1}^{i_{j+1}} b_{j+1}\{h_{i-1}([s_1/a_{i_j+1}\cdots a_{i-1}]) - h_{i-1}([s_2/a_{i_j+1}\cdots a_{i-1}])\}/a_i$$

is congruent to zero modulo b_{j+1}. This implies

$$[s_1/a_{i_j+1}\cdots a_{i-1}] \equiv [s_2/a_{i_j+1}\cdots a_{i-1}] \pmod{a_i}$$

for each $i=i_j+1, \cdots, i_{j+1}$, which in turn implies $s_1=s_2$. This completes the proof that h_j' is a mod b_{j+1} projection map.

Since the a_i appearing in b_j are relatively prime in pairs we know $L=L_q$. Therefore

$$\begin{aligned} w(S_k') &= \sum_{j=0}^{k-1} Lh_j'([w/b_1\cdots b_j])/b_{j+1} \\ &= \sum_{j=0}^{k-1} L\sum_{i=i_j+1}^{i_{j+1}} b_{j+1}h_{i-1}([w/a_1\cdots a_{i-1}])/a_ib_{j+1} \\ &= \sum_{i=1}^{m} Lh_{i-1}([w/a_1\cdots a_{i-1}])/a_i = w(S_m), \end{aligned}$$

and (i) is proved.

(ii) *not*-III *implies not*-I. Let $h_0, \cdots, h_{m-1}$ be any sequence of modular projection maps associated with q and $h_0', \cdots, h_{k-1}'$ be such a sequence associated with q_1. We shall show that the corresponding splitting sequences are not equivalent when III is violated. If $L_q \neq L_{q_1}$ there is nothing to prove. Therefore we suppose $L = L_q = L_{q_1}$.

Let b_j be the first of $b_1, \cdots, b_k$ whose factors, according to (11), are not relatively prime in pairs and suppose $a_{i_{j-1}+t}$ and $a_{i_{j-1}+r}$, $1 \leqq t < r$, are two factors of b_j with greatest common divisor $d > 1$. Choose f_1 and f_2 such that

$$0 < f_1 < a_{i_{j-1}+t}, \qquad h_{i_{j-1}+t-1}(f_1) = a_{i_{j-1}+t}/d,$$
$$0 < f_2 < a_{i_{j-1}+r}, \qquad h_{i_{j-1}+r-1}(f_2) = a_{i_{j-1}+r}/d.$$

Now choose v such that $0 \leqq v < a_1 \cdots a_{i_{j-1}+t-1}$ and put

$$w_1 = v + f_1 a_1 \cdots a_{i_{j-1}+t-1}, \qquad w_2 = v + f_2 a_1 \cdots a_{i_{j-1}+r-1}.$$

Then, using Lemma 1 twice, we find

$$\begin{aligned} w_1(S_m) &\equiv v(S_m) + L h_{i_{j-1}+t-1}(f_1)/a_{i_{j-1}+t} \equiv v(S_m) + L/d \\ &\equiv v(S_m) + L h_{i_{j-1}+r-1}(f_2)/a_{i_{j-1}+r} \equiv w_2(S_m) \pmod{L}. \end{aligned}$$

On the other hand

$$\begin{aligned} w_1(S_k') &\equiv \sum_{i=1}^{k} L h_{i-1}'([(v + f_1 a_1 \cdots a_{i_{j-1}+t-1})/b_1 \cdots b_{i-1}])/b_i \\ &\equiv \sum_{i=1}^{j} L h_{i-1}'([v/b_1 \cdots b_{i-1}] + f_1 a_1 \cdots a_{i_{j-1}+t-1}/b_1 \cdots b_{i-1})/b_i \\ &\equiv \sum_{i=1}^{j-1} L h_{i-1}'([v/b_1 \cdots b_{i-1}]) \\ &\quad + L h_{j-1}'([v/b_1 \cdots b_{j-1}] + f_1 a_1 \cdots a_{i_{j-1}+t-1}/b_1 \cdots b_{j-1})/b_j \end{aligned}$$

modulo L. If we change f_1 to f_2 and t to r on the right side of this expression we get a similar expression for $w_2(S_k')$. Using these expressions we see that if $w_1(S_k') - w_2(S_k')$ were divisible by L then b_j would divide the left, and therefore the right, side of the equation

$$\begin{aligned} &(f_1 a_1 \cdots a_{i_{j-1}+t-1} - f_2 a_1 \cdots a_{i_{j-1}+r-1})/b_1 \cdots b_{j-1} \\ &\qquad = a_{i_{j-1}+1} \cdots a_{i_{j-1}+t-1}(f_1 - f_2 a_{i_{j-1}+t} \cdots a_{i_{j-1}+r-1}). \end{aligned}$$

But this right side is not zero and is, in absolute value, smaller than b_j. Therefore

$$w_1(S_k') \not\equiv w_2(S_k') \pmod{L}.$$

This, along with $w_1(S_m) \equiv w_2(S_m) \pmod{L}$, completes the proof of (ii) and therefore of the theorem.

This theorem merely touches upon the considerably wider problem of giving

a complete characterization of the equivalence classes of splitting sequences associated with factorizations $q \in Q_n$ with given L_q.

5. A small generalization of Theorem 3. In proving Theorem 3 in section 2 we made use of the fact that if a polynomial of Lth degree vanishes at each of the Lth roots of unity other than 1 then all coefficients of the polynomial are equal. There is a very simple generalization of this fact that we set out in a lemma.

LEMMA 2. *Let $P(x) = \sum_{n=0}^{ab-1} c_n x^n$. Then the following two statements are equivalent.*

(i) *$c_r = c_s$ for every r and s such that $r \equiv s$* (mod a);

(ii) *$\exp(2\pi si/ab)$ is a zero of P for all s satisfying $0 \leqq s \leqq ab-1$ and $s \not\equiv 0$* (mod b).

According to an argument suggested by Prof. E. M. Wright, (ii) is equivalent to the statement that $P(x)$ has the factor

$$(1 - x^{ab})/(1 - x^a) = \sum_{j=0}^{b-1} x^{ja}$$

and this is clearly equivalent to (i).

Using this lemma and (8) we are able to state the following generalization of Theorem 3.

THEOREM 3'. *Suppose a is a divisor of L and $0 \leqq r < u < L$ with $r \equiv u$* (mod a). *Further, let $\bar{\nu}'$ be the maximum number of $n_1, \cdots, n_m$ that divide an s satisfying $0 < s < L$ and $s \not\equiv 0$* (mod L/a). *Then*

$$\sum_{n \in C_r} n^t = \sum_{n \in C_u} n^t \qquad \text{for } 0 \leqq t < m - \bar{\nu}'.$$

When $a = 1$ this theorem reduces to Theorem 3. The theorem is of some interest since $\bar{\nu}'$ is often smaller than $\bar{\nu}$. This increases the range of values for t. Of course we pay the price in that we are no longer splitting *all* non-negative integers from 0 to $n-1$ into classes with the same set of power sums.

One can, through further refinements of Lemma 2, get even more information from (8). However the point of diminishing returns is soon reached and we have not yet discovered a proper further generalization.

6. An application. Consider the congruence system

$$\begin{aligned} &Lx_1/n_1 + \cdots + Lx_m/n_m \equiv r \pmod{L} \\ &0 \leqq x_1 < n_1, \cdots, 0 \leqq x_m < n_m, \end{aligned} \tag{12}$$

where L is the least common multiple of the n_i.

If $X = (x_1, \cdots, x_m)$ is a solution of (12) define X' by:

$$X' = x_1 p_0 + x_2 p_1 + x_3 p_2 + \cdots + x_m p_{m-1}, \tag{13}$$

where the p_i are as in section 2. Then $0 \leqq X' < p_m$ and $X' \in C_r$ where C_r is com-

puted from the factorization $\{n_1, \cdots, n_m\}$ of $n_1 \cdots n_m$ using the canonical maps. Conversely if $X' \in C_r$ (again using the canonical maps) the corresponding X is a solution of (12).

If we write X^t for the t-th power of X' and call it the t-th power of the vector X then we can (in the case where canonical maps are used) rephrase Theorem 3′ (and hence Theorem 3) as follows.

THEOREM 6. *Suppose a is a divisor of L and $\bar{\nu}'$ is the maximum number of $n_1, \cdots, n_m$ that divide an s lying strictly between* 0 *and L and not divisible by L/a. Then the sum of the t-th powers of the solutions of* (12) *is dependent only upon the residue class of r modulo a for all values of t satisfying* $0 \leqq t < m - \bar{\nu}'$.

In the particular case $a = 1$ and $t = 0$ the conclusion tells us that the number of solutions of (12) is independent of r; in fact there are $n_1 \cdots n_m/L$ solutions.

If the greatest common divisor of the n_i is unity then L is also the least common multiple of $L/n_1, \cdots, L/n_m$. Thus, in this case, the same analysis may be applied to what we call the *complementary system* to (12):

$$\begin{aligned} &n_1x_1 + \cdots + n_mx_m \equiv s \pmod{L} \\ &0 \leqq x_1 < L/n_1, \cdots, 0 \leqq x_m < L/n_m. \end{aligned} \tag{14}$$

Returning to (12) and letting S_m be the splitting sequence of the factorization $\{n_1, \cdots, n_m\}$ relative to the canonical maps we have, as already observed,

$$X \text{ is a solution of (12) if and only if } X'(S_m) = r.$$

This enables us to give an algorithmic solution of (12) (and also of (14) when the n_i have greatest common divisor unity). As an example consider the system

$$\begin{aligned} &6x_1 + 10x_2 + 15x_3 \equiv 1 \pmod{30} \\ &0 \leqq x_1 < 5, \quad 0 \leqq x_2 < 3, \quad 0 \leqq x_3 < 2. \end{aligned}$$

The number of solutions is $5 \cdot 3 \cdot 2/30 = 1$. The splitting sequence S_3 of the factorization $\{5, 3, 2\}$ of 30 with the canonical maps is:

0, 6, 12, 18, 24, 10, 16, 22, 28, 4, 20, 26, 2, 8, 14,
15, 21, 27, 3, 9, 25, 1, 7, 13, 19, 5, 11, 17, 23, 29.

The only w, $0 \leqq w < 30$, with $w(S_3) = 1$ is 21. Since

$$21 \equiv 1, \ [21/5] \equiv 1, \ [21/3 \cdot 5] \equiv 1 \ \text{(modulo 5, 3, 2 respectively)}.$$

the only solution is $X = (1, 1, 1)$.

It should be noted that the method is particularly useful when there are many solutions and one needs to find them all or when one wishes to find solutions for many values of r.

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References

1. D. H. Lehmer, The Tarry-Escott problem, Scripta Math., 13 (1947) 37–41.
2. M. E. Prouhet, Mémoire sur quelques relations entre les puissances des nombres, C. R. Acad. Sci. Paris, 33 (1851) 225.
3. J. B. Roberts, A new proof of a theorem of Lehmer, Canad. J. Math. 10 (1958) 191–4.
4. ——— Polynomial identities, Proc. AMS, 11 (1960) 723–30.
5. E. M. Wright, Equal sums of like powers, Edinburgh Math. Proc., 8 (1949) 138–42.
6. ——— Equal sums of like powers, Bull. AMS, 54 (1948) 755–7.
7. ——— Prouhet's 1851 solution of the Tarry-Escott problem of 1910, this MONTHLY 66 (1959) 199–201.

CONCENTRIC POLYGONS

PAUL J. KELLY AND DAVID MERRIELL, University of California, Santa Barbara

1. Introduction. This paper brings together from the literature a number of properties of concentric polygons. The writers feel that these results deserve to be better known, first for their intrinsic geometric interest, and second because the proofs are an excellent illustration of the power of complex variables in geometry and at a level suitable for undergraduates.

Edward Kasner in [7, 8, 9] generalized the theorem that the midpoints of the sides of any quadrilateral are vertices of a parallelogram. In these papers a polygon is taken to be a set of n points, not necessarily distinct, together with a cyclic ordering. Jesse Douglas in [4, 5] placed this generalization in a wider context and introduced the notion of representing the vertices by complex coordinates.

Let $\pi=(P_1, P_2, \cdots, P_n)$ be an n-gon. If the Cartesian coordinates of P_j are (a_j, b_j), then P_j can also be represented in the complex plane by a_j+ib_j. We use P_j to designate both the point and its complex representation. The centroid of π is

$$\frac{1}{n}\sum_{j=1}^{n} P_j.$$

Given any complex number $c=\lambda+i\mu$, we define the *c-derived* polygon of π to be a polygon $\pi_c=(Q_1, Q_2, \cdots, Q_n)$ where $Q_j=(1-c)P_j+cP_{j+1}$ and $P_{n+1}=P_1$. The point Q_j is obtained by moving from P_j along the line P_jP_{j+1} a distance which is λ times the length of segment P_jP_{j+1} and then moving 90° to the left a distance which is μ times the length of this segment. Thus the triangles (P_j, P_{j+1}, Q_j) are all similar to the triangle $(0, 1, c)$. This construction of the derived polygon is designated in [4] by $S(a)$, where $a=-c/(1-c)$.

Since $\Sigma Q_j=\Sigma P_j+c\Sigma(P_{j+1}-P_j)=\Sigma P_j$, the derived polygon π_c has the same centroid as π. Hence we say π and π_c are *concentric* polygons.

Let $\pi_{c,d}=(\pi_c)_d=(R_1, R_2, \cdots, R_n)$. Then

$$R_j = (1-d)Q_j + dQ_{j+1} = (1-c-d+cd)P_j + (c+d-2cd)P_{j+1} + cdP_{j+2}.$$

Since the coefficients are symmetric in c and d, $\pi_{c,d}=\pi_{d,c}$. An extension of this argument shows that if one forms successive derived polygons using complex constants $c_1, c_2, \cdots, c_n$, the final polygon is the same no matter in what order the constants are used.

We define the *alternating midpoint polygon* of π to be the polygon $M(\pi) =(M_1, M_2, \cdots, M_n)$ where $M_j=\frac{1}{2}(P_j+P_{j+2})$, the midpoint of the segment P_jP_{j+2}, indices greater than n being reduced modulo n. If $c=\frac{1}{2}(1+i)$ and $\bar{c}$ is the conjugate of c, then $M(\pi)=\pi_{c,\bar{c}}$. For, in this case $1-c-\bar{c}+c\bar{c}=\frac{1}{2}$, $c+\bar{c}-2c\bar{c}=0$, and $c\bar{c}=\frac{1}{2}$. As a corollary of the remarks about the order of the constants, if d is any constant, $M(\pi)_d=M(\pi_d)$.

2. Douglas' Theorem. On the sides of any n-gon π, construct isosceles triangles with vertex to the right and vertex-angle $2\pi/n$. The resulting vertices form a derived polygon π_{c_1}, where

$$c_k = \frac{1}{2} - \frac{i}{2}\cot\frac{\pi k}{n}\cdot$$

On the sides of this polygon construct isosceles triangles with vertex to the right and vertex-angle $4\pi/n$. Continue this construction using vertex-angles $\alpha=6\pi/n$, $8\pi/n, \cdots, 2(n-1)\pi/n$, where the triangles are constructed to the left with vertex angle $2\pi-\alpha$ when $\alpha>\pi$. Jesse Douglas proved in 1938 that the vertices of the final polygon $\pi_{c_1,c_2,\ldots,c_{n-1}}$ coincide in the centroid of π. This remarkable theorem was proved independently by B. H. Neumann in 1940. An elementary proof is given in [11]. If the construction using c_{n-p} is omitted, the final polygon is regular in the sense that its vertices occur at equal angular intervals of $2p\pi/n$ on the circumference of a circle. If $p=1$, in which case the final construction is omitted, the polygon is regular in the ordinary sense. The constants c_k may be written in the form $1/(1-\omega_k)$ where the ω_k are the $n-1$ roots of $x^{n-1}+x^{n-2}+\cdots+x+1=0$.

3. Triangles. By the special case of Douglas' Theorem for $n=3$, the vertices of the derived polygons for

$$c=\frac{1}{2}\pm\frac{i\sqrt{3}}{6}$$

form equilateral triangles (possibly of side 0). This is equivalent to the well-known theorem that the centers of equilateral triangles constructed either outwardly or inwardly on the sides of any triangle are vertices of an equilateral triangle.

A famous problem of Fermat was to find the point having minimum sum of distances to the vertices of a triangle. If the triangle is $\pi=(P_1, P_2, P_3)$ and

$\pi_c=(Q_1, Q_2, Q_3)$ is the derived triangle for $c=\frac{1}{2}(1+i\sqrt{3})$, then the lines P_1Q_2, P_2Q_3, P_3Q_1 are concurrent in the Fermat point [3]. The proof of concurrency is contained in the following more general result.

THEOREM 1. *If π_c is the derived triangle of π for $c=\frac{1}{2}+i\mu$, then the lines P_jQ_{j+1} are concurrent.*

Proof. The condition for concurrency of lines AB, CD, and EF is that

$$\Delta = \begin{vmatrix} B-A & D-C & F-E \\ \overline{B}-\overline{A} & \overline{D}-\overline{C} & \overline{F}-\overline{E} \\ B\overline{A}-A\overline{B} & D\overline{C}-C\overline{D} & F\overline{E}-E\overline{F} \end{vmatrix} = 0,$$

where $\overline{P}$ is the complex conjugate of P [13]. Since $1-c=\bar{c}$, $Q_j=\bar{c}P_j+cP_{j+1}$ so for the lines P_jQ_{j+1},

$$\Delta = \begin{vmatrix} \bar{c}P_2+cP_3-P_1 & \bar{c}P_3+cP_1-P_2 & \bar{c}P_1+cP_2-P_3 \\ c\overline{P}_2+\bar{c}\overline{P}_3-\overline{P}_1 & c\overline{P}_3+\bar{c}\overline{P}_1-\overline{P}_2 & c\overline{P}_1+\bar{c}\overline{P}_2-\overline{P}_3 \\ (\bar{c}\overline{P}_1P_2+c\overline{P}_1P_3 & (\bar{c}\overline{P}_2P_3+cP_1\overline{P}_2 & (\bar{c}P_1\overline{P}_3+cP_2\overline{P}_3 \\ -cP_1\overline{P}_2-\bar{c}P_1\overline{P}_3) & -cP_2\overline{P}_3-\bar{c}\overline{P}_1P_2) & -c\overline{P}_1P_3-\bar{c}\overline{P}_2P_3) \end{vmatrix}.$$

Adding the second and third columns to the first gives a column of zeros so $\Delta=0$.

For $\mu=0$, the point of concurrency is the centroid of π, for $\mu=\frac{1}{2}\sqrt{3}$ it is the Fermat point, and as $\mu\rightarrow\pm\infty$ the limiting position is the orthocenter.

THEOREM 2. *The locus of the points of concurrency of the lines P_jQ_{j+1} as μ varies is a hyperbola which passes through the centroid, the Fermat point, the orthocenter, and the three vertices of π.*

Proof. Let a, b, c be real numbers with a and c not zero. Choose $P_1=2a$, $P_2=0$ and $P_3=2b+2ci$. Then $Q_1=a-2a\mu i$, $Q_2=b-2c\mu+(c+2b\mu)i$ and $Q_3=a+b+2c\mu+[c+2(a-b)\mu]i$. The Cartesian equations of P_1Q_2 and P_2Q_3 are

$$y=\frac{c+2b\mu}{b-2c\mu-2a}(x-2a) \quad\text{and}\quad y=\frac{c+2(a-b)\mu}{a+b+2c\mu}x.$$

Eliminating μ, one obtains

$$c(2b-a)(x^2-y^2)+2(ab-a^2-b^2+c^2)xy+2ac(a-2b)x+2a(ab+b^2-c^2)y=0.$$

The discriminant is nonnegative so the conic is a hyperbola which is degenerate when $a=2b$. One can easily check that the vertices lie on the curve. This hyperbola and its properties were discussed by Casey in [2].

Since the perpendiculars from Q_j onto the sides P_jP_{j+1} meet in the circumcenter of π, the triangles π and π_c are orthologic [6]. It follows that the perpendiculars from P_j onto Q_jQ_{j-1} are concurrent. For $\mu=0$,the point of concurrency is the orthocenter of π, for $\mu=\infty$ it is the centroid, and for $\mu=\frac{1}{2}$ it coincides with the point of concurrency of the lines P_jQ_{j+1}.

COROLLARY. *The locus of the points of concurrency of the perpendiculars from P_j onto Q_jQ_{j-1} as μ varies is the hyperbola of Theorem* 2.

Proof. From the proof of the theorem, the perpendiculars from P_2 onto Q_2Q_1 and P_1 onto Q_1Q_3 have equations

$$y = \frac{a - b + 2c\mu}{c + 2(a + b)\mu} x \quad \text{and} \quad y = \frac{b + 2c\mu}{2b\mu - c - 4a\mu}(x - 2a).$$

If one chooses $\mu = 1/4\nu$, these equations reduce to those for P_2Q_3 and P_1Q_2 with μ replaced by ν. Hence the elimination of ν yields the same hyperbola.

4. Quadrilaterals. For $n=4$, the constants of Douglas' Theorem are $\frac{1}{2}-\frac{i}{2}$, $\frac{1}{2}$, and $\frac{1}{2}+\frac{i}{2}$. The construction of successive derived polygons using in either order $\frac{1}{2}-\frac{i}{2}$ and $\frac{1}{2}$, or $\frac{1}{2}$ and $\frac{1}{2}+\frac{i}{2}$, yields vertices of a square. The construction using $\frac{1}{2}\pm\frac{i}{2}$ yields $M(\pi)$ consisting of the midpoints of the diagonals of π.

The next theorem is a generalization of Problem 24 of [12].

THEOREM 3. *Let $c=\frac{1}{2}+i\mu$ and let Q_j and Q_j' denote the vertices of π_c and $\pi_{\bar{c}}$ respectively where π is any quadrilateral. Then (Q_1, Q_2', Q_3, Q_4') and (Q_1', Q_2, Q_3', Q_4) form parallelograms.*

Proof. Since $c=\frac{1}{2}+i\mu$, $\bar{c}=1-c$. The midpoints of Q_1Q_3 and $Q_2'Q_4'$ coincide in $\frac{1}{2}[\bar{c}(P_1+P_3)+c(P_2+P_4)]$. Similarly, the midpoints of Q_2Q_4 and $Q_1'Q_3'$ coincide.

A theorem of van Aubel [6] also proved in [4] and [12] can be rephrased: If π is any quadrilateral and $c=\frac{1}{2}\pm\frac{i}{2}$, the diagonals of π_c are equal in length and, if the length is not zero, the lines of the diagonals are perpendicular.

THEOREM 4. *Let π be any quadrilateral and $c=\frac{1}{2}+\frac{i}{2}$. The circle whose center is the centroid of π and whose diameter is the segment joining the midpoints of the diagonals of π passes through the midpoints and intersection of the diagonals of π_c. The four midpoints are vertices of a square.*

Proof. The midpoints of the diagonals of π and π_c are the vertices of $M(\pi)$ and $M(\pi_c)$ respectively. Since $M(\pi_c)=M(\pi)_c$ and $c=\frac{1}{2}+\frac{i}{2}$, the four midpoints form a square. The pairs of midpoints are ends of perpendicular diameters of the circle. By van Aubel's theorem, the diagonals of π_c are perpendicular, so the two midpoints of the diagonals together with the point of intersection P form a triangle with right angle at P. Hence the circle, which has the midpoint segment of π_c as diameter, passes through P.

References

1. H. F. Baker, A remark on polygons, J. London Math. Soc., 17 (1942) 162–164.

2. J. Casey, A treatise on the analytical geometry of the point, line, circle, and conic sections, Hodges Figgis, Dublin; Longmans Green, London, 1893.

3. H. S. M. Coxeter, Introduction to geometry, Wiley, New York, 1961.

4. Jesse Douglas, Geometry of polygons in the complex plane, J. Math. Phys., 19 (1940) 93–130.
5. ———, On linear polygon transformations, Bull. Amer. Math. Soc., 46 (1940) 551–560.
6. H. G. Forder, The calculus of extension, Cambridge University Press, London, 1941; Chelsea, New York, 1960.
7. E. Kasner, The group generated by central symmetries, with applications to polygons, this MONTHLY, 10 (1903) 57–63.
8. E. Kasner, E. Jonas, K. Way, and G. Comenetz, Centroidal polygons and groups, Scripta Math., 2 (1934) 131–138.
9. E. Kasner and G. Comenetz, Groups of multipoint transformations with applications to polygons, Scripta Math., 4 (1936) 37–49.
10. B. H. Neumann, Some remarks on polygons, J. London Math. Soc., 16 (1941) 230–245.
11. ———, A remark on polygons, J. London Math Soc., 17 (1942) 165–166.
12. I. M. Yaglom, Geometric transformations, Random House, New York, 1962.
13. C. Zwikker, Advanced plane geometry, North-Holland, Amsterdam, 1950.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables 46, Florida

SUMMATION OF CERTAIN SERIES

L. CARLITZ, Duke University

1. Making use of some results from the theory of elliptic functions, Watson [3] proved the following two formulas of Ramanujan:

$$(1)\qquad \begin{aligned} &\sum_{n=1}^{\infty} \frac{n}{e^{2n\pi}-1} = \frac{1}{24} - \frac{1}{8\pi}, \\ &\sum_{n=1}^{\infty} \frac{n^{4m+1}}{e^{2n\pi}-1} = \frac{B_{4m+2}}{8m+4} \qquad (m = 1, 2, 3, \cdots), \end{aligned}$$

where B_{2m} denotes a Bernoulli number in the even suffix notation. Hardy [1] has given two proofs of the more general result (also due to Ramanujan)

$$(2)\qquad \alpha^k\left(\frac{1}{2}\zeta(1-2k) + \sum_{n=1}^{\infty}\frac{n^{2k-1}}{e^{2n}-1}\right) = (-1)^k\beta^k\left(\frac{1}{2}\zeta(1-2k) + \sum_{n=1}^{\infty}\frac{n^{2k-1}}{e^{2n}-1}\right),$$

where $\zeta(1-2k)$ is the Riemann zeta-function, k is an integer greater than 1 and α, β are positive numbers such that $\alpha\beta=\pi^2$. When k is odd and $\alpha=\beta$, both sides of (2) vanish and we get (1).

It may be of interest to mention the formulas

$$\alpha^k \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^{2k-1}}{e^{n\alpha} - e^{-n\alpha}} = (-1)^k \beta^k \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^{2k-1}}{e^{n\beta} - e^{-n\beta}} \tag{3}$$

where k is an integer greater than 1 and $\alpha\beta = \pi^2$. When $k = 1$ we have

$$\alpha \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{e^{n\alpha} - e^{-n\alpha}} = \frac{1}{4} - \beta \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{e^{n\beta} - e^{-n\beta}}, \tag{4}$$

where $\alpha\beta = \pi^2$. The proof of these formulas is similar to Hardy's first proof. Put

$$f_k(\alpha) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^{2k-1}}{e^{n\alpha} - e^{-n\alpha}}.$$

Then we have

$$f_k(\alpha) = \frac{1}{2\pi i} \int_{(2k+\delta)} \Gamma(s) \alpha^{-s} \eta(s) \xi(s - 2k + 1) ds, \tag{5}$$

where $k \geqq 1$, $0 < \delta < 1$,

$$\eta(s) = \left(1 - \frac{1}{2^s}\right) \zeta(s), \qquad \xi(s) = \left(1 - \frac{2}{2^s}\right) \zeta(s)$$

and $\int_{(\beta)} = \int_{\beta - \infty i}^{\beta + \infty i}$. We now move the line of integration in (5) to the left into the position $(-\delta - \infty i, -\delta + \infty i)$. When $k < 1$ we find that the integrand has no singularities in the strip $-\delta \leqq \sigma \leqq 2k + \delta$; when $k = 1$ we have a pole at $s = 1$ with residue $(4\alpha)^{-1}$. It follows that

$$f_k(\alpha) = c_k + \frac{1}{2\pi i} \int_{(-\delta)}, \tag{6}$$

where

$$c_k = \begin{cases} 0 & (k > 1) \\ \dfrac{1}{4\alpha} & (k = 1). \end{cases}$$

Now replace s by $2k - s$ and use the functional equation for $\zeta(s)$:

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1 - s).$$

Then (6) becomes $f_k(\alpha) = c_k + (-1)^k (\pi/\alpha)^{2k} f_k(\pi^2/\alpha)$, and (3) and (4) follow immediately.

If we take $\alpha = \beta = \pi$, k odd (3) and (4) reduce to

$$(7) \qquad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^{4k+1}}{e^{n\pi} - e^{-n\pi}} = 0 \qquad (k = 1, 2, 3, \cdots),$$

$$(8) \qquad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{e^{n\pi} - e^{-n\pi}} = \frac{1}{8\pi},$$

respectively.

In this connection we note that Phillips [2] has proved that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{4m-1}(e^{n\pi} - e^{-n\pi})} = -\frac{\pi^{4m-1}}{2(4_m)!} \left\{ (2^{4m-1} - 1)B_{4m} - \sum_{n=1}^{2m-1} (-1)^n \binom{4m}{2n} (2^{2n-1} - 1)(2^{4m-2n-1} - 1)B_{4m-2n}B_{2n} \right\} (m = 1, 2, 3, \cdots).$$

2. Jacobi has obtained the expansion [4, p. 520, ex. 5]

$$(9) \qquad \frac{k^2K^2}{4\pi^2} \operatorname{sn}^2(2Kx) = \frac{K(k - E)}{4\pi^2} - \frac{1}{2} \sum_{1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos 2n\pi x.$$

If x is replaced by $x+\frac{1}{2}$, (9) becomes

$$(10) \qquad \frac{k^2K^2}{4\pi^2} \frac{\operatorname{cn}^2(2Kx)}{\operatorname{dn}^2(2Kx)} = \frac{K(K - E)}{4\pi^2} + \frac{1}{2} \sum_{1}^{\infty} (-1)^{n-1} \frac{nq^n}{1 - q^{2n}} \cos 2n\pi x.$$

We now take $k = 1/\sqrt{2}$ so that $q = e^{-\pi}$ and [4, p. 524]

$$(11) \qquad K = \tfrac{1}{4}\pi^{-1/2}(\Gamma(\tfrac{1}{4}))^2, \qquad 2E - K = 2\pi^{3/2}(\Gamma(\tfrac{1}{4}))^{-2}.$$

In particular, when $x = 0$, (10) reduces to (8), while (9) becomes

$$(12) \qquad \sum_{1}^{\infty} \frac{n}{e^{n\pi} - e^{-n\pi}} = \frac{(\Gamma(\frac{1}{4}))^2}{64\pi^3} - \frac{1}{8\pi}.$$

It is not difficult to show that (10) implies (7). Indeed when $k = 1/\sqrt{2}$ we have

$$\operatorname{cn} ix = \frac{1}{\operatorname{cn} x}, \qquad \operatorname{dn} ix = \frac{\operatorname{dn} x}{\operatorname{cn} x}.$$

Thus (10) yields, since $\operatorname{dn}^2 x = \frac{1}{2}(1 + \operatorname{cn}^2 x)$,

$$\frac{K(K - E)}{4\pi^2} + \frac{1}{4} \sum_{1}^{\infty} (-1)^{n-1} \frac{ne^{-n\pi}}{1 - e^{-2n\pi}} (\cos 2n\pi x + \cosh 2n\pi x) = \frac{K^2}{8\pi^2};$$

making use of (11) we get

$$\frac{1}{2} \sum_{1}^{\infty} \frac{ne^{-n\pi}}{1 - e^{-2n\pi}} (\cos 2n\pi x + \cosh 2n\pi x) = \frac{1}{8\pi}.$$

This evidently contains both (7) and (8).

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References

1. G. H. Hardy, A formula of Ramanujan, J. London Math. Soc., 3 (1928) 238–240.
2. E. G. Phillips, Note on summation of series, *ibid.*, 4 (1929) 114–116.
3. G. N. Watson, Theorems stated by Ramanujan, II: theorems on summation of series, *ibid.*, 3 (1928) 216–225.
4. E. T. Whittaker and G. N. Watson, Modern analysis, 4th ed., Cambridge, 1927.

INVERSION FORMULAE FOR CHARACTERISTIC FUNCTIONS OF ABSOLUTELY CONTINUOUS DISTRIBUTIONS

EUGENE LUKACS, The Catholic University of America, Washington, D. C.

Let $F(x)$ be a distribution function, that is, a nondecreasing function which is continuous to the right and for which $F(-\infty)=\lim_{x\to-\infty} F(x)=0$ while $F(+\infty)=\lim_{x\to\infty} F(x)=1$. If a distribution function $F(x)$ is absolutely continuous then it has a Lebesgue integrable density function $p(x)$ and $F(x)=\int_{-\infty}^{x}p(y)dy$. A density function $p(x)$ is characterized by the following two properties: (i) $p(x)\geqq 0$ for all real x, and (ii) $\int_{-\infty}^{\infty}p(x)dx=1$. The Fourier transform

$$f(t) = \int_{-\infty}^{\infty} e^{itx}dF(x) \tag{1}$$

of a distribution function $F(x)$ is called its characteristic function. If a distribution function $F(x)$ is absolutely continuous with density function $p(x)=F'(x)$, then its characteristic function is given by

$$f(t) = \int_{-\infty}^{\infty} e^{itx}p(x)dx. \tag{2}$$

We mention next several properties of density functions and characteristic functions which will be used later.

Let $F_1(x)$ and $F_2(x)$ be two distribution functions. Then the function

$$F(x) = \int_{-\infty}^{\infty} F_1(x-y)dF_2(y) = \int_{-\infty}^{\infty} F_2(x-y)dF_1(y) \tag{3}$$

is also a distribution function. We say that F is the convolution of F_1 and F_2 and write

$$F = F_1 * F_2.$$

It is known ([2] p. 45, the convolution theorem) that a distribution F is the convolution of two distributions F_1 and F_2 if and only if the corresponding characteristic functions satisfy the equation

$$f(t) = f_1(t)f_2(t). \tag{4}$$

In the following we shall be concerned with the case in which $F_1(x)$ as well as $F_2(x)$ are absolutely continuous with density functions $p_1(x)$ and $p_2(x)$ respectively. Then $F(x)$ is also absolutely continuous and has a density function $p(x)$ and (3) reduces to

$$p(x) = \int_{-\infty}^{\infty} p_1(x-y)p_2(y)dy = \int_{-\infty}^{\infty} p_2(x-y)p_1(y)dy. \tag{3a}$$

We write then again $p = p_1 * p_2$.

We see from (1) that a distribution function $F(x)$ determines the characteristic function. Moreover, Paul Lévy ([1] p. 166) gave a general inversion formula

$$F(x+h) - F(x) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{1-e^{-ith}}{it} e^{-itx}f(t)dt \tag{5}$$

which is valid if x and $x+h$ are continuity points of $F(x)$. Since a distribution function is completely determined by the values at its continuity points we can say that $F(x)$ and $f(t)$ determine each other uniquely. We conclude from this uniqueness property and the convolution theorem that the product of two characteristic functions is always a characteristic function.

If a characteristic function $f(t)$ is absolutely integrable over $(-\infty, +\infty)$ then the corresponding distribution function is absolutely continuous. Its density function $p(x) = F'(x)$ is then given by the formula

$$p(x) = F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}f(t)dt. \tag{6}$$

It is well known (see for instance [2] p. 41 and 72) that the absolute integrability of $f(t)$ is a sufficient, but not a necessary condition for the validity of the inversion formula (6). For instance, formula (6) holds also for all Pólya-type characteristic functions even if they are not absolutely integrable.

The purpose of this note is to derive inversion formulae which are valid for all absolutely continuous distribution functions $F(x)$.

Let $G(x)$ be an absolutely continuous distribution function whose characteristic function $g(t)$ is absolutely integrable over $(-\infty, +\infty)$ and write $q(x) = G'(x)$ for its frequency function. According to formula (6)

$$q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}g(t)dt.$$

For any T, $g_T(t) = g(t/T)$ is also a characteristic function which is absolutely integrable over $(-\infty, +\infty)$ and it is easily seen that its frequency function $q_T(x)$ is given by

$$q_T(x) = Tq(Tx). \tag{7}$$

Let $p(x)$ be again the frequency function corresponding to $f(t)$ and remember that $f(t)$ was assumed to belong to an absolutely continuous distribution but is

not supposed to be absolutely integrable. We consider the function

$$h_T(t) = f(t)g_T(t) = f(t)g\left(\frac{t}{T}\right). \tag{8}$$

We note that $h_T(t)$ is the product of two characteristic functions and is therefore also a characteristic function. We see from (1) that $|f(t)| \leqq 1$. Since $g(t)$ is absolutely integrable over $(-\infty, +\infty)$ we conclude from (8) that $h_T(t)$ is absolutely integrable over $(-\infty, +\infty)$. Therefore $h_T(t)$ belongs to an absolutely continuous distribution $H_T(x)$. The density function $H_T'(x)$ of this distribution can be determined by means of the inversion formula (6) and is given by

$$H_T'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}g_T(t)f(t)dt. \tag{9}$$

On the other hand it follows from (8) and (3a) that

$$H_T'(x) = p * q_T = T\int_{-\infty}^{\infty} p(x - z)q(Tz)dz = \int_{-\infty}^{\infty} p\left(x - \frac{y}{T}\right)q(y)dy.$$

Since $\int_{-\infty}^{\infty} q(y)dy = 1$ we have

$$\left| H_T'(x) - p(x) \right| = \left|\int_{-\infty}^{\infty}\left[p\left(x - \frac{y}{T}\right) - p(x)\right]q(y)dy\right|. \tag{10}$$

Let x be a continuity point of $p(x)$ and let $\epsilon > 0$ be an arbitrarily small number. We note that for all sufficiently large T

$$\left| p\left(x - \frac{y}{T}\right) - p(x) \right| \leqq \epsilon,$$

and from (10) that also $|H_T'(x) - p(x)| \leqq \epsilon$, so that

$$\lim_{T\to\infty} H_T'(x) = p(x). \tag{11}$$

We combine (9) and (11) and obtain

$$F'(x) = p(x) = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}g\left(\frac{t}{T}\right)f(t)dt. \tag{12}$$

This inversion formula is valid for any absolutely continuous distribution function $F(x)$, provided that x is a continuity point of the frequency function $p(x)$.

Formula (12) can also be interpreted by stating that the inversion formula (6) is valid for all continuity points of an arbitrary frequency function if the integration is considered in the sense of a summability method with summability factor $g(t/T)$, where $g(t)$ is an absolutely integrable characteristic function.

For instance if we put

$$g(t) = \begin{cases} 1 - |t| & \text{for } |t| \leqq 1 \\ 0 & \text{for } |t| \geqq 1 \end{cases} \tag{13}$$

then (12) becomes

$$F'(x) = p(x) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} e^{-itx}\left(1 - \frac{|t|}{T}\right) f(t)dt. \tag{14}$$

This means that formula (6) is valid in the sense of $(C, 1)$ summability.

If on the other hand we set

$$g(t) = e^{-|t|}, \tag{15}$$

then we obtain from (12) the inversion formula

$$F'(x) = p(x) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) \exp\left[-itx + |t|/T\right]dt. \tag{16}$$

This means that (6) is valid in the sense of Abel summability. Formula (16) is also given by H. Richter [3].

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References

1. P. Lévy, Calcul des probabilités, Gauthier-Villars, Paris, 1925.
2. E. Lukacs, Characteristic Functions, London, 1960.
3. H. Richter, Wahrscheinlichkeitsrechnung, Springer Verlag, Berlin, 1956.

THE NEAR RINGS ON A FINITE CYCLIC GROUP

JAMES R. CLAY, University of Washington

The triple $(G, +, *)$ is said to be a *left near ring* if $(G, +)$ is a group, $(G, *)$ is a semi-group, and $*$ is left distributive over $+$, i.e. for $a, b, c \in G$, $a * (b+c) = a * b + a * c$. The definition and theory for right near rings are similar to that of left near rings. In the following, the term "near ring" will mean "left near ring."

In this paper, we derive the machinery that is necessary and sufficient to construct all the near rings whose additive group is finite and cyclic.

Since each cyclic group of order n is isomorphic to the integers modulo n, throughout this paper we shall let our set $G = Z_n = \{0, 1, 2, \cdots, n-1\}$, $1 < n < \infty$. Also, $+$ and $\cdot$ will denote addition and multiplication modulo n, respectively, on our set G, and π will denote a function whose domain is Z_n and whose range is contained in Z_n.

The operation $+$ on G is one of the n^{n^2} distinct operations that can be defined on G. If the trivial group $\{0\}$ is excepted, then $+$ is not the only operation. We shall see that of the remaining $n^{n^2}-1$ operations, exactly n^n of them

are left distributive over $+$. These n^n operations will be put into a 1-1 correspondence with the set of all functions π mapping Z_n into itself. Then, necessary and sufficient conditions will be placed upon the π such that the corresponding operations will be associative. Finally, necessary and sufficient conditions will be placed upon the set of π defining associative operations to determine isomorphism, thus defining an equivalence relation on the family of all near rings having the same additive cyclic group.

THEOREM I. *Let $*$ be a binary operation on Z_n. The operation $*$ is left distributive over $+$ iff there is a function π such that for all $p, q \in Z_n$, $p * q = \pi(p) \cdot q$.*

Proof. If $*$ is a binary operation, and $p \in Z_n$, then $p * 1 = s$, $s \in Z_n$. Define $\pi(p) = s$. This defines our map π. For p, $q \in Z_n$, note that since $*$ is left distributive over $+$,

$$p*q = p*(\underbrace{1 + 1 + \cdots + 1}_{q \text{ terms}}) = \underbrace{p*1 + p*1 + \cdots + p*1}_{q \text{ terms}} = \pi(p) \cdot q.$$

Let π be such that for all p, $q \in Z_n$, $p * q = \pi(p) \cdot q$. Then for p, q, $r \in Z_n$,

$$p*(q + r) = \pi(p) \cdot (q + r) = \pi(p) \cdot q + \pi(p) \cdot r = p*q + p*r.$$

COROLLARY. *There are exactly n^n left distributive operations defined on Z_n.*

Proof. Certainly there are n^n distinct functions π with domain Z_n and range contained in Z_n. Let π_1 and π_2 be any two such functions such that $\pi_1 \neq \pi_2$. Since $\pi_1 \neq \pi_2$, there is a $p \in Z_n$ such that $\pi_1(p) \neq \pi_2(p)$. If π_1 and π_2 both defined the same operation $*$, then we would have

$$0 \neq \pi_1(p) - \pi_2(p) = \pi_1(p) \cdot 1 - \pi_2(p) \cdot 1 = p*1 - p*1 = 0.$$

THEOREM II. *A necessary and sufficient condition for a function π to define an associative left distributive operation is that $\pi(p) \cdot \pi(q) = \pi(p \cdot \pi(q))$ for all p, $q \in Z_n$.*

Proof. Assume that the operation $*$ defined by π is associative. Then for p, q, $r \in Z_n$

$$q*(p*r) = q*(\pi(p) \cdot r) = \pi(q) \cdot \pi(p) \cdot r = \pi(p) \cdot \pi(q) \cdot r$$

and

$$(q*p)*r = (\pi(q) \cdot p)*r = \pi(p \cdot \pi(q)) \cdot r.$$

Hence, when $r = 1$ we have $\pi(p) \cdot \pi(q) = \pi(p \cdot \pi(q))$.

For the converse, note that for all $r \in Z_n$ and for p, $q \in Z_n$ arbitrary, $r \cdot \pi(p) \cdot \pi(q) = r \cdot \pi(p \cdot \pi(q))$. But as shown above, $r \cdot \pi(p) \cdot \pi(q) = q * (p * r)$ and $r \cdot \pi(p \cdot \pi(q)) = (q * p) * r$.

THEOREM III. *Let f be a group automorphism on $(Z_n, +)$, and suppose that $f(1) = s$. Assume π_1 and π_2 define the two associative operations $*_1$ and $*_2$ respectively. Then f is a near ring isomorphism iff $\pi_1(p) = \pi_2(p \cdot s)$ for all $p \in Z_n$.*

Proof. First note that since f is an automorphism, $(s, n)=1$, i.e. s and n are relatively prime.

Assume f is a near ring isomorphism between $(Z_n, +, *_1)$ and $(Z_n, +, *_2)$. Then $f(p *_1 q)=f(p) *_2 f(q)$. But

$$f(p*_1q) = f(\pi_1(p)\cdot q) = (\pi_1(p)\cdot q)\cdot f(1) = \pi_1(p)\cdot q\cdot s$$

and

$$f(p)*_2f(q) = (p\cdot f(1))*_2(q\cdot f(1)) = (p\cdot s)*_2(q\cdot s) = \pi_2(p\cdot s)\cdot q\cdot s.$$

Since $(s, n)=1$ we have $\pi_1(p)\cdot q=\pi_2(p\cdot s)\cdot q$. For $q=1$ we have $\pi_1(p)=\pi_2(p\cdot s)$.

Conversely, assume that $\pi_1(p)=\pi_2(p\cdot s)$ for all $p\in Z_n$. Certainly $\pi_1(p)\cdot q\cdot s = \pi_2(p\cdot s)\cdot q\cdot s$, and as shown above $\pi_1(p)\cdot q\cdot s = f(p *_1 q)$ and $\pi_2(p\cdot s)\cdot q\cdot s = f(p) *_2 f(q)$. Since $p, q\in Z_n$ are arbitrary, f is an isomorphism.

The following are some immediate consequences of the above results.

We will need the fact that $k * 0=0$ for all k Z_n if $*$ is an associative left distributive operation. Note that $0 * k$ need not equal 0.

THEOREM IV. *If $(Z_n, +, *)$ is a near ring of prime order p, and if π is the function defined by $*$, then $\pi(0)\in\{0, 1\}$. If $\pi(0)=1$, then $\pi(k)=1$ for all $k\in Z_p$.*

Proof. Let $q=k$ and $p=0$ in the results of Theorem II, and then $\pi(0) =\pi(k)\cdot\pi(0)$. If $\pi(0)=0$ our results hold trivially. Assume $\pi(0)=t\neq 0$. Then $t=\pi(k)\cdot t$ holds for all $k\in Z_p$. Since p is a prime, there is a multiplicative inverse t^{-1} with respect to $\cdot$, so multiplying on the right by t^{-1} gives $1=\pi(k)$ for all k including $k=1$.

It is also true that $\pi(0)=1$ implies $\pi(k)=1$ for all $k\in Z_n$ even if n is not a prime.

THEOREM V. *If $\pi(0)=0$ and $\pi(k)\in\{0, 1\}$ for all $k\in Z_n$, then $*$ defined by π is associative.*

Proof. Theorem II implies that if $\pi(p)\cdot\pi(q)=\pi(p\cdot\pi(q))$ for all p, $q\in Z_n$, then $*$ is associative.

Checking each of the four cases below yields the theorem

	$\pi(p)$	$\pi(q)$
Case 1	0	0
Case 2	0	1
Case 3	1	0
Case 4	1	1.

Recall that there are n distinct operations $*$ on Z_n such that $(Z_n, +, *)$ is a ring. It is easy to see that each of these operations defines an endomorphism on Z_n, and conversely. Also, the function π defined by $\pi(k)=1$ for all $k\in Z_n$ defines an associative operation on Z_n. Using these two facts and the results of Theorem V, it is easy to see that there are at least $2^{n-1}+n$ left near rings that can be constructed from a finite cyclic group of order $n>1$.

For $n=2$ and $n=3$, there are exactly $2^{n-1}+n$, but for $n>3$ there are many

more. Indeed, as a result of programming the results of Theorem II for the IBM 7090 computer, it was found that for $n=4$ there are 17 distinct near rings on Z_n. For $n=5$, there are 29; for $n=6$, there are 98; and for $n=7$, there are 112. The results of Theorem III can also be programmed, and as a result, for any integer $n>1$, all the near rings on Z_n can be constructed and be separated into equivalence classes by isomorphism.

Reference

1. D. W. Blackett, Simple and semi-simple near rings, Proc. Amer. Math. Soc., 4 (1953) 772-785.

A NOTE ON SYMMETRIC FUNCTIONS

A. O. MORRIS, University College of Wales, Aberystwyth

Given a set of n variables $\alpha_1, \alpha_2, \cdots, \alpha_n$, their elementary symmetric functions a_r, their homogeneous product sums h_r, and their power sums s_r may be defined by the generating functions $A(x)$, $H(x)$ and $S(x)$ as follows:

$$A(x) = \prod_{i=1}^{n} (1 - \alpha_i x) = 1 + \sum_{r=1}^{n} a_r(-x)^r, \tag{1}$$

$$H(x) = \prod_{i=1}^{n} (1 - \alpha_i x)^{-1} = 1 + \sum_{r=1}^{\infty} h_r x^r, \tag{2}$$

$$S(x) = \log H(x) = \sum_{r=1}^{\infty} s_r x^r / r. \tag{3}$$

Recently Littlewood [1] has defined a more general type of symmetric function $q_r(t)$ by means of a generating function $Q(x, t)$ as

$$Q(x, t) = \prod_{i=1}^{n} (1 - \alpha_i x t)/(1 - \alpha_i x) = 1 + \sum_{r=1}^{\infty} q_r(t) x^r. \tag{4}$$

It is of interest to find formulae expressing the $q_r(t)$ in terms of the a_r, h_r or s_r, and vice versa. Perron [2] has obtained such formulae in the special case $t=-1$.

First we note the following identities connecting $Q(x, t)$, $A(x)$, $H(x)$ and $S(x)$:

$$Q(x, t) = A(xt)/A(x) = H(x)/H(xt) = \exp\{S(x) - S(xt)\}. \tag{5}$$

By comparing the coefficients of x^r on both sides of the equations

$$\begin{aligned} Q(x, t)A(x) &= A(xt), \\ Q(x, t)H(xt) &= H(x), \\ Q(x, t)\{S'(x) - S_x(xt)\} &= Q_x(x, t), \end{aligned}$$

we obtain the following sets of relations

$$\sum_{R=0}^{r-1} q_{r-R}(-1)^R a_R + (1 - t^r)a_r = 0, \tag{6}$$

$$\sum_{R=0}^{r-1} q_{r-R}t^R h_R + (t^r - 1)h_r = 0, \tag{7}$$

$$\sum_{R=1}^{r} q_{r-R}(1 - t^R)s_R + (-r)q_r = 0. \tag{8}$$

Taking $r=1, 2, \cdots, p$ we obtain from each of the relations (6), (7) and (8) a system of p linear equations in p variables. These may be solved by determinants to express q_p in terms of a_r, h_r and s_r respectively, or to express a_p, h_p or s_p in terms of the q_r.

If $\phi_p(t)=(1-t)(1-t^2)\cdots(1-t^p)$, we find that

$$q_p = \begin{vmatrix} a_1(1-t) & 1 & 0 & \cdots & 0 \\ a_2(1-t^2) & a_1 & 1 & \cdots & 0 \\ a_3(1-t^3) & a_2 & a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ a_p(1-t^p) & a_{p-1} & a_{p-2} & \cdots & a_1 \end{vmatrix} \tag{9}$$

$$\phi_p(t)a_p = \begin{vmatrix} q_1 & 1-t & 0 & \cdots 0 & 0 \\ q_2 & q_1 & 1-t^2 & \cdots & \cdots \\ q_3 & q_2 & q_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 1-t^{p-1} \\ q_p & q_{p-1} & q_{p-2} & \cdots & q_1 \end{vmatrix} \tag{10}$$

$$q_p = (-1)^p \begin{vmatrix} (t-1)h_1 & 1 & 0 & \cdots & 0 \\ (t^2-1)h_2 & th_1 & 1 & \cdots & 0 \\ (t^3-1)h_3 & t^2h_2 & th_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 \\ (t^p-1)h_p & t^{p-1}h_{p-1} & t^{p-2}h_{p-2} & \cdots & th_1 \end{vmatrix} \tag{11}$$

$$(12)\qquad \phi_p(t)h_p = \begin{vmatrix} q_1 & t-1 & 0 & \cdots & 0 \\ q_2 & tq_1 & t^2-1 & \cdots & 0 \\ q_3 & tq_2 & t^2q_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & t^{p-1}-1 \\ q_p & tq_{p-1} & t^2q_{p-2} & \cdots & t^{p-1}q_1 \end{vmatrix}$$

$$(13)\qquad p!q_p = \begin{vmatrix} (1-t)s_1 & -1 & 0 & \cdots & 0 \\ (1-t^2)s_2 & (1-t)s_1 & -2 & \cdots & 0 \\ (1-t^3)s_3 & (1-t^2)s_2 & (1-t)s_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -(p-1) \\ (1-t^p)s_p & (1-t^{p-1})s_{p-1} & (1-t^{p-2})s_{p-2} & \cdots & (1-t)s_1 \end{vmatrix}$$

$$(14)\qquad (1-t^p)s_p = (-1)^{p-1}\begin{vmatrix} q_1 & 1 & 0 & \cdots & 0 \\ 2q_2 & q_1 & 1 & \cdots & 0 \\ 3q_3 & q_2 & q_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 \\ pq_p & q_{p-1} & q_{p-2} & \cdots & q_1 \end{vmatrix}.$$

(13) can alternatively be obtained in a more convenient form as follows.

From (5) we see that

$$\begin{aligned} Q(x,t) &= \exp\left\{\sum_{r=1}^{\infty}\frac{(1-t^r)s_rx^r}{r}\right\} \\ &= \prod_{r=1}^{\infty}\left\{\exp\frac{(1-t^r)s_rx^r}{r}\right\} \\ &= \prod_{r=1}^{\infty}\left\{\sum_{\alpha_r=0}^{\infty}\frac{1}{\alpha_r!}(1-t^r)^{\alpha_r}\left(\frac{s_r}{r}\right)^{r\alpha_r}x^{\alpha_r}\right\} \\ &= \sum_{m=0}^{\infty}\left\{\sum_{(\rho)}\frac{1}{z_\rho}(1-t)^{\rho_1}(1-t^2)^{\rho_2}\cdots S_\rho\right\}x^m, \end{aligned}$$

where the summation is for all partitions $(\rho)=(1^{\rho_1}2^{\rho_2}3^{\rho_3}\cdots)$ of m such that $\rho_1+2\rho_2+3\rho_3+\cdots=m$, $s_\rho=s_1^{\rho_1}s_2^{\rho_2}s_3^{\rho_3}\cdots$ and $z_\rho=1^{\rho_1}\rho_1!\,2^{\rho_2}\rho_2!\,3^{\rho_3}\rho_3!\cdots$. Hence, we see that

$$(15)\qquad q_m(t) = \sum_{(\rho)}\frac{1}{z_\rho}(1-t)^{\rho_1}(1-t^2)^{\rho_2}\cdots S_\rho.$$

I wish to thank the referee for his comments.

References

1. D. E. Littlewood, On certain symmetric functions, Proc. London Math. Soc., (3) 11 (1961) 485–498.

2. O. Perron, Über die Abhängigkeit von Potenzsummen und einen Satz von Polya, Math. Z., 63 (1955) 19–30

A SIMPLE NORM INEQUALITY

C. F. Dunkl and K. S. Williams, University of Toronto

The following useful norm inequality seems to be new. For nonzero vectors x and y in a normed linear space,

$$(*)\qquad \|x-y\| \geqq \tfrac{1}{4}(\|x\|+\|y\|)\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|.$$

It yields, for example, a simple proof that if X is a Banach Space, and M_1, M_2 are closed independent subspaces of X, then if there exists a $d>0$ such that $\|x_1-x_2\|\geqq d$, whenever $x_1\in M_1$, $x_2\in M_2$ and $\|x_1\|=\|x_2\|=1$, then $M_1\oplus M_2$ is closed.

Proof of Inequality:

$$(1)\qquad \begin{aligned}\|x\|\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| &\leqq \|x\|\left\|\frac{x}{\|x\|}-\frac{y}{\|x\|}\right\|+\|x\|\left\|\frac{y}{\|x\|}-\frac{y}{\|y\|}\right\| \leqq \|x-y\| \\ &+\frac{\|(\|y\|-\|x\|)y\|}{\|y\|}\leqq \|x-y\|+\big|\,\|y\|-\|x\|\,\big| \leqq 2\|x-y\|.\end{aligned}$$

Similarly by adding and subtracting $x/\|y\|$ we have:

$$(2)\qquad \|y\|\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\leqq 2\|x-y\|.$$

The result now follows by adding (1) and (2).

The authors originally conjectured the inequality (*) in the stronger form

$$\|x-y\| \geqq \tfrac{1}{2}(\|x\|+\|y\|)\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|,$$

but this is not true in general. Consider, for example, the normed linear space consisting of ordered pairs of real numbers, say (x_1, x_2), with norm equal to $|x_1|+|x_2|$. Take $x=(1,\epsilon)$ and $y=(1,0)$, where ϵ is positive and small. Then the inequality (*) becomes

$$\epsilon \geqq \frac{4(1+\epsilon/2)}{4(1+\epsilon)}\,\epsilon.$$

Thus it is obvious that the constant $\frac{1}{4}$ is the best possible. We have been unable, however, to answer this question: Does equality ever hold if $\|x\|+\|y\|\neq 0$?

We show that in a complex inner-product space with $\sqrt{\{(x, x)\}}$ as norm the inequality does hold with $\frac{1}{2}$ instead of $\frac{1}{4}$. For then

$$\begin{aligned}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2 &= \left(\frac{x}{\|x\|}-\frac{y}{\|y\|}, \frac{x}{\|x\|}-\frac{y}{\|y\|}\right) \\ &= 2 - 2\operatorname{Re}\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right) \\ &= \frac{1}{\|x\|\,\|y\|}[2\|x\|\,\|y\| - 2\operatorname{Re}(x, y)] \\ &= \frac{1}{\|x\|\,\|y\|}[2\|x\|\,\|y\| - (\|x\|^2+\|y\|^2-\|x-y\|^2)] \\ &= \frac{[\|x-y\|^2-(\|x\|-\|y\|)^2]}{\|x\|\,\|y\|}.\end{aligned}$$

Hence

$$\|x-y\|^2 - \left(\frac{\|x\|+\|y\|}{2}\right)^2\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2 = \frac{[\|x\|-\|y\|]^2}{4\|x\|\,\|y\|}[(\|x\|+\|y\|)^2-\|x-y\|^2] \geqq 0.$$

One further question, is the converse true—Does the inequality with the constant $\frac{1}{2}$ imply that X is an inner-product space?

ON A NOTE BY Q. G. MOHAMMAD

H. GUGGENHEIMER, University of Minnesota

Theorem 2 in the Note [1] is based on the fact that all roots of $p(z)=z^n+a_qz^{n-q}+\cdots+a_n$ are also roots of $P(z)=(z^q-a_q)p(z)=z^{n+q}+b_{q+1}z^{n-1}+\cdots+b_{n+q}$, and on an estimate of the influence of the gap-size q on bounds for the roots. Dr. Mohammad's argument on top of p. 903 leads to an improvement of my statements in the Note [2] in the same issue of the MONTHLY. Especially, the bound 2 is now replaced by a bound 1.

Let be $m(k, M)$ the unique root >1 of

$$x^k - x^{k-1} - M = 0$$

defined for positive integer k and positive M. In

$$z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0 \tag{1}$$

let be a_q the first nonzero coefficient, a_p the coefficient greatest in absolute value. *All roots of* (1) *lie in the circle* $|z| < m(q, |a_p|)$.

If (1) has real coefficients, let be a_r the first negative coefficient, a_s the negative coefficient of greatest absolute value. *All roots of* (1) *are* $<m(r, |a_s|)$.

A table of $m(k, M)$ is given below. For practical purposes, approximations are quickly found in a table of powers.

TABLE of $m(k, M)$

M \ $k=$	2	3	4	5	6
0.5	1.37	1.30	1.26	1.23	1.22
1	1.62	1.48	1.39	1.33	1.29
2	2.00	1.73	1.55	1.46	1.40
3	2.31	1.87	1.66	1.54	1.46
4	2.57	2.00	1.76	1.61	1.51
5	2.80	2.12	1.84	1.66	1.56
6	3.00	2.22	1.90	1.71	1.60
7	3.20	2.32	1.95	1.75	1.63
8	3.38	2.40	2.00	1.79	1.66
9	3.55	2.48	2.05	1.83	1.68
10	3.71	2.55	2.10	1.86	1.71
20	5.00	3.10	2.41	2.09	1.88
40	6.85	3.80	2.81	2.34	2.07
60	8.27	4.28	3.08	2.52	2.20
80	9.46	4.67	3.29	2.68	2.29
100	10.52	5.00	3.45	2.75	2.38
500	22.87	8.29	5.00	3.70	3.03
1000	32.13	10.40	5.90	4.21	3.39

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References

1. Q. G. Mohammad, On the zeros of polynomials, this MONTHLY, 69 (1962) 901–904.

2. H. Guggenheimer, Bounds for roots of algebraic equations, this MONTHLY, 69 (1962) 915–916.

SUMS OF LOGARITHMS OF BINOMIAL COEFFICIENTS

H. W. GOULD, West Virginia University

The object of this note is to establish two general limits involving sums of logarithms of binomial coefficients.

Let

$$P_n = \sum_{k=0}^{n} \log \binom{pn}{pk}, \qquad S_n = \frac{1}{n^2} P_n, \tag{1}$$

and

$$Q_n = \sum_{k=0}^{n} (-1)^k \log \binom{pn}{pk}, \qquad T_n = \frac{1}{n} Q_n, \tag{2}$$

where p is any positive integer. We shall show that

$$\lim_{n\to\infty} S_n = \frac{p}{2}, \tag{3}$$

and

$$\lim_{n\to\infty} T_{2n} = 0. \tag{4}$$

It is clear that $T_{2n-1}=0$ for all positive integers n.

We shall need to note the binomial coefficient identity

$$\binom{pn}{pk} = \binom{pn-p}{pk-p}\binom{pn}{p}\binom{pk}{p}^{-1}. \tag{5}$$

In the case of P_n we proceed as follows. By means of (5) we have, when $n\geqq 1$,

$$P_n = \sum_{k=1}^{n} \log\binom{pn}{pk} = \sum_{k=1}^{n} \log\binom{pn-p}{pk-p} + \sum_{k=1}^{n} \log\binom{pn}{p} - \sum_{k=1}^{n} \log\binom{pk}{p},$$

or

$$P_n - P_{n-1} = \log\left[\binom{pn}{p}^n \prod_{k=1}^{n}\binom{pk}{p}^{-1}\right] = \log\frac{(pn)!^n}{(pn-p)!^n(pn)!}. \tag{6}$$

Now by a general limit theorem of Stolz [1, 414] if b_n increases steadily to ∞ then

$$\lim_{n\to\infty} \frac{P_n}{b_n} = \lim_{n\to\infty} \frac{P_n - P_{n-1}}{b_n - b_{n-1}}, \tag{7}$$

provided that the second limit exists. This is a kind of finite difference analogue of the familiar rule of l'Hospital. Cauchy gave the case $b_n=n$. To apply the theorem here we choose $b_n=n^2$, so that $b_n-b_{n-1}=2n-1=n(2-1/n)$. Then we have

$$\frac{P_n - P_{n-1}}{b_n - b_{n-1}} = \frac{1}{2-1/n}\log\frac{(pn)!}{(pn-p)!(pn)!^{1/n}}.$$

Since the outside factor tends to $\frac{1}{2}$ we shall have finished the proof of (3) if we can show that the ratio of factorials tends to e^p. To show this we need the familiar limit

$$\lim_{r\to\infty} \frac{r}{r!^{1/r}} = e. \tag{8}$$

Indeed we have

$$\lim_{n\to\infty}\frac{(pn)!}{(pn-p)!(pn)!^{1/n}}=\lim_{n\to\infty}\frac{pn(pn-1)\cdots(pn-p+1)}{[(pn)!^{1/pn}]^p}$$
$$=\lim_{r\to\infty}\frac{r(r-1)\cdots(r-p+1)}{(r!^{1/r})^p}$$
$$=\lim_{r\to\infty}\left(\frac{r}{r!^{1/r}}\right)^p\cdot\left(1-\frac{1}{r}\right)\left(1-\frac{2}{r}\right)\cdots\left(1-\frac{p-1}{r}\right)=e^p,$$

for $p\geqq 2$, and it is clearly also correct when $p=1$.

The simplest case, $p=1$, was the subject of a recently proposed problem [2]. In the case of Q_n we find by means of (5) that

$$Q_n=\sum_{k=1}^{n}(-1)^k\log\binom{pn-p}{pk-p}+\sum_{k=1}^{n}(-1)^k\log\binom{pn}{p}-\sum_{k=1}^{n}(-1)^k\log\binom{pk}{p},$$

or

$$Q_n+Q_{n-1}=-\frac{1-(-1)^n}{2}\log\binom{pn}{p}-\sum_{k=1}^{n}(-1)^k\log\binom{pk}{p},\qquad n\geqq 1. \tag{9}$$

Since $Q_n=0$ when n is odd we have then

$$\begin{aligned}Q_{2n}&=-\sum_{k=1}^{2n}(-1)^k\log\binom{pk}{p}=\sum_{k=1}^{n}\log\binom{(2k-1)p}{p}-\sum_{k=1}^{n}\log\binom{2kp}{p}\\&=\log\prod_{k=1}^{n}\binom{2kp-p}{p}\binom{2kp}{p}^{-1}=\log\prod_{k=1}^{n}\prod_{j=1}^{p}\frac{2pk-p-j+1}{2pk-j+1}.\end{aligned} \tag{10}$$

Now to show that T_{2n} tends to 0 as $n\to\infty$ we should have to show that the nth root of the product tends to 1. For example, when $p=3$ we should have to show that

$$\lim_{n\to\infty}\left\{\prod_{k=1}^{n}\frac{(6k-3)(6k-4)(6k-5)}{6k(6k-1)(6k-2)}\right\}^{1/n}=1.$$

But in general, if $a>b>c\geqq 0$, then

$$\lim_{n\to\infty}\left\{\prod_{k=1}^{n}\frac{ak-b}{ak-c}\right\}^{1/n}=1. \tag{11}$$

This is immediate from the Cauchy limit theorem, to the effect that $\lim_{n\to\infty}a_n^{1/n}$ exists and has the same value as $\lim_{n\to\infty}a_{n+1}/a_n$ provided this limit exists. Thus by (11) any product of a finite number of such factors has the same property, so we see that T_{2n} tends to zero as n increases. The basic technique we have employed may be applied to a wide variety of similar series.

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References

1. T. J. I'A. Bromwich, An Introduction to the Theory of Infinite Series, 2nd ed., Macmillan, London, 1926.
2. D. R. Hayes, Problem E 1557, this MONTHLY, 70 (1963) 92.

CENTRAL-FORCE LAWS FOR AN ELLIPTIC ORBIT

KURT TOMAN, Air Force Cambridge Research Laboratory, L. G. Hanscom Field, Bedford, Mass.

An ellipse can represent the trajectory of a particle of mass m which moves under the influence of a central-force field. If the origin of the central-force field lies at the center of the elliptic trajectory or at the focal point respectively, the well-known force laws of harmonic and gravitational motion are obtained. Other force laws result for different locations of the origin of a central force. It is the purpose of this analysis to illustrate the influence of the location of the origin of a central force upon force laws which yield the same elliptic trajectory. In this study the origin of the force is displaced in discrete steps from the center of the ellipse to the apex along the major semi-axis.

The vector of a central force $\mathbf{F}$ is, by definition, everywhere directed toward the same center such that $\mathbf{F}\times\mathbf{r}=0$, where $\mathbf{r}$ is the position vector of the particle. From Newton's second law of motion $\mathbf{F}=m\ddot{\mathbf{r}}$ it follows that $\ddot{\mathbf{r}}\times\mathbf{r}=0$. Through integration one obtains $\mathbf{r}\times\dot{\mathbf{r}}=\mathbf{c}$. Consequently, $\mathbf{r}\cdot(\mathbf{r}\times\dot{\mathbf{r}})=\mathbf{r}\cdot\mathbf{c}=0$ and the trajectory of the particle is a plane curve.

In a cylindrical coordinate system the equation of motion of a particle whose position vector is $\mathbf{r}(t)$ can be written as

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\phi}^2)\boldsymbol{\rho} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\boldsymbol{\omega} + \ddot{z}\mathbf{z}, \tag{1}$$

where $\boldsymbol{\rho}$, $\boldsymbol{\omega}$, $\mathbf{z}$ are the unit vectors of the acceleration components corresponding to the range r, azimuth ϕ, and elevation z, respectively. The dots indicate differentiation with respect to time. Since the trajectory of the particle is a plane curve this coordinate system can be chosen so that $\ddot{z}=0$. Since $\mathbf{F}\times\mathbf{r}=0$ there exists no acceleration in the direction of $\boldsymbol{\omega}$ and $r\ddot{\phi}+2\dot{r}\dot{\phi}=0$ in (1), from which follows the constancy of the areal velocity $r^2\dot{\phi}/2=f$. Thus, (1) reduces to

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\phi}^2)\boldsymbol{\rho}. \tag{2}$$

Combining (2) with Newton's second law of motion one obtains the scalar equation for the force

$$F = m(\ddot{r} - r\dot{\phi}^2).$$

The substitution of space- for time-derivatives yields

$$F = 4mf^2r^{-5}(rr'' - 2r'^2 - r^2), \tag{3}$$

where

$$r = r(\phi), \qquad r' = dr/d\phi, \qquad r'' = d^2r/d\phi^2.$$

The force law of (3) is determined if $r(\phi)$ is known.

Let us choose a Cartesian coordinate system with its origin at the center of an ellipse whose major semi-axis of length a lies along the abscissa $x(y=0)$ and whose minor semi-axis of length b lies along the ordinate $y(x=0)$. The center of the ellipse has the coordinates $x_0=y_0=0$. First, let us assume that this center of the ellipse contains the location of the origin $P_{\mathrm{I}}(x=0, y=0)$ of a central force under whose influence a particle of mass m is moving along a trajectory which is specified by this ellipse. For this origin $r(\phi)$ can be written as

$$r = b(1 - \epsilon^2 \cos^2 \phi)^{-1/2} \Big|_{P_{\mathrm{I}}(x=y=0)} \tag{4}$$

where ϵ is the eccentricity of the ellipse. Obtaining r', r'' from (4) the force law can be determined from (3) as

$$F = -(4mf^2p^{-1}a^{-3})r = -(4\pi^2 mT^{-2})r,$$

where $p=a(1-\epsilon^2)$ and T is the period of revolution. This well-known force law describes the harmonic, isochronous motion ($F \propto r$).

If the focal point of the same ellipse represents the location of the origin $P_{\mathrm{V}}(x=\epsilon a, y=0)$ of a central force and the position vector of the particle is now measured from P_{V}, $r(\phi)$ becomes

$$r = p(1 + \epsilon \cos \phi)^{-1} \Big|_{P_{\mathrm{V}}(x=\epsilon a, y=0)}. \tag{5}$$

Substituting r, r', r'' from (5) into (3) one obtains the central-force law

$$F = -(4mf^2p^{-1})r^{-2} = -(4\pi^2 ma^3T^{-2})r^{-2}. \tag{6}$$

This well-known force law describes the gravitational motion ($F \propto r^{-2}$).

Another simple analytical solution of (3) is obtained if the origin of the central force is located at the apex of the ellipse. For this origin $r(\phi)$ has the form

$$r = -2p \cos \phi(1 - \epsilon^2 \cos^2 \phi)^{-1} \Big|_{P(x=a, y=0)}$$

and the solution of (3) is obtained as

$$F = (4mf^2\epsilon^6p^{-4})r(1 - [1 + r^2\epsilon^2p^{-2}]^{1/2})^{-3}$$

which reduces to

$$F = -(4mf^2\epsilon^3p^{-1})r^{-2} \qquad \text{for } r\epsilon p^{-1} \ggg 1,$$

$$F = -(32mf^2p^2)r^{-5} \qquad \text{for } r\epsilon p^{-1} \lll 1.$$

For other locations of the origin of a central force solutions of (3) become rather involved and must be computed. For a particular ellipse ($a=250$, $b=125$) computations include the following positions ($y=0$) of the origin of the central force relative to the center of the ellipse.

x_P	0	62.5	140	187.5	$\epsilon a=216.5$	220	240	249
curve	I	II	III	IV	V	VI	VII	VIII

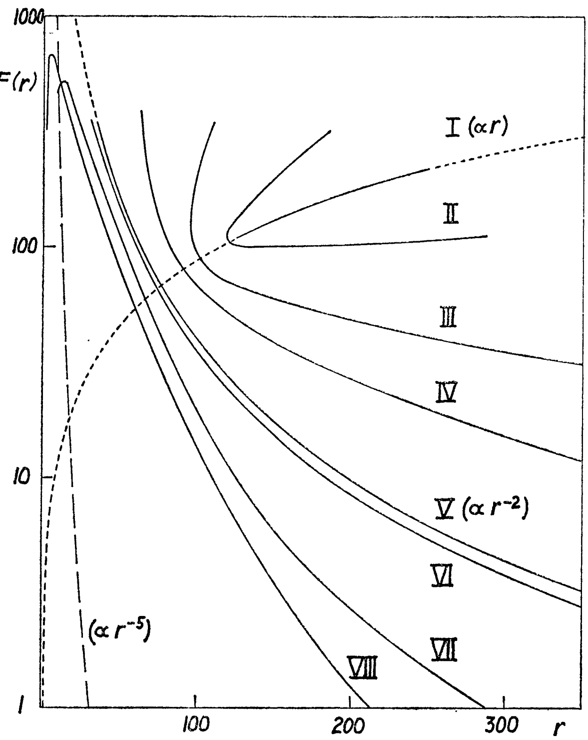


FIG. 1. Central-force laws for an elliptic trajectory.

The Roman numerals correspond to the curves of Fig. 1. For each curve a conveniently normalized force F, read on a log-scale, is plotted along the ordinate. The range r, read on a linear scale, is plotted along the abscissa. For $x_P=0$ the force law $F\propto r$ is represented by curve I. A displacement of the origin of the central force from $x=0$ to $x>0$ unfolds curve I into II, III, etc. For $x=a-b^2/a=187.5$ (IV) the force law approaches $F\propto r^{-2}$ of curve V. For $x>\epsilon a$ curves VI to VIII are obtained. A ($F\propto r^{-5}$)-distribution is shown for comparison.

Fig. 1 depicts members of a family of central-force laws. It is seen that $F(r)$ can be double valued. This circumstance reveals that F is also a function of ϕ. The ϕ-dependence of F is a property of all force laws whose origin is neither the center nor the focal point of the elliptic trajectory (see [1]) although it is not discernible if the origin of the force lies in the neighborhood of the focal point (e.g. curve VI).

Reference

1. K. Stumpff, Himmelsmechanik, Bd.1; VEB Deutscher Verlag der Wissenschaften, Berlin, 1959, pp. 141–144.

SELF PRODUCING SEQUENCES OF DIGITS

J. H. JORDAN, Washington State University

D. Kaprekar [1] noticed that if one takes any four digits, not all the same, in the base ten, and lets A represent the largest number expressible in these four digits, and lets B represent the smallest number expressible in these four digits, and then considers the four digits produced by $A-B$ as a new sequence of four digits, and repeats the operation, then within seven repetitions he will obtain the sequence of digits $\{7, 6, 4, 1\}$. Further if the operation is repeated on $\{7, 6, 4, 1\}$ the resulting sequence will be $\{7, 6, 4, 1\}$. This interesting phenomenon motivated this paper.

Let n be a positive integer and let R_n be a sequence of digits of the base n not all the same. Two sequences will be considered the same if one is merely a permutation of the other.

DEFINITION. *The Kaprekar Operation on a sequence of digits, $K(R_n)$, is as described above except that R_n may have any number of elements, not just four.*

DEFINITION. *A sequence R_n is self producing if $K(R_n)=R_n$.*

Digits larger than nine will be denoted by representing the digits in the base ten and underscoring. All numbers will be represented to base ten for convenience.

LEMMA. *If $n>2$ and if R_n is a sequence of digits with more than two elements then any number that can be represented by $K(R_n)$ is composite. Indeed, by a generalization of the rule of casting out nines, it is divisible by $n-1$.*

The following are examples of self producing sequences of digits:

$$R_4 = \{3, 2, 1, 0\}; \quad R_5 = \{3, 3, 2, 0\}; \quad R_8 = \{5, 2\}; \quad R_{10} = \{9, 5, 4\};$$
$$R_{10} = \{7, 6, 4, 1\}; \quad R_{10} = \{9, 8, 7, 6, 5, 4, 3, 2, 1\};$$

and

$$R_{13} = \{\underline{10}, 9, 7, 5, 4, 1\}.$$

THEOREM 1. *Given $m>1$, there are infinitely many positive integers n that have a sequence of m digits that are self producing.*

Proof. Case 1: If $m=2K-1$, $K>1$, let n be any multiple of K, say $K\cdot g=n$, then

$$R_n = \{n-1, (K-1)g, (K-1)g-1, (K-2)g, \cdots, g, g-1\}$$

is self producing.

Case 2: If $m=2K$, and if $K=1$, then for $n=3g+2$, $R_n=\{2g+1, g\}$ is self producing.

If $K>1$ and $n=10+15g$, then

$$R_n=\{7+12g,\ 6+10g, \cdots, 6+10g,\ 6+9g,\ 4+6g,\ 3+5g, \cdots, 3+5g,\ 1+3g\},$$

where the $6+10g$ and $3+5g$ each occur $K-2$ times, is self producing.

Case 1 of the proof exhibits sequences in which all digits are different when $n \geqq 2K$. This is in some ways better than the sequences exhibited in Case 2 of the proof. It seems reasonable to raise the question: For m even, do there exist infinitely many self producing sequences of m different digits?

The author has found two other parametrizations that could have been used in the proof of Case 2. They are:

For $n=13+21g$ and $K>2$

$$R_n = \{10 + 18g, 9 + 15g, 8 + 14g, \cdots, 8 + 14g, 7 + 12g, 5 + 9g,\ 4 + 7g, \cdots, 4 + 6g, 1 + 3g\},$$

where $8+14g$ and $4+7g$ occur $K-3$ times.

For $n=28+51g$ and $K>3$

$$R_n = \{22 + 42g, 19 + 36g, 18 + 34g, \cdots, 18 + 34g, 18 + 33g, 16 + 30g,\ 11 + 21g, 10 + 18g, 9 + 17g, \cdots, 9 + 17g, 8 + 15g, 4 + 9g\},$$

where $18+34g$ and $9+17g$, occur $K-4$ times.

A general formula would be desirable.

COROLLARY 1. *The following bases have at least one sequence of $2K$ digits that are self producing:*

i) $n \equiv 10 \pmod{15}$ and $K > 1$

ii) $n \equiv 13 \pmod{21}$ and $K > 2$

iii) $n \equiv 28 \pmod{51}$ and $K > 3$.

COROLLARY 2. *The following bases have at least two distinct sequences of $2K$ digits that are self producing:*

i) $n \equiv 55 \pmod{105}$ and $K > 2$

ii) $n \equiv 130 \pmod{255}$ and $K > 3$

iii) $n \equiv 181 \pmod{357}$ and $K > 3$.

COROLLARY 3. *If $n \equiv 895 \pmod{1785}$ then n has at least 3 sequences of $2K$ digits that are self producing, for $K>3$.*

Example for Corollary 2: Let $n=55$, then two self producing sequences of six digits are:

$$\{\underline{43}, \underline{36}, \underline{33}, \underline{22}, \underline{18}, \underline{10}\} \quad \text{and} \quad \{\underline{46}, \underline{39}, \underline{31}, \underline{23}, \underline{16}, 7\}.$$

COROLLARY 4. *There is a self producing sequence of m elements to the base ten if and only if $m \neq 1, 2, 5$ or 7.*

Proof. If $m \neq 1, 2, 5$ or 7 and if m is even the proof is imbedded in Case 2 of the proof of Theorem 1; if $m=3$ or 9 the examples suffice; if $m=2K-1$ for $K>5$ then

$$R_{10} = \{9, 8, 7, 6, \cdots, 6, 5, 4, 3, \cdots, 3, 2, 1\},$$

where 6 and 3 occur $K-4$ times, is self producing. It is easy to see that there are no self producing sequences for $m=1$ and $m=2$.

To show there is no self producing sequence for $m=5$ we assume that there is a self producing sequence $\{9, a, b, c, d\}$ and consider:

$$\begin{array}{ccccc} 9 & a & b & c & d \\ -\,d & c & b & a & 9 \\ \hline 9-d & x & 9 & y & d+1. \end{array}$$

Notice that $9-d>x$, $x+y=8$ and either $c=d$ or $c=d+1$. If $c=d$ then $x=y=d$ and $a=9-d$; but $a=2d+1$ so $3d=8$ which is impossible. If $c=d+1$ then $y \geqq d+1$, so $x=d$ and $y=8-d$, so $a=9-d$, but $a=2d+2$ so $3d=7$, which is impossible.

To show that there is no self producing sequence for $m=7$ we assume that there is a self producing sequence $\{9, a, b, c, d, e, f\}$ and consider:

$$\begin{array}{ccccccc} 9 & a & b & c & d & e & f \\ -f & e & d & c & b & a & 9 \\ \hline 9-f & x & y & 9 & z & w & f+1. \end{array}$$

Notice that $9-f \geqq x>y$, $z \geqq w$, $x+w=9$, $y+z=8$ and either $e=f$ or $e=f+1$. If $e=f$ then f must occur at least twice in the answer. From the inequalities, $w=f$ and either z or $y=f$. Now $w=f \Rightarrow x=9-f$, but $x=a-e=a-f$ $\therefore$ $a=9$. Since $z \leqq z+y=8$, $a=9-f$; $\therefore$ $f=0$, whence $x=9$, which implies that $b=9$, since there are three 9's in the answer. $b=9 \Rightarrow z=d$, leaving c and y as the only unmatched elements; $\therefore$ $c=y$. Since $c \geqq d$, $y \geqq z$. Since one of y or z must be 0 it must be $z=0=d$. Since $d=e=f=0$, three 0's are required in the answer; but $x+y+z+w=17$, which is impossible with three of them zeros. If $e=f+1$ then (by considering size) $y=f$. It now follows that $z=8-f$. Since a is next to the largest, $a=9-f$. Now $x=a-e=8-2f$. By considering sizes we see that $b=8-f=z$. Now $d=b-1-y=7-2f$. But d and w are the only unmatched digits, and therefore $d=w=2f+1$. Combining the two expressions for d we have $4f=6$, which is impossible.

Other corollaries of this type are possible to other bases.

THEOREM 2. *The base n possesses a sequence of two digits that is self producing if and only if $n=3g+2$.*

Proof. If $n=3g+2$ the proof is imbedded in Case 2 of the proof of Theorem 1. Let $\{a, b\}$, with $a>b$, be self producing to the base n; then $an+b-(bn+a)=bn+a$ or $2b+1=a$ and $2a-n=b$, whence $n=3b+2$.

DEFINITION. *A self producing sequence of digits is unanimous if it is eventually obtained from any other sequence of digits, not all the same, by repeated Kaprekar Operations.*

Examples. $R_5=\{3, 3, 2, 0\}$; $R_{10}=\{7, 6, 4, 1\}$; $R_{10}=\{9, 5, 4\}$; and $R_2=\{1, 0\}$ are unanimous.

Necessary and sufficient conditions are desired to tell when a base n has a unanimous sequence of m digits. An obvious necessary condition is that the base n must have exactly one self producing sequence of m digits. This condition is not sufficient since $R_4=\{3, 2, 1, 0\}$ and $R_8=\{5, 2\}$ are not unanimous. An immediate consequence of this condition is that if n satisfies the conditions of Corollary 2 then it would not possess a unanimous set of m digits for m even and $m>6$.

As observed by the referee, there is a positive result in this direction.

THEOREM 3. *The base n possesses a unanimous sequence of 3 digits if and only if $n=2K$.*

Proof. If $n=2K$ then $\{n-1, K, K-1\}$ is self producing by the proof of Theorem 1. Now since the Kaprekar Operation on 3 digits in the base n always yields $n-1$ as one of its digits and the sum of the other two equals $n-1$, the sequence after one Kaprekar Operation would be $\{n-1, a, n-1-a\}$, where $a\geqq K$. Repeating the Kaprekar Operation yields $\{n-1, a-1, n-a\}$; thus $a-K$ repetitions of the Kaprekar Operation will result in $\{n-1, K, K-1\}$ which is then unanimous. If $n=2K+1$, a self producing sequence would be of the form $\{2K, a, 2K-a\}$, with $a\geqq K$. Applying the Kaprekar Operation to this set we obtain $\{2K, a-1, 2K-a+1\}$, thus $2a-1=2K$, which is impossible; hence a self producing sequence does not exist, and therefore a unanimous sequence is impossible.

THEOREM 4. *If R_2 has $h\geqq 1$ ones and $m-h\geqq 1$ zeros then $K(R_2)$ has* $\max(h, m-h)$ *ones.*

Proof. Let $p=\max(h, m-h)$ and $q=\min(h, m-h)$. The largest number formed by the digits of R_2 is $A=2^{m+1}-2^{m-h+1}$ and the smallest $B=2^{h+1}-1$. Hence

$$\begin{aligned} A-B &= 2^{m+1}-2^{p+1}-2^{q+1}+1 \\ &= 2^m+2^{m-1}+\cdots+2^{p+2}+2^{p+1}-2^{q+1}+1 \\ &= 2^m+2^{m-1}+\cdots+2^{p+2}+2^p+2^{p-1}+\cdots+2^{q+1}+1 \end{aligned}$$

which is the desired result.

COROLLARY 1. *$K(R_2)=R_2$ if and only if $h\geqq m-h$.*

COROLLARY 2. *The base two has a unanimous sequence of m digits if and only if $m=2$ or 3.*

Are there infinitely many unanimous sequences for $m=2$ or $m=4$? Are there any unanimous sequences for $m\geqq 5$?

Reference

1. D. R. Kaprekar, An Interesting Property of the Number 6174, Scripta Math., 21 (1955) 304.

EXPECTATION FOR SOLITAIRE

R. A. JACOBSON, South Dakota State College

In this paper we discuss a generalization of problem E 1544 [1]. A solitaire player shuffles a deck of cards numbered from 1 to n inclusive, and lays them face up, one at a time. Every time he turns up a card, he scores a number of points, to be determined in one of the two following methods:

(i) the score is based on the card just turned up and all preceding cards,

(ii) the score is based on the card just turned up and its immediate predecessor only.

In either case, the first card turned over contributes a score of zero. Our problem is to find the expectation of his score, assuming that all arrangements of the deck are equiprobable.

For example, the games might be scored as follows:

Game 1. Score 1 point for *each* preceding card which is less and 0 points for *each* preceding card which is greater than the card just turned up.

Game 2. Score the positive difference between the card just turned up and its immediate predecessor if the predecessor is greater. If the predecessor is smaller, score zero.

Game 3. If the card just turned up is greater than its immediate predecessor, score two points if their sum is even and zero points if their sum is odd. If the predecessor is greater, score minus one.

Game 4. Score two points for *each* odd preceding card less than the card just turned up, and score zero points for all other cards.

It is evident that games 1 and 4 are scored by method (i) while games 2 and 3 are examples of method (ii).

In the following, $f(a, b)$ is the score achieved if card $[a]$ precedes card $[b]$. The expectation and the sum of the scores for all $n!$ equiprobable outcomes for a deck of n cards are denoted by E_n and S_n, respectively; that is, $E_n = S_n/n!$.

THEOREM 1. *In case* (i), *the expected value is given by the recursion relation*

$$E_{n+1} = E_n + \frac{1}{2} \sum_{q=1}^{n} \{f(q, n+1) + f(n+1, q)\}; \qquad E_1 = 0. \tag{1}$$

Proof. Consider the $n!$ by n array (matrix) of equiprobable arrangements of a deck of n cards where each row indicates one ordering in which the cards are turned face up, one at a time. Let s_k be the score associated with row k, so that $S_n = \sum_{k=1}^{n!} s_k$. The $(n+1)!$ equiprobable arrangements and the total score, S_{n+1}, for a deck of $n+1$ cards can be found by inserting card $[n+1]$ in the above array. We place card $[n+1]$ between the p and $(p+1)$-st card of row k and compute the resulting score for this arrangement. The scoring would be unchanged until card $[n+1]$ is turned up. Turning up card $[n+1]$, we score $\sum_{i=1}^{p} f(q_i, n+1)$

points, where cards $[q_i]$ are the cards which precede $[n+1]$. Thereafter, upon turning up *each* card $[q_j]$, $j=p+1, \cdots, n$, we would score the same as in the previous array plus the score $f(n+1, q_j)$. Thus, the total score given by inserting $[n+1]$ at this point in row k is

$$s_k + \sum_{i=1}^{p} f(q_i, n+1) + \sum_{j=p+1}^{n} f(n+1, q_j).$$

Since each card $[q]$ appears $p(n-1)!$ times in the original array in columns $1, 2, \cdots, p$; it follows that the insertion of card $[n+1]$ in each of the $n!$ rows between cards of column p and column $p+1$ yields a total score of

$$\sum_{k=1}^{n!} s_k + \sum_{q=1}^{n} p(n-1)! f(q, n+1) + \sum_{q=1}^{n} (n-p)(n-1)! f(n+1, q). \tag{2}$$

Sum (2) also holds if $p=0$, n; in other words, if card $[n+1]$ is first or last.

Thus, we find that the total sum of all $(n+1)!$ equiprobable scores is

$$\begin{aligned} S_{n+1} &= \sum_{p=0}^{n} \left[S_n + (n-1)! \sum_{q=1}^{n} \{ p f(q, n+1) + (n-p) f(n+1, q) \} \right] \\ &= (n+1)S_n + (n-1)! \frac{n(n+1)}{2} \sum_{q=1}^{n} \{ f(q, n+1) + f(n+1, q) \} \\ &= (n+1)S_n + \frac{(n+1)!}{2} \sum_{q=1}^{n} \{ f(q, n+1) + f(n+1, q) \}. \end{aligned} \tag{3}$$

The result (1) is then obtained by substituting $S_n = n!E_n$, $S_{n+1} = (n+1)!E_{n+1}$ in (3) and simplifying.

For game 1, $f(a, b) = 1$, $a<b$; $f(a, b) = 0$, $a>b$ and (1) becomes the difference equation

$$\begin{aligned} E_{n+1} &= E_n + \frac{1}{2} \sum_{q=1}^{n} (1), \\ &= E_n + \frac{n}{2}, \qquad E_1 = 0; \end{aligned}$$

and thus $E_n = n(n-1)/4$, $n = 1, 2, \cdots$.

For game 4, $f(a, b) = 1-(-1)^a$, $a<b$; $f(a, b) = 0$, $a>b$ and (1) becomes the difference equation

$$\begin{aligned} E_{n+1} &= E_n + \frac{1}{2} \sum_{q=1}^{n} \{ 1 - (-1)^q \}, \\ &= E_n + \frac{1}{2}\left(n + \frac{1-(-1)^n}{2} \right), \qquad E_1 = 0; \end{aligned}$$

and thus

$$E_n = \frac{2n^2 - 1 + (-1)^n}{8}, \qquad n = 1, 2, \cdots.$$

THEOREM 2. *In case* (ii), *the expected value is given by the recursion relation*

$$E_{n+1} = E_n + \frac{1}{n+1}\Bigg[\sum_{q=1}^{n}\{f(q, n+1) + f(n+1, q)\} - \frac{1}{n}\sum_{q=2}^{n}\sum_{r=1}^{q-1}\{f(q, r) + f(r, q)\}\Bigg], \qquad E_1 = 0. \tag{4}$$

Preceding as before, we find the total sum S_{n+1} by inserting card $[n+1]$ in the original array of equiprobable outcomes for a deck of n cards. If card $[n+1]$ is placed between card $[a]$ and $[b]$ of row k in the original array, we get the score $s_k+f(a, n+1)-f(a, b)+f(n+1, b)$. A particular pair of cards $[q]$, $[r]$ occur in $(n-2)!$ rows of the original array in columns p and $(p+1)$ in both the orders $[q]$, $[r]$ and $[r]$, $[q]$. It follows that the sum of the scores obtained by inserting card $[n+1]$ between columns p and $(p+1)$ in these $(n-2)!$ rows would equal the sum obtained from these rows in the original array plus the sum

$$(n-2)!\{f(q, n+1) - f(q, r) + f(n+1, r)\}$$

for the rows in which $[q]$ precedes $[r]$, and the sum

$$(n-2)!\{f(r, n+1) - f(r, q) + f(n+1, q)\}$$

for the rows in which $[r]$ precedes $[q]$. Utilizing all possible pairs of cards, we see that the sum of the scores obtained by inserting card $[n+1]$ between columns p and $(p+1)$ is

$$\begin{aligned} S_n + (n-2)!\sum_{q=2}^{n}\sum_{r=1}^{q-1}\{f(q, n+1) - f(q, r) + f(n+1, r) + f(r, n+1) \\ - f(r, q) + f(n+1, q)\} \\ = S_n + (n-1)!\sum_{q=1}^{n}\{f(q, n+1) + f(n+1, q)\} \\ - (n-2)!\sum_{q=2}^{n}\sum_{r=1}^{q-1}\{f(q, r) + f(r, q)\}. \end{aligned} \tag{5}$$

If card $[n+1]$ is first in a row, the score is increased by $f(n+1, q)$ where $[q]$ was previously the first card in the row. Since card $[q]$ is first in $(n-1)!$ rows of the original array for each q, the resulting sum for all arrangements in which card $[n+1]$ is now first is

$$S_n + (n-1)!\sum_{q=1}^{n} f(n+1, q). \tag{6}$$

Similarly, the scores obtained when card $[n+1]$ is last in all rows add up to

$$S_n + (n-1)!\sum_{q=1}^{n} f(q, n+1). \tag{7}$$

Noting that card $[n+1]$ can be placed between two adjacent columns in $n-1$ ways, we combine (5), (6) and (7) to find that the total sum for all $(n+1)!$ equiprobable outcomes is

$$\begin{aligned} S_{n+1} = (n-1)\Big[S_n &+ (n-1)!\sum_{q=1}^{n}\{f(q, n+1) + f(n+1, q)\} \\ &- (n-2)!\sum_{q=2}^{n}\sum_{r=1}^{q-1}\{f(q,r) + f(r,q)\}\Big] \\ &+ \Big[S_n + (n-1)!\sum_{q=1}^{n} f(n+1, q)\Big] + \Big[S_n + (n-1)!\sum_{q=1}^{n} f(q, n+1)\Big], \end{aligned}$$

$$\begin{aligned} S_{n+1} = (n+1)S_n &+ n!\sum_{q=1}^{n}\{f(q, n+1) + f(n+1, q)\} \\ &- (n-1)!\sum_{q=2}^{n}\sum_{r=1}^{q-1}\{f(q,r) + f(r,q)\}. \end{aligned} \tag{8}$$

Substituting $S_n = n!\,E_n$ and $S_{n+1} = (n+1)!\,E_{n+1}$ in (8) and simplifying, we have equation (4).

In game 2, $f(a, b) = 0$, $a < b$; $f(a, b) = a - b$, $a > b$ and the expectation from (4) is

$$\begin{aligned} E_{n+1} &= E_n + \frac{1}{n+1}\left[\sum_{q=1}^{n}(n+1-q) - \frac{1}{n}\sum_{q=2}^{n}\sum_{r=1}^{q-1}(q-r)\right], \\ &= E_n + \frac{1}{n+1}\left[\frac{n(n+1)}{2} - \frac{1}{n}\sum_{q=2}^{n}\frac{q(q-1)}{2}\right], \\ &= E_n + \frac{2n+1}{6}, \qquad E_1 = 0; \end{aligned}$$

and thus $E_n = (n^2-1)/6$, $n = 1, 2, \cdots$.

In game 3, $f(a, b) = 1 + (-1)^{a+b}$, $a < b$, $f(a, b) = -1$, $a > b$ and the expectation from (4) is

$$\begin{aligned} E_{n+1} = E_n &+ \frac{1}{n+1}\Big[\sum_{q=1}^{n}\{1 + (-1)^{q+n+1} + (-1)\} \\ &\qquad - \frac{1}{n}\sum_{q=2}^{n}\sum_{r=1}^{q-1}\{(-1) + 1 + (-1)^{r+q}\}\Big] \\ = E_n &+ \frac{1}{n+1}\left[\sum_{q=1}^{n}(-1)^{q+n+1} - \frac{1}{n}\sum_{q=2}^{n}\sum_{r=1}^{q-1}(-1)^{r+q}\right] \end{aligned}$$

$$= E_n + \frac{1}{n+1}\left[\frac{(-1)^n - 1}{2} + \frac{1}{n}\sum_{q=2}^{n}\frac{(-1)^q + 1}{2}\right]$$

$$= E_n + \frac{(2n+1)(-1)^n - 1}{4n(n+1)}, \qquad E_1 = 0,$$

and thus $E_n = \{1 - 2n - (-1)^n\}/4n$, $n = 1, 2, \cdots$.

Reference

1. H. W. Hickey, this MONTHLY, 69 (1962) 920.

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland.

REMARKS ABOUT QUOTIENTS

RAYMOND M. REDHEFFER, University of California, Los Angeles

1. Introduction. In [1] Thomas Mott establishes an interesting result for nonnegative functions f and g, under the assumption that f and $-g$ are nondecreasing. Specifically, if

$$u(x) = \int_a^x f(s)ds \quad \text{and} \quad v(x) = \int_a^x g(s)ds \tag{1}$$

then u/v is nondecreasing at points where $v \neq 0$.

Because of the monotony of f and g the set where u and v are positive is an interval, outside of which both f and g are 0. It suffices, therefore, to prove the theorem under the assumption that both u and v are positive.

If the monotony fails, so that $u(x_1)v(x_2) > u(x_2)v(x_1)$ for some $x_1 < x_2$, then we can approximate f and g by continuous positive increasing functions, f° and g°, in such a way that the same inequality holds for the corresponding integrals u° and v°. Thus, it suffices to establish the result for f and g continuous.

The theorem for this case requires nothing beyond first-semester calculus. Let $u = e^U$ and $v = e^V$. We would like to show that $\log(u/v) = U - V$ is increasing or, equivalently, that $U' \geqq V'$. The mean-value theorem gives

$$U' = \frac{f(x)}{u(x)} = \frac{f(x)}{(x-a)f(\xi)} \geqq \frac{1}{x-a}$$

when we recall that $f(x) \geqq f(\xi)$ for $a < \xi < x$. Similarly, $V' \leqq 1/(x-a)$ and hence, the proof is complete.

2. A differential inequality. As a rule, the difference of two increasing functions U and V is only of bounded variation; $U-V$ can have infinitely many maxima and minima. Nevertheless the function $U-V$ of the preceding discussion turned out to be monotone! Why?

The reason is that u and v almost satisfy a certain differential inequality. Indeed, we could have approximated f and $-g$ by strictly increasing differentiable functions. Then $u'' > 0$, $v'' < 0$, and hence, at points where $u > 0$ and $v > 0$, we have $u^{-1}u'' > v^{-1}v''$.

More generally, let an operator T be defined by $T0 = 0$, and by

$$Tu = \phi(x) - f(x, u^{-1}u', u^{-1}u''), \tag{2}$$

whenever $u \neq 0$. At an interior point P, where $u > 0$ and $v > 0$, suppose $Tu < Tv$, and suppose

$$f(x, u^{-1}u', s) \quad \text{or} \quad f(x, v^{-1}v', s) \qquad (x = P) \tag{3}$$

is monotone nondecreasing as a function of s. Then u/v cannot have a relative maximum at P. For proof set $u = e^U$ as before, so that $Tu = \tilde{T}U$, where

$$\tilde{T}U = \phi(x) - f[x, U', U'' + (U')^2].$$

At P we have a relative maximum for $U-V$, hence $U' = V'$, $U'' \leqq V''$, and the assumed relation $\tilde{T}U < \tilde{T}V$ fails. The special case $Tu = -u^{-1}u''$ gives another proof of Mott's theorem.

3. A kind of duality. In an interval (a, b) let T have the form (2), let $u(a) = u(b) = 0$, and let u be continuous for $a \leqq x \leqq b$. Suppose that, at each point where $uv \neq 0$, one of the functions (3) is nondecreasing in s. Then the following conditions are incompatible:

$$u \not\equiv 0, \qquad Tu < Tv, \qquad \inf |v| > 0 \qquad (a < x < b).$$

Indeed, if all three hold we can assume $\sup u > 0$, since $T(-u) = Tu$, and similarly we can assume $\inf v > 0$. Then u/v has an interior maximum, in contradiction to the foregoing considerations.

The above remark can be used in three ways. If there exists a v satisfying $Tu < Tv$ and $\inf v > 0$, then $u \equiv 0$. This is a uniqueness theorem for u.

If there exists a $u \not\equiv 0$ such that $Tu < Tv$ then $\inf |v| = 0$. This is a Sturmian theorem for v.

If there exists a solution $u \not\equiv 0$ of the equation $Tu = \lambda$, where λ is a constant, and if v is any $C^{(2)}$ function satisfying $\inf |v| > 0$, then it is not the case that $Tu < Tv$. This yields a lower bound for the characteristic value λ; namely, $\lambda \geqq \inf_x Tv$. (The estimate is often sharp, because the choice $v = u$ would give equality.)

4. A more general setting. The same methods apply to many partial differential equations. As an illustration, consider the operator

$$Tu = \phi(x) - f(x, u^{-1} \operatorname{grad} u, u^{-1}\Delta u), \tag{4}$$

where x is a point of Euclidean n-space, Δu is the Laplacian, and the monotony condition analogous to (3) holds. The interval (a, b) is now replaced by a bounded region B. An appropriate boundary operator (also admissible for the one-dimensional case) is

$$Ru = \kappa(x) - k(x, u^{-1}u_\nu), \qquad (x \in \partial B),$$

where $k(x, s)$ is nondecreasing in s, and where u_ν denotes the normal derivative. At each boundary point we need: either $u=0$, or $Ru<Rv$.

Discussion of this more general problem is closely analogous to the discussion of (2), as the reader can verify. It turns out, also, that mild continuity conditions, coupled with strict monotony of f, enable us to replace the condition $Tu<Tv$ by $Tu \leqq Tv$.

As an application consider the theory of functions of a complex variable. By the Cauchy-Riemann equations the function $u=|f(z)|^2$ satisfies $Tu=0$, where

$$Tu = |u^{-1} \operatorname{grad} u|^2 - u^{-1}\Delta u.$$

If $v=|g(z)|^2$ then $Tu \leqq Tv$ and the general theory applies. The assertion that u/v satisfies the maximum principle is the ordinary maximum principle when $v=1$, it is the Schwarz lemma when $v=|z|$, and it is essentially the three-circles theorem of Hadamard when $v=|z|^\alpha$. Analogs of these useful tools of function theory are thus available for broad classes of nonlinear partial differential equations.

5. Concluding remark. It is pleasant to find unity in seeming diversity. Maximum principles, uniqueness, Sturmian theorems, lower bounds for characteristic values, the Schwarz lemma from function theory, and the theorem of Mott on integrals not only belong in the same ball park, but they are, so to speak, members of the same team.

References

1. Thomas E. Mott, On the quotient of monotone functions, this MONTHLY, 70 (1963) 195–199.

2. Raymond M. Redheffer, Maximum principles and duality, Monatsh. Math., 62 (1958) 56–75.

Mathematical Swifties

"It's an equivalence class of rational Cauchy sequences," Tom said realistically.
"Is the closure of this set equal to the whole space?" Tom asked densely.
"That line is perpendicular to the plane," Tom stated normally.
"Of course that open cover has a finite subcover," Tom observed compactly.
"Is that expansion taylor-made?" Tom asked seriously.
"Those sets have no elements in common," Tom remarked disjointedly.

R. T. SMYTHE

LEFT AND RIGHT IDEALS IN THE RING OF 2×2 MATRICES

HENRYK MINC, University of Florida and University of California, Santa Barbara

One of the most illuminating examples for a first course on rings and ideals is the ring M_2 of 2×2 matrices over, say, the real field. Although all properties of left and right ideals of this ring and of ideals of its subrings are well known, yet surprisingly enough they are not discussed in any of the standard text-books. In this note we characterize all the left and all the right ideals of M_2 and thus show that M_2 is simple. We examine in the same manner the ring of the upper triangular 2×2 matrices and find all its ideals. We use the terminology of [1].

It is well known that if a left (right) ideal S of a ring R with identity 1 contains a left (right) regular element then $S=R$. For, if $a\in S$ and $ba=1$ for some $b\in R$ then $ba=1\in RS=S$ and therefore $x=x1\in S$ for any $x\in R$. In particular, if any left or right ideal S of M_2 contains a non-singular matrix then $S=M_2$. We now prove that every left (right) ideal of M_2 is a principal left (right) ideal and thus determine the structure of matrices in such ideals.

THEOREM 1. *The principal left ideal generated by a singular nonzero matrix*

$$A = \begin{pmatrix} ax & bx \\ ay & by \end{pmatrix}$$

consists of all matrices of the form

$$\begin{pmatrix} ar_1 & br_1 \\ ar_2 & br_2 \end{pmatrix}$$

where r_1 and r_2 run over all real numbers, i.e. of all matrices whose rows are scalar multiples of the vector $(a\ b)$. The principal right ideal generated by the matrix A consists of all matrices whose columns are scalar multiples of the vector $\binom{x}{y}$.

Proof. Let $C=(c_{ij})$ be any matrix in M_2. It suffices to prove that the rows of CA are scalar multiples of $(a\ b)$ and the columns of AC are scalar multiples of $\binom{x}{y}$.

$$CA = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}(a\ b) = \begin{pmatrix} c_{11}x + c_{12}y \\ c_{21}x + c_{22}y \end{pmatrix}(a\ b)$$

and

$$AC = \begin{pmatrix} x \\ y \end{pmatrix}(a\ b)\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}(ac_{11} + bc_{21} \quad ac_{12} + bc_{22});$$

i.e.

$$CA = \begin{pmatrix} ar_1 & br_1 \\ ar_2 & br_2 \end{pmatrix} \quad \text{and} \quad AC = \begin{pmatrix} xs_1 & xs_2 \\ ys_1 & ys_2 \end{pmatrix},$$

where

$$r_1 = c_{11}x + c_{12}y, \qquad r_2 = c_{21}x + c_{22}y, \qquad s_1 = ac_{11} + bc_{21}, \qquad s_2 = ac_{12} + bc_{22}.$$

THEOREM 2. *Every left (right) ideal of M_2 is a principal left (right) ideal.*

Proof. We prove the theorem for left ideals. The ring M_2 is a principal left ideal generated by any nonsingular matrix. In view of Theorem 1 it suffices to prove that if a left ideal S of M_2 contains matrices A and B such that $A_{(i)}$, the ith row of A, and $B_{(j)}$, the jth row of B, are linearly independent, then $S = M_2$. Let E_{rs} denote the 2×2 matrix with 1 in the (r, s) position and zeros elsewhere. Then the matrix

$$E_{1i}A + E_{2j}B = \begin{pmatrix} A_{(i)} \\ B_{(j)} \end{pmatrix}$$

belongs to S and has linearly independent rows, i.e. it is nonsingular. Hence $S = M_2$.

We prove similarly that every right ideal of M_2 is principal.

COROLLARY 1. *A left ideal of M_2 is proper if and only if it consists of all matrices whose rows are scalar multiples of a fixed nonzero vector.*

COROLLARY 2. *A right ideal of M_2 is proper if and only if it consists of all matrices whose columns are scalar multiples of a fixed nonzero vector.*

THEOREM 3. *The ring M_2 is simple.*

For, if S is a proper left ideal of M_2 then, by Corollary 1, it contains nonzero matrices

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix},$$

where a and b are some real numbers, not both zero. But then Corollary 2 implies that S cannot be a proper right ideal.

Note. The usual proof of simplicity of M_n, the ring of $n \times n$ matrices, can of course be used to prove that M_2 is simple. Thus let S be a nonzero ideal of M_2 and suppose that $A = (a_{ij}) \in S$ and $a_{rs} \neq 0$. Since S is a two-sided ideal, $XAY \in S$ for any X and Y in M_2. In particular, the nonsingular matrix

$$E_{1r}AE_{s1} + E_{2r}AE_{s2} = \begin{pmatrix} a_{rs} & 0 \\ 0 & a_{rs} \end{pmatrix}$$

belongs to S. Hence $S = M_2$. Thus the only nonzero ideal of M_2 is the ring itself and therefore M_2 is simple.

We now investigate left and right ideals in T_2, the ring of 2×2 upper triangular matrices, and show that this subring of M_2 contains proper two-sided ideals.

THEOREM 4. *If S is a left ideal of T_2 and $S \neq T_2$ then either* (i) *S consists of all scalar multiples of $a_{11}E_{11}+a_{12}E_{12}$ for some fixed scalars a_{11} and a_{12}, or* (ii) *S consists of all 2×2 matrices whose second row is zero, or* (iii) *S consists of all 2×2 matrices whose first column is zero.*

Proof. Since $S\neq T_2$ the left ideal S cannot contain a nonsingular matrix. Hence if $A=(a_{ij})\in S$ then either $a_{11}=a_{21}=0$ or $a_{21}=a_{22}=0$. Moreover, S cannot contain matrices $B=(b_{ij})$ and $C=(c_{ij})$ such that $b_{21}=b_{22}=0$, $b_{11}\neq 0$ and $c_{11}=c_{21}=0$, $c_{22}\neq 0$, for then $B+C$ is nonsingular. It follows that all matrices in S have either a zero row or a zero column. Suppose now that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}$$

is a nonzero matrix in S and let $Z=(z_{ij})$ be any matrix in T_2. Then $ZA=z_{11}A$ and thus the principal left ideal generated by A consists of all scalar multiples of A. If S contains also a matrix

$$B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & 0 \end{pmatrix}$$

which is not a scalar multiple of A then it must contain all matrices whose second row is zero. For, $(a_{11}\, a_{12})$ and $(b_{11}\, b_{12})$ are then linearly independent and, if

$$H = \begin{pmatrix} h_{11} & h_{12} \\ 0 & 0 \end{pmatrix}$$

is any matrix whose second row is zero, there exist scalars x and y such that $x(a_{11}\, a_{12})+y(b_{11}\, b_{12})=(h_{11}\, h_{12})$, i.e., $XA+YB=H$, where X and Y are any matrices in T_2 with x and y as their (1, 1) entries respectively. Hence $H\in S$.

Now suppose that a left ideal S of T_2 contains a matrix

$$C = \begin{pmatrix} 0 & c_{12} \\ 0 & c_{22} \end{pmatrix}, \qquad c_{22} \neq 0,$$

and $X=(x_{ij})$ is any matrix in T_2. Then

$$XC = \begin{pmatrix} 0 & x_{11}c_{12}+x_{12}c_{22} \\ 0 & x_{22}c_{22} \end{pmatrix}$$

and clearly S contains all 2×2 matrices whose first column is zero.

THEOREM 5. *If S is a right ideal of T_2 and $S\neq T_2$ then either* (i) *S consists of all scalar multiples of $a_{12}E_{12}+a_{22}E_{22}$ for some fixed scalars a_{12} and a_{22} or* (ii) *S consists of all 2×2 matrices whose second row is zero, or* (iii) *S consists of all 2×2 matrices whose first column is zero.*

The proof is similar to that of Theorem 4.

The following result is an immediate consequence of Theorems 4 and 5 and of the definition of an ideal (two-sided).

THEOREM 6. *The ring* T_2 *contains exactly three proper nonzero ideals*: (i) *the ideal consisting of all scalar multiples of* E_{12}, (ii) *the ideal consisting of all* 2×2 *matrices whose second row is zero*, (iii) *the ideal consisting of all* 2×2 *matrices whose first column is zero.*

Reference

1. N. Jacobson, Lectures on Abstract Algebra, I, II, Van Nostrand, Princeton, 1951/53.

SOME RECURSIVE FORMULAS FOR EVALUATION OF A CLASS OF DEFINITE INTEGRALS

M. L. GLASSER, Battelle Memorial Institute

We consider the class of integrals $\int_0^\infty f(x)(\log x)^n dx$, where $f(x)$ is a rational function having no real poles and a zero of at least second order at infinity. Log x denotes the principal branch of the logarithm function:

$$\log z = \log \rho + i\theta, 0 \leqq \theta < 2\pi,$$

where $z = \rho e^{i\theta}$. Let $\sigma[f(z)]$ denote the sum of the residues at the poles of f in the upper half plane.

THEOREM I [1].

$$\int_0^\infty [f(x) + f(-x)] \log x dx + \pi i \int_0^\infty f(x) dx = 2\pi i \sigma[f(-z) \log z].$$

Proof. Let C be the positively oriented boundary of the region between two semicircles, about the origin in the upper $z = x + iy$ plane, of radii $P > p$. Let P be large enough and p small enough so that C encloses all the poles of f in the upper half plane. Then, breaking up

$$\int_C f(-z) \log z dz$$

into integrals over the various segments of the contour, we find that the integrals over the semicircles vanish at $p \to 0$, $P \to \infty$; e.g., over the smaller semicircle,

$$\left| \int f(-z) \log z dz \right| \leqq \int_0^\pi | f(-pe^{i\theta}) p [\log p + i\theta] | d\theta$$

which vanishes as $p \to 0$. By a simple change of variable the remaining two integrals may be written

$$\int_p^P f(-x) \log x dx + \int_p^P f(x)(\log x + i\pi) dx.$$

The theorem follows upon applying Cauchy's theorem and letting $p\to 0$, $P\to\infty$.

COROLLARY. *If f is odd then* $\int_0^\infty f(x)dx=-2\sigma[f(z)\log z]$.

If f is even then $\int_0^\infty f(x)\log x dx=i\pi\sigma[f(z)\log z]+\frac{1}{2}\pi^2\sigma[f(z)]$.

THEOREM II.

$$\int_0^\infty (\log x)^n[f(z)+f(-z)]dz+\sum_{k=1}^n\binom{n}{k}(\pi i)^k\int_0^\infty(\log x)^{n-k}f(x)dx$$
$$=2\pi i\sigma[(\log z)^n f(-z)].$$

Proof. Let C be the above contour. Breaking up $\int_C(\log z)^n f(-z)dz$ as before, we again find that as $P\to\infty$, $p\to 0$ the integrals along the semicircles vanish and we have left

$$\int_p^P(\log x)^n f(-x)dx+\int_p^P(\log x+\pi i)^n f(x)dx.$$

The theorem follows from expanding the second integral by the binomial theorem, applying Cauchy's theorem and letting $p\to 0$, $P\to\infty$.

Let I_j denote $\int_0^\infty(\log x)^{j-1}f(x)dx$, $j=1, 2, \cdots$; and $\sigma_k=\sigma[(\log z)^k f(-z)]$. Then, immediately, we have the following

COROLLARY. *If f is odd then*

$$I_n=\frac{1}{n}\left[2\sigma_n-\sum_{k=2}^n\binom{n}{k}(\pi i)^{k-1}I_{n+1-k}\right].$$

If f is even then

$$I_{n+1}=\frac{\pi i}{2}\left[2\sigma_n-\sum_{k=1}^n\binom{n}{k}(\pi i)^{k-1}I_{n+1-k}\right].$$

As an example we consider

$$I_3=\int_0^\infty\frac{(\log x)^2}{x^2+1}dx,$$

where $f(z)=(z^2+1)^{-1}$ and is even. There is only one pole in the upper half plane, at $z=i$. We easily find $\sigma_0=1/2i$, $\sigma_1=\pi/4$, $\sigma_2=\pi^2 i/8$. Thus $I_1=\pi/2$, $I_2=0$, $I_3=\pi^3/8$.

Reference

1. E. T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford, University Press, London, 1935, p. 154, Ex. 24.

ON AN ITERATIVE METHOD

TOSIHIRO HOMMA, Osaka Gakugei University

The following theorem is an example of the power of iterative methods:

SCHAEFER'S THEOREM [2]. *If a completely continuous (not necessarily linear) map A maps a bounded closed convex set C of a Hilbert space into itself so that*

$$\|Ax - Ay\| \leqq \|x - y\| \tag{1}$$

for $x, y \in C$, then the sequence

$$x_{n+1} = aAx_n + (1 - a)x_n \qquad (n = 0, 1, 2, \cdots)$$

converges to a fixed point of A for any $x_0 \in C$ and any a such that $0 < a < 1$.

The success of an iterative method may be a simple consequence of a theorem in general topology. Cheney and Goldstein [1] show that their theorem contains Schaefer's when A satisfies the additional condition that $\|Ax - Ay\| < \|x - y\|$ unless $x = y$.

THEOREM OF CHENEY AND GOLDSTEIN. *Let B be a map of a metric linear space into itself satisfying* (1),

$$\|Bx - B^2x\| < \|x - Bx\| \text{ unless } Bx = x, \tag{2}$$

and

$$\{B^nx: n = 0, 1, 2, \cdots\} \text{ has a cluster point,} \tag{3}$$

then B^nx converges to a fixed point of B.

Since Schaefer's original proof uses either Schauder's fixed point theorem or the spectral theorem, whereas Cheney-Goldstein's proof is completely elementary, it may be observed that Schaefer's theorem is implied by Cheney-Goldstein's theorem if A is assumed to be linear.

PROPOSITION. *Cheney-Goldstein's theorem implies Schaefer's if A is linear, but not if A is nonlinear.*

Proof. Suppose that A is linear and $B = aA + (1-a)I$, where I is the identity operator. Since $x_n = B^nx_0$, x_n is contained in the convex closure D of $\{A^nx_0: n = 0, 1, 2, \cdots\}$ which is compact according to the complete continuity of A. Moreover, D is invariant under A. Hence (3) is satisfied by B.

Now, assume $Ax_0 \neq x_0$. If $A(Bx_0 - x_0) = Bx_0 - x_0$, then $A^2x_0 - Ax_0 = Ax_0 - x_0$ since $Bx - x = a(Ax - x)$, so that we have $A^nx_0 - A^{n-1}x_0 = Ax_0 - x_0$ for $n = 1, 2, \cdots$, which contradicts (3). Therefore $A(Bx_0 - x_0) \neq Bx_0 - x_0$ unless $Bx_0 = x_0$. Since A and consequently B satisfy (1) and $\|Bx\| < \|x\|$ unless $Ax = Bx = x$, B satisfies (2).

Therefore the first half of the proposition is proved. To prove the remainder, consider the real line as a Hilbert space. Let us define a completely continuous

nonlinear map A which maps $[0, 1]$ into itself:

$$Ax = \begin{cases} x - \frac{1}{4} & \text{for } x \geqq \frac{1}{4}, \\ 0, & \text{for } x < \frac{1}{4}. \end{cases} \tag{4}$$

Clearly A satisfies the hypothesis of Schaefer's theorem, whereas (2) is not satisfied by B since $\|B^2x - Bx\| = \|Bx - x\|$ for $x = 1$.

References

1. W. Cheney and A. Goldstein, A proximity map for convex sets, Proc. Amer. Math. Soc., 10 (1959) 448–450.

2. H. Schaefer, Über die Methode sukzessiver Approximationen, Jber. Deutsch. Math. Verein., 59 (1957) 131–140.

ANOTHER COMPLETENESS PROPERTY

ROBERT R. CHRISTIAN, The University of British Columbia

It is well known [see, for example, page 163 of *Universal Mathematics*, CUPM, 1958] that the only convex subsets of the real line are intervals: degenerate, finite, or infinite. That is, if an ordered field is complete, then its only convex subsets are intervals. It is not so widely recognized that the converse is also true:

THEOREM. *An ordered field F whose only convex sets are intervals (degenerate, finite, infinite), is complete.*

Proof. It will be sufficient to show that every nonempty set M that is bounded above has a least upper bound. Let I be the set of all upper bounds of M. Then I is a nonempty convex set with no number as right end point. Thus I is an infinite interval. Since $I \neq F$, I must have a left end point b (which may or may not be in I). This number is the least upper bound of M.

Evidently, the property given in the theorem can serve as a completeness axiom in defining the real number system. In some ways this new property seems more intuitively evident than other completeness properties, and so may commend itself for classroom instruction. Finally, the property in question may be stated in terms of betweenness, and hence may be of some use in geometric contexts.

Query: If $\Gamma(x)$ is the Eulerian Gamma Function, it is not hard to calculate an approximate value of $\int_0^\infty dx/\Gamma(x)$. Does anyone know of an evaluation in closed form? Address all answers to Mr. Lowell T. Van Tassel, 5550 Lodi Street, San Diego 17, California.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C., 20005

A COLLEGE PROGRAM IN MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS

ROBERT L. POE, Kansas State Teachers College of Emporia

The changes now taking place in the mathematics programs of many elementary schools have brought into prominence the question of what is the best way to strengthen the mathematical training of the teachers and supervisors of grade-school mathematics. In many colleges and universities, the content of mathematics courses for teachers is determined by the department of mathematics; and this poses a new problem for those departments of mathematics that in the past have offered only "standard courses" which all students must take, regardless of the students' major fields. There is strong evidence that such a universal approach (i.e. the same mathematics courses for *all* students) is not the most effective program. In fact, the universal approach has this danger: the needs of elementary teachers are not met in a "standard course"; and, further, a reaction to "standard courses" can, and often does, result in the creation of special "courses for teachers" that are badly planned and offer only a greatly weakened content with no compensating strengths.

The situation is made more acute by a further consideration. In the past, teachers of mathematics in elementary schools have usually not been required to have special mathematical training; but today, more and more principals and superintendents of elementary schools are asking for teachers with a strong mathematical background. That is to say, there appears to be a decided movement toward having teachers in elementary schools who would qualify as specialists in mathematics, even though these same teachers are also to teach other subjects such as geography, history, or penmanship.

At the Kansas State Teachers College of Emporia, the Department of Mathematics determines what mathematics courses the elementary teacher must take and what the mathematical content of these courses shall be. A brief description of these courses follows.

Courses required of all who wish to teach in the elementary grades. These courses are required of the prospective elementary teacher regardless of major academic subject; that is, all prospective elementary teachers, even those who are *not* planning to teach mathematics, are required to take these courses.

1. *Fundamentals of Mathematics* (two semester hours, freshman year).
2. *Structure of Arithmetic* (two semester hours, freshman or sophomore year).

The first course, *Fundamentals of Mathematics*, covers numbers, numerals, number systems, the concepts of algebra based on the laws of a number field,

elementary mathematical logic, intuitive and deductive geometry, and the theory of measurement. Although this course provides only "two hours of credit," intensive work is demanded of the student.

In the second course, *Structure of Arithmetic*, the concepts and principles of arithmetic are developed from the properties of the field of real numbers. The treatment is thorough. The insight of the student is reinforced by numerous problems that he is required to work: exercises in bases other than base 10, ratio and percentage problems, decimal representation, approximation of irrational numbers by rational numbers, etc. Again, the "two-hours credit" is misleading: the amount of work required of the student is quite heavy.

This concludes the description of the *mathematics* courses required of all prospective elementary teachers. There is, in addition, a required course (dealing with methods of teaching mathematics) given under the auspices of the Department of Education; it should be mentioned that this "methods" course requires either current registration in or prior credit for the *Structure of Arithmetic* course given by the Department of Mathematics.

Courses required for an area of concentration in mathematics. After completion of the basic program, the student is eligible to take additional mathematics courses. Many students feel that competence in mathematics is of growing importance to the elementary teacher; they believe that elementary school administrators will soon come to realize the value of properly trained teachers of mathematics. As mentioned earlier, an increasing number of principals and superintendents of elementary schools are asking for teachers with a strong background in mathematics; doubtless the students are correct in their prediction. However this may be, more and more students are choosing course work that will prepare them to be better teachers of mathematics at the grade school level. To ensure that such additional course work does in fact do what it is supposed to do (i.e. to supply the mathematical background needed by a *competent* teacher of grade school mathematics), the Department of Mathematics has designed the courses listed below; the prospective teacher who chooses this area of concentration is required to have 12 hours selected from among these courses. All these courses are taught by the staff of the Department of Mathematics.

1. Algebra for Elementary Teachers (3 semester hours)
2. Geometry for Elementary Teachers (3 sem. hrs.)
3. Astronomy (2 sem. hrs.)

After completion of Algebra and Geometry, the student is eligible to take:

4. Number Systems (3 sem. hrs.)
5. Introduction to Deductive Mathematics (3 sem. hrs.)
6. Introduction to Inductive Mathematics (3 sem. hrs.)
7. Applications of Arithmetic (A Seminar—3 sem. hrs.)

A brief description of these courses follows:

Algebra for Elementary Teachers resembles the "standard" college algebra course, except that not all topics of a "standard" course are covered. Emphasis is

placed upon those parts of algebra that are considered important to the teacher of elementary mathematics. In this way, the student thoroughly masters the material that *is* covered. Problem-solving is an important part of this course.

Some students are allowed to substitute the regular "standard" algebra course, and a few students are allowed to by-pass algebra altogether; the Department of Mathematics decides which students qualify for this substitution or waiver on the basis of high school transcripts.

Geometry for Elementary Teachers is designed to acquaint the student with the concepts, principles, and applications of geometry that are most useful to the elementary teacher. Portions of plane and solid Euclidean geometry, plane trigonometry, direct and indirect measurement, are among the topics covered.

Astronomy is taught by the staff of the Department of Mathematics. Presentation of much of the material is made vivid by lectures given in the new Planetarium located on campus. In addition to general descriptive astronomy, the student learns about eclipses, tides, the concept of time, and the construction of calendars. An elementary account is given of the theory of star evolution, star distances, star brightness, and the classification of stars.

Number Systems—an algebraic construction of the real number system, beginning with Peano's Postulates for the natural numbers. The *algebraic* aspect of number systems is emphasized: ordered pairs of numbers, algebraic extensions, equivalence relations, isomorphisms, etc. Specific examples are used extensively; e.g., modulo arithmetics are used to illustrate integral domains and fields.

Introduction to Deductive Mathematics deals with what is meant by *mathematical proof*. Various kinds of proof, such as *reductio ad absurdum*, mathematical induction, logical inference, enumeration of cases, the method of exhaustion, the exclusion principle, are treated; and illustrations are afforded by specific examples. The student becomes familiar with the working terms used in serious mathematics, such as: If . . . , then . . . ; If and only if . . . ; A necessary but not a sufficient condition . . . .

Introduction to Inductive Mathematics introduces probability and statistics. The treatment is reasonably thorough for a course offered at this level. Among the things discussed and illustrated by examples are, e.g., random sampling and discrete sample spaces. The student makes calculations using the mean and the standard deviation. Absurdities that can result from improper use of statistics are also discussed at some length.

Applications of Arithmetic is actually a seminar in *problem-solving*. The student utilizes everything he has learned from previous mathematics courses (algebra, geometry, etc.). In numerical problems, the student learns to *estimate* the answer as well as *calculate* the answer, and then to compare the two (as an aid in detecting mistakes).

Other Courses. Students with an especially strong background in mathematics are sometimes allowed to substitute more advanced courses (such as Calculus) for some of the courses listed above.

AN APPEAL FROM CUPM

As part of its effort to improve mathematics education in the colleges and universities of our nation, the Committee on the Undergraduate Program in Mathematics (CUPM) has produced proposals for recommended curricula and outlines for specific courses in several broad areas. Some of the CUPM recommended courses are of such a nature that they can be taught using existing textbooks or combinations of available books. Many of them, however, either contain material not presently available in texts at the undergraduate level or are organized in such a way as to make them difficult or impossible to teach without the preparation of new texts.

In some areas, where present publications and experience are particularly thin, CUPM has directly sponsored the writing of text materials. However, the Committee does not exist for the purpose of producing textbooks; it is determined, in fact, to do as little of this as possible. It exists for the purpose of proposing changes in mathematical curricula and stimulating the production, by individual authors, of texts reflecting its recommendations. At the same time, the Committee and its Panels are always eager to learn of forward-looking ideas in mathematics education which may be at large today and may influence their deliberations and conclusions.

For this reason, the Committee issues in this note a strong appeal to all persons associated with its work or acquainted with its recommendations for information leading to the discovery of manuscripts or sets of notes which reflect these recommendations. It also issues an appeal to the mathematical community at large for clues leading to any existing notes for experimental undergraduate mathematics courses which appear to be promising and merit wider circulation. Such notes may be extremely helpful to the Committee and might even effect major changes in some of its views.

Any person who is reached by this appeal and who knows of the existence of any such manuscript or rough set of notes will contribute significantly to the work of CUPM by sending this information to:

CUPM Central Office, P.O. Box 1024, Berkeley 1, California.

Such assistance will be greatly appreciated by CUPM and may indeed be of great benefit to mathematics education now and in the future.

THE MATHEMATICS CURRICULUM OF THE JUNIOR COLLEGES, COLLEGES, AND UNIVERSITIES IN WEST VIRGINIA 1962–63

ANDREW N. AHEART, West Virginia State College, Institute

J. William Drew [1] recently made a study of the mathematics curriculums of 27 fully accredited colleges and universities from 13 states, mostly east of the Mississippi, but no college from the state of West Virginia was included in this study. To fill this gap is the object of this note.

We present a comprehensive survey of the mathematical courses and instructional personnel of every junior college, college, and university—both public and private—in the state of West Virginia. A tabular summary is given of the 1962–63 catalog offerings in mathematics of the eight public state colleges (Bluefield State College, Concord College, Fairmont State College, Glenville State College, Shepherd College, West Liberty State College, West Virginia Institute of Technology, and West Virginia State College), the seven private colleges (Alderson-Broaddus College, Bethany College, Davis and Elkins College, Morris Harvey College, Salem College, West Virginia Wesleyan College, and Wheeling College), the three junior colleges (Beckley College, Greenbrier College for Women, and Potomac State College), and the two state universities (Marshall University and West Virginia University) plus the Kanawha Valley Graduate Center of the West Virginia University.

The following explanatory comments are in order before presenting the tabulated data.

1. There are 17 four-year institutions of higher education plus the Kanawha Valley Graduate Center and three junior colleges included in this report. These 21 institutions are accredited under the North Central Association of Colleges and Junior Colleges and other agencies of accreditation.

2. The course data and personnel for the junior colleges are tabulated separately.

3. The course offerings studied were listed under the following headings: Department of Mathematics, Department of Mathematics and Physics, and Department of Mathematics and Astronomy. Unless the course was specifically listed under one of the three headings, it is not included in the tabulations. A number of courses such as plane surveying, engineering drawing, descriptive geometry, business mathematics, statistics, and teaching methods were offered in many institutions under a wide assortment of departments such as physics, technical science, engineering, economics, business, and education.

4. In the various institutions many courses were listed under a wide variety of titles. We have attempted in the tables to group such courses under some common characteristic title.

5. Some of the courses listed were designated as being offered in alternate semesters, alternate years, or summers only.

6. A large number of the teaching personnel had pursued advanced study beyond the highest degree attained, but I was unable to arrive at an exact percentage.

7. The first year of calculus is offered in a number of different sequences and different course titles. These offerings have been grouped as calculus in Table I.

8. In Table I, only those courses are included which are offered by at least five of the colleges and universities.

TABLE I. MATHEMATICS OFFERINGS IN SEVENTEEN COLLEGES AND UNIVERSITIES AND THE KANAWHA VALLEY GRADUATE CENTER

Courses	Number of Schools Listing	Credit Hours
Algebra (Intermediate)	14	0–3
College Algebra	15	3–4
Plane Trigonometry with or without spherical	15	3–6
Arithmetic or Mathematics for Teachers (content) (See comment 3.)	12	2–3
Solid Geometry and/or Space Geometry	10	½ unit – 3
Plane Analytic Geometry	12	3–4
Calculus (See comment 7.)	17	6–12
Solid Analytic Geometry	6	2–3
Differential Equations (Ordinary)	17	3–6
Introduction to Mathematics or Introduction to Analysis (modern ideas such as logic, sets, etc.)	9	3–6
Modern Algebra	11	3–6
Seminars and/or Independent Study	7	1–6
Mathematics for Public Schools, Modern Mathematics for High School Teachers, Special Topics for Teachers (See comment 3.)	5	3–6
Theory of Equations	11	3
Advanced Calculus	12	3–8
Elementary Statistics and/or Business-Economic Statistics (See comment 3.)	6	3
Mathematical Statistics	8	3–6
Theory of Probability and Statistics and/or Mathematical Probability	5	3–4
Numerical Analysis or Finite Differences	5	3
Vector Analysis and/or Tensor Analysis	10	3
Development of Mathematics and/or History of Mathematics	5	2–3
Modern Geometry for Teachers or College Geometry	8	3
Descriptive Geometry (See comment 3.)	5	2–3

TABLE II. SUMMARY OF COURSE OFFERINGS AND MAJOR REQUIREMENTS

Total number of credit hours offered in all schools	1177
Total number of noncredit courses	10
Total number of courses offered with high school units of credit	4
Average number of credit hours required for a major	31.7
Average number of credit hours required for a minor	19.8
Average number of credit hours required for a teaching field	24.5

TABLE III. DEGREES OF MATHEMATICS STAFF

Highest Degree Attained	Number	Percent
Bachelors	4	4.4
Masters	72	79.1
Doctorates	15	16.5
Total number of teachers	91	

TABLE IV. MATHEMATICS OFFERINGS IN THREE JUNIOR COLLEGES

Courses	Number of Schools Listing	Credit Hours
General Mathematics (Arithmetic and Algebra)	1	3
Elementary Algebra	1	1 unit
Intermediate Algebra	2	3
College Algebra	3	3
Algebra (Accelerated Course)	1	0
Plane Geometry	2	1 unit
Solid Geometry	3	1 unit to 3 hours
Plane Trigonometry	3	1 unit to 3 hours
Mathematics for Teachers	2	3
Business Mathematics	2	3
Consumer Mathematics	1	3
Analytic Geometry	2	4
Calculus (Integrated differential and integral)	2	8
Calculus (Integrated plane and solid analytics, differential and integral)	2	6 to 12 hours
Slide Rule	1	0

Reference

1. J. William Drew, The Mathematics Curriculum in the Small College. This MONTHLY, 69 (1962) 664.

SCIENCE IN THE PRIMARY GRADES

ARTHUR H. LIVERMORE, Reed College and The American Association for the Advancement of Science

One of the responsibilities of the Commission on Science Education of the American Association for the Advancement of Science is to encourage the development of science materials for elementary schools. In partial fulfillment of this responsibility the Commission sponsored an eight-week writing session at Stanford University during the summer of 1963. The work at Stanford was directed by John R. Mayor and Arthur H. Livermore. Financial support was provided by a grant from the National Science Foundation.

Members of the writing group came from various disciplines and educational levels. The majority of the writers were college and university people from the fields of astronomy, biology, chemistry, geology, mathematics, physics, psychology, and science education. The rest of the group were elementary school teachers and supervisors. In all, 35 individuals cooperated in the work. There was close association with the SMSG writing group which worked in an adjacent dormitory. Members of both groups ate lunch together and had many opportunities for informal discussions.

The product of the writing group is a series of 84 exercises to be used in grades K through 3. These have been published in four parts entitled *Science, a process approach*. It is expected that fifteen or twenty more exercises will be completed this fall. These will be published in loose leaf form.

At two $8\frac{1}{2}$-day conferences held at Cornell and at the University of Wisconsin in the summer of 1962, it was decided to adopt a "process approach" to teaching science in the early grades. During the winter and spring of this year a panel under the chairmanship of Burton H. Colvin of Boeing Scientific Laboratories (Seattle) made detailed plans for the preparation of a science sequence in which the process approach would be followed. The summer writing group was presented at the start with a list of the processes and with some suggestions from the panel for materials that might be written. Though the processes were fleshed-out with a rich variety of material from various fields of science, the fact that the processes were kept continually in mind prevented the writing group from producing mere "bits and pieces."

Even in the earliest grades the effort is made through the activities provided to develop in children the abilities to make precise observations, to measure, to communicate, to recognize space-time relationships and to use numbers. Later other processes are added—classification, inference, prediction.

As an example of the way in which abilities are gradually developed through the materials we might look at a series of exercises on Observation in Part One. In this series the children use various senses for observation. The exercises are entitled: *Observing Color, Shape, Size, and Texture; Perception of Sound; Perception of Odor; Observing Temperature; Observing Hardness; Observation of Color and Color Changes in Plants.* Then there is a culminating exercise called *Observation Using Several of the Senses.*

In the series of exercises directed toward developing the ability to make appropriate measurements are two—*Making Comparisons Using a Balance* and *Measuring Forces with Springs.* Here, in first grade, children not only learn the process of measurement, but are introduced to the concept of mass. The distinction between mass and weight is considered in more detail in later exercises (third grade).

In this sequence of science activities, recognizing and using numbers and number relations has been considered one of the basic abilities necessary for the study of science. Writers of the materials were encouraged to make the exercises quantitative, where appropriate, and not to delay the teaching of one of the

process skills only because children might not be prepared to use the number relations involved. Exercises on counting, adding, and multiplying were also introduced. Through the number exercises it is expected not only that the children will develop the facility with numbers that they will need in the science exercises, but also that, because of the way in which numbers are taught, they will understand the algorithms involved in using numbers.

With attention directed toward the processes of science, it is possible to define clearly the objectives of each exercise. In one exercise at the kindergarten—first grade level we expect the child to learn to recognize three-dimensional shapes by observing the shadows which they cast. In another the objective is to begin to develop the child's ability to use graphs as a means of communication. With the objectives clearly defined, the abilities developed can be checked. For this purpose, at the end of each exercise there is an appraisal activity. In the appraisal of the introductory exercise on graphing the child is asked to make a bar graph to tell how many of each kind of toy there is in a mixture of three types of toys. It is suggested to the teacher that a typical response of the child should be, "I divided the toys into three piles: trucks, cars, and motorcycles. For each truck I put a red square in this 'line' of squares, for each car a blue square in this 'line' of squares, and for each motorcycle a yellow square in this 'line' of squares. There are six cars, four trucks, and two motorcycles."

The writers believe that the abilities developed in the early grades will make it possible for the children to learn the facts and the concepts more easily in later grades. There it should be possible to assume that the child will use his process abilities as he studies more of the "content" of science. This assumption is being made in some additional exercises which are being provided in unbound form for teachers to use in the later part of the third grade. These include exercises on force and motion, mass and acceleration, and solubility.

The materials are being tried out this year in four grade levels at twelve centers—four in the west, two in the south, two in the midwest, and four in the east. There are approximately ten teachers at each center. A science consultant meets with the teachers every two weeks to assist them in their work. An evaluation program has been established. The evaluation results will be used as a guide as the materials are rewritten and extended into grades four and five in the summer of 1964.

NSF SUMMER INSTITUTES FOR HIGH SCHOOL STUDENTS

ROBERT SPIRA, University of California, Berkeley

As part of the national scientific effort, the NSF each year conducts Institutes in mathematics for high school students. This article discusses the methods used at the University of California at Berkeley during the last three years.

The aim of the NSF program is to inform young people about the possibilities and attractiveness of a career in mathematics. At Berkeley, the program tries to reproduce the atmosphere of productivity of professional mathemati-

cians. Along with this, a great deal of information on professional opportunities is imparted. The emphasis is on interest and self-discipline.

The Institute is of great professional benefit to the participants. There are six weeks of intense activity, and then a year for reflection and personal action. The participants are started on a mathematical career.

Selection of the participants is one of the critical problems. More students should learn of the programs and apply for them. For selection, most attention is paid to the written statement of the applicant and the MAA high school test score. We seek the students who put out extra effort and have started projects on their own. The MAA tests are an excellent guide.

For the most part, the participants are around 16 years old, going into their senior year. A high school graduate can be selected, and there are usually some juniors and freshmen. The younger participants can return the following year for a second six weeks. These returnees lend continuity to the program, and also set examples of what can be accomplished. It is difficult to tell which participants will "take hold," so a fair effort is made to develop each person. For reasons of class stability, two to four of the best girls have been selected each year, even though their records are not as good as the boys'.

One of the initial problems is that students who have been best in their school are brought into contact for the first time with professionally equal or superior companions. We attempt to develop a working philosophy which deals with this ego-shattering experience.

The six weeks of the program are filled with hard problems, research projects, two courses of study and visiting lecturers. The learning of material is not emphasized, so that a small number of topics are taken up and discussed very thoroughly. The topics are independent, so that a student who gets lost has a fresh chance around once a week.

An essential part of the program is weekly problem sheets. These consist of 6 to 8 problems of varying difficulty. Typical problems are:

Sum $\frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}+\cdots+\frac{n}{10^n}+\cdots$ (first week)

Find numbers $x = 2^n \cdot 3 \cdot P$ (P a prime) such that the sum of divisors of x equals $3x$. (third week)

Let $Q = 1 + x + x^4 + x^9 + \cdots$. Find Q^4. (fifth week)

Some problems have definite answers, while others are rather indefinite, requiring a limiting down or changing to become reasonable, the usual case in any mathematical research. Problems in the current MONTHLY are also used, and if solutions are obtained they are submitted.

Meetings are held three times a week to discuss the problems. During this time there are discussions of mathematical techniques, psychological attitudes, organization of solution efforts, organization and technique of thinking, and the solution or efforts at solution of the problems. There is a weekly session of

the returnees to make up the problems for the following week. Sometimes problems come from the students.

Most of the participants subscribe to the MONTHLY at the conclusion of the Institute, so that they may continue to receive the stimulation of the Problem sections. A few of the boys make remarkable progress during the ensuing year on the MONTHLY problems; one young man, for instance, solved 12 elementary and 5 advanced problems from the MONTHLY.

The course work usually centers around a series of topics. These have been taken from: Number theory, geometry, theory of groups, and the theory of graphs. The best results seem to occur when the instructor has only a little previous knowledge of the subject material. Then the lectures have a great deal of floundering about and troublesome points in proofs. Since the amount of material covered is immaterial, a great deal of the technique of mathematical thinking can be observed. This is not to say that the best method is to watch an incompetent person perpetrate illogic. Rather, everything is made clear during the course of a proof, and if a troublesome point or doubt appears, it is brought out in the open and analyzed. If it cannot be settled on the spot, it is put off for further thought, and returned to at the next lecture. Sometimes a proof has been returned to in four different lectures. To sum up, we try to go through the proof, not the motions.

For research problems, the ones thought up by the students themselves are the best. For those students not at the point of thinking up problems, we have assigned problems. These come from investigations of the lecturers, generalizations of MONTHLY problems, etc. C. S. Ogilvy's book *Tomorrow's Math* is also very valuable for this. Typical research problems are:

What kind of rectangles can be formed from T-shaped tetrominoes?

Which polyominoes can form rectangles?

How far outside a given triangle is the Erdös-Mordell inequality satisfied?

This last year, some of the participants started work on very hard problems involving graph theory, such as the four color problem. The research problems should be started in the first week so that there is a reasonable time to work on them. It is a good idea to lay out a plan of research with each student, and to spend some joint time attacking the problem. The problems should be genuine. It is an excellent idea to have the lecturers perform mathematical research themselves and to give reports every few days on their own progress.

In conclusion, the program at Berkeley attempts to give a fair picture of the attractiveness of the life of a mathematician. A careful effort is made to develop each participant to the maximum degree.

The NSF mathematics program at Berkeley is under the direction of Frantisek Wolf.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

COLLABORATING EDITOR: C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1651. *Proposed by Azriel Rosenfeld, Budd Electronics, Long Island City, N. Y.*

Prove that no multidigit integer is equal to the sum of the squares of its digits.

E 1652. *Proposed by Erwin Just, Bronx Community College*

Prove that $\prod_{k=0}^{m-1} k! > (m/e)^m$ for all positive integral m.

E 1653. *Proposed by Arthur Engel, Stuttgart, Germany*

There are given $p_n = 1 + [en!]$ points in space. Each pair of these points is connected by a line, and each line is colored with one of n different colors. Show that there is at least one triangle all of whose sides are of the same color.

E 1654. *Proposed by Ralph Greenberg, University of Pennsylvania*

A set of numbers is said to be *special* if the sum of the numbers is zero. Let $N(r)$ denote the number of special proper subsets of the set of rth roots of unity. Show that $N(r) = 0$ if and only if r is prime.

E 1655. *Proposed by A. J. Goldman, National Bureau of Standards*

Determine the validity of the following asserted algorithm for finding the least common multiple L of a finite sequence $X = (x_1, x_2, \cdots, x_n)$ of positive integers. Beginning with $X^{(1)} = X$, at the mth step one has a finite sequence

$$X^{(m)} = (x_1^{(m)}, x_2^{(m)}, \cdots, x_n^{(m)}).$$

If all components of $X^{(m)}$ are equal, their common value is L and the algorithm stops. If not, choose a minimum component $x_k^{(m)}$ of $X^{(m)}$ and form $X^{(m+1)}$ by $x_k^{(m+1)} = x_k^{(m)} + x_k$, $x_i^{(m+1)} = x_i^{(m)}$ for $i \neq k$.

E 1656. *Proposed by G. A. Heuer and D. B. Erickson, Concordia College*

Let $\{R; +, \cdot\}$ be a system such that $\{R; +\}$ is a cancellation semigroup, $\{R; \cdot\}$ is a semigroup, and "$\cdot$" is right and left distributive over "$+$". Let $z \in R$ be such that $zx = xz = z$ for all $x \in R$. Is z an additive identity?

E 1657. *Proposed by Michael Gemignani, University of Notre Dame*

Let G be any group and A a subgroup of G. Let $x \in G$, $x \notin A$. We say x *augments* A if $A_x = A \cup \{x, x^{-1}\}$ is also a subgroup of G. Suppose x augments A. Show that A_x is cyclic of order 2, 3, or 4.

E 1658. *Proposed by D. L. Silverman, Beverly Hills, Calif.*

Points are selected at random on the circumference of a circle until they form the vertices of an inscribed polygon which encloses the center of the circle. Prove that the "expected polygon" is a pentagon.

E 1659. *Proposed by José Gallego-Diaz, Universidad del Zulia, Maracaibo, Venezuela*

A parabola has the property that the circumcircle of the triangle formed by three tangents to the curve passes through a fixed point (the focus). Does this property characterize the parabola?

E 1660. *Proposed by Seymour Kass, Illinois Institute of Technology*

Give an example of a strongly partially ordered set which has the property that every pair of unrelated elements has a sup and inf, while every pair of related elements has neither. (Strong partial order: antireflexive and transitive.)

SOLUTIONS

Horological Interchangeability

E 1571 [1963, 330]. *Proposed by J. L. Pietenpol, Columbia University*

How many times in a twelve hour period are the hands of a clock interchangeable (i.e., such that interchanging the positions of the hands yields a possible clock reading)?

Solution by D. C. B. Marsh, Colorado School of Mines. Measured in degrees clockwise from 12:00, at M minutes after H o'clock, the locations of the minute and hour hands are M and $5H+M/12$ respectively. For "interchangeability" we must satisfy

$$M' = 5H + M/12, \qquad 5H' + M'/12 = M,$$

with $H, H' \in \{1, 2, \cdots, 12\}$ and $0 \leqq M, M' < 60$. Solving

$$M = 60(H + 12H')/143, \qquad M' = 60(H' + 12H)/143,$$

we note that all pairs of H, H' yield solutions with only $H = H' = 1$ and $H = H' = 12$ giving the same reading—spaced 12 hours apart. Thus, during a twelve hour period there are $12^2 - 1 = 143$ times when the clock hands are interchangeable; of these, 11 are self-corresponding while the others form 66 dual pairs.

Also solved by J. C. Abad, R. H. Anglin, W. D. Anscher, K. F. Bailie, Frank Dapkus, Monte Dernham, P. J. Erdelsky, Robert Feinerman, Michael Goldberg, J. A. H. Hunter, A. R. Hyde, R. A. Jacobson, Emmett Keeler and Richard Zeckhauser (jointly), P. L. Kingston, E. F. Lang,

Harry Langman, Coline Makepeace, Helen M. Marston, J. E. Morriello, P. N. Muller, J. B. Muskat, P. R. Nolan, E. T. Ordman, Stanton Philipp. S. J. Ryan, Jean-Pierre Sampson, D. L. Silverman, Guy Torchinelli, B. R. Toskey, Gary Venter, Andy Vince, Julius Vogel, W. C. Waterhouse, S. E. Weinstein, K. L. Yocom, A. R. Zingher, and the proposer.

Attention was called to Problem 61 in H. E. Dudeney's *Amusements in Mathematics* (Dover Publications, Inc., 1958) and to Exercise 19 of Chapter II in Harry Langman's *Play Mathematics* (Hafner Publishing Company, 1962).

Squares and Rectangles on a Chess Board

E 1572 [1963, 330]. *Proposed by Anders Bager, Hjørring, Denmark*

Enumerate the number of (1) squares, (2) rectangles, on an $n\times n$ "chess" board.

Solution by R. A. Jacobson, South Dakota State College. The number of squares of dimension $j\times j$ on an $n\times n$ chessboard is $(n+1-j)^2$. Hence the total number of squares is given by

$$\sum_{j=1}^{n} (n+1-j)^2 = n(n+1)(2n+1)/6.$$

The number of rectangles of dimension $j\times k$ on an $n\times n$ chess board is $(n+1-j)(n+1-k)$. It follows that the total number of rectangles is

$$\sum_{j=1}^{n}\sum_{k=1}^{n} (n+1-j)(n+1-k) = n^2(n+1)^2/4.$$

Also solved by J. C. Abad, R. G. Albert, J. A. Andrews and W. C. Waterhouse (jointly), R. H. Anglin, W. D. Anscher, Joseph Arkin, K. F. Bailie, B. W. Banks and J. R. Fall and Lawrence Lessner (jointly), M. J. Behr and William Roughead (jointly), W. G. Brady, Julian Braun, Robert Brooks, R. E. Brown, A. W. Brunson, D. I. A. Cohen, Charles Conlin, Frank Dapkus, J. F. Dillon, P. J. Erdelsky, Bruce Erickson, J. A. Faucher, S. T. Fisk, E. T. Frankel, C. M. Frye, Michael Gemignani, Michael Goldberg, Ralph Greenberg, R. E. Greenwood, J. C. Hennessey, K. D. Herr, J. A. H. Hunter, A. R. Hyde, Roman Kaluzniacki, Emmett Keeler and Richard Zeckhauser (jointly), C. L. Krueger, Joel Kugelmass, G. J. Kurowski and J. D. Watson (jointly), Harry Langman, H. R. Leifer, S. B. Leonard, Robert Maas, Coline Makepeace, Andrezj Makowski, C. F. Marion, D. C. B. Marsh, Helen M. Marston, R. A. Melter, Stephen Montague, J. E. Morriello, P. N. Muller, Amos Nannini, Sam Newman, L. S. Nicholson, E. T. Ordman, R. R. Perez, Stanton Philipp, E. M. Scheuer, R. R. Seeber, R. L. Syverson, Ronald Tannenwald, Rory Thompson, Dmitri Thoro, Guy Torchinelli, B. R. Toskey, Gary Venter, Andy Vince, Julius Vogel, S. E. Weinstein, Ron Wilder, and the proposer.

Editorial Note. Attention was called to J. A. H. Hunter and J. S. Madachy, *Mathematical Diversions* (D. Van Nostrand Col, Inc., 1963), p. 129, to H. E. Dudeney, *Amusements in Mathematics* (Dover Publications, Inc., 1958), Problem 347, to *Scripta Mathematica*, 1949, p. 100, to *School Science and Mathematics*, Problem 2859, and to this MONTHLY, Problem 1127 [1955, 183].

It is interesting that the number of squares on an $n\times n$ chess board is $\sum_{i=1}^{n} i^2$ and the number of rectangles is $\sum_{i=1}^{n} i^3$. The number of squares on an $m\times n$ chess board, $m\geqq n$, is

$$n(n+1)(3m+1-n)/6,$$

and the number of rectangles is

$$mn(m+1)(n+1)/4.$$

The number of cubes in an $m\times n\times p$ three-dimensional chess board, $m\geqq p$, $n\geqq p$, is

$$p(p+1)[6mn-(p-1)(2m+2n-p)]/12,$$

and the number of rectangular parallelepipeds is

$$mnp(m+1)(n+1)(p+1)/8.$$

The last two formulas reduce to $[n(n+1)/2]^2$ and $[n(n+1)/2]^3$ for $p=m=n$. It follows that the number of cubes on an $n\times n\times n$ three-dimensional chess board is equal to the number of rectangles on an $n\times n$ planar chess board.

In his *Play Mathematics* (Hafner Publishing Co., 1962, p. 36), Harry Langman shows that there are

$$(n-1)n(n+1)(2m-n)/12$$

squares with vertices at points of an $m\times n$ rectangular lattice, $m\geqq n$.

A Triangle Inequality Involving the Angle Bisectors

E 1573 [1963, 331]. *Proposed by Franz Leuenberger, Zuoz, Switzerland*

Prove that the arithmetic mean of the angle bisectors of a triangle T never exceeds the sum of the distances of the circumcenter from the three sides of T, with equality if and only if T is equilateral.

Solution by Leonard Carlitz, Duke University. Let α, β, γ denote the angles, a, b, c the sides, and t_a, t_b, t_c the internal angle bisectors of the triangle. It is well known that $t_a=[2bc\cos(\alpha/2)]/(b+c)$. Since $4bc\leqq(b+c)^2$, with equality if and only if $b=c$, we get

$$t_a^2\leqq bc\cos^2(\alpha/2). \tag{1}$$

Then, by Cauchy's inequality,

$$\left(\sum t_a\right)^2\leqq\sum bc\sum\cos^2(\alpha/2). \tag{2}$$

Since $\sum\cos^2(\alpha/2)\leqq 9/4$ (the maximum is attained when $\alpha=\beta=\gamma=60°$), (2) becomes

$$\left(\sum t_a\right)^2\leqq(9/4)\sum bc.$$

Thus, to prove the stated inequality, it will suffice to show that

$$\sum bc\leqq 4(R+r)^2, \tag{3}$$

where R is the circumradius and r the inradius. Since $a=2R\sin\alpha$ and (by Carnot's theorem)

$$\sum R\cos\alpha=R+r,$$

(3) is equivalent to

$$\sum\sin\beta\sin\gamma\leqq\left(\sum\cos\alpha\right)^2. \tag{4}$$

Now set

$$\cos\alpha=x,\qquad\cos\beta=y,\qquad\cos\gamma=(1-x^2)^{1/2}(1-y^2)^{1/2}-xy,$$

so that (4) becomes

$$(1 - x^2)^{1/2}(1 - y^2)^{1/2} + [(1 - x^2)^{1/2} + (1 - y^2)^{1/2}][y(1 - x^2)^{1/2} + x(1 - y^2)^{1/2}]$$
$$\leqq [x + y - xy + (1 - x^2)^{1/2}(1 - y^2)^{1/2}]^2.$$

This reduces to

$$(1 - x - y + 2xy)(1 - x^2)^{1/2}(1 - y^2)^{1/2}$$
$$\leqq 1 - x - y + 2xy - x^2y - xy^2 + 2x^2y^2.$$

Since $1-x-y+2xy>0$ when $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, we get

$$(5) \qquad (1 - x^2)^{1/2}(1 - y^2)^{1/2} \leqq 1 + xy - xy/(1 - x - y + 2xy).$$

Now

$$(6) \qquad (1 - x^2)^{1/2}(1 - y^2)^{1/2} \leqq 1 - xy$$

and

$$1 - x - y + 2xy \geqq 1/2,$$

so that

$$xy/(1 - x - y + 2xy) \leqq 2xy.$$

Hence (6) implies (5). This completes the proof of (3).

Also solved by A. N. Aheart and the proposer.

Editorial Note. One wonders if there isn't a nicer way to establish the trigonometric inequality (4).

A Concurrency Condition

E 1574 [1963, 331] Corrected. *Proposed by Simon Vatriquant, Brussels, Belgium*

If, in triangle ABC, $\sin A \cos B = \sin C \cos C$, show that the circumdiameter through A, the median through B, and the angle bisector through C are concurrent.

I. *Solution by P. R. Nolan, Department of Education, Dublin, Ireland.* Let the circumdiameter AA' cut BC in D; let the median through B cut CA in E; let the angle bisector through C cut AB in F. Then, since $\measuredangle ABA' = 90°$, $\measuredangle BAD = 90° - \measuredangle C$. Similarly, $\measuredangle CAD = 90° - \measuredangle B$. It follows that

$$BD/DC = c \cos C/b \cos B = \sin C \cos C/\sin B \cos B.$$

Also, $CE/EA = 1$, $AF/FB = b/a = \sin B/\sin A$. Therefore

$$(BD/DC)(CE/EA)(AF/FB) = \sin C \cos C/\sin A \cos B = 1,$$

and AD, BE, CF are concurrent by Ceva's theorem.

II. *Solution by K. W. Crain, Purdue University.* In trilinear coordinates the equations of the concerned lines through A, B, C respectively are

$$\beta \cos C - \gamma \cos B = 0,$$

$$\alpha \sin A - \gamma \sin C = 0,$$

$$\alpha - \beta = 0.$$

Now a necessary and sufficient condition that these lines be concurrent is that the determinant of the system of equations be zero, or

$$\sin A \cos B - \sin C \cos C = 0.$$

Also solved by A. N. Aheart, José Gallego-Diaz, D. C. B. Marsh, and the proposer.

Solutions of $\phi(m) = m/p$

E 1575 [1963, 331]. *Proposed by Ronald Alter, University of Pennsylvania*

Find all pairs of positive integers m, p such that $\phi(m) = m/p$, where p is prime and ϕ is the Euler function.

Solution by L. R. Heinen and W. C. Waterhouse, Harvard University. We shall not require in advance that p be prime. Let p_i, $i = 1, \cdots, k$, be the primes dividing m. Then $\phi(m) = m \prod_{i=1}^{k} (p_i - 1)/p_i$, so we must solve

$$\prod_{i=1}^{k} (p_i - 1)/p_i = 1/p.$$

Let p_k be the greatest of the p_i; then clearly p_k divides p, so $1/p_k \geqq 1/p$. But

$$\prod_{i=1}^{k} (p_i - 1)/p_i \geqq \prod_{s=2}^{p_k} (s - 1)/s = 1/p_k.$$

Hence $p_k = p$, and all integers between 1 and p_k must be primes dividing m. Thus the only solutions are $p = 2$, $m = 2^a$ and $p = 3$, $m = 2^a 3^b$, for positive a and b.

Also solved by J. C. Abad, R. G. Albert, W. D. Anscher, Joseph Arkin, K. F. Bailie, Merrill Barnebey, Marjorie R. Bicknell, D. A. Blaeuer, W. R. Boland and J. C. Smith, Jr. (jointly), W. H. Bonney, Leonard Carlitz, S. R. Cavior, Martin Cohen, Bruce Erickson, G. J. Etgen, Stephen Fisk, Michael Goldberg, Jerry Goodman, Ralph Greenberg, Michael Heiberg, J. A. H. Hunter, R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), Roman Kaluzniacki, Joel Kugelmass, Douglas Lind, N. F. Lindquist, Barry Litvack, A. E. Livingston, Andrzej Makowski, C. F. Marion, D. C. B. Marsh, Helen M. Marston, Stephen Montague, J. B. Muskat, E. T. Ordman, Stanton Philipp, Henry Ricardo, W. H. Richardson, J. A. Schumaker, R. Sibson, Jr., D. L. Silverman, Guy Torchinelli, B. R. Toskey, Dennis Travis, Andy Vince, P. O. Wood, Jr., K. L. Yocom, and the proposer.

Makowski pointed out that a proof, by A. Schinzel, of the more general result established above can be found in W. Sierpiński, *Theory of Numbers* (in Polish), Warsaw, 1949, pp. 196–7.

Solutions of $n^{\phi(s)}=s$

E 1576 [1963, 331]. *Proposed by L. R. Heinen and W. C. Waterhouse, Harvard University*

Find all pairs of positive integers n, s such that $n^{\phi(s)}=s$, where ϕ is the Euler function.

Solution by D. L. Silverman, Beverly Hills, Calif. Let $\phi(s)=m$. For $n>2$, $m=\phi(n^{\phi(s)})>n^{\phi(s)-1}=n^{m-1}$, contradicting the inequality $n^{m-1}\geqq m$ for $n>2$ and m arbitrary. Hence $n\leqq 2$, and we get the obvious and unique solutions (n, s) $=(1, 1)$, $(2, 2)$, or $(2, 4)$.

Also solved by J. C. Abad, Merrill Barnebey, W. H. Bonney, Leonard Carlitz, S. R. Cavior, Martin Cohen, G. J. Etgen, Stephen Fisk, Michael Goldberg, Jerry Goodman, Ralph Greenberg, R. A. Jacobson, Roman Kaluzniacki, Joel Kugelmass, A. E. Livingston, Andrzej Makowski, C. F. Marion, D. C. B. Marsh, Stanton Philipp, W. H. Richardson, J. C. Smith, Jr., Guy Torchinelli, B. R. Toskey, Andy Vince, P. O. Wood, Jr., and the proposers.

Stancliff Determinants

E 1577 [1963, 331]. *Proposed by Marjorie Bicknell, San Jose State College*

Show that for any integer n one can construct a symmetric fourth order determinant whose elements are ten consecutive integers and whose value is n.

Solution by Cornelius Groenewoud, Cornell Aeronautical Laboratory, Buffalo, N. Y. The determinant

$$\begin{vmatrix} n+1 & n-8 & n & n-7 \\ n-8 & n-1 & n-6 & n-2 \\ n & n-6 & n-5 & n-3 \\ n-7 & n-2 & n-3 & n-4 \end{vmatrix}$$

is composed of the ten consecutive integers $n-8, n-7, \cdots, n-1, n, n+1$, is symmetric, and has the value n.

Remarks by the proposer. This problem appears in the unpublished notes of the late Fenton S. Stancliff, who studied extensively the effect of adding a constant to each element of a matrix. The problem has many essentially different solutions. If we are restricted to use only two consecutive integers, we have the following two matrices,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

for each of which the determinant value decreases by k when k is added to each element.

Also solved by Brother U. Alfred, R. A. Jacobson, J. F. Latimer, D. C. B. Marsh, Stanton Philipp, and K. L. Yocom.

A Divergent Series

E 1578 [1963, 331]. *Proposed by E. O. Thorp, New Mexico State University*

Consider the series $\sum_{n=1}^{\infty} |\sin n|^a$, where a is an arbitrary positive number. For which values of a does the series converge and for which values does it diverge?

Solution by Julius Vogel, The Prudential Insurance Company of America, Newark, N. J. The series diverges for all positive a, because $|\sin n|^a$ does not approach zero with increasing n. In fact, given any integer n_0, however large, define t as an integer falling in the interval

$$2\pi n_0 + \pi/6 < t < 2\pi n_0 + 5\pi/6.$$

Then $t > n_0$ and $|\sin t|^a > (1/2)^a$.

Also solved by E. R. Barnes, D. I. A. Cohen, Frank Dapkus, Michael Gemignani, Michael Goldberg, Ralph Greenberg, T. J. Grilliot, R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), Joel Kugelmass, A. E. Livingston, D. C. B. Marsh, D. J. Peterson, Stanton Philipp, D. L. Silverman, Guy Torchinelli, W. C. Waterhouse, and the proposer.

Livingston showed, more generally, that $\sum_{n=1}^{\infty} |\sin nx|^a$ diverges for all real a and $x \not\equiv 0 \pmod{\pi}$.

Supplemented, Strongly Normal, and Normal Subgroups

E 1579 [1963, 331]. *Proposed by Azriel Rosenfeld, Yeshiva University*

Call a subgroup H of a group G *strongly normal* if every subgroup K of H which is normal in H is normal in G. Prove that supplemented implies strongly normal implies normal, but that neither reverse implication holds.

Solution by W. H. Bonney, New Mexico State University. Let H be a supplemented subgroup of G so that $G = H \otimes K$, where $\otimes$ means direct product. Now given a normal subgroup N of H, $n \in N$, $g = hk \in G$, we have

$$(hk)^{-1}n(hk) = k^{-1}h^{-1}nhk = k^{-1}n_1k,$$

with $n_1 \in N$ since N is normal in H. But $k^{-1}n_1k = n_1$, since every element in K commutes with every element in H. Thus strongly normal is implied by supplemented. Clearly strongly normal implies normal.

In the permutation group S_3 on three letters, the subgroup of even permutations has three elements, so is strongly normal, being simple and of index 2. It is not supplemented since otherwise S_3 would be abelian.

Let $G = G_1 \otimes G_2$ where $G_1 \cong G_2$ and G has at least two elements. Then G is normal in its holomorph H and G_1 is normal in G. On the other hand, by the definition of the holomorph of a group, there exists an $h \in H$ such that $h^{-1}G_1h = G_2$. Hence normal does not imply strongly normal.

Also solved by the proposer.

Wieferich Squares

E 1580 [1963, 331]. *Proposed by Guy Torchinelli, State University College at Buffalo*

Prove or disprove: If p is prime then $2^{p-1} \not\equiv 1$, mod p^2.

Disproved, by an appropriate reference, by H. L. Alder, W. R. Boland, Leonard Carlitz, J. D. Cloud, D. I. A. Cohen, Martin Cohen, W. A. Edelstein, Stephen Fisk, Michael Gemignani, Michael Goldberg, Jerry Goodman, Ralph Greenberg, A. W. Johnson, Jr., Edgar Karst, Sidney Kravitz, Douglas Lind, Barry Litvack, Andrzej Makowski, J. B. Muskat, K. K. Norton, Stanton Philipp, H. J. Ricardo, R. R. Seeber, R. E. Shafer, D. Suryanararyana, Dmitri Thoro, Andy Vince, W. C. Waterhouse, K. S. Williams, P. O. Wood, Jr., and the proposer.

Editorial Note. It was conjectured for some time that $2^{p-1}-1$, where p is a prime, though divisible by p, is never divisible by p^2. It was shown, in 1909 by A. Wieferich, that if this were true, then we cannot have positive integers x, y, z, p such that $x^p+y^p=z^p$, where p is prime and x, y, z are prime to p, and an important case of "Fermat's Last Theorem" would be established. In 1913, however, W. Meissner showed that the conjecture is false for $p=1093$, and, in 1922, N. G. W. H. Beeger showed it false for $p=3511$. A simple verification of the first counterexample can be found in Hardy and Wright, *An Introduction to the Theory of Numbers*, p. 73. Calling the square of a prime p such that p^2 divides $2^{p-1}-1$ a *Wieferich square*, it is not known if there are infinitely many Wieferich squares. The best result so far along this line was given (following some previous weaker results) by Erna H. Pearson in *Math. Comp.*, 17 (1963), pp. 194–5, where it is shown that 1093 and 3511 are the only primes $p \leqq 200{,}183$ (the 18,000th prime) with $2^{p-1} \equiv 1 \pmod{p^2}$. In this same paper it is also shown that 5, 13, and 563 are the only primes $p \leqq 200{,}183$ with $(p-1)! \equiv -1 \pmod{p^2}$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

COLLABORATING EDITORS: L. CARLITZ, Duke University; H. S. M. COXETER, University of Toronto; and A. WILANSKY, Lehigh University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editors. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5161. *Proposed by Seth Warner, Duke University*

Let K be a finite field and K^* the multiplicative group of all nonzero elements of K. Show that $H \cup \{0\}$ is a subfield of K for every subgroup H of K^* if and only if the order of K^* is a Mersenne prime.

5162. *Proposed by W. E. Briggs, University of Colorado*

Show that $\sum_{n \leq x} \sigma^2(n)/n^2$ is asymptotic to Ax, and find the constant A.

5163. *Proposed by S. D. Chatterji, University of New South Wales, Australia*

Let $(S, \sum, m)$ be a totally-finite positive measure space and let $f(s)$ be a real-valued integrable function on S such that $f(s)>0$ almost everywhere with respect to the measure m. Let T be a one to one measure-preserving transformation of S onto itself. Prove that

$$\int_S \frac{f(s)}{f(Ts)}\, dm(s) \geqq m(S).$$

5164. *Proposed by Gregory Thompson and Peter Truenfels, Minneapolis-Honeywell Regulator Co.*

Find the Taylor series expansion of $e^{-x}J_0(ix)$ about the origin.

5165. *Proposed by A. S. Galbraith, Army Research Office, Durham, N. C.*

Let p_k and q_k, $k=1, 2, \cdots$, be the terms of two arithmetic progressions. Evaluate the determinant whose element in the ith row and jth column is p_{i+j}/q_{i+j}.

5166. *Proposed by Peter Rejto and Charles Conley, New York University*

Let A be a closed and densely defined operator on a Hilbert space such that the range of A is contained in its domain, and such that A^2 is the identity on the domain of A. Does it follow that the spectrum of A includes at most the points 1 and -1?

5167. *Proposed by P. D. Barry and W. K. Hayman, University of London*

Let $0<u_1\leqq u_2\leqq \cdots, u_n\leqq n, S_n=\sum_{i=1}^n u_i$. Show that $\sum(u_n/S_n)^\alpha$ converges for all $\alpha>1$.

5168. *Proposed by A. Himmelfarb, Fordham University*

All subrings of Z_m, the ring of integers modulo m, are of the form dZ_m, where d is a divisor of m. (dZ_m denotes the set of all ring multiples of d.) For a given m, determine (a) those d for which dZ_m has a unity element, (b) the unity element for such d, and (c) the number of such d.

5169. *Proposed by Sándor Lajos, University of Economics, Budapest, Hungary*

A subsemigroup B of a semigroup S is called bi-ideal of S if $BSB\subseteq B$ (see: A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, v. I, p. 84.) Prove that a subset A of a semigroup S is a bi-ideal in S if and only if A is a left [right] ideal of a right [left] ideal of S.

5170. *Proposed by Stanley Franklin, University of California at Los Angeles*

For any infinite cardinal $\aleph$, there exists a compact Hausdorff space E of cardinality $\aleph$ which is not the Stone-Cech compactification of any of its proper dense subspaces.

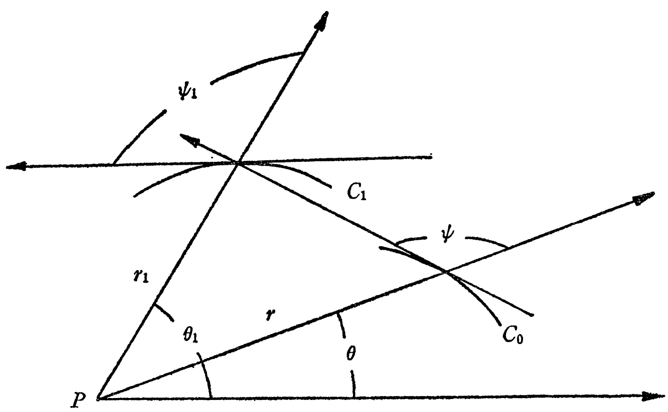
SOLUTIONS

A Geometric Limit

5047 [1962, 812]. *Proposed by M. S. Knebelman, Bucknell University*

Given a plane curve C_0 and a fixed point P. C_n is the pedal of P with respect to C_{n-1}, $n=1, 2, \cdots$. Prove that as $n\to\infty$, lim C_n consists of all circles with center at P, each tangent to C_0. (Circles of zero or infinite radius not excluded.)

Solution by the proposer. We choose P as the pole of a polar coordinate system and $r=r(\theta)$ as the polar equation of C_0. We assume that $r(\theta)$ is at least piecewise differentiable in some θ domain. If the equation of C_n is $r_n=f_n(\theta_n)$, the problem is to find $\lim_{n\to\infty} f_n(\theta_n)$.



From the figure it is evident that C_1 has the parametric equations

$$(1)\qquad \begin{aligned} r_1 &= r(\theta)\sin\psi(\theta)\\ \theta_1 &= \theta + (\psi(\theta) - \tfrac{1}{2}\pi),\end{aligned}$$

where $\psi(\theta)$ is defined by $\tan\psi = r/r'$, $0\leqq\psi\leqq\pi$. From (1) we have $dr_1/d\theta = r\cos\psi\cdot\psi' + r'\sin\psi = r\cos\psi\cdot(1+\psi')$ and $d\theta_1/d\theta = 1+\psi'$, so that $\tan\psi_1 = \tan\psi$ except possibly in case $\psi(\theta) = -1$, but in this case C_0 is a straight line and will be considered separately. Thus $\psi_1(\theta_1) = \psi(\theta)$. Similarly we shall find that

$$(2)\qquad \begin{aligned} r_n &= r(\theta)\sin^n\psi(\theta)\\ \theta_n &= \theta + n(\psi(\theta) - \tfrac{1}{2}\pi)\end{aligned}$$

as the parametric equations of C_n.

In order to consider $\lim_{n\to\infty} C_n$ it is convenient to introduce $\phi(\theta) = \psi(\theta) - \frac{1}{2}\pi$. Now suppose there is a point on C_0 at which $\phi(\theta) = 0$. Choosing the line through P and this point as the initial line, (2) becomes

$$(3)\qquad \begin{aligned} r_n &= r(\theta)\cos^n\phi(\theta)\\ \theta_n &= \theta + n\phi(\theta), \qquad \phi(0) = 0.\end{aligned}$$

We further assume that there is an open θ interval $(0, \alpha)$ in which $\phi(\theta) \neq 0$. For otherwise the part of C_0 on this interval would be a circle of radius $r(\theta)$ and center at P and would be its own pedal curve. To obtain a point on C_n we let θ_n be some finite angle and compute $r(\theta_n)$. But from the second equation in (3) $\phi(0) \to 0$ as well as $\theta \to 0$ as $n \to \infty$. Hence, since $\phi_n(\theta_n) = \phi(\theta)$, $\phi_n(\theta_n) \to 0$ and $r(\theta_n) \to r(0)$ so that $\lim_{n\to\infty} r_n = r(0)$, which is the equation of a circle center at P and tangent to C_0 at $\theta = 0$. Each point on C_0, where $\psi = \frac{1}{2}\pi$ may be treated the same way, thus yielding a number of tangent circles. If there is no point on C_0 at which $\psi = \frac{1}{2}\pi$ then corresponding to any θ, $\theta_n \to \infty$ and $\lim_{n\to\infty} C_n$ does not exist.

It only remains to consider the case of a straight line $r \cos \theta = a$. Its first pedal is $r_1 = a\theta_1 = 0$ which is a point. If this point is treated as a circle of zero radius its pedal is $r = a \cos \theta$ and $r_n = a \cos^{n+1} (\theta_n/(n+1))$, so that $\lim_{n\to\infty} r_n = a$, a circle of radius a.

Ideal Bases in Algebraic Number Fields

5068 [1963, 97]. *Proposed by Peter Flor, Mathematics Institute of the University of Vienna, Austria*

Let K be a quadratic field over the rationals with discriminant d. For any irrational number $t \in K$, let $at^2 + bt + c = 0$, $(a, b, c) = 1$ (a, b, c rational integers) and let $D(t) = b^2 - 4ac$. Let (α, β) be any pair of numbers of K which are linearly independent over the rationals. Then it can be shown easily that (α, β) is the basis of some (integral or fractional) ideal in K if and only if $D(\alpha/\beta) = d$. Can this result be extended in any way to characterize ideal bases in algebraic number fields of arbitrary degree?

Solution by Johann Cigler, University of Vienna. Let K be an algebraic number field of degree n which is identical with its conjugate fields. Let d be its discriminant. Suppose that $\alpha_1, \cdots, \alpha_n$ are n elements of K. These elements generate an ideal A whose norm will be denoted by $N(A)$. Let $(\beta_1, \cdots, \beta_n)$ be a basis of A. Then it is known (cf. E. Hecke, *Vorlesungen über die Theorie der algebraischen Zahlen*) that $(\alpha_1, \cdots, \alpha_n)$ is a basis of A if and only if

$$(1) \qquad \begin{vmatrix} \alpha_1 & \cdots & \alpha_n \\ \alpha_1^{(2)} & \cdots & \alpha_n^{(2)} \\ \cdot & \cdot & \cdot \\ \alpha_1^{(n)} & \cdots & \alpha_n^{(n)} \end{vmatrix}^2 = \begin{vmatrix} \beta_1 & \cdots & \beta_n \\ \beta_1^{(2)} & \cdots & \beta_n^{(2)} \\ \cdot & \cdot & \cdot \\ \beta_1^{(n)} & \cdots & \beta_n^{(n)} \end{vmatrix}^2.$$

Now the relation

$$(2) \qquad \frac{1}{N(A)^2} \begin{vmatrix} \beta_1 & \cdots & \beta_n \\ \cdot & \cdot & \cdot \\ \beta_1^{(n)} & \cdots & \beta_n^{(n)} \end{vmatrix}^2 = d$$

holds. On the other hand, if we form the expansion

$$\sum c_i x^i = \prod_{\nu=1}^{n} (\alpha_1^{(\nu)} + \alpha_2^{(\nu)} x + \cdots + \alpha_n^{(\nu)} x^{n-1}), \tag{3}$$

then $N(A)$ is equal to g, the greatest common divisor of the coefficients c_i. (See loc. cit., p. 107, Satz 88.) Combining (1), (2) and (3) we get therefore:

The numbers $\alpha_1, \cdots, \alpha_n$ form a basis of an ideal A in K if and only if

$$\frac{1}{g^2}\begin{vmatrix} \alpha_1 & \cdots & \alpha_n \\ \cdot & \cdots & \cdot \\ \alpha_1^{(n)} & \cdots & \alpha_n^{(n)} \end{vmatrix}^2 = d. \tag{4}$$

When $n = 2$, the discriminant $D(\alpha_1/\alpha_2)$ of the normed polynomial $c(x-\alpha_1/\alpha_2)(x-\alpha_1^{(2)}/\alpha_2^{(2)})$ is exactly the lefthand side of (4). (A polynomial is said to be normed if all coefficients are integers without common divisor.)

Weakly Continuous Mapping in Hausdorff Space

5069 [1963, 97]. *Proposed by D. R. Andrew, University of Southwestern Louisiana*

Prove or disprove the following statement. If S is a Hausdorff space and $f: S\rightarrow T$ is a weakly continuous one-to-one mapping of S onto the space T such that $f^{-1}: T\rightarrow S$ is weakly continuous, then T is necessarily a Hausdorff space. (See Norman Levine, *A Decomposition of Continuity in Topological Spaces*, this MONTHLY, 1961, pp. 44–46, for the definition of a weakly continuous function.)

Solution by S. M. Robinson, Smith College. The following example provides a negative answer. Let us recall first that $f: S\rightarrow T$ is weakly continuous at $s\in S$ if for every neighborhood V of $f(s)$ in T, there is a neighborhood U of s in S such that $f(U)\subseteq \mathrm{cl}_T V$. f is weakly continuous if it is weakly continuous at each point of S.

Now let S be the closed unit interval $[0, 1]$ with a topology described by the following neighborhood systems:

For $s\neq 0$, $s\neq 1$, let the local base at s be the same as is given by the usual topology.

For $s=0$, let the local base be the family of sets

$$[0, \alpha) \cup (\beta, 1) - \{x_n\}_{n=1}^{\infty}, \quad \text{where} \quad 0 < \alpha < \beta < x_n < 1$$

for each n, and where $x_n\rightarrow 1$ in the usual sense.

For $s=1$, the base is defined in the following way. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary fixed sequence satisfying $a_n<1$ and $a_n\rightarrow 1$ in the usual sense; then, the local base at 1 will be the family of sets of the form $\{A_n\}$, $A_n=\{1, a_{n+1}, a_{n+2}, \cdots\}$. The topology of S is clearly a Hausdorff topology.

The space T will again be the closed unit interval and for each $t\neq 0$, t will have the same local base as in S. At 0, however, we will take as the local base

the sets $[0, \alpha)\cup(\beta, 1)$. That T is not Hausdorff can be seen by examining the neighborhoods of 0 and 1.

Let f be the identity map from S onto T. Since the topology for S is stronger than that for T, f must be continuous and therefore is weakly continuous. Also, since for each $t\neq 0$, t has the same neighborhood base in each space, f^{-1} is continuous at each point not equal to 0. We claim that f^{-1} is weakly continuous at 0. Let $V=[0, \alpha)\cup(\beta, 1)-\{x_n\}_{n=1}^{\infty}$ be a typical basic neighborhood of 0 in S. Then $U=\mathrm{cl}_S V$ is a neighborhood of 0 in T, for $U\supseteq[0, \alpha)\cup(\beta, 1)$. Thus we have that $f^{-1}(U)=\mathrm{cl}_S V$. This proves the final assertion and establishes the counterexample.

Compound Events in a Probability Space

5070 [1963, 97]. *Proposed by S. Birnbaum, New York City*

Let

(1) $A, B, \cdots, K$ be k events in a probability space, and let

(2) $$P(A \cup B \cup \cdots \cup K) + P(A \cup B \cup \cdots \cup \overline{K}) \cup \cdots + P(A \cup \overline{B} \cup \cdots \cup \overline{K})$$

be denoted by $\sum P$. Show that $\sum P=2^k-1$.

Editorial Note. Several readers noted the ambiguity in (2) and suggested the following revised statement: Let $A_1, A_2, \cdots, A_k$ be k events in a probability space. Show that $P(Z_1 \cup Z_2 \cup \cdots \cup Z_k) = 2^k - 1$, where the sum is to be taken over all 2^k possible distinct sequences $Z_1, \cdots, Z_k$ such that Z_j is either A_j or $\overline{A}_j$ $(j=1, \cdots, k)$.

Solution by E. S. Keeping, University of Alberta. Let E_i stand for one of the 2^k compound events mentioned in the problem, and let $\overline{E}_i$ be the complement of E_i. Then $\sum_i \{P(E_i)+P(\overline{E}_i)\}=2^k$.

Since the complement of $A_1\cup A_2\cup \cdots \cup A_k$ is $\overline{A}_1\cap\overline{A}_2\cap \cdots \cap\overline{A}_k$, and similarly for the others, the $\overline{E}_i$ are disjoint and together make up the whole probability space, so that $\sum_i P(\overline{E}_i)=P(\sum_i \overline{E}_i)=1$. Hence $\sum_i P(E_i)=2^k-1$.

Also solved by M. T. Bird, A. J. Bosch, W. E. Bonnide, D. Z. Djoković, N. J. Fine, John B. Kelly, Max Klicker, Alfred Lehman, S. G. Mohanty, Vivian Pessin, E. M. Scheuer, Donna J. Seaman, Zbyněk Sidák, D. L. Silverman, W. A. O'N. Waugh, John Weissman, and the proposer.

A Set of Transcendental Numbers

5072 [1963, 215]. *Proposed by W. E. Briggs, University of Colorado*

Prove that

$$g(\theta) = \sum_{n=1}^{\infty} \frac{\theta}{n(n+\theta)}$$

is irrational for infinitely many rational θ on $(0, 1)$.

Solution by Sidney Heller, Brookhaven National Laboratory, Upton, N. Y. We have

$$g(\theta) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\theta}\right), \qquad g(1-\theta) = 1 + \sum_{n=2}^{\infty}\left(\frac{1}{n} - \frac{1}{n-\theta}\right).$$

Hence there follow

$$g(1-\theta) - g(\theta) - \frac{1}{1-\theta} = \sum_{n=1}^{\infty}\frac{2\theta}{\theta^2 - n^2},$$

$$g(1-\theta) - g(\theta) - \frac{1}{1-\theta} + \frac{1}{\theta} = \pi\cot\pi\theta.$$

For rational θ, $\cot\pi\theta$ is algebraic, whence $\pi\cot\pi\theta$ is transcendental, and therefore at least one of $g(\theta)$, $g(1-\theta)$ is transcendental. This implies the result.

Also solved by R. O. Atkinson, I. N. Baker, George Bergman, L. Carlitz, S. Chowla and S. L. Segal, N. J. Fine, Stephen Fisk, Ralph Greenberg, Stanton Philipp, J. H. van Lint, and the proposer.

An Extended Chord Theorem

5073 [1963, 215]. *Proposed by D. J. Newman, Yeshiva University*

Let $f(x)$ be continuous on $[0, n]$ and suppose that $f(0)=f(n)=0$. Show that $f(x)=f(y)$ has at least n different solutions with $x-y$ a positive integer. (The well-known chord theorem asserts that there is at least one solution with $x-y=1$.)

Solution by N. J. Fine, University of Pennsylvania. A proof can be given by induction on n. The case $n=1$ is trivial. Given f continuous on $[0, n]$, with $f(0)=f(n)$, by the chord theorem there exist α, $\alpha+1\in[0, n]$ such that $f(\alpha)=f(\alpha+1)$. Define g on $[0, n-1]$ by

$$g(x) = f(x) \quad (0 \leqq x \leqq \alpha), \qquad g(x) = f(x+1) \quad (\alpha \leqq x \leqq n-1).$$

Then g is continuous and $g(0)=g(n-1)$. By the inductive hypothesis there are $n-1$ solutions (x_i, y_i) to the equation $g(x)=g(y)$ with $y-x$ a positive integer. Since $0=g(y_i)-g(x_i)$ is equal to one of the differences

$$\begin{aligned} &f(y_i) - f(x_i), && \text{if } y_i < \alpha,\\ &f(y_i+1) - f(x_i), && \text{if } x_i \leqq \alpha \leqq y_i,\\ &f(y_i+1) - f(x_i+1), && \text{if } \alpha < x_i, \end{aligned}$$

we have $n-1$ (distinct) solutions to $f(x)=f(y)$ with $y-x$ a positive integer. Adjoining to these the new solution $(\alpha, \alpha+1)$, we have n solutions and the induction is complete.

Also solved by I. N. Baker, D. Z. Djoković, R. A. Jacobson, Vivian Pessin, H. Shniad, Jean Tzembalario, and the proposer.

Subsequence of a Sequence of Measurable Sets in (0, 1)

5074 [1963, 215]. *Proposed by Paul Erdös, University College, London*

Let E_n, $1 \leqq n < \infty$ be an infinite sequence of measurable sets in (0, 1), each E_n having measure $\geqq c$. It is well known that there exists an infinite subsequence E_{n_k}, $1 \leqq k < \infty$ such that $\bigcap_{k=1}^{\infty} E_{n_k}$ is not empty. Prove that the sequence $n_1 < n_2 < \cdots$ can be chosen so as to have upper density c, and that this result is the best possible.

Solution by Richard Scoville, Duke University. Let $E_i(x)$ be the characteristic function of the set E_i, and let $a_n(x) = \{E_1(x) + E_2(x) + \cdots + E_n(x)\}/n$. Let $f_n(x) = \sup_{m \geqq n} a_m(x)$ and $f(x) = \limsup a_n(x) = \lim f_n(x)$. We must show, then, that for some x, $f(x) \geqq c$. Suppose, on the contrary, that $f(x) < c$ almost everywhere. Then

$$c > \int f = \lim \int f_n \geqq \limsup \int a_n \geqq c,$$

a contradiction. This proves the conclusion of the problem. But we note that, for our argument to be valid, it is only necessary that $\limsup \{\mu(E_1) + \mu(E_2) + \cdots + \mu(E_n)\}/n \geqq c$.

To show that the conclusion is the best possible, we let $\{r_n\}$ be a sequence of positive numbers decreasing to zero, to be determined later. Let $F_n = [0, r_n + \frac{1}{2}] - \{\frac{1}{2}\}$ if n is odd, and $F_n = [\frac{1}{2} - r_n, 1]$ if n is even. Then $\lim \{F_1(x) + F_2(x) + \cdots + F_n(x)\}/n \equiv \frac{1}{2}$. Now, we can choose the sequence $\{r_n\}$ in such a way that for each n, there is an interval C_n in F_n of length r_n, the sequence $\{C_n\}$ satisfying $\limsup \{C_1(x) + C_2(x) + \cdots + C_n(x)\}/n \equiv \frac{1}{2}$. Then if we put $E_n = F_n - C_n$, we have $\mu(E_n) = \frac{1}{2}$, $f(x) \equiv \frac{1}{2}$, and $\liminf a_n(x) \equiv 0$. For any other $c \neq \frac{1}{2}$ a similar construction is possible.

An Irrational Number

5075 [1963, 215]. *Proposed by Paul Erdös, University College, London*

Let $n_1 < n_2 < \cdots$ be a sequence of integers such that $\limsup n_k/n_1 n_2 \cdots n_{k-1} = \infty$ and $\liminf n_k/n_{k-1} > 1$. Prove $\sum 1/n_k$ is irrational. (See also problem 4773 [1958, 782].)

Solution by G. M. Bergman, Harvard University. Suppose that $\sum 1/n_k = p/q$, p and q positive integers. Then for any k_1,

$$\sum_{k=k_1}^{\infty} \frac{1}{n_k} = \frac{p}{q} - \sum_{k=1}^{k_1-1} \frac{1}{n_k}.$$

This must have denominator $\leqq q n_1 n_2 \cdots n_{k_1-1}$, and so must be $\geqq 1/q n_1 n_2 \cdots n_{k_1-1}$.

Now since $\liminf n_k/n_{k-1} > 1$, we can find $\alpha > 1$ and an integer k_0 such that $k > k_0$ implies $n_k/n_{k-1} \geqq \alpha$. By the "lim sup" condition, we can find $k_1 > k_0$ such

that $n_{k_1}/n_1 \cdots n_{k_1-1} > \alpha q/(\alpha-1)$. Then

$$\sum_{k=k_1}^{\infty} \frac{1}{n_k} \leqq \sum_{j=0}^{\infty} \frac{1}{\alpha^j n_{k_1}} = \frac{\alpha}{(\alpha-1)n_{k_1}} < \frac{1}{q n_1 \cdots n_{k_1-1}},$$

a contradiction.

More generally, if we require, further, that $\limsup n_k/(n_1 \cdots n_{k-1})^r = \infty$, then the sum cannot be algebraic of degree r. The proof is essentially the same, but we must replace the step $p/q - \sum_{k=1}^{k_1-1} = \sum_{k=k_1}^{\infty}$ by $P(\sum_{k=1}^{k_1-1}) \leqq B \sum_{k=k_1}^{\infty}$, where B is a bound on the derivative of the polynomial P (and we consider the denominator of the left-hand term). If this condition is satisfied at once for all r, the result is equivalent to $\limsup (\log n_k)/(\log n_1 \cdots n_{k-1}) = \infty$, which will therefore make the sum transcendental.

Also solved by Robert Breusch, Leopold Flatto, Ralph Greenberg, and J. H. van Lint.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, Carleton College, and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. Programmed Materials: K. O. May, Carleton College, Northfield, Minn. Films: E. P. Vance, Oberlin College, Oberlin, Ohio.

Cohomology Operations. Lectures by N. E. Steenrod, written and revised by D. B. A. Epstein. Annals of Mathematics Studies No. 50, Princeton University Press, Princeton, N. J., 1962. x+139 pp. $3.00.

The study of primary cohomology operations, especially the Steenrod squares and reduced powers, has proved very fruitful in algebraic topology and its applications. This book, in its first four chapters, gives a good sampling of applications of the Steenrod squares to such problems as embedding spaces in spheres and the vector field problem. This is done using axioms for the mod 2 Steenrod algebra. Then, in the last five chapters, the reduced powers are constructed in a "new and more perspicuous" manner and shown to satisfy the axioms.

Anyone familiar with the papers or lectures of N. E. Steenrod will recognize the lucid style and the very helpful intuitive discussions as in the first section of chapter seven. This book should be readable by anyone who has had a good one year course in algebraic topology. It will clearly become the standard text in its field. This reviewer gives it his highest recommendation.

F. P. PETERSON, Massachusetts Inst. of Technology

Numerical Analysis, with Emphasis on the Application of Numerical Techniques to Problems of Infinitesimal Calculus in Single Variable, 2nd ed. By Zdeněk Kopal. Wiley, New York, 1961. 594 pp. $12.00.

The second edition of Professor Kopal's well known and useful Numerical Analysis textbook is generally unchanged from that of the first edition, the major exception being that what had been Appendix I has now been greatly expanded to form Chapter 9 on "Operational Methods in Numerical Analysis." This chapter is particularly interesting in that the rational approximations of Padé are now treated, and shown to be very useful in numerical differentiation, numerical integration, and interpolation. Regrettably, the author has decided not to bring the bibliography up to date, and has instead recommended the use of the Mathematical Reviews for listings of current research papers.

Professor Kopal's *Numerical Analysis*, with its excellent selection of topics, will continue to be one of the leading books on this subject.

R. S. VARGA, Case Institute of Technology

Proceedings of the Second International Conference on Operational Research. Edited by J. Banbury and J. Maitland, Wiley, New York, 1961, 830 pp. $15.00.

One of the livelier scientific novelties of the last 25 years has been the development of operations research, a field of science that is seeking to account for the behavior of man-machine systems operating in a natural environment. The present volume, the proceedings of the conference held at Aix-en-Provence in 1960, gives an overview of current interests of workers in operations research. There were sessions on methodological aspects of operations research, computers, the measurement of human factors, new mathematical methods, control of production, inventory control, and mathematical programming, as well as the application of operations research methods to the steel industry, the oil industry, atomic and electric power, military problems, mining, transport, and the problems of local and national governments.

Since over 80 papers were given at the conference, strict length limitations were observed in order to keep the proceedings volume down to its present size; thus the papers are generally less than ten pages, giving them frequently a rather hasty synoptic character. Nevertheless, mathematicians interested in what is going on in operations research will find this book of interest; the brevity of the contributions, in fact, permits easy sampling. In particular, the reviewer recommends Moody on "Production Allocation in the Beet Sugar Industry" (p.237), Lombaers on "Determining the Ore-Unloading Capacity of a Harbour Installation by Simulation on a Computer" (p. 328), Gillams on "Central Planning of an Atomic Energy Industry" (p. 470), Wohlstetter on "National Decisions Concerning Defense" (p. 517), and Hicks and Houlden on "Operational Research in the British Coal Industry" (p. 609).

H. J. MISER, The Mitre Corporation

Mathematics for Quantum Mechanics. By J. D. Jackson. Benjamin, New York, 1962. x+97 pp. $3.50 (paper), $4.75 (cloth).

In this short book, emphasis is on the unity of the mathematical methods of quantum mechanics. There are chapters on eigenvalue problems in classical physics, orthogonal functions and expansions, Sturm-Liouville theory and linear operators on functions, and on linear vector spaces, as well as useful appendices on Bessel functions, Legendre functions and spherical harmonics. The material is lucidly presented with no attempt at mathematical rigor. Unfortunately, many definitions are stated without essential qualifications, which could easily have been supplied without greatly adding to the length of the text. The text may be used for introductory lectures or for supplemental reading to bring beginning students, deficient in applied mathematics, to an adequate level of knowledge in the mathematics of quantum mechanics.

ERNEST IKENBERRY, Auburn University

An Introduction to Information Theory. By Fazlollah M. Reza. McGraw-Hill, New York, xxi+496 pp. $13.50.

This book is offered as an introduction to probability, information theory, and coding theory. The author sets no prerequisites beyond the usual mathematics included in an engineering or science program. The book is organized into four sections: Discrete Schemes Without Memory, Continuum Without Memory, Schemes With Memory, and Some Recent Developments. There are seven short appendices, seven tables, a good bibliography, and an index.

It will be convenient to discuss the author's three main topics separately. First, the five chapters on probability. This material is by now completely standard, and it is not unreasonable to expect the author to avoid the more obvious pitfalls; but, alas! we find probabilities confused with frequencies, a completely inadequate definition of independence, events that are "independent but not necessarily mutually exclusive," and Bayes' theorem linked to causality. This part of the book cannot survive comparison with any one of several recent texts.

The five chapters on information theory are relatively more successful. The author lists several alternative derivations of the entropy functional, and carries the development up to a heuristic treatment of the fundamental theorem for the continuous noisy memoryless channel, and a rigorous treatment of the discrete case. Here, the least successful sections are consistently those in which the author has added his own explanations to those of the original workers in the field.

Finally, in the two chapters on coding, the author gets as far as the simpler error-correcting codes, and goes into some detail on group codes. Much of this material has been taken almost directly from the original papers.

In a disarming touch, the author expresses the hope that this work will provide "an existence proof of Shannon's fundamental theorem that information can be transmitted . . . despite all forms of noise." This is undoubtedly the case;

but I would hesitate to recommend the book as a text without warning of the need for constant vigilance against varying notation, inaccuracy of expression, and diagrams that are badly drawn, wrongly labelled, or are actively misleading. Those on pages 47, 90, and 92 are particularly unfortunate. The noise level is indeed high; and so is the price.

C. L. MALLOWS, Bell Telephone Laboratories

The Summation of Series. By Harold T. Davis. The Principia Press of Trinity University, San Antonio, 1962. x+140 pp. $6.00.

The author declares in the preface: "The purpose of this small volume is to advance the reader's understanding of the problem of the summation of series, with special emphasis upon the case of finite limits." A systematic development of the calculus of finite differences and its relationship to or its analogy with the infinitesimal calculus form the contents of Chapter 1. The next two chapters are devoted to special methods of summing series, and these involve for the most part the use of gamma and psi functions, or Lubbock's summation formula or the Euler-Maclaurin formula. Chapter 4 is concerned with describing some of the techniques employed in the construction of "Tables of finite sums." Such a table plays a rôle analogous to that of a table of integrals. The author's "central exhibit" is a table of this type, which includes a fairly exhaustive collection and forms the appendix to the book. He illustrates (in Chapters 4 and 5) the use of these tables in actual summation problems. In the final chapter he discusses, among other topics, the general convergence tests for infinite series as also certain general methods of finding the sums of such series.

The virtue of the book consists in the number of illustrative examples, followed by examples to be solved, and in the very detailed table of finite sums. The book is essentially one on the summation of series employing the methods of the calculus of finite differences and could prove useful to those who need to compute such sums.

M. S. RAMANUJAN, University of Madras, India

A Guide to FORTRAN Programming. By Daniel D. McCracken. Wiley, New York, and London, 1961. 88 pp.

McCracken's guide to FORTRAN programming may be just the book you are looking for. There seems to be a rash of FORTRAN books on the market now. Each user should certainly examine the crop before making his decision. Your reviewer likes this volume. It is well presented and has good problems, some of which are short enough to permit the student to concentrate on a single point, while others (called case studies) are realistic enough to provide the reader some feeling for what programming is really about. The author has limited his attention to FORTRAN Programming and has not attempted to indicate the related mathematics. This seems a wise decision although your reviewer would have preferred at least some footnote references and/or suggestions on where the interested reader might look for up-to-date mathematical

treatments of related problems.

The author concentrates on the IBM 709-7090 versions of FORTRAN and uses an appendix to point out the needed variations for other computers including the IBM 650, 1620, 705, 7070, as well as the Honeywell Algebraic Compiler, Philco 2000 Altac, and Central Data Corp. 1604 FORTRAN.

Since the IBM 1620 seems currently to be a popular computer in both industrial and university circles, it might have been appropriate for the author to discuss the available 1620 versions in more detail. The reader would not, for example, be aware that GOTRAN is faster than FORTRAN in teaching and other situations where many short programs are run once or twice each. No mention is made of the excellent FORGO program, nor of the now widely used AFIT FORTRAN program. Perhaps the publisher will permit the author to include such comments in the next issue and also, we hope, to fill the half pages at the ends of chapters with some more of D. D. McCracken's excellent problems. A few extras like "Write a program to read three numbers and print them out on one line in descending order" are always welcome.

R. V. ANDREE, University of Oklahoma

Complex Numbers and Functions. By T. Esterman. Athlone Press, London, 1962. viii+250 pp. $6.75.

This book presents a first course in the theory of analytic functions of a complex variable and covers the usual material from the concept of complex numbers through Cauchy's integral theorem to residue calculus, series developments, and special functions. The presentation is characterized by a purely arithmetical approach and the exclusion of geometric intuition in argument and proof. The spirit of the book is that of Hardy's "A Course of Pure Mathematics" and Landau's "Foundations of Analysis." This yields a rigorous and logical development of the theory which is supported by the clear exposition and the lucid style of the author. On the other hand, a certain amount of insight and motivation is lost by the exclusion of the geometric background of the theory; Riemann's mapping theorem is not reached and the amount of material on conformal mapping remains below the average of the standard text books. In general, however, the author succeeds in compressing into his book more information than its size would lead one to expect by relegating a part of the derivations into the exercises, which seem to be skillfully graded and well selected. Two less usual features of the book appear in its last two chapters. In one, the author uses the concepts of curve and winding number, derived in connection with the complex integral, to give an elegant proof of the Jordan curve theorem without topological-geometric argument. No figure is used in this chapter. Picard's theorem is proved by means of the modular function which is explicitly constructed by Eisenstein series and very carefully discussed. The student who is exposed to this derivation will be convinced that Picard's theorem is valid but he might be left wondering how one could guess this complicated way to prove it. This example shows the main characteristic of the author's approach. He is excellent

in logical and rigorous argument but somewhat negligent on motivation and intuition. The book will be of interest to students with a more abstract outlook and preference for close argument and be most valuable to teachers of a course in complex variables who wish to present elegant but precise proofs for theorems which are often treated superficially in the conventional text books.

M. M. SCHIFFER, Stanford University

Introduction to Calculus. By K. Kuratowski. Addison-Wesley, Reading, Mass., 1962. 315 pp. $5.00.

This book is based on lectures given at the University of Warsaw. It develops the subject systematically and efficiently, treating many topics in greater depth than is common in a first course at an American university. There is an abundance of examples, and these are extremely well chosen to illustrate both the content of the theorems and the classical techniques of mathematical analysis.

Proofs of theorems are rigorous, but the informal style of the exposition does not display the structure of the subject too sharply. Simply listing clearly the mathematical information assumed known by the reader would help. The problems are too few in number, and many will be quite difficult for beginning students. It is assumed that the efficient development of the subject will be adequate motivation for the reader; no guiding comments are inserted on the historical development of the subject or on the power, generality, and use of the result being discussed.

The book is divided into four chapters as follows: I, Sequences and Series (68 pp.); II, Functions (56 pp.); III, Differential Calculus (61 pp.); IV, Integral Calculus (108 pp.).

Chapter I introduces the real number system and discusses the properties of sequences and series in some detail, closing with the derivation of standard (d'Alembert, Cauchy, Raabe, Kummer) convergence criteria and a discussion of convergence properties of products of series and infinite products. Chapter II discusses primarily the notions of limit and continuity for functions and sequences and series of functions, culminating in treatments of power series and approximation theorems for continuous functions. A last section introduces the notation of mathematical logic but does not discuss the axiomatic development of the subject. Chapter III is a straightforward discussion of differential calculus, giving geometrical interpretations of the principal results and concluding with a detailed treatment of Taylor's theorem. The notion of a differential is discussed but not used extensively. Chapter IV establishes the properties of the indefinite integral and introduces definite integrals in terms of them. Standard calculational methods are discussed, and geometric applications are considered. The Riemann integral is defined and related to the previous concept of integral. A final section is devoted to the properties of improper integrals and of Fourier series. Mastery of the material presented will equip the student with a good working knowledge of the calculus of real valued functions of one real variable.

The misprints in the text are minor and should cause no difficulty. The book is recommended as supplementary reading for the superior student and as a text when the instructor is willing to spend considerable effort amplifying the material.

R. C. Mjolsness, General Electric Company

BRIEF MENTION

Coding Theorems of Information Theory. By J. Wolfowitz. Prentice-Hall, Englewood Cliffs, N. J., 1962. 126 pp. $7.00.

Types of Formalization in Small-Group Research. By J. Berger, B. Cohen, J. L. Snell, and M. Zelditch, Jr., Houghton Mifflin, Boston, 1962. x+159 pp. $4.50.

A study of the relation of mathematical models to theory-building in the behavioral sciences, written by one mathematician and three sociologists.

The Theory of Transonic Flow. By K. G. Guderley. Translated from the German by J. R. Moszynski. Addison Wesley, Reading, Mass., 1962. $9.00.

Elasticity, Fracture and Flow with Engineering and Geological Applications, 2nd ed. By J. C. Jaeger, Wiley, New York, 1962. viii+208 pp. $3.00.

Statistical Theory of Reliability. Proceedings of an Advanced Seminar Conducted by the Mathematics Research Center, United States Army at the University of Wisconsin, Madison, May 8–10, 1962. Edited by Marvin Zelen. The University of Wisconsin Press, Madison, 1963. xvii+166 pp. $5.00.

An Introduction to Dimensional Method. By E. W. Jupp. Cleaver-Hume Press, London, 1962. 89 pp. 12/6 d.

Intended "for students of physics and engineering; for the General Certificate Examination students, at A level, and the National Certificate student, at the A2 level."

Signal Flow Graphs and Applications. A research monograph on a group of original studies of the analysis of linear physical systems. By Louis P. A. Robichaud, Maurice Boisvert and Jean Robert. Prentice-Hall, Englewood Cliffs, N. J., 1962. xiv+214 pp. $6.75.

Encyclopaedic Dictionary of Physics, vol. 7. (Stellar Magnitude to Zwitter Ion.) Edited by J. Thewlis. Macmillan, New York, 1963. ix+866 pp. $298.00 for 8 volume set.

The Computing and Data Processing Society of Canada: Proceedings of 3rd Conference, June 1962. Editor, H. S. Gellman. University of Toronto Press, Toronto, 1963. vi+293 pp. $6.00.

Gravitative Stosswellen Nichtanalytische Wellenlösungen der Einsteinschen Gravitationsgleichungen. By Hans Jürgen Treder. Akademie-Verlag, Berlin, 1962. 143 pp. DM38.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern. Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor M. K. Fort, University of Georgia, represented the Association at the inauguration of Sanford S. Atwood as President of Emory University on November 15, 1963.

Professor Henry Sharp, Emory University, represented the Association at the Seventy-Fifth Anniversary Convocation of the Georgia Institute of Technology on October 7, 1963.

Professor R. A. Moore, Carnegie Institute of Technology, represented the Association at the inauguration of Bennett M. Rich as President and the installation of Paul R. Stewart as Chancellor of Waynesburg College on October 12, 1963.

University of Alberta: Professor M. V. Subbarao, University of Missouri, has been appointed Associate Professor; Associate Professor John McNamee has been promoted to Professor; Professor Max Wyman has been appointed Dean of the Faculty of Science.

Andrews University: Assistant Professor R. A. Jorgensen has been promoted to Associate Professor; Associate Professor H. T. Jones has been promoted to Professor.

Brigham Young University: Messrs. R. D. Jamison, University of Utah, and R. J. Egbert, University of Arizona, have been appointed Assistant Professors; Assistant Professor Paul Yearout has been promoted to Associate Professor; Dr. K. L. Hillam has been appointed Chairman of the Department of Mathematics.

University of California, Los Angeles: Dr. G. L. Seever, University of California, Berkeley, has been appointed Lecturer; Associate Professor R. H. Sorgenfrey has been promoted to Professor; Assistant Professors T. S. Ferguson and Basil Gordon have been promoted to Associate Professors.

Clarkson College of Technology: Visiting Professor Harry Langman, Ball State Teachers College, has been appointed Visiting Professor; Assistant Professor A. G. Davis has been promoted to Associate Professor.

Dartmouth College: Professor Ernst Snapper, Indiana University, has been appointed Professor; Assistant Professors R. H. Crowell and R. E. Williamson have been promoted to Associate Professors.

Duke University: Drs. J. A. Kelingos, University of Michigan, and O. P. Stackelberg, University of Minnesota, have been appointed Assistant Professors; Assistant Professor T. D. Reynolds has been promoted to Associate Professor.

Emory University: Assistant Professor P. D. Hill, Auburn University, has been appointed Associate Professor; Mrs. Dora H. Skypek, University of Wisconsin, has been appointed Assistant Professor.

Florida State University: Assistant Professors J. E. Snover and R. D. McWilliams have been promoted to Associate Professors.

Illinois Institute of Technology: Associate Professor W. B. Caton, DePaul University, has been appointed Associate Professor; Associate Professor L. A. Kokoris has been promoted to Professor.

State College of Iowa: Mr. John Cross, University of Illinois, has been appointed Assistant Professor; Associate Professor Augusta Schurrer has been promoted to Professor.

Iowa State University: Dr. E. J. Peake, New Mexico State University, has been appointed Assistant Professor; Professor G. P. Weeg, Michigan State University, has been appointed Visiting Professor.

Kansas State Teachers College: Mr. John Gerriets, Utah State University, has been appointed Assistant Professor; Professor O. J. Peterson retired May 31, 1963.

University of Manitoba: Dr. W. J. Jonsson and Mr. H. C. Finlayson have been promoted to Assistant Professors; Professor N. S. Mendelsohn has been appointed Head of the Department of Mathematics and Astronomy.

Mankato State College: Assistant Professor V. D. Turner has returned from leave at the University of Minnesota; Assistant Professor Francis Hatfield has returned from leave at the University of Florida and has been promoted to Associate Professor; Professor Carey Jensen retired June 1963.

Mercer University: Professor Riley Plymale has retired as Chairman of the Mathematics Department but will continue as a faculty member; Associate Professor Sherwood Ebey has been appointed Chairman of the Mathematics Department.

University of Michigan: Assistant Professor Arlen Brown, Rice University, has been appointed Associate Professor; Assistant Professors W. M. Kincaid and K. B. Leisenring have been promoted to Associate Professors.

Mississippi State University: Assistant Professor T. W. Daniel, Arkansas State College, has been appointed Assistant Professor; Mr. S. J. Tramel has been promoted to Assistant Professor; Professor Arthur Ollivier retired May 31, 1963 with the title of Professor Emeritus.

Missouri School of Mines and Metallurgy: Messrs. T. B. Baird, J. W. Joiner, H. D. Pyron and F. G. Walters have been promoted to Assistant Professors; Professor R. M. Rankin retired August 1963 with the title of Professor Emeritus; Professor D. H. Erkiletian, Jr. has been appointed Chairman of the Mathematics Department.

Montana State University: Dr. L. A. Schmittroth, Phillips Petroleum Company, Idaho Falls, Idaho, has been appointed Visiting Lecturer; Associate Professor William Myers has been promoted to Professor.

Montclair State College: Mr. T. F. Carroll, Sefton Fibre Can Company, New Orleans, Louisiana, and Mr. J. F. Dillon, Naval Ordnance Laboratory, Silver Spring, Maryland, have been appointed Assistant Professors; Professor P. C. Clifford has been appointed Chairman of the Mathematics Department.

Newark College of Engineering: Assistant Professor Michael Lione has been promoted to Associate Professor; Dr. E. C. Molina has retired.

Ohio State University: Professor A. E. Ross, University of Notre Dame, has been appointed Chairman of the Mathematics Department; Professor Stefan Drobot, University of Notre Dame, has been appointed Professor; Dr. Kenneth Leland, Louisiana State University, has been appointed Assistant Professor; Professor Hans Zassenhaus, University of Notre Dame, has been appointed Visiting Mershon Professor; Mr. R. T. Barnes has been promoted to Assistant Professor.

University of Pittsburgh: Associate Professor F. G. Asenjo, Southern Illinois University, and Dr. Chong-Yun Chao, IBM Research Center, Yorktown Heights, New York, have been appointed Associate Professors; Assistant Professor H. A. Gindler, San Diego State College, has been appointed Assistant Professor.

Sacramento State College: Dr. Nancy J. Poxon, University of Illinois, and Mr. Rudolph Merkel, University of California, Davis, have been appointed Assistant Professors; Mr. Robert Alves has been promoted to Assistant Professor.

San Jose State College: Dr. K. S. Davis, Electronic Specialty Company, Los Angeles, California, and Mr. Bruce Trumbo, Knox College, have been appointed Assistant Professors; Assistant Professor C. M. Larsen has been promoted to Associate Professor; Associate Professors V. E. Hoggatt and Robert Wrede have been promoted to Professors.

South Dakota School of Mines and Technology: Assistant Professor H. J. Biesterfeldt, Lebanon Valley College, has been appointed Assistant Professor; Assistant Professor B. L. McAllister has been promoted to Associate Professor; Associate Professors D. C. Benson and R. E. Doutt have been promoted to Professors.

Southeastern Louisiana College: Assistant Professors L. H. Davis and H. R. Moore have been promoted to Associate Professors.

State University of New York at Albany: Messrs. Arthur Hadley and Roland Minch have been promoted to Assistant Professors; Assistant Professor P. T. Schaefer has been promoted to Associate Professor.

State University of New York at Buffalo: Professor F. D. Parker, University of Alaska, has been appointed Professor; Assistant Professor Aaron Siegel, Drexel Institute of Technology, Drs. K. D. Magill, Pennsylvania State University, Sidney Penner, Illinois Institute of Technology, and J. M. Scandura, Syracuse University, have been appointed Assistant Professors; Messrs. R. B. Hooper, Trinity-Pawling School, Pawling, New York, and R. F. Shortt, Mansfield State College, have been appointed Lecturers.

Virginia Polytechnic Institute: Assistant Professor Ronson Warne, Louisiana State University, has been appointed Associate Professor; Professor Leonard McFadden has been appointed Acting Head of the Mathematics Department.

University of Washington: Assistant Professors R. J. Nunke and R. W. Richardson have been promoted to Associate Professors; Dr. Robert G. Thompson, University of Colorado, has been appointed Visiting Lecturer.

Whittier College: Dr. P. B. Norman, Autonetics, Anaheim, California, has been appointed Visiting Professor; Mr. R. A. Newcomb has been appointed Acting Chairman of the Department of Mathematics.

Wisconsin State College, Eau Claire: Mr. T. L. Vickrey, University of Texas, has been appointed Assistant Professor; Assistant Professor M. E. Wick has been promoted to Associate Professor.

Assistant Professor George Baldwin, New Mexico State University, has been appointed Associate Professor at Eastern New Mexico University.

Mr. W. P. Banks, University of Michigan, has been appointed Assistant Professor at Illinois State Normal University.

Assistant Professor W. A. Beck, Chatham College, has been promoted to Associate Professor.

Professor R. R. Bernard, Davidson College, has returned after a year's leave of absence at Dartmouth University.

Dr. Alfred Blumstein is on leave from the Institute for Defense Analyses as Visiting Associate Professor at Cornell University for the 1963–64 academic year.

Dr. Sylvan Burgstahler, University of Minnesota, Duluth, has been promoted to Assistant Professor.

Associate Professor R. K. Butz, Auburn University, has been promoted to Professor.

Mr. S. D. Calkins, Florida State University, has been appointed Assistant Professor at Winthrop College.

Rev. R. I. Canavan, S.J., St. Peter's College, has been appointed Chairman of the Mathematics Department.

Associate Professor Philip Cooperman, Fairleigh Dickinson University, has been promoted to Professor and appointed Acting Chairman of the Mathematics Department.

Assistant Professor R. A. Dobyns, McNeese State College, has been promoted to Associate Professor.

Assistant Professor Walter Ehrenpreis, Trenton State College, has been promoted to Associate Professor.

Dr. L. R. Ford, Jr., CEIR, Los Angeles, California, has joined the Defense Research Corporation of Santa Barbara, California, as Director of its recently organized Mathematics Group.

Professor S. W. Hahn, Wittenberg University, has been appointed Associate Dean of the College.

Associate Professor A. P. Hillman, University of Santa Clara, has been appointed Acting Chairman of the Mathematics Department.

Assistant Professor N. W. Johnson, Geneva College, has been appointed Assistant Professor at Michigan State University.

Dr. F. W. Lawvere, Columbia University, has been appointed Assistant Professor at Reed College.

Visiting Assistant Professor C. W. Lytle, Drew University, has been appointed Assistant Professor.

Assistant Professor D. M. Olson, Huron College, has been appointed Assistant Professor at Hartwick College.

Professor Emma Olson, Kent State University, retired July 20, 1963 with the title of Professor Emeritus.

Assistant Professor M. J. Poliferno, Trinity College, has been promoted to Associate Professor.

Professor R. A. Rankin, University of Glasgow, Scotland, has been appointed Visiting Professor at Indiana University.

Assistant Professor Rodrigo Restrepo, University of British Columbia, has been promoted to Associate Professor.

Dr. James H. Rollins, University of Illinois, has been appointed Assistant Professor at North Texas State University.

Mr. E. A. Schreiner, Wayne State University, has been appointed Assistant Professor at Western Michigan University.

Associate Professor Gaston Smith, University of Southern Mississippi, has returned from the University of Alabama and has been promoted to Professor.

Assistant Professor B. R. Toskey, Seattle University, has been promoted to Associate Professor.

Assistant Professor D. H. Tucker, University of Utah, has been promoted to Associate Professor.

Assistant Professor L. H. Turner, University of Minnesota, has been appointed Associate Professor at the University of Tennessee.

Associate Professor H. R. van der Vaart, North Carolina State College, has been promoted to Professor.

Professor Richard Varga, Case Institute of Technology, has been appointed Visiting Associate Professor at the California Institute of Technology.

Associate Professor E. A. Walker, New Mexico State University, has been promoted to Professor.

Associate Professor Lina R. Walter, Paterson State College, has been promoted to Professor.

Professor S. E. Warschawski, University of Minnesota, has been appointed Professor and Chairman of the Department of Mathematics at the University of California, San Diego.

Associate Professor Guido Weiss, Washington University, has been promoted to Professor.

Miss Laura E. Christman, retired from Senn High School, Chicago, Illinois, died on July 6, 1963. She was a member of the Association for 32 years.

Associate Professor R. E. Fullerton, University of Maryland, died on May 21, 1963. He was a member of the Association for 10 years.

Professor Raphael Salem, Institute Henri Poincaré, France, died on June 20, 1963. He was a member of the Association for 20 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

JUNE MEETING OF THE NORTHEASTERN SECTION

The spring meeting of the Northeastern Section of the Mathematical Association of America was held at the University of Maine, Orono, Maine on June 21 and 22, 1963. Professor Evans Munroe, Chairman of the Section, presided at all meetings.

There were 77 people registered for the meetings including 65 members of the Association.

The following papers were presented:

1. *On the foundations of combinatorial analysis*, by Professor Gian-Carlo Rota, Massachusetts Institute of Technology.

2. Reports from Undergraduate Independent Study Programs:

(i) *Some theorems on non-commutative Noetherian rings*, by Mr. Walter J. Savitch, University of New Hampshire, introduced by Professor Edward H. Batho.

(ii) *An undergraduate view of algebraic geometry*, by Mr. Joseph C. Bodenrader, College of the Holy Cross, introduced by Professor Patrick Shanahan.

(iii) *Normal spaces*, by Mr. George A. Kozlowski, Jr., Wesleyan University, introduced by Professor G. Phillip Johnson.

(iv) *Topology, algebra, and duality*, by Mr. Thomas J. Kyrouz, Bowdoin College, introduced by Professor Reinhard Korgen.

3. *Computers on the college campus*, by Professor Robert J. Walker, Cornell University. (By invitation)

High speed digital computers are affecting the mathematics curriculum in a variety of ways—changing emphasis on present topics, suggesting new ones, generating interest in allied fields, and providing jobs for mathematicians. In addition to a critical look at all their offerings, undergraduate teachers should give special attention to the needs of two groups of students, the mathematics majors who go into computing (there is a surprisingly large number of these) and the embryo "computer scientists." Such special courses as numerical analysis and probability for the former, and these plus logic and algebraic structures for the latter are recommended.

4. *On variation diminishing transformations*, by Professor I. J. Schoenberg, University of Pennsylvania. (By invitation)

At the third annual meeting of the Association of Maine College Teachers of Mathematics, held at Orono, Maine, on December 14, 1940, the speaker gave a paper *On the Descartes' rule of signs* in which he described his results which appeared in the Math. Zeitschrift, vol. 38 (1934), pp. 546–574. The present lecture is in the nature of a progress report on Descartes' rule. The ideas were traced which led to the determination of the variation diminishing convolution transformation by Albert Edrei and the speaker in the early fifties.

R. S. PIETERS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fifth Summer Meeting, University of Massachusetts, Amherst, August 24–26, 1964.

Forty-eighth Annual Meeting, Denver-Hilton Hotel, Denver, Colorado, January 28–30, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pa., May 2, 1964.

ILLINOIS, Bradley University, Peoria, May 8–9, 1964; Southern Illinois University, Carbondale, May 14–15, 1965.

INDIANA, Butler University, Indianapolis, May 2, 1964.

IOWA, Luther College, Decorah, April 17–18, 1964.

KANSAS, Kansas State University, Manhattan, April 18, 1964.

KENTUCKY, University of Kentucky, Lexington, Spring 1964.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 14–15, 1964.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Spring 1964.

MICHIGAN, Michigan State University, East Lansing, March 28, 1964.

MINNESOTA

MISSOURI, University of Missouri, Columbia, April 18, 1964.

NEBRASKA, University of Nebraska, Lincoln, May 1–2, 1964.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 7, 1964.

NORTHEASTERN, Worcester Polytechnic Institute, Worcester, Mass., November 28, 1964.

NORTHERN CALIFORNIA, Stanford University, February 1, 1964.

OHIO, University of Akron, May 9, 1964.

OKLAHOMA, East Central State College, Ada, Oklahoma, April 10–11, 1964.

PACIFIC NORTHWEST, Washington State University, Pullman, Washington, June 19, 1964.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado College, Colorado Springs, Colorado, May 1–2, 1964.

SOUTHEASTERN, The Citadel, Charleston, South Carolina, March 20–21, 1964.

SOUTHERN CALIFORNIA, San Fernando Valley State College, Northridge, March 14, 1964.

SOUTHWESTERN, New Mexico State University, University Park, March 1964.

TEXAS, Texas Technological College, Lubbock, April 10–11, 1964.

UPPER NEW YORK STATE, New York State Education Department, Albany, May 16, 1964.

WISCONSIN, Wisconsin State College, Whitewater, May 2, 1964.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN MATHEMATICAL SOCIETY, New York, New York, February 29, 1964.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Maine, Orono, June 22–26, 1964.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Detroit, November 26–28, 1964.

INSTITUTE OF MATHEMATICAL STATISTICS, Berne, Switzerland, September 14–16, 1964.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Miami Beach, Florida, April 22–25, 1964.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Queen Elizabeth Hotel, Montreal, Canada, May 27–29, 1964.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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CONTENTS

FEBRUARY 1964

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10053.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For nonmembers the subscription price is $8.00.

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SYMMETRIC DIFFERENTIAL OPERATORS

ROBERT McKELVEY, University of Colorado

The self-adjoint differential boundary value problem has served from the beginning as a touchstone in the theory of symmetric operators in Hilbert space. Questions of extension and of spectral resolution may be cast in particularly simple form for ordinary differential operators, in terms of boundary conditions, spectral matrices, and eigenfunction expansions. It is my intent to give here an account of the theory of symmetric differential operators, taking advantage of these simplifications in order to describe Naimark's theory of extensions into transcendent Hilbert spaces, and the qualitative characteristics of the spectra so obtained.

Some definitions will be necessary at the outset. We shall deal with a Hilbert space H having inner product (u, v) between pairs of its elements. Usually this will be the space L_2 of functions quadratically integrable on an interval $I=(\alpha, \beta)$ and with inner product

$$(u, v) = \int_\alpha^\beta u(x)\bar{v}(x)dx, \qquad u, v \in L_2.$$

A linear operator T will be said to act in H when both its domain and range are linear manifolds *in* H. An operator T^+ in H is called *an extension* of T when the domain of T^+ contains the domain of T, and the operators coincide where they both apply:

$$T^+ \supseteq T \text{ iff } D_{T^+} \supseteq D_T, \qquad Tu = T^+u \quad \text{for } u \in D_T.$$

In his studies of symmetric operators, the Soviet mathematician M. A. Naimark [1] has found it useful to broaden this definition to include extensions which *transcend* the space H. That is, the operator T^+ no longer necessarily acts in H, but has domain and range in some Hilbert space H^+ in which H is possibly only a proper subspace. One thinks of H as given, and of H^+ as any one of its many enlargements. We will distinguish between the two concepts of extension by referring to *extensions in H* on the one hand, and *Naimark extensions* on the other.

If D_T is dense in H, then the *adjoint* T^* of T is defined as the most extensive operator acting in H for which

$$(Tu, v) = (u, T^*v)$$

for all $u \in D_T$ and all $v \in D_{T^*}$. The operator T is *symmetric* when contained in its adjoint: $T \subseteq T^*$ or, equivalently, when

$$(Tu, v) = (u, Tv), \qquad u, v \in D_T.$$

If not only $T \subseteq T^*$ but also $T = T^*$, then T is called *self-adjoint.*

Since the 1923 memoir of Torsten Carleman [2] on singular integral equations, and the early papers of von Neumann [3] on abstract Hilbert space, it has been recognized that, among symmetric operators, it is the self-adjoint which is the true analogue, in infinite dimensional spaces, of the Hermitian-symmetric matrix. It is the self-adjoint which induces a natural decomposition of the space in which it acts, and which can be represented in a canonical diagonal form.

The distinction between symmetric and self-adjoint is of crucial importance to our subject, and will remain central throughout our discussion.

Probably the most important symmetric ordinary differential operators are those associated with the Sturm-Liouville (SL) differential equation

$$\frac{d^2u}{dx^2} + [\lambda - q(x)]u = 0 \qquad (x \in I),$$

with $q(x)$ real and continuous on the open or closed, finite or infinite real interval I. This equation, when associated with a compact interval, is called ***regular*** and is the equation treated originally by Sturm and Liouville [4]. The *singular* equation evidently admits of the possibility of erratic behavior of $q(x)$, and hence of $u(x)$, at an open end point or at infinity. A treatment of the singular SL problem was one of the great early accomplishments of Hermann Weyl [5].

Weyl's work showed that the proper setting for the consideration of the SL boundary value problem is in the context of the Hilbert space L_2. The definitive formulation of the problem in that setting is due to Marshall Stone [6]. One introduces a differential operator Q acting in L_2 by the formula

$$Qu = -\frac{d^2u}{dx^2} + q(x)u \qquad x \in I.$$

Its domain D_Q consists of those functions $u \in L_2$ for which Qu exists and is also in L_2.

For elements $u, v \in D_Q$ the Lagrange identity

$$\bar{v}Qu - u\overline{Q\bar{v}} = u'\bar{v} - u\bar{v}'$$

holds in the integrated form

$$(Qu, v) - (u, Qv) = \lim_{\substack{b \nearrow \beta \\ a \searrow \alpha}} \mathop{\triangle}_{x=a}^{b} [u(x)\bar{v}'(x) - u'(x)\bar{v}(x)].$$

(The explicit limit takes account of the possibility that the interval I may be open.)

The operator Q evidently has the largest domain compatible with an L_2 setting. Normally, though, some restriction of this domain will be required to insure the vanishing of the right side of Lagrange's identity and so to produce

an operator that is symmetric. It is certainly sufficient to restrict the operator to the manifold D_o, where $u \in D_o$ iff

$$(Qu, v) - (u, Qv) = 0 \qquad \text{for } all\ v \in D_Q.$$

This formula holds in particular for any $v \in D_o$. If one denotes by Q_o the restriction of Q to D_o, then the formula becomes

$$(Q_o u, v) - (u, Q_o v) = 0, \qquad u, v \in D_o;$$

that is to say, Q_o is symmetric.

The manifold $D_o = D_{Q_0}$ is, in a way, the smallest domain with which it is natural to associate an SL operator. A glance at the Lagrange identity shows that, in the regular case (of compact I), D_{Q_0} coincides with the manifold of functions $u(x)$ in D_Q which vanish, along with derivatives at the ends of I:

$$u(\alpha) = u'(\alpha) = u(\beta) = u'(\beta) = 0.$$

In the singular case D_{Q_0} may be characterized less simply as a kind of completion of the manifold of smooth functions which vanish identically near the ends of I.

All these remarks about the SL differential operator apply in straightforward analogy to much more general symmetric differential operators [7], [8]. Not merely is the symmetric Q_o contained in Q, but its adjoint is exactly Q:

$$Q_o \subseteq Q = Q_o^*.$$

It may even happen that Q_o coincides with Q, and so is self-adjoint. In general, though, Q_o is properly included in Q, and between them lie infinitely many intermediate operators. These necessarily include all the symmetric and especially all the self-adjoint extensions in L_2 of Q_o.

In fact, as an operator is extended, so its adjoint is restricted; if $Q_1 \supseteq Q_o$, then $Q_1^* \subseteq Q$. In particular, if Q_1 is a symmetric extension of Q_o, then the multiple inclusion holds that

$$Q_o \subseteq Q_1 \subseteq Q_1^* \subseteq Q.$$

Thus, it may happen that, by extension, the relation of inclusion is refined to equality, and the condition of symmetry of self-adjointness.

While extension in L_2 to the self-adjoint is always possible for the SL operator, either regular or singular, and also for many other symmetric differential operators, nevertheless this is not universally the case. As J. von Neumann [3] has shown in detail, there are symmetric operators which do not have self-adjoint extensions in the space, and even some which do not have proper symmetric extensions in the space. These operators are by no means pathological, and include in particular a great many ordinary differential operators. If, on the other hand, one follows Naimark in considering the broader class of self-adjoint extensions which may transcend the space, then one finds a different, and

simpler situation: *Every* symmetric operator has at least one self-adjoint Naimark extension.

At least as regards differential operators, it may at first appear inappropriate to consider Naimark extensions. A self-adjoint extension Q^+ which acts in L_2 is, by the preceding formula, contained in Q, and hence acts by a concrete process of differentiation applied in a certain manifold of functions in D. A Naimark extension, on the other hand, may contain in its domain vectors belonging to an abstract space, to which it is applied by a process remote from that of differentiation. Actually, as we shall see, the Naimark extensions have a concrete interpretation. In any case, adequate justification for considering them lies in the information which they provide about their common generator Q_0, whose credentials as a differential operator are unassailable.

In harmony with this point of view, we shall always require of a self-adjoint Naimark extension Q^+ of Q_0 that it be a *minimal* extension of Q_0, i.e., that no other self-adjoint operator shall lie between Q_0 and Q^+. Furthermore, we will not distinguish between extensions which appear the same when viewed from L_2, i.e., which are unitarily equivalent under a mapping which leaves fixed the elements of L_2. With these understandings, we agree to call a self-adjoint Naimark extension Q^+ of Q_0 a self-adjoint *differential* operator. Later on we show how these operators can be given concrete specification in terms of boundary conditions.

Up to now I have been describing the mode of generation of symmetric and self-adjoint differential operators. Now I shall turn to a consideration of the eigenfunction expansions which these operators induce. Fundamental to this is the property of the self-adjoint operator that it can be represented in a canonical diagonal form.

Applied to the present circumstances, this means that the self-adjoint Q^+ in H^+ is unitarily equivalent to a certain canonical self-adjoint operator T^+ in a space K^+. In particular, there exists an isometric mapping V of L_2 onto a subspace K of K^+, the mapping carrying Q_0 over to a symmetric restriction of T^+. The mapping is onto K^+ itself whenever Q^+ acts in L_2.

I will now describe a canonical operator, one both adequate to and appropriate for the representation of the differential operator Q^+. It is, namely, the operator M of multiplication by the variable, in a vector-valued Lebesgue space $L_2(\sigma)$.

Here

$$\sigma(\lambda) = \left\|\sigma_{jk}(\lambda)\right\|_{j,k=1}^{n} \qquad (-\infty < \lambda < \infty)$$

is a monotone increasing matrix function of λ; that is to say, $(\sigma(\lambda)\mathfrak{n}, \mathfrak{n})$ is monotone increasing for each n-vector $\mathfrak{n} = \left\|\eta_k\right\|_{k=1}^{n}$. Consider the linear manifold C_0 of vector-valued functions

$$\mathfrak{f}(\lambda) = \left\|f_j(\lambda)\right\|_{j=1}^{n} \qquad (-\infty < \lambda < \infty)$$

with components $f_j(\lambda)$ which are continuous functions of compact support. One defines an inner product on C_0 by

$$(\boldsymbol{f}, \boldsymbol{g}) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} f_j(\lambda)\bar{g}_k(\lambda)d\sigma_{jk}(\lambda),$$

the monotonicity of σ guaranteeing that $(\boldsymbol{f}, \boldsymbol{f}) \geqq 0$. The Hilbert space $L_2(\sigma)$ is now defined as the completion of C_0 relative to this inner product. The multiplication operator M is given by

$$M\boldsymbol{f}(\lambda) = \lambda \boldsymbol{f}(\lambda)$$

with its domain consisting of those $\boldsymbol{f}(\lambda) \in L_2(\sigma)$ for which it is also true that $\lambda \boldsymbol{f}(\lambda) \in L_2(\sigma)$.

It has been proved for very general ordinary differential operators that the self-adjoint Q^+ is unitarily equivalent to the multiplication operator M in a space $L_2(\sigma)$, with the order n of the matrix σ equal to that of the differential operator. The isometric mapping V of L_2 onto a subspace of $L_2(\sigma)$ can be given as a kind of Fourier transform involving a basis $s_1(x, \lambda), \cdots, s_n(x, \lambda)$ of solutions (not necessarily in L_2) of the differential equation $Qs = \lambda s$. If $f(x) \in L_2$ and

$$Vf(x) = \|f_j(\lambda)\|_{j=1}^{n}$$

then the transform and inverse transform relations are

$$f_j(\lambda) = \int_{\alpha}^{\beta} f(x)s_j(x, \lambda)dx \qquad (j = 1, 2 \cdots n)$$

$$f(x) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} f_j(\lambda)s_k(x, \lambda)d\sigma_{jk}(\lambda).$$

The integrals converge, respectively, in the norms of $L_2(\sigma)$ and L_2.

The pair of transform formulas is said to constitute an *eigenfunction expansion* for a function $f \in L_2$. These formulas do not in themselves express the equivalence of Q^+ to M; that fact is shown in the formula

$$Q^+f = \int_{-\infty}^{\infty} \lambda \sum_{j,k=1}^{n} f_j(\lambda)s_k(x, \lambda)d\sigma_{jk}(\lambda)$$

valid for those $f \in D_{Q^+}$ for which both f and Q^+f are in the subspace L_2. In particular, when Q^+ is an extension in L_2 of Q_0, the formula applies to all $f \in D_{Q^+}$. In every case it holds for such f as belong to D_{Q_0}, and, since $Q_0 \subseteq Q^+$, implies that

$$Q_0 f = \int_{-\infty}^{\infty} \lambda \sum_{j,k=1}^{n} f_j(\lambda)s_k(x, \lambda)d\sigma_{jk}(\lambda) \qquad (f \in D_{L_0}).$$

In this sense the Fourier transform relations can be said to yield a diagonalization of the symmetric operator Q_0, and so may be said to constitute an eigenfunction expansion for Q_0.

The results which I have been describing are the work of many hands. Concerning extensions in L_2 they were proved for the SL operator by Weyl and Marshall Stone. Generalization to wider classes of operators proved difficult, but ultimately was achieved by a number of authors using varied approaches: I will mention K. Kodaira [9], M. Krein [10], N. Levinson [11], and E. A. Coddington [12]. The results for Naimark extensions are due to A. V. Štraus [13] and E. A. Coddington [14], [15]. A quite direct proof can be achieved by applying a method of L. Gårding [16], based upon the direct-integral formulation of the spectral theorem. (The treatment by this method of extensions confined to L_2 is contained in work of F. Brauer [17].)

The results quoted are evidently deficient in one important respect: they give no specific information about the *spectral matrix function* $\sigma(\lambda)$. Such information can be deduced by a study of the *generalized resolvent operator* R_ξ. Let P be the operator acting in H^+ which projects orthogonally onto the subspace L_2. Then R_ξ is defined on L_2 by [1]

$$R_\xi = P[Q^+ - \xi I]^{-1}.$$

The complex number ξ takes values for which the inverse exists; in particular, ξ can be any nonreal number.

One can immediately write down an expression for R_ξ in term of $\sigma(\lambda)$. That follows from the chain of equations

$$\begin{aligned}(R_\xi f, f) &= ([Q^+ - \xi]^{-1} f, f) = ([M - \xi]^{-1} f, f) \\ &= \int_{-\infty}^{\infty} [\lambda - \xi]^{-1} \sum_{j,k=1}^{n} f_j(\lambda)\bar{f}_k(\lambda) d\sigma_{jk}(\lambda).\end{aligned}$$

By Stieltjes inversion one obtains reciprocally an expression for $\sigma(\lambda)$ in terms of R_ξ.

The general principle of the inversion, in Hilbert space theory, has been known for a long time, was used in fact by Hellinger [18]. Strangely enough, the explicit form which it assumes for differential operators was not discovered until much later, when E. C. Titchmarsh [19] and K. Kodaira [20] came upon it independently.

To Hermann Weyl, delivering the 1948 Gibbs lecture [21], this delay illustrated "the degree to which mathematicians . . . got absorbed in abstract generalizations and lost sight of their task of finishing up . . . concrete problems of undeniable importance."

I will not set down the Titchmarsh-Kodaira formula; what I wish to emphasize instead is that from knowledge of the resolvent operator R_ξ for nonreal ξ one may recover the spectral function σ, and hence Q^+ itself.

Evidently, the operator R_ξ acts in L_2. The Russian mathematician A. V. Štraus [22] has shown how to characterize it directly in this setting. Immediately from the definition one finds that R_ξ serves as a left inverse for $Q_0 - \xi I$ and, simultaneously, a right inverse for $Q - \xi I$:

$$R_\xi(Q_0 - \xi I)u = u; \qquad (Q - \xi I)R_\xi u = u.$$

Since $(R_\xi)^{-1}$ also exists, these relations imply that R_ξ can be expressed in terms of an operator Q_ξ lying between Q_0 and Q, $Q_0 \subseteq Q_\xi \subseteq Q$, by the formula

$$R_\xi = [Q_\xi - \xi I]^{-1} \qquad \operatorname{Im} \xi \neq 0.$$

In the special case where Q^+ is an operator acting in L_2, the projection P becomes the identity, and Q_ξ reduces to Q^+ itself, independent of ξ. In general, however, Q_ξ is not self-adjoint, but belongs to a broader class of operators called *maximal dissipative*, characterized by

$$\operatorname{Im}(Q_\xi u, u) \geqq 0, \qquad Q_\xi D(Q_\xi) = L_2, \qquad \text{for } \operatorname{Im} \xi > 0.$$

We note that, from the definition, $R_{\bar\xi} = (R_\xi)^*$; hence $Q_{\bar\xi} = Q_\xi^*$. Thus Q_ξ for $\operatorname{Im} \xi < 0$ is determined by Q_ξ for $\operatorname{Im} \xi > 0$. For an investigation of the operator Q_ξ for *real* ξ, see [23].

Štraus' chief discovery is that every self-adjoint extension Q^+ of a symmetric Q_0 generates in this way a maximal dissipative extension Q_ξ of Q_0, which is *analytic* in ξ in the nonreal plane. Moreover, every extension Q_ξ of Q_0 with the stated properties is thus obtainable. This result leads to an enumeration and classification of all self-adjoint extensions Q^+ of Q_0. For our purposes the most useful form of this classification is in terms of boundary conditions specifying the differential operator Q_ξ. (It is an interesting exercise in hindsight to discover in the pages of Carleman's Uppsala memoir, and even more clearly in Marshall Stone's AMS colloquium volume, certain unmistakable foreshadowings of Štraus' theory. The generalized resolvent was present from the beginning, but only as an operator in L_2, with its important relation to the Naimark extensions still undiscovered.)

An enlightening perspective upon the classification procedure is obtained by introducing the bilinear form, or indefinite inner product $\{u, v\}$, defined for pairs of elements in D_Q by

$$\{u, v\} = \frac{1}{2i}\left[(Qu, v) - (u, Qv)\right].$$

This inner product is highly degenerate; indeed, relative to it, the manifold D_{Q_0} is orthogonal to itself and to *all* of D_Q. If $Q_0 \subseteq Q_1 \subseteq Q$ then also $Q_0 \subseteq Q_1^* \subseteq Q$, and the domains of Q_1 and Q_1^* are orthogonal complements. Hence, in order for Q_1 to be symmetric, its domain must be self-orthogonal; in order for Q_1 to be self-adjoint, its domain must be self-complementary. The condition that Q_1 be dissipative is that $\{u, u\} \geqq 0$ in D_{Q_1}.

For an ordinary differential operator, the relative dimension of D_Q, modulo D_{Q_0}, is always finite. Consequently, the domain of an operator Q_1, lying between Q_0 and Q, is characterized by a finite set of conditions

$$\{u, v_j\} = 0 \qquad j = 1, \cdots, m,$$

expressing the orthogonality of $u \in D_{Q_1}$ to a relative basis $v_1, \cdots, v_m$ of $D_{Q^*}(\text{mod } D_{Q_0})$. By the Lagrange identity, these orthogonality conditions

amount to boundary conditions: in the SL case, of the form

$$\{u, v_j\} = \frac{1}{2i} \lim_{\substack{b \nearrow \beta \\ a \searrow \alpha}} \mathop{\Delta}_{x=a}^{b} [u(x)\bar{v}_j'(x) - u'(x)\bar{v}_j(x)] = 0, \qquad j = 1, 2, \cdots, m.$$

It follows that the domain of the dissipative operator-valued function Q_ξ is specified for nonreal ξ by a variable set of boundary conditions

$$\{u, v_j(\xi)\} = 0 \qquad\qquad j = 1, 2, \cdots, m;$$

since Q_ξ is dissipative, these boundary conditions will also be called *dissipative*. The conditions may be *self-adjoint* for nonreal ξ, i.e., may specify a self-adjoint Q_ξ, only in the case already described where Q^+ acts in L_2. In that case $v_1, \cdots, v_m$ are *constant* elements of D_Q, and specify the domain of Q^+ itself. It should be noted that the boundary conditions take on an especially simple form for the regular SL operator, since in this case u, u', v_j, v_j' are all continuous on the *closed* interval (α, β). Thus, for example, the pairs of conditions

$$\begin{cases} u(\alpha) = 0 \\ u(\beta) = 0 \end{cases} \quad \text{or} \quad \begin{cases} u(\alpha) = u(\beta) \\ u'(\alpha) = u'(\beta) \end{cases}$$

specify the domains of self-adjoint operators containing Q_0. An example of dissipative boundary conditions is the pair

$$\begin{cases} u(\alpha) = 0 \\ u'(\beta) + H(\xi)u(\beta) = 0, \end{cases}$$

where $H(\xi)$ is analytic on the upper half plane and

$$\operatorname{Im} H(\xi) \geqq 0 \qquad \text{when } \operatorname{Im} \xi > 0.$$

It will be useful for us to single out certain special categories of boundary conditions, in addition to the self-adjoint. These will have in common the property that the $v_k(\xi)$ tend to continuous limits on a *real* interval Δ. If, on Δ, Q_ξ is self-adjoint, then the corresponding boundary conditions will be called *quasi-self-adjoint* on Δ. On the other hand, if, on Δ, Q_ξ neither is self-adjoint nor even contains a proper symmetric extension of Q_0, then the boundary conditions will be called *strictly dissipative* on Δ.

The significance of these categories of boundary conditions is in connection with the qualitative properties of the spectral matrix. They are useful in describing and comparing the various spectral matrix functions $\sigma(\lambda)$ which are generated by one and the same symmetric differential operator Q_0. Most known theorems along this line have to do with the *spectrum* S of $\sigma(\lambda)$, that is the set of its points of increase.

The earliest result of this kind is due to Weyl and deals with the *essential spectrum* S', namely, the set of cluster points of S. Weyl discovered that the

essential spectrum is the same point set for all self-adjoint extensions Q^+ in L_2, thus, for all self-adjoint boundary conditions.

It can be shown (see [23]) that this invariance of essential spectrum extends even to families of boundary conditions which are only quasi-self-adjoint on a real interval Δ. That is, the essential spectrum is the same in Δ for all such boundary conditions. For an *arbitrary* family of boundary conditions, the essential spectrum must contain this basic invariant part, but may be a much larger set. In fact, if the boundary conditions are strictly dissipative on the interval Δ, that entire interval will belong to S.

The phenomenon is perhaps most strikingly illustrated by an operator Q_0 having at least one extension with a purely discrete spectrum—that is, the spectrum consists of isolated eigenvalues. This is so, in particular, for any regular differential operator. It is clear that in such cases the invariant part of the essential spectrum must be void. Consequently, *all* self-adjoint and quasi-self-adjoint boundary conditions generate discrete spectra. On the other hand, boundary conditions which are strictly dissipative on some Δ generate a spectral function $\sigma(\lambda)$ which increases strictly and smoothly there—a fact first announced for *regular* operators by Coddington and Gilbert ([15]). (Actually a more precise statement is possible. See [15] and [23].)

A behavior remarkably similar to that of essential spectrum is shown by a second point set called the *absolutely continuous spectrum* of σ. Lebesgue decomposition of the components of $\sigma(\lambda)$ breaks this matrix function into a sum

$$\sigma(\lambda) = \sigma_a(\lambda) + \sigma_s(\lambda)$$

of increasing matrix functions—the first absolutely continuous, the second purely singular. The *absolutely continuous spectrum* S_a of σ is defined as the set of points of increase of σ_a. The invariant character of absolutely continuous spectrum was first recognized by Aronszajn ([24]); earlier investigators from the time of Weyl had incorrectly supposed that the similarly defined *continuous* spectrum would show this behavior. Just as for essential spectrum, the invariance of S_a can be shown to extend to quasi-self-adjoint boundary conditions. For arbitrary boundary conditions S_a will contain the invariant part, though in general, properly. Where the boundary conditions are strictly dissipative, there the spectral function is everywhere increasing, and furthermore is absolutely continuous. (These assertions have been proved so far only for the SL operator; see [23].)

I wish to conclude my discussion of differential boundary problems by stating a final qualitative property, this time a property of the *point* spectrum of σ. The result is analogous to a theorem of R. Nevanlinna ([25], p. 60), regarding the indefinite Hamburger moment problem.

We consider the class of SL problems which allow a nonvoid invariant part in the essential spectrum and for which, therefore as is well known, the spectral matrix σ is essentially scalar. (This is Weyl's limit point-limit circle case.)

Let λ_0 be a point of jump of a σ_0 which corresponds to *self-adjoint* boundary conditions. Then the Nevanlinna property is the following:

> the mass concentrated at λ_0 by σ_0 exceeds the mass which is concentrated there by any other spectral function of the operator.

A proof of the Nevanlinna property can be constructed along the lines of [**23**], Section 6.

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References

1. M. A. Naimark, Spectral functions of a symmetric operator (Russian) Izv. Akad. Nauk SSSR, ser. mat., 4 (1940) 277–318.

2. Torsten Carleman, Sur les équations intégrales singulières à noyau réel et symétrique, Uppsala, 1923.

3. John v. Neumann, Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, Math. Ann., 102 (1929) 370–427.

4. E. L. Ince, Ordinary differential equations, Chp. 10, 11, Dover, 1926.

5. H. Weyl, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen, Math. Ann., 68 (1910) 220–269.

6. Marshall Stone, Linear transformations in Hilbert space, AMS Colloquium Publications, 15 (1932).

7. I. M. Glazman, On the theory of singular differential operators, Amer. Math. Soc. Translation, 1953.

8. E. A. Coddington, The spectral representation of ordinary self-adjoint differential operators, Ann. Math., 60 (1954) 192–211.

9. K. Kodaira, On ordinary differential equations of any even order and the corresponding eigenfunction expansions, Amer. J. Math., 72 (1950) 502–544.

10. N. I. Achieser, I. M. Glasmann, Theorie der Linearen Operatoren im Hilbert-Raum, Anhang 2, Berlin (1960).

11. N. Levinson, The expansion theorem for singular self-adjoint linear differential operators, Ann. Math., ser. 2, 59 (1954) 300–315; Transform and inverse transform expansions for singular self-adjoint differential operators, Illinois J. Math., 2 (1958) 224–235.

12. E. A. Coddington, On self-adjoint ordinary differential operators, Math. Scand., 4 (1956) 9–21.

13. A. V. Štraus, On spectral functions of an even order differential operator, Dokl. Akad. Nauk SSSR (N.S.) 115 (1957) 67–70.

14. E. A. Coddington, Generalized resolutions of the identity for symmetric ordinary differential operators, Ann. of Math., 68 (1958) 378–392.

15. E. A. Coddington and R. C. Gilbert, Generalized resolvents of ordinary differential operators, Trans. Amer. Math. Soc., 93 (1959) 216–241.

16. L. Gårding, Applications of the theory of direct integrals of Hilbert spaces to some integral and differential operators, Inst. for Fluid Dym. and Appl. Math., Univ. Md., 1954.

17. F. Brauer, Spectral theory for the differential equation $Lu=\lambda Mu$, Canad. J. Math., 10 (1958) 413–428.

18. E. Hellinger, Journal für Mathematik, 136 (1909) 210–271.

19. E. C. Titchmarsch, Eigenfunction expansions associated with second order differential equations, I, Oxford, 1956.

20. K. Kodaira, The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices, Amer. J. Math., 71 (1949) 921–945.

21. H. Weyl, Ramifications, old and new, of the eigenvalue problem, Bull. Amer. Math. Soc., 56 (1950).

22. A. V. Štraus, Generalized resolvents of symmetric operators, Izv. Akad. Nauk SSSR, Ser. Mat., **18** (1954) 51–86.

23. R. McKelvey, The spectra of minimal self-adjoint extensions of a symmetric operator, Pacific J. Math., 12 (1962) 1003–1022.

24. N. Aronszajn, On a problem of Weyl in the theory of singular Sturm-Liouville equations, Amer. J. Math., **79** (1957) 597–608.

25. J. A. Shohat and J. D. Tamarkin, The problem of moments, Surveys Amer. Math. Soc., No. I (1948).

EUCLIDEAN GEOMETRY AND MINKOWSKIAN CHRONOMETRY

WALTER NOLL, Carnegie Institute of Technology

1. Introduction. The term "Minkowskian Chronometry" is used here for the study of the structure of space-time appropriate to Einstein's special relativity. This term suggests a parallel with Euclidean geometry. Originally, geometry was an empirical science which dealt with measuring of distances on earth. The Greeks transformed this science into a beautiful mathematical discipline. Chronometry is the science of the measurement of time intervals. Einstein's first paper was a critical study of time measurements. It was Minkowski, however, who made a geometrical discipline out of Einstein's chronometry. More recently, Synge ([2], [3]) has been an able advocate of the geometrical point of view in relativity.

The present paper is an attempt to axiomatize Minkowskian chronometry using direct coordinate-free methods. I believe that the coordinate-free approach fosters the cultivation of intuition, a scarce commodity in relativity because the phenomena this theory is intended to describe are as yet rather remote from our daily experience. I hope, moreover, that Minkowskian chronometry will become, as Euclidean geometry did, a branch of mathematics that is of interest purely for its esthetic value.

Section 2 contains a collection of definitions and results on vector spaces with an inner product that is not necessarily positive definite. We give a new proof of the inertia theorem of Sylvester, a proof that remains valid for infinite dimensional spaces.

In Section 3 we present certain inequalities valid in inner product spaces of index one. The most notable of these are the "reversed Schwarz inequality" and the "reversed triangle inequality."

Section 4 consists of an axiomatic introduction to pseudo-Euclidean geometry, based on the fact that the structure of a pseudo-Euclidean space is determined by its "separation function," which in the Euclidean case is identical to the square of the metric. Associated with each pseudo-Euclidean space is a unique "translation space," which is a vector space with inner product.

"Inner product" is just another term for "nondegenerate bilinear form." Usually the term is used, however, only when the form is positive definite. But indefinite inner products can be treated in the same geometric spirit as definite inner products usually are, and we adopt here the terminology that has become standard so far only for the case of definite inner products (see, e.g. Halmos [5]). The theory of bilinear forms is extensive, much of it going back to the nineteenth century. In most textbooks, however, the subject is treated in analytical language with components. A notable exception is the treatise of Bourbaki [1], to which we refer for further information on the topics treated in Sections 2 and 4.

When the index of the translation space is zero, pseudo-Euclidean geometry reduces to Euclidean geometry, and when the index is one, it reduces to Minkowskian chronometry. From a physical point of view it is reasonable to use distance as a primitive notion in Euclidean geometry, because there are yardsticks to measure distances. In this case, the separation of two points is just the square of their distance. There are no "separation meters," however, to measure the separation of two arbitrary events in relativistic space-time. In Section 5 we give an axiomatic introduction to Minkowskian chronometry based on primitive notions that have a more direct physical meaning: observers, clock-readings, signals. We show—and this is the least trivial part of the paper—that these primitive data determine the separation function uniquely. The fact that the index of the translation space is one is not an assumption of the theory but comes out as a theorem.

Section 6 deals with temporal order, i.e. with the possibility of distinguishing future from past. It turns out that such order, in Minkowskian chronometry, is related to the distinction between emission and reception of signals. We note that in classical space-time, such distinction is not sufficient to determine temporal order. The axioms given here may answer a problem proposed by Suppes ([4], Sect. 4).

We do not impose anywhere a restriction on the dimension, which may even be infinite. Of course, in the presently known physical applications of Minkowskian chronometry the dimension is 4.

2. Inner product spaces. (For details see [1].) Let $\mathcal{V}$ be a real vector space. A nondegenerate symmetric bilinear form on $\mathcal{V}$ will be called an inner product. The inner product of $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ will be denoted by $\mathbf{u}\cdot\mathbf{v}$, and we will abbreviate $\mathbf{u}\cdot\mathbf{u} = \mathbf{u}^2$. Nondegeneracy means that

$$(2.1) \qquad \mathbf{u}\cdot\mathbf{v} = 0 \text{ for all } \mathbf{v} \in \mathcal{V} \text{ implies } \mathbf{u} = \mathbf{0}.$$

A *quadratic form* Φ on $\mathcal{V}$ is a function $\Phi\colon \mathcal{V} \to \mathcal{R}$ ($\mathcal{R}$ = set of real numbers) such that $\Phi(\mathbf{v}) = \mathbf{v}\cdot\mathbf{v}$ for some inner product. The quadratic form determines the inner product uniquely.

An *inner product space* $\mathcal{V}$ is a vector space with an additional structure defined by an inner product. If $\mathcal{U}$ is a subspace of $\mathcal{V}$, then $\mathcal{U}^\perp = \{\mathbf{v} \mid \mathbf{v}\cdot\mathbf{u} = 0$ for all

$\mathbf{u}\in\mathcal{U}\}$ is called the *orthogonal complement* of $\mathcal{U}$. If dim $\mathcal{U}$ is finite, then $\mathcal{U}^{\perp\perp}=\mathcal{U}$ and dim $\mathcal{U}+$dim $\mathcal{U}^{\perp}=$dim $\mathcal{V}$. A subspace $\mathcal{U}$ is called *regular* if $\mathcal{U}\cap\mathcal{U}^{\perp}=\{\mathbf{0}\}$ and *singular* if $\mathcal{U}\cap\mathcal{U}^{\perp}\neq\{\mathbf{0}\}$. If dim $\mathcal{U}$ is finite and $\mathcal{U}$ is regular, then $\mathcal{V}$ has the direct decomposition $\mathcal{V}=\mathcal{U}\oplus\mathcal{U}^{\perp}$.

A linear transformation $Q\colon\mathcal{V}\to\mathcal{V}$ is called *orthogonal* if it preserves the inner product. In view of (2.1), Q is then also one-to-one and hence an automorphism of the inner product space $\mathcal{V}$.

The following terminology is suggested by the physical applications. The sets of vectors

$$(2.2)\qquad \begin{aligned}\mathcal{V}_+ &= \{\mathbf{v}\,|\,\mathbf{v}^2>0 \text{ or } \mathbf{v}=\mathbf{0}\},\\ \mathcal{V}_- &= \{\mathbf{v}\,|\,\mathbf{v}^2<0 \text{ or } \mathbf{v}=\mathbf{0}\},\\ \mathcal{V}_0 &= \{\mathbf{v}\,|\,\mathbf{v}^2=0\}\end{aligned}$$

will be called the *space-cone*, the *time-cone*, and the *signal-cone*, respectively. These cones have only the zero vector $\mathbf{0}$ in common. A vector $\mathbf{v}$ is said to be *space-like*, *time-like*, or a *signal vector* depending on whether $\mathbf{v}\in\mathcal{V}_+$, $\mathbf{v}\in\mathcal{V}_-$, or $\mathbf{v}\in\mathcal{V}_0$. The maximal dimension of the time-like subspaces of $\mathcal{V}$, (i.e. subspaces contained in $\mathcal{V}_-$) will be called the *index* of $\mathcal{V}$; it will be denoted by $i=$ind $\mathcal{V}$. (The customary definition is ind $\mathcal{V}=$maximal dimension of the signal subspaces of $\mathcal{V}$. Our definition will give the same value if the sign of the inner product is properly adjusted, as is shown in Theorem 2 below.)

THEOREM 1. *If $\mathcal{U}$ is a time-like subspace of maximal dimension $i=$ind $\mathcal{V}$ and if $i<\infty$, then the complement $\mathcal{U}^{\perp}$ is space-like and $\mathcal{V}$ has the direct decomposition*

$$(2.3)\qquad \mathcal{V}=\mathcal{U}\oplus\mathcal{U}^{\perp},\qquad \mathcal{U}\subset\mathcal{V}_-,\ \mathcal{U}^{\perp}\subset\mathcal{V}_+.$$

In addition, for any decomposition of the type (2.3), *we have* dim $\mathcal{U}=i$.

Proof. Let $\mathcal{U}\subset\mathcal{V}_-$, dim $\mathcal{U}=i$. $\mathcal{U}$ is then regular and $\mathcal{V}$ has the direct decomposition $\mathcal{V}=\mathcal{U}\oplus\mathcal{U}^{\perp}$. Assume that $\mathcal{U}^{\perp}$ is not space-like. Then there is a vector $\mathbf{w}\in\mathcal{U}^{\perp}$ such that $\mathbf{w}^2\leqq 0$, $\mathbf{w}\neq\mathbf{0}$. By (2.1), there is a vector $\mathbf{v}\in\mathcal{V}$ such that $\mathbf{w}\cdot\mathbf{v}=\alpha\neq 0$. Let

$$\mathbf{v}=\mathbf{u}+\mathbf{z},\qquad \mathbf{u}\in\mathcal{U},\ \mathbf{z}\in\mathcal{U}^{\perp}$$

be the decomposition of $\mathbf{v}$. We have $\alpha=\mathbf{w}\cdot\mathbf{v}=\mathbf{w}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{z}=\mathbf{w}\cdot\mathbf{z}$. Since $\mathbf{w}$, $\mathbf{z}\in\mathcal{U}^{\perp}$, the vector $\mathbf{p}=\mathbf{z}+\beta\mathbf{w}$ also belongs to $\mathcal{U}^{\perp}$ for any choice of β. Now,

$$\mathbf{p}^2=\mathbf{z}^2+2\beta\mathbf{w}\cdot\mathbf{z}+\beta^2\mathbf{w}^2\leqq\mathbf{z}^2+2\beta\alpha.$$

Since $\alpha\neq 0$, we can adjust β such that $\mathbf{p}^2<0$. If $\mathbf{x}\in\mathcal{U}$ and $\mathbf{y}=\mathbf{x}+\lambda\mathbf{p}$, then $\mathbf{y}^2=\mathbf{x}^2+\lambda^2\mathbf{p}^2<0$, unless $\mathbf{y}=\mathbf{0}$. Therefore, the space $\mathcal{U}'$ spanned by $\mathcal{U}$ and $\mathbf{p}$ is time-like and of dimension $i+1$, which contradicts the assumption that $\mathcal{U}$ is of maximal dimension.

Consider now an arbitrary decomposition of the form (2.3) and assume that $\hat{\mathcal{U}}$ is a time-like subspace of maximal dimension i. Every $\hat{\mathbf{u}}\in\hat{\mathcal{U}}$ has a unique

decomposition $\hat{\mathbf{u}}=\mathbf{u}+\mathbf{z}$, $\mathbf{u}\in\mathcal{U}$, $\mathbf{z}\in\mathcal{U}^\perp$. This decomposition defines a linear transformation $\hat{\mathbf{u}}\rightarrow\mathbf{u}=L\hat{\mathbf{u}}$ of $\hat{\mathcal{U}}$ into $\mathcal{U}$. If $\mathbf{u}=\mathbf{0}$, then $\hat{\mathbf{u}}=\mathbf{0}+\mathbf{z}\in\mathcal{U}^\perp\subset\mathcal{V}_+$, and hence $\hat{\mathbf{u}}=\mathbf{0}$. Thus L is one-to-one, which implies $i=\dim\hat{\mathcal{U}}\leqq\dim\mathcal{U}$. On the other hand, $\dim\mathcal{U}\leqq\dim\hat{\mathcal{U}}=i$ by definition of i. Q.E.D.

In the case when $\dim\mathcal{V}$ is finite, Theorem 1 reduces to a version of the classical inertia theorem of Sylvester (cf. [1], Chapter 7, no. 2).

Lincoln E. Bragg has disclosed to me a counterexample which shows that the conclusion of Theorem 1 may be false when $i=\infty$. From now on we assume that the index i of $\mathcal{V}$ is finite ($\dim\mathcal{V}$ need not be finite).

COROLLARY. *Let a decomposition of the type* (2.3) *be given. Then any other decomposition of the same type has the form*

$$\mathcal{V} = Q(\mathcal{U}) \oplus Q(\mathcal{U}^\perp), \tag{2.4}$$

where Q is orthogonal.

THEOREM 2. *If $\dim\mathcal{V}-i\geqq i$, then the maximal dimension of the signal subspaces of $\mathcal{V}$ is also given by i.*

Proof. Let $\mathcal{S}$ be a signal subspace of maximal dimension i', and consider a decomposition of the form (2.3). By Theorem 1 we have $\dim\mathcal{U}=i$, $\dim\mathcal{U}^\perp=\dim\mathcal{V}-i\geqq i$. The unique decomposition $\mathbf{s}=\mathbf{u}+\mathbf{z}$, $\mathbf{u}\in\mathcal{U}$, $\mathbf{z}\in\mathcal{U}^\perp$ defines two linear transformations $\mathbf{s}\rightarrow\mathbf{u}=L\mathbf{s}$ and $\mathbf{s}\rightarrow\mathbf{z}=N\mathbf{s}$ of $\mathcal{S}$ into $\mathcal{U}$ and $\mathcal{U}^\perp$, respectively. It is easily seen that both L and N are one-to-one. It follows that $i'=\dim\mathcal{S}\leqq\dim\mathcal{U}=i$. If $i'<i$, $L(\mathcal{S})$ and $N(\mathcal{S})$ must be proper subspaces of $\mathcal{U}$ and $\mathcal{U}^\perp$, respectively. It is then possible to find $\mathbf{x}\in\mathcal{U}$ and $\mathbf{y}\in\mathcal{U}^\perp$ such that $\mathbf{x}$ is orthogonal to $L(\mathcal{S})$ and $\mathbf{y}$ is orthogonal to $N(\mathcal{S})$. If $\mathbf{x}$ and $\mathbf{y}$ are normalized such that $\mathbf{x}^2=-1$, $\mathbf{y}^2=+1$, the space spanned by $\mathcal{S}$ and $\mathbf{x}+\mathbf{y}$ is easily seen to be a signal subspace of dimension $i'+1$, which contradicts the definition of i'. Q.E.D.

3. Spaces of index one. From now on we assume that $\mathcal{V}$ is an inner product space of index one. The following two theorems are corollaries of the results of the previous section.

THEOREM 1. *Let $\mathbf{l}$ be a time-like unit vector ($\mathbf{l}^2=-1$). Then every vector $\mathbf{v}\in\mathcal{V}$ has unique decomposition of the form*

$$\mathbf{v} = \xi\mathbf{l} + \mathbf{w}, \qquad \mathbf{w}\cdot\mathbf{l} = 0, \qquad \mathbf{w} \in \mathcal{V}_+. \tag{3.1}$$

THEOREM 2. *If a vector is orthogonal to a nonzero time-like vector then it must be space-like.*

In an indefinite inner product space ($\operatorname{ind}\mathcal{V}\neq 0$), Schwarz's inequality is of course not valid. When $\operatorname{ind}\mathcal{V}=1$ the following two theorems partially replace the Schwarz inequality:

THEOREM 3 (reversed Schwarz inequality). *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are both time-like, then*

$$(\mathbf{u}\cdot\mathbf{v})^2 \geqq \mathbf{u}^2\mathbf{v}^2, \tag{3.2}$$

and equality holds only when $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly dependent.*

THEOREM 4. *For any three nonzero time-like vectors* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$,

$$(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{u}) < 0. \tag{3.3}$$

Proof. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}\in\mathcal{V}_-$, all $\neq\mathbf{0}$. Consider

$$\mathbf{z} = \alpha\mathbf{u} - \beta\mathbf{v}, \qquad \alpha = \mathbf{v}\cdot\mathbf{w}, \qquad \beta = \mathbf{u}\cdot\mathbf{w}.$$

We have $\mathbf{z}\cdot\mathbf{w}=\alpha\beta-\beta\alpha=0$. It follows from Theorem 2 that

$$0 \leqq \mathbf{z}^2 = \alpha^2\mathbf{u}^2 + \beta^2\mathbf{v}^2 - 2\alpha\beta\mathbf{u}\cdot\mathbf{v},$$

i.e.

$$2(\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{v}) \leqq \alpha^2\mathbf{u}^2 + \beta^2\mathbf{v}^2. \tag{3.4}$$

Here equality can hold only when $\mathbf{z}=\mathbf{0}$, i.e. when $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. Since $\mathbf{w}$ is time-like and $\neq\mathbf{0}$, Theorem 2 shows that $\alpha=\mathbf{v}\cdot\mathbf{w}$ and $\beta=\mathbf{u}\cdot\mathbf{w}$ cannot be zero. Therefore, the right-hand side of (3.4) is negative, which proves Theorem 4. Theorem 3 is trivial when $\mathbf{u}=\mathbf{0}$ or $\mathbf{v}=\mathbf{0}$. Otherwise it follows from (3.4) when we put $\mathbf{w}=\mathbf{u}$ and observe that $\mathbf{u}^2<0$. Q.E.D.

Consider the relation $\mathbf{u}\cdot\mathbf{v}<0$ within the set $\hat{\mathcal{V}}_-$ of all nonzero time-like vectors. Theorem 4 shows that this relation is transitive. Clearly, it is also reflexive and symmetric. Hence $\mathbf{u}\cdot\mathbf{v}<0$ is an equivalence relation in $\hat{\mathcal{V}}_-$. The vectors $\mathbf{u}\in\hat{\mathcal{V}}_-$ and $-\mathbf{u}\in\hat{\mathcal{V}}_-$ belong to different equivalence classes, hence there are at least two such classes. But there are no more than two classes, because if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ belonged to three different classes, then the inner products on the left side of (3.3) would be all positive, which contradicts (3.3). We adjoin the zero vector $\mathbf{0}$ to each of the two equivalence classes and denote the resulting sets by $\mathcal{V}_-^1$ and $\mathcal{V}_-^2$, respectively. It is easily verified that $\mathcal{V}_-^1$ and $\mathcal{V}_-^2$ are in fact convex cones. We summarize:

THEOREM 5. *The time cone* $\mathcal{V}_-$ *is the union of two convex cones* $\mathcal{V}_-^1$ *and* $\mathcal{V}_-^2$ *which have only the zero vector in common. Two time-like vectors* $\mathbf{u}$, $\mathbf{v}$ *belong to the same cone if and only if* $\mathbf{u}\cdot\mathbf{v}\leqq 0$. *If* $\mathbf{u}\in\mathcal{V}_-$ *belongs to one of the cones, then* $-\mathbf{u}$ *belongs to the other, i.e.,* $\mathcal{V}_-^1 = -\mathcal{V}_-^2$.

We call $\mathcal{V}_-^1$ and $\mathcal{V}_-^2$ the two *directed time cones.*

If $\mathbf{v}$ is time-like, we call $\tau(\mathbf{v})=\sqrt{-\mathbf{v}^2}$ the *duration* of $\mathbf{v}$. We have $\tau(\mathbf{v})\geqq 0$, and $\tau(\mathbf{v})=0$ only if $\mathbf{v}=\mathbf{0}$. The following "reversed triangle inequality" is a consequence of Theorems 3 and 5. The proof is analogous to that of the ordinary triangle inequality.

THEOREM 6 (reversed triangle inequality). *If the two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *both belong to one of the directed time cones* ($\mathbf{u}\cdot\mathbf{v}\leqq 0$), *then*

$$\tau(\mathbf{u}+\mathbf{v}) \geqq \tau(\mathbf{u}) + \tau(\mathbf{v}), \tag{3.5}$$

and equality holds only when $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly dependent.*

Remark. The reversed triangle inequality expresses what is often misleadingly called "the relativistic clock paradox."

4. **Pseudo-Euclidean Geometry.** Let a set $\mathcal{E}$ of points $x, y, \cdots$ and a function

$$\sigma: \mathcal{E} \times \mathcal{E} \to \mathcal{R} \tag{4.1}$$

be given, where $\mathcal{R}$ denotes the real line. The function σ defines a certain structure on $\mathcal{E}$. The automorphisms of this structure are the one-to-one mappings $\mathfrak{a}$ of $\mathcal{E}$ onto itself which satisfy

$$\sigma(\mathfrak{a}(x), \mathfrak{a}(y)) = \sigma(x, y) \tag{4.2}$$

for all $x, y\in\mathcal{E}$. We denote by $\mathcal{A}$ the group of all these automorphisms. We impose restrictive conditions on σ by assuming that $\mathcal{A}$ contains a subgroup $\mathcal{V}$ which satisfies the axioms (E_1)–(E_4) stated below.

(E_1) *$\mathcal{V}$ is commutative.*

(E_2) *$\mathcal{V}$ is transitive.*

(E_3) *If* $\mathbf{v}\in\mathcal{V}$ *maps some point* $x\in\mathcal{E}$ *onto itself, then* $\mathbf{v}$ *is the identity mapping.*

We write the group operation in $\mathcal{V}$ additively and denote the identity mapping by $\mathbf{0}$. We write $x+\mathbf{v}\in\mathcal{E}$ for the image $\mathbf{v}(x)$ of $x\in\mathcal{E}$ under $\mathbf{v}\in\mathcal{V}$. It follows from (E_2) and (E_3) that for any two points $x, y\in\mathcal{E}$ there is a *unique* $\mathbf{v}\in\mathcal{V}$ which maps x onto y. The mapping $\mathbf{v}$ determined in this way will be denoted by $\mathbf{v}=y-x$. The "sum" $x+\mathbf{v}$ and "point-difference" $y-x$ thus defined obey the rules suggested by the notation.

The following proposition is easily established: The value $\sigma(x, y)$ depends only on the point difference $y-x$; i.e., there is a function $\Phi: \mathcal{V}\to\mathcal{R}$ such that

$$\Phi(y - x) = \sigma(x, y) \tag{4.3}$$

for all $x, y\in\mathcal{E}$.

The last condition required of $\mathcal{V}$ is the following.

(E_4) *$\mathcal{V}$ is the underlying additive group of a real vector space and Φ is a nondegenerate quadratic form on $\mathcal{V}$; i.e., $\mathcal{V}$ can be given the structure of an inner product space such that*

$$(y - x)^2 = (y - x)\cdot(y - x) = \sigma(x, y). \tag{4.4}$$

In view of (E_4) we refer to the automorphism $\mathbf{v}$ in $\mathcal{V}$ as *vectors*.

UNIQUENESS THEOREM. *There is at most one subgroup $\mathcal{V}$ of the automorphism group $\mathcal{A}$ such that $\mathcal{V}$ satisfies the axioms* (E_1)–(E_4). *If such a subgroup exists then its structure as an inner product space, as required for* (E_4), *is unique.*

Proof. Assume that $\mathcal{V}$ and $\hat{\mathcal{V}}$ are subgroups of $\mathcal{A}$ which both satisfy (E_1)–(E_4). If two points $x, y \in \mathcal{E}$ are given, we denote by $y-x$ the unique vector in $\mathcal{V}$ which maps x onto y and by $y \mathbin{\hat{-}} x$ the unique vector in $\hat{\mathcal{V}}$ which maps x onto y.

We choose a fixed point $q \in \mathcal{E}$ and define a mapping $\mathfrak{f}$ of $\mathcal{V}$ into $\hat{\mathcal{V}}$ by

$$\hat{\mathbf{v}} = \mathfrak{f}(\mathbf{v}) = (q + \mathbf{v}) \mathbin{\hat{-}} q, \tag{4.5}$$

so that

$$q + \mathbf{v} = q + \mathfrak{f}(\mathbf{v}) = q + \hat{\mathbf{v}} \tag{4.6}$$

for all $\mathbf{v} \in \mathcal{V}$. It is clear that $\mathfrak{f}$ is one-to-one and onto, and that $\mathfrak{f}(\mathbf{0}) = \hat{\mathbf{0}} = \mathbf{0}$.

By (4.4) we have

$$\sigma(x, y) = (y - x)^2 = (y \mathbin{\hat{-}} x)^2. \tag{4.7}$$

Substituting $x = q + \mathbf{u}$ and $y = q + \mathbf{v}$ into (4.7) we find that

$$(\mathbf{v} - \mathbf{u})^2 = (\hat{\mathbf{v}} - \hat{\mathbf{u}})^2 \tag{4.8}$$

holds for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. In particular, when $\mathbf{u} = \mathbf{0} = \hat{\mathbf{0}} = \hat{\mathbf{u}}$, (4.8) shows that $\mathbf{v}^2 = \hat{\mathbf{v}}^2$ for all $\mathbf{v} \in \mathcal{V}$. Hence, if we expand (4.8),

$$\mathbf{v}^2 - 2\mathbf{v}\cdot\mathbf{u} + \mathbf{u}^2 = \hat{\mathbf{v}}^2 - 2\hat{\mathbf{v}}\cdot\hat{\mathbf{u}} + \hat{\mathbf{u}}^2, \tag{4.9}$$

the square terms cancel and we obtain

$$\mathbf{v}\cdot\mathbf{u} = \hat{\mathbf{v}}\cdot\hat{\mathbf{u}} = \mathfrak{f}(\mathbf{v})\cdot\hat{\mathbf{u}}. \tag{4.10}$$

Repeated use of (4.10) shows that

$$\begin{aligned} \mathfrak{f}(\alpha\mathbf{v} + \beta\mathbf{w})\cdot\hat{\mathbf{u}} &= (\alpha\mathbf{v} + \beta\mathbf{w})\cdot\mathbf{u} = \alpha(\mathbf{v}\cdot\mathbf{u}) + \beta(\mathbf{w}\cdot\mathbf{u}) \\ &= [\alpha\mathfrak{f}(\mathbf{v}) + \beta\mathfrak{f}(\mathbf{w})]\cdot\hat{\mathbf{u}}. \end{aligned}$$

Hence

$$[\alpha\mathfrak{f}(\mathbf{v}) + \beta\mathfrak{f}(\mathbf{w}) - \mathfrak{f}(\alpha\mathbf{v} + \beta\mathbf{w})]\cdot\hat{\mathbf{u}} = 0,$$

which holds for all $\hat{\mathbf{u}} \in \hat{\mathcal{V}}$. Since ($E_4$) requires that the inner product is non-degenerate it follows from (2.1) that

$$\alpha\mathfrak{f}(\mathbf{v}) + \beta\mathfrak{f}(\mathbf{w}) = \mathfrak{f}(\alpha\mathbf{v} + \beta\mathbf{w}) \tag{4.11}$$

is valid for all $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ and all real α, β. It is the content of (4.10), (4.11) and the remark after (4.6) that $\mathfrak{f}$ is an inner product space isomorphism of $\mathcal{V}$ onto $\hat{\mathcal{V}}$.

Let $x \in \mathcal{E}$ and $\mathbf{v} \in \mathcal{V}$ be given. Put $\mathbf{u} = x - q$ so that $x = q + \mathbf{u}$. Using (4.6) twice and (4.11) once we derive

$$x + \mathbf{v} = q + \mathbf{u} + \mathbf{v} = q + \mathfrak{f}(\mathbf{u} + \mathbf{v}) = q + \mathfrak{f}(\mathbf{u}) + \mathfrak{f}(\mathbf{v}) = q + \mathbf{u} + \mathfrak{f}(\mathbf{v}) = x + \mathfrak{f}(\mathbf{v}).$$

Thus, $x + \mathbf{v} = x + \mathfrak{f}(\mathbf{v})$ is valid for all $x \in \mathcal{E}$, which shows that the mappings $\mathbf{v}$ and $\mathfrak{f}(\mathbf{v})$ of $\mathcal{E}$ onto $\mathcal{E}$ are the same. Therefore, $\mathfrak{f}$ is the identity mapping of $\mathcal{V}$ onto itself, which completes the proof that $\mathcal{V}$ and $\hat{\mathcal{V}}$ coincide as inner product spaces.

DEFINITION. *A set* $\mathcal{E}$ *which is endowed with a structure defined by a real valued function* σ *on* $\mathcal{E}\times\mathcal{E}$ *is called a pseudo-Euclidean space if the axioms* (E_1)–(E_4) *are satisfied. The function* σ *will be called the separation function of* $\mathcal{E}$. *The inner product space* $\mathcal{V}$ *determined by* σ *is called the translation space of* $\mathcal{E}$. (The definition given here differs from that of Bourbaki ([1] Chapter 6, no. 6) in that the translation space $\mathcal{V}$ is *not* regarded as part of the defining structure.)

The uniqueness theorem insures that the translation space is well defined.

If the separation function σ is nonnegative and if the translation space $\mathcal{V}$ is finite-dimensional, then $\mathcal{E}$ may be regarded as a Euclidean space in the ordinary sense.

The following is a corollary of the uniqueness theorem.

REPRESENTATION THEOREM. *Let* $\mathcal{E}$ *be a pseudo-Euclidean space and* q *a point in* $\mathcal{E}$. *Then every automorphism* $\mathfrak{a}$ *of* $\mathcal{E}$ *has a unique representation of the form*

$$\mathfrak{a}(x) = \mathfrak{a}(q) + Q(x - q), \tag{4.12}$$

where Q *is an orthogonal transformation of the translation space* $\mathcal{V}$.

Proof. The mapping Q defined by

$$Q(\mathbf{v}) = \mathfrak{a} \circ \mathbf{v} \circ \mathfrak{a}^{-1} \tag{4.13}$$

is an isomorphism of the subgroup $\mathcal{V}$ of $\mathcal{A}$ onto its conjugate $\mathcal{V}^* = \mathfrak{a} \circ \mathcal{V} \circ \mathfrak{a}^{-1}$ in $\mathcal{A}$. The mapping Q may be used to transport the inner product space structure of $\mathcal{V}$ to $\mathcal{V}^*$. It is clear that $\mathcal{V}^*$ then satisfies the conditions (E_1)–(E_4). Hence, by the uniqueness theorem, it must coincide with $\mathcal{V}$, and Q must be an automorphism of the inner product space $\mathcal{V}$. Since we use the notation $\mathbf{u}(q) = q + \mathbf{u}$ when $\mathbf{u} \in \mathcal{V}$, the image of $q \in \mathcal{E}$ under the mapping $Q\mathbf{v} \circ \mathfrak{a} = \mathfrak{a} \circ \mathbf{v}$ (see (4.13)) is given by

$$\mathfrak{a}(q) + Q\mathbf{v} = \mathfrak{a}(q + \mathbf{v}). \tag{4.14}$$

We obtain (4.12) by substituting $\mathbf{v} = x - q$ into (4.14).

Remark I. In the Euclidean case (4.12) is the well-known formula for rigid displacements. Many textbooks, however, derive this formula under unnecessary a priori assumptions of smoothness or even linearity of $\mathfrak{a}$; a theorem that does not require such assumptions appears as Exercise 21a, chapter 6, in Bourbaki [1]. The uniqueness theorem and the representation theorem, including proofs, remain valid when the field of real numbers is replaced by an arbitrary commutative ring $\mathcal{R}$ of characteristic $\neq 2$. In this case, the translation space is a module over $\mathcal{R}$.

Remark II. One may be tempted to define a subspace $\mathcal{F}$ of a pseudo-Euclidean space $\mathcal{E}$ by the condition that the restriction $\sigma_{\mathcal{F}}$ to $\mathcal{F}$ of the separation function σ of $\mathcal{E}$ satisfy the axioms (E_1)–(E_4) and hence give $\mathcal{F}$ the structure of a pseudo-Euclidean space. Unfortunately, it may happen that such a "subspace" $\mathcal{F}$ is

not a subspace in the customary sense, i.e., a set of the form

$$\mathfrak{F} = \{x \mid x = p + \mathbf{u}, \mathbf{u} \in \mathfrak{U}\}, \tag{4.15}$$

where $\mathfrak{U}$ is a subspace of the translation space of $\mathcal{E}$. For example, let $\mathcal{E} = \mathcal{R} \times \mathcal{R} \times \mathcal{R}$ with σ defined by

$$\sigma(x, y) = (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2. \tag{4.16}$$

The set $\mathfrak{F}$ of all triples of the form $(\phi(\xi), \phi(\xi), \xi)$, where ϕ is an arbitrary function $\phi: \mathcal{R} \to \mathcal{R}$, is a "subspace," but not of the type (4.15) when ϕ is not linear. But for this fact Theorem 1 of the following section would be trivial.

Remark III. The uniqueness theorem given in this Section shows that the complete structure of a pseudo-Euclidean space, including the translation space, is determined by a knowledge of the separation function σ alone. One may ask whether σ itself may not be uniquely determined by the prescription of even less information. A result in this direction was found by Suppes [2]. Under the assumption that the translation space has index 1 and dimension 4, he showed that σ is uniquely determined by its values $\sigma(x, y)$ for all pairs (x, y) such that $\sigma(x, y) < 0$. This result is easily obtained by an adaptation of the reasoning leading from (5.10) to (5.14) given in the following section. The uniqueness theorem of the following section is a different result of the same type.

5. Minkowskian Chronometry. We assume that the following data are given:

(a) a set $\mathcal{E}$, whose elements $x, y, \cdots$ will be called *events*;

(b) a family Ω of subsets of $\mathcal{E}$ which covers $\mathcal{E}$. The members $\mathcal{L}$ of Ω will be called *observers*;

(c) for each observer $\mathcal{L} \in \Omega$, a nonpositive separation function $\sigma_{\mathcal{L}}$ on $\mathcal{L}$ which gives $\mathcal{L}$ the structure of a one-dimensional pseudo-Euclidean space;

(d) a symmetric binary relation $\sim$ on $\mathcal{E}$ with the following property: Given any observer $\mathcal{L}$ and any event $x \notin \mathcal{L}$, there are at least two events y_1 and y_2 in $\mathcal{L}$ that are related to x. The relation $\sim$ will be called the *signal relation* and a pair (x, y) of events related by $\sim$ will be called a *signal.*

Remarks on physical interpretation: $x \in \mathcal{L}$ means physically that x is an event which is experienced by the observer $\mathcal{L}$. We imagine that each observer is equipped with a clock. If $\tau_{\mathcal{L}}(x, y)$ is the time-difference of the two clock readings at the events x and y of $\mathcal{L}$, then the separation $\sigma_{\mathcal{L}}(x, y)$ of $x, y \in \mathcal{L}$ is assumed to be given by $\sigma_{\mathcal{L}}(x, y) = -(\tau_{\mathcal{L}}(x, y))^2$. The two events of a signal (x, y) are interpreted to be the emission and reception of a light, radio, or other electromagnetic signal. If $x \notin \mathcal{L}$, we may imagine x to be the event of reflection of a signal which is sent out by $\mathcal{L}$ at y_1 and returns to $\mathcal{L}$ at y_2.

The data described under (a)–(d) define a certain structure on $\mathcal{E}$. We impose restrictions on this structure by assuming that there exists a separation function σ on all of $\mathcal{E}$ which endows $\mathcal{E}$ with the structure of a pseudo-Euclidean space and which satisfies the following two axioms:

(M_1) *σ is an extension of the separation function $\sigma_{\mathcal{L}}$ for each observer $\mathcal{L}\in\Omega$. In other words,*

$$\sigma(x, y) = \sigma_{\mathcal{L}}(x, y) \tag{5.1}$$

holds whenever $x, y\in\mathcal{L}$.

(M_2) *The pair (x, y) is a signal if and only if the separation of x and y is zero, i.e. $\sigma(x, y)=0$ if and only if $y\sim x$.*

UNIQUENESS THEOREM. *There is at most one separation function σ which satisfies the axioms* (M_1) *and* (M_2).

We first prove a number of preliminary theorems, assuming that some separation function σ satisfying (M_1) and (M_2) on $\mathcal{E}$ is given. The corresponding translation space is denoted by $\mathcal{V}$, as in Section 4.

THEOREM 1. *Every observer $\mathcal{L}$ is a time-like straight line in $\mathcal{E}$. More precisely: There is a vector $l\in\mathcal{V}$ with the following properties*

(i) *l is a time-like unit vector*, i.e. $l^2=-1$.

(ii) *if $q\in\mathcal{L}$ is given, then $x\in\mathcal{L}$ if and only if $x=q+\xi l$, and $\xi\in\mathcal{R}$.*

A vector l with the properties (i) and (ii) will be called a *direction vector* of the observer $\mathcal{L}$. It is clear that if l is a direction vector, then $-l$ is also one and there can be no others.

Proof. Let $\mathcal{V}'$ be the one-dimensional translation space corresponding to $\sigma_{\mathcal{L}}$. The event-difference in $\mathcal{V}'$ of the two events $x, y\in\mathcal{L}$ will be denoted by $y \stackrel{\prime}{-} x$. This difference must be carefully distinguished from the event difference $y-x$ in $\mathcal{V}$, which corresponds to the separation function σ. It follows from axiom (M_1) and from (4.3) that

$$(y - x)^2 = (y \stackrel{\prime}{-} x)^2 = \sigma(x, y) = \sigma_{\mathcal{L}}\cdot(x, y) \tag{5.2}$$

for $x, y\in\mathcal{L}$. Since $\mathcal{V}'$ is assumed to be one-dimensional and $\sigma_{\mathcal{L}}$ nonpositive, $\mathcal{L}$ can be represented in the form

$$\mathcal{L} = \{x \mid x = q + \xi l', \xi \in \mathcal{R}\},$$

where q is a fixed event in $\mathcal{L}$ and $l'\in\mathcal{V}'$ is such that $l'^2=-1$.

We now define a mapping $\mathbf{f}$ of $\mathcal{R}$ into $\mathcal{V}$ by $\mathbf{f}(\xi)=(q+\xi l')-q$. $\mathcal{L}$ is then the set of events x that are of the form

$$x = q + \xi l' = q + \mathbf{f}(\xi), \qquad \xi \in \mathcal{R}. \tag{5.3}$$

The same argument as the one that led from (4.5) to (4.10) shows that for all $\xi, \eta\in\mathcal{R}$ we must have

$$-\xi\eta = \mathbf{f}(\xi)\cdot\mathbf{f}(\eta). \tag{5.4}$$

We fix ξ and η and consider the vector $\mathbf{s}\in\mathcal{V}$ given by

$$\mathbf{s} = \eta\mathbf{f}(\xi) - \xi\mathbf{f}(\eta). \tag{5.5}$$

It follows from (5.4) that

$$\mathbf{s}^2 = \eta^2(-\xi^2) - 2\xi\eta(-\xi\eta) + \xi^2(-\eta^2) = 0,$$

i.e. that $\mathbf{s}$ is a signal vector. Consider the event $z = q + \mathbf{s}$. By (5.2) we have

$$\mathbf{s}^2 = (z - q)^2 = \sigma(q, z) = 0. \tag{5.6}$$

If $z \in \mathcal{L}$ then $\mathbf{s} = z - q$ must be of the form $\mathbf{s} = \mathbf{f}(\zeta)$. By (5.4) it then follows that $0 = \mathbf{s}^2 = -\zeta^2$ and hence that $\mathbf{s} = \mathbf{f}(0) = \mathbf{0}$.

Assume now that $\mathbf{s} \neq \mathbf{0}$, in which case $z \notin \mathcal{L}$. Axiom (M_2) and (5.6) imply that $q \in \mathcal{L}$ and $z \notin \mathcal{L}$ must be related by a signal. The signal relation has the property that there must be another event $p \in \mathcal{L}$, $p \neq q$, which is also related to z. Using axiom (M_2) again, we find

$$0 = \sigma(p, z) = (z - p)^2 = (\mathbf{s} + \mathbf{v})^2 = 2\mathbf{s}\cdot\mathbf{v} + \mathbf{v}^2, \tag{5.7}$$

where $\mathbf{v} = q - p$. Since $p \in \mathcal{L}$, $\mathbf{v}$ must be of the form $\mathbf{v} = \mathbf{f}(\lambda)$. By (5.4) and (5.5), we obtain

$$\mathbf{s}\cdot\mathbf{v} = \eta(-\xi\lambda) - \xi(-\eta\lambda) = 0$$

and hence, by (5.7), $0 = \mathbf{v}^2 = -\lambda^2$. Consequently, $\mathbf{v} = \mathbf{f}(0) = \mathbf{0} = q - p$, i.e. $p = q$, which contradicts $p \neq q$.

We conclude that always $\mathbf{s} = \mathbf{0}$ and hence, by (5.5), that

$$\eta\, \mathbf{f}(\xi) = \xi\, \mathbf{f}(\eta) \tag{5.8}$$

holds for all real ξ and η. Putting $\eta = 1$ and $\mathbf{f}(1) = \boldsymbol{l} \in \mathcal{V}$, we see that (5.8) gives $\mathbf{f}(\xi) = \xi\boldsymbol{l}$. It follows from (5.3) that $\mathcal{L}$ is the set of events of the form $x = q + \xi\boldsymbol{l}$. The relation $\boldsymbol{l}^2 = -1$ is a consequence of (5.4). Hence $\boldsymbol{l}$ has the properties (i) and (ii). Q.E.D.

THEOREM 2. *Let $\mathcal{L}$ be an observer with direction vector $\boldsymbol{l}$ and let $q \in \mathcal{L}$, $x \in \mathcal{E}$. An event $y = q + \eta\boldsymbol{l} \in \mathcal{L}$ is then related to x by a signal if and only if η is a root of the equation*

$$\eta^2 + \boldsymbol{l}\cdot(x - q)\eta - (x - q)^2 = 0. \tag{5.9}$$

When $x \notin \mathcal{L}$, there are exactly two events

$$y_1 = q + \eta_1\boldsymbol{l}, \qquad y_2 = q + \eta_2\boldsymbol{l} \tag{5.10}$$

in $\mathcal{L}$ that are related to x by a signal and we have

$$\sigma(q, x) = (x - q)^2 = -\eta_1\eta_2, \qquad (x - q)\cdot\boldsymbol{l} = -(\eta_1 + \eta_2), \tag{5.11}$$

$$[\boldsymbol{l}\cdot(x - q)]^2 + 4(x - q)^2 > 0. \tag{5.12}$$

Proof. By axiom (M_2), $y = q + \eta\boldsymbol{l}$ is related to x by a signal if and only if

$$\sigma(x, y) = (y - x)^2 = (q - x + \eta\boldsymbol{l})^2 = -\eta^2 - \boldsymbol{l}\cdot(x - q)\eta + (x - q)^2 = 0,$$

which proves the first part of the theorem.

When $x \notin \mathcal{L}$, the property of the signal relation described in (d) requires that the roots of (5.9) must be real and distinct. (5.12) is a necessary and sufficient condition for this requirement. The roots of (5.9) always satisfy (5.11).

Proof of the uniqueness theorem. Let q and x be any two events in $\mathcal{E}$. Since the family of all observers covers $\mathcal{E}$, there is at least one observer such that $q \in \mathcal{L}$. If $x \in \mathcal{L}$ also, then by axiom (M_1)

$$\sigma(q, x) = \sigma_{\mathcal{L}}(q, x). \tag{5.13}$$

If $x \notin \mathcal{L}$, we consider the events $y_1, y_2 \in \mathcal{L}$, determined as in Theorem 2. With the notation of Theorem 2, we have

$$\begin{aligned} \sigma(q, y_i) &= (y_i - q)^2 = \eta_i^2 l^2 = -\eta_i^2, & i = 1, 2 \\ \sigma(y_1, y_2) &= (y_2 - y_1)^2 = (\eta_2 - \eta_1)^2 l^2 = -\eta_2^2 + 2\eta_1\eta_2 - \eta_1^2. \end{aligned}$$

It follows that

$$\eta_1\eta_2 = \tfrac{1}{2}[\sigma(y_1, y_2) - \sigma(q, y_1) - \sigma(q, y_2)].$$

Noting that $y_1, y_2, q \in \mathcal{L}$, we infer from axiom (M_1) and $(5.11)_1$ that

$$\sigma(q, x) = -\tfrac{1}{2}[\sigma_{\mathcal{L}}(y_1, y_2) - \sigma_{\mathcal{L}}(q, y_1) - \sigma_{\mathcal{L}}(y, y_2)]. \tag{5.14}$$

Equations (5.13) and (5.14) show that $\sigma(q, x)$ is uniquely determined when $\sigma_{\mathcal{L}}$ is given. Q.E.D.

DEFINITION. *A set $\mathcal{E}$ which is endowed with a structure defined by Ω, $\{\sigma_{\mathcal{L}} \mid \mathcal{L} \in \Omega\}$, and $\sim$ as described under* (a)–(d) *is called a Minkowskian domain if the axioms* (M_1) *and* (M_2) *are satisfied.*

The uniqueness theorem shows that a Minkowskian domain is also endowed, in a canonical way, with the structure of a pseudo-Euclidean space.

THEOREM 3. *The translation space $\mathcal{V}$ of a Minkowskian domain has index* 1.

Proof. Let $\mathcal{L}$ be an observer with direction vector l, and let $\mathcal{U} = \{\mathbf{u} \mid \mathbf{u} = \xi l, \xi \in \mathcal{R}\}$ be the one-dimensional subspace of $\mathcal{V}$ generated by l. Consider a nonzero vector $\mathbf{v} \in \mathcal{U}^\perp$, which then satisfies $l \cdot \mathbf{v} = 0$. Choose an event $q \in \mathcal{L}$ and put $x = q + \mathbf{v}$. It is clear that $x \notin \mathcal{L}$. Since $l \cdot \mathbf{v} = l \cdot (x - q) = 0$, (5.12) states that $(x-q)^2 = \mathbf{v}^2 > 0$, which shows that $\mathbf{v}$ is space-like. It follows that $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$ is a direct decomposition with the property that $\mathcal{U} \subset \mathcal{V}_-$, $\mathcal{U}^\perp \subset \mathcal{V}_+$, i.e., it is a decomposition of the type (2.3). Theorem 1 of Section 2 implies that ind $\mathcal{V} = \dim \mathcal{U} = 1$.

Remark. As is shown by Theorem 1, the family Ω of observers is a congruence of straight lines in $\mathcal{E}$. Since Ω covers $\mathcal{E}$, there must be at least one line through each event. For example, Ω could be the set of all straight lines parallel to a given time-like direction. Unless Ω contains "very many" observers, a knowledge of $\sigma(x, y)$ for all pairs (x, y) such that x, y belong to the same observer is *not* sufficient to determine σ. It is the interconnection of σ with the signal relation, as provided by axiom (M_2), which renders σ unique.

6. Temporal order. As in the previous section we assume that data as described under (a), (b), and (c) are given. Instead of (d), we suppose that we have

(d′) a binary relation $\rightharpoonup$ on $\mathcal{E}$ with the following property: Given any observer $\mathcal{L}$ and any event $x \in \mathcal{E}$, there is an event $y_1 \in \mathcal{L}$ such that $y_1 \rightharpoonup x$ and there is an event $y_2 \in \mathcal{L}$ such that $x \rightharpoonup y_2$. Moreover, $x \rightharpoonup y$ and $y \rightharpoonup x$ can both be valid only if $x = y$. The relation $\rightharpoonup$ will be called the *directed signal relation* and the relation defined by

$$(6.1) \qquad x \sim y \text{ if and only if } x \rightharpoonup y \text{ or } y \rightharpoonup x$$

the (undirected) signal relation associated with $\rightharpoonup$.

It is clear that $\sim$ has the property described under (d) of the previous section.

Remark on physical interpretation. The change from (d) to (d′) corresponds to introducing the possibility of distinguishing the emission x from the reception y of a signal $x \rightharpoonup y$.

We require here the existence of a separation function σ on $\mathcal{E}$ which satisfies not only the axioms (M_1) and (M_2) but also

(M_3) *If $\mathcal{L}$ is an observer and if y_1, y_2 and z_1, z_2 are events in $\mathcal{L}$ such that*

$$(6.2) \qquad y_1 \rightharpoonup p \rightharpoonup y_2, \qquad z_1 \rightharpoonup q \rightharpoonup z_2$$

for some p, $q \in \mathcal{E}$, then

$$(6.3) \qquad \sigma_{\mathcal{L}}(y_1, z_1) + \sigma_{\mathcal{L}}(y_2, z_2) - \sigma_{\mathcal{L}}(y_1, z_2) - \sigma_{\mathcal{L}}(y_2, z_1) \leqq 0.$$

DEFINITION. *A set $\mathcal{E}$ which is endowed with a structure defined by Ω, $\{\sigma_{\mathcal{L}} \mid \mathcal{L} \in \Omega\}$, and $\rightharpoonup$ as described under* (a)–(c) *and* (d′) *is called a directed Minkowskian domain if the axioms* (M_1), (M_2), *and* (M_3) *are satisfied.*

From now on we shall exclude the trivial case when the domain $\mathcal{E}$ coincides with one of the observers. The dimension of the translation space $\mathcal{V}$ of $\mathcal{E}$ is then at least two.

DEFINITION. *We say that the event $x \in \mathcal{E}$ is earlier than the event $y \in \mathcal{E}$, and we write $x \prec y$, if there is an event p such that $x \rightharpoonup p \rightharpoonup y$.*

THEOREM 1. *Every observer $\mathcal{L}$ has a unique direction vector $\boldsymbol{l}$ with the following property: For any two events x, $y \in \mathcal{E}$ the relation $x \prec y$ holds if and only if*

$$(6.4) \qquad (y - x)^2 \leqq 0, \qquad (y - x) \cdot \boldsymbol{l} \leqq 0.$$

The direction vector $\boldsymbol{l}$ determined by the condition (6.4) will be called the *proper direction vector* of $\mathcal{L}$.

LEMMA. *Let $\mathcal{L}$ be an observer with direction vector $\boldsymbol{l}$ and let x_1, $x_2 \in \mathcal{E}$. Assume that $q \in \mathcal{L}$ and $z_i = q + \xi_i \boldsymbol{l}$ are the two events related to x_i by a signal* ($i = 1, 2$). *Then*

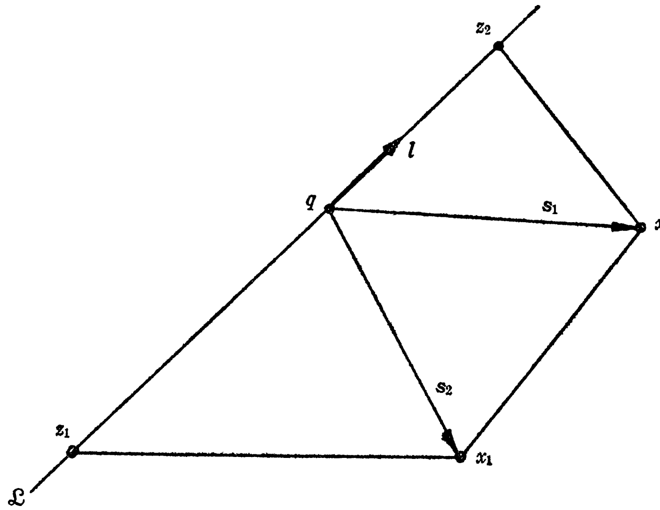
$$(6.5) \qquad (x_2 - x_1) \cdot \boldsymbol{l} = (\xi_1 - \xi_2), \qquad \xi_1 \xi_2 (x_2 - x_1)^2 \geqq 0,$$

and $(6.5)_2$ *can reduce to equality only if* $(x_2 - x_1)^2 = 0$.

Proof. Theorem 2 of Section 5 applies when we replace there q, x, y_1, y_2, η_1, η_2 by q, x_i, q, z_i, 0, ξ_i, respectively. $(5.11)_2$ then yields

$$\mathbf{s}_i \cdot \boldsymbol{l} = -\xi_i, \tag{6.6}$$

where $\mathbf{s}_i = x_i - q$. Hence $(x_2 - x_1)\cdot \boldsymbol{l} = (\mathbf{s}_2 - \mathbf{s}_1)\cdot \boldsymbol{l} = -(\xi_2 - \xi_1)$, which proves $(6.5)_1$.



Consider the vector $\mathbf{u} = \xi_2 \mathbf{s}_1 - \xi_1 \mathbf{s}_2$. It follows from (6.6) that $\mathbf{u}\cdot \boldsymbol{l} = 0$. Therefore, by Theorem 3 of Section 5 and Theorem 2 of Section 3, $\mathbf{u}$ must be space-like. Since $\mathbf{s}_i^2 = 0$, we obtain

$$\mathbf{u}^2 = -2\xi_1\xi_2 \mathbf{s}_1 \cdot \mathbf{s}_2 \geqq 0. \tag{6.7}$$

On the other hand we have

$$(x_2 - x_1)^2 = (\mathbf{s}_2 - \mathbf{s}_1)^2 = -2\mathbf{s}_1 \cdot \mathbf{s}_2,$$

which shows that (6.7) is equivalent to $(6.5)_2$. Equality can hold in (6.7) only if $\mathbf{u} = \xi_2\mathbf{s}_1 - \xi_1 \mathbf{s}_2 = \mathbf{0}$, in which case $x_2 - x_1 = \mathbf{s}_2 - \mathbf{s}_1$ must be a signal vector. Q.E.D.

Proof of Theorem 1. The proof will be carried out in four steps.

I. Assume first that $x, y \in \mathfrak{L}$. Let $\boldsymbol{l}'$ be one of the two direction vectors of $\mathfrak{L}$ (see Theorem 1 of the previous section) and let $y - x = \alpha \boldsymbol{l}'$. Since the translation space $\mathcal{V}$ of $\mathcal{E}$ is such that ind $\mathcal{V} = 1$, dim $\mathcal{V} \geqq 2$, there is a space-like unit vector orthogonal to $\boldsymbol{l}'$, i.e. a vector $\mathbf{u}$ such that $\mathbf{u}\cdot \boldsymbol{l}' = 0$, $\mathbf{u}^2 = 1$. Put $q = x + \frac{1}{2}\alpha\, \boldsymbol{l}' \in \mathfrak{L}$ and $z = q + \frac{1}{2}\alpha \mathbf{u}$, so that

$$x = q - \frac{\alpha}{2}\boldsymbol{l}', \qquad y = q + \frac{\alpha}{2}\boldsymbol{l}', \qquad z - q = \frac{\alpha}{2}\mathbf{u}.$$

The two roots of the equation

$$\eta^2 + \boldsymbol{l}'\cdot(z - q)\eta - (z - q)^2 = \eta^2 - \left(\frac{\alpha}{2}\right)^2$$

are $\eta = \pm\frac{1}{2}\alpha$. Hence, by Theorem 2 of Section 5, x and y are the events in $\mathcal{L}$ that are related to z by a signal. Therefore x and y, in some order, must be the two events in $\mathcal{L}$ that correspond to z as described in (d′), and we must have either $x \rightharpoonup z \rightharpoonup y$, i.e. $x \prec y$, or $y \rightharpoonup z \rightharpoonup x$, i.e. $y \prec x$. Both $x \prec y$ and $y \prec x$ can hold only when $x = y = z$.

II. We choose two events $z_1, z_2 \in \mathcal{L}$ such that $(z_2 - z_1)^2 = -1$. By the result of I we may assume that z_1, z_2 are arranged such that $z_1 \prec z_2$. We put $\boldsymbol{l} = z_2 - z_1$. Let $x, y \in \mathcal{L}$ and assume $x \prec y$. An easy calculation, starting from (6.3) with y_1, y_2 replaced by x, y, shows that

$$\sigma_{\mathcal{L}}(x, z_1) + \sigma_{\mathcal{L}}(y, z_2) - \sigma_{\mathcal{L}}(x, z_2) - \sigma_{\mathcal{L}}(y, z_1) = -2(y - x)\cdot\boldsymbol{l} \leqq 0.$$

Under the restrictive hypothesis $x, y \in \mathcal{L}$ the conclusion of the theorem now follows. We note that if $x, y \in \mathcal{L}$, then

$$y - x = \xi\,\boldsymbol{l}, \quad \text{where} \quad \begin{array}{l} \xi \geqq 0 \text{ if } x \prec y \\ \xi \leqq 0 \text{ if } y \prec x. \end{array} \tag{6.8}$$

(6.8) characterizes the proper direction vector of $\mathcal{L}$.

III. Assume $x \prec y$ holds for two events $x, y \in \mathcal{E}$, i.e. $x \rightharpoonup q \rightharpoonup y$ for some $q \in \mathcal{E}$. Let $\mathcal{L}'$ be an observer passing through q and let $\boldsymbol{l}'$ be the proper direction vector of $\mathcal{L}'$. According to the requirements described in (d′), there are events $z_1, z_2 \in \mathcal{L}'$ such that

$$z_1 \rightharpoonup x \rightharpoonup q \quad \text{and} \quad q \rightharpoonup y \rightharpoonup z_2,$$

i.e. such that $z_1 \prec q \prec z_2$. By II, (6.8) applies with x, y replaced by z_1, q or q, z_2. Hence

$$z_i = q + \xi_i \boldsymbol{l}', \qquad \xi_1 \leqq 0, \qquad \xi_2 \geqq 0. \tag{6.9}$$

The lemma and (6.9) show that

$$(y - x)\cdot\boldsymbol{l}' \leqq 0, \qquad (y - x)^2 \leqq 0. \tag{6.10}$$

Therefore the conclusion (6.4) holds for the particular observer $\mathcal{L}'$.

Let $\mathcal{L}$ be another observer and let $q' \in \mathcal{L}'$ be chosen arbitrarily. It is clear that there are events $x_1, x_2 \in \mathcal{L}$ and $z_1', z_2' \in \mathcal{L}'$ such that $z_1' \rightharpoonup x_1 \rightharpoonup q' \rightharpoonup x_2 \rightharpoonup z_2'$. If $\mathcal{L}' \neq \mathcal{L}$ we must have $x_1 \neq x_2$ and hence

$$x_2 - x_1 = \alpha\,\boldsymbol{l}, \qquad \alpha > 0, \tag{6.11}$$

where $\boldsymbol{l}$ is the proper direction vector of $\mathcal{L}$. Also, (6.9) remains valid when q, z_i, ξ_i are replaced by q', z_i', ξ_i'. Thus, the lemma applies again and we find, using (6.11), that

$$(x_2 - x_1)\cdot\boldsymbol{l}' = \alpha\,\boldsymbol{l}\cdot\boldsymbol{l}' \leqq 0, \qquad \boldsymbol{l}\cdot\boldsymbol{l}' \leqq 0. \tag{6.12}$$

If $y \neq x$, it follows from (6.10), (6.12) and Theorem 2 of Section 3 that actually

$$(y - x)^2 \leqq 0, \qquad (y - x)\cdot\boldsymbol{l}' < 0, \qquad \boldsymbol{l}\cdot\boldsymbol{l}' < 0.$$

Theorem 4 of Section 3 shows that we must have $(y-x)\cdot \boldsymbol{l}<0$. If $y=x$, $(y-x)\cdot \boldsymbol{l}=0$ is trivially true. Hence (6.4) holds for the proper direction vector $\boldsymbol{l}$ of an *arbitrary* observer.

IV. Assume that $(y-x)^2\leqq 0$. It is not hard to see that it is then possible to choose an observer $\mathcal{L}$ and events q, z_1, $z_2\in\mathcal{L}$ such that the hypotheses of the lemma are satisfied for $x_2=y$, $x_1=x$. If $(y-x)^2<0$, $(6.5)_2$ implies $\xi_1\xi_2<0$, which states that ξ_1 and ξ_2 have opposite sign. From part II we conclude that either $z_1\rightharpoonup x\rightharpoonup q\rightharpoonup y\rightharpoonup z_2$ or $z_2\rightharpoonup y\rightharpoonup q\rightharpoonup x\rightharpoonup z_1$. Hence, we must have either $x\prec y$ or $y\prec x$. If $(y-x)^2=0$, axiom (M_2) shows that x and y are related by a signal and hence that $x\prec y$ or $y\prec x$, trivially. Thus, x and y are comparable. This observation, together with the result of III, completes the proof of Theorem 1. Q.E.D.

Remark. The number $-(y-x)\cdot \boldsymbol{l}$ is the "time-difference" of the events x and y relative to the observer $\mathcal{L}$. Theorem 1 states, therefore, that x is earlier than y if and only if it is "earlier" for *every* observer.

An argument based on Theorem 1 and Theorem 4 of Section 3 will prove

THEOREM 2. *The relation $\prec$ is a partial order on $\mathcal{E}$, which has the property that x and y are comparable if and only if $\sigma(x, y)\leqq 0$.*

One of the two directed time-cones described in Theorem 5 of Section 3, say $\mathcal{V}_-^1$, is singled out by the property that $y-x\in\mathcal{V}_-^1$ implies $x\prec y$. We may call $\mathcal{V}_-^1$ the *future time-cone* and $\mathcal{V}_-^2=-\mathcal{V}_-^1$ the *past time-cone.*

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References

1. N. Bourbaki, Algèbre, Chap. 9, Hermann, Paris, 1959.
2. J. L. Synge, Relativity, The Special Theory, North Holland, Amsterdam, 1956.
3. ———, Relativity, The General Theory, North Holland, Amsterdam, 1960.
4. P. Suppes, Axioms for Relativistic Kinematics with or without Parity. Symposium on the Axiomatic Method, with special reference to geometry and physics, North Holland, Amsterdam, 1959, p. 291–307.
5. P. R. Halmos, Finite Dimensional Vector Spaces, Van Nostrand, Princeton, 1958.

SPIN INTEGRALS IN DYNAMICS

D. G. PARKYN, University of Cape Town

The solution of standard problems in rigid dynamics depends upon the existence of simple integrals. In the case of the rolling motion of a sphere it is common to find "spin integrals" (cf. Unthank [1], Milne [2]). It will be shown that spin integrals can exist when the fixed surface s on which the sphere rolls has the form of a plane, a sphere, a right circular cone or a cylinder, the axes of the last two being vertical. Spin integrals are limited to two distinct types,

but except in the case of a plane, only one can occur for a given rolling surface. Finally the use which can be made of such integrals is demonstrated in a discussion of the problem of a sphere rolling on the inner surface of a circular cylinder with vertical axis.

The main aim of dynamics is to obtain integrals of the equations of motion. In the field of nonholonomic problems this is often a matter of some difficulty and very few general classes of integrals are known. The following analysis delimits precisely the surfaces for which a certain class of integrals will exist in the case of the rolling motion of a sphere and therefore contributes to the theory of nonholonomic systems. While it is true that in any particular problem in which a spin integral exists it can be determined *ab initio*, the value of the prior knowledge of its existence should not be underestimated.

Consider a homogeneous sphere of radius a, centre G, rolling and spinning on a fixed rough surface, the point of contact being A. Let the position vector of A with respect to a fixed point O be $\mathbf{r}$, the unit vertical vector be $\mathbf{k}$ and the unit vector along the normal AG be $\mathbf{n}$. Then, denoting differentiation with respect to the time by dots, the equations of motion may be written

$$m(\ddot{\mathbf{r}} + a\ddot{\mathbf{n}}) = \mathbf{R} - mg\mathbf{k}, \tag{1}$$

$$C\dot{\mathbf{\Omega}} = -(a\mathbf{n}) \times \mathbf{R}, \tag{2}$$

where m is the mass, C the diametral moment of inertia and $\mathbf{\Omega}$ the angular velocity of the sphere, $\mathbf{R}$ being the total reaction at the point of contact A.

Since the velocity of G is $(\dot{\mathbf{r}}+a\dot{\mathbf{n}})$ and the velocity of A relative to G is given by $\mathbf{\Omega}\times(-a\mathbf{n})$ it follows that the velocity of A as a point of the sphere is

$$\dot{\mathbf{r}} + a\dot{\mathbf{n}} - a\mathbf{\Omega} \times \mathbf{n}.$$

If pure rolling takes place, with no slipping at the point of contact between the sphere and the fixed surface, we obtain the rolling condition

$$\dot{\mathbf{r}} + a\dot{\mathbf{n}} - a\mathbf{\Omega} \times \mathbf{n} = \mathbf{O}. \tag{3}$$

We proceed to eliminate the reaction $\mathbf{R}$ and the position vector $\mathbf{r}$ from the above equations. Thus from equation (3)

$$\ddot{\mathbf{r}} + a\ddot{\mathbf{n}} = a\dot{\mathbf{\Omega}} \times \mathbf{n} + a\mathbf{\Omega} \times \dot{\mathbf{n}}, \tag{4}$$

whence from equation (1)

$$\mathbf{R} = mg\mathbf{k} + ma\{\dot{\mathbf{\Omega}} \times \mathbf{n} + \mathbf{\Omega} \times \dot{\mathbf{n}}\}. \tag{5}$$

Substituting this result in equation (2) we obtain

$$C\dot{\mathbf{\Omega}} = -mga\mathbf{n} \times \mathbf{k} - ma^2\mathbf{n} \times \{\dot{\mathbf{\Omega}} \times \mathbf{n} + \mathbf{\Omega} \times \dot{\mathbf{n}}\}$$

whence, on expanding the triple vector products,

$$(6)\qquad (C + ma^2)\dot{\mathbf{\Omega}} = -mga\mathbf{n} \times \mathbf{k} + ma^2\{(\mathbf{n}\cdot\dot{\mathbf{\Omega}})\mathbf{n} - (\mathbf{n}\cdot\dot{\mathbf{n}})\mathbf{\Omega} + (\mathbf{n}\cdot\mathbf{\Omega})\dot{\mathbf{n}}\}.$$

From equation (2), however,

$$(7)\qquad \mathbf{n}\cdot\dot{\mathbf{\Omega}} = 0,$$

and since $\mathbf{n}$ is a unit vector

$$(8)\qquad \mathbf{n}\cdot\dot{\mathbf{n}} = 0;$$

hence equation (6) reduces to

$$(9)\qquad (C + ma^2)\dot{\mathbf{\Omega}} = -mga\mathbf{n} \times \mathbf{k} + ma^2(\mathbf{n}\cdot\mathbf{\Omega})\dot{\mathbf{n}}.$$

Equation (9) is not in general integrable. For certain surfaces, however, it admits scalar integrals of the form

$$(10)\qquad (\alpha\mathbf{n} + \beta\mathbf{k})\cdot\mathbf{\Omega} = \text{Constant},$$

where α, β are constants, not both zero. Integrals of this form will be termed spin-integrals. We proceed to investigate the nature of the rolling surfaces for which such spin integrals exist.

Thus, differentiating equation (10) we obtain

$$(11)\qquad \alpha\dot{\mathbf{n}}\cdot\mathbf{\Omega} + (\alpha\mathbf{n} + \beta\mathbf{k})\cdot\dot{\mathbf{\Omega}} = 0.$$

Substituting for $\dot{\mathbf{\Omega}}$ from equation (9) and making use of relations (7) and (8), we see that equation (11) may be written

$$(12)\qquad (C + ma^2)\alpha\dot{\mathbf{n}}\cdot\mathbf{\Omega} + \beta ma^2(\mathbf{n}\cdot\mathbf{\Omega})(\mathbf{k}\cdot\dot{\mathbf{n}}) = 0.$$

But on multiplying the rolling condition (3) vectorially by $\mathbf{n}$ and the result scalarly by $\dot{\mathbf{n}}$ we obtain the relation

$$(13)\qquad a\dot{\mathbf{n}}\cdot\mathbf{\Omega} = \dot{\mathbf{n}}\cdot\mathbf{n} \times \dot{\mathbf{r}}$$

whence, on substitution, equation (12) becomes

$$(14)\qquad (C + ma^2)\underset{*}{\alpha}(\dot{\mathbf{n}}\cdot\mathbf{n} \times \dot{\mathbf{r}}) + \beta ma^3(\mathbf{n}\cdot\mathbf{\Omega})(\mathbf{k}\cdot\dot{\mathbf{n}}) = 0.$$

If a spin integral is to exist, equation (14) must hold for arbitrary $\mathbf{\Omega}$, since $\mathbf{\Omega}$ can be assigned at any point as an initial condition. Hence the existence of a spin integral implies that both

$$(15)\qquad \alpha(\dot{\mathbf{n}}\cdot\mathbf{n} \times \dot{\mathbf{r}}) = 0$$

and

$$(16)\qquad \beta(\mathbf{k}\cdot\dot{\mathbf{n}}) = 0.$$

From equation (15) either

$$\alpha = 0 \quad \text{or} \quad \dot{\mathbf{n}}\cdot\mathbf{n}\times\dot{\mathbf{r}} = 0. \tag{17}$$

If $\alpha=0$ then necessarily $\beta\neq 0$ and $\mathbf{k}\cdot\dot{\mathbf{n}}=0$ whence

$$\mathbf{k}\cdot\mathbf{n} = c, \text{ a constant,} \tag{18}$$

a condition which is satisfied if the fixed surface s has the form of a plane of any inclination, or a right circular cone or cylinder with vertical axis. In any of these cases a spin integral of the form

$$\mathbf{k}\cdot\boldsymbol{\Omega} = K \tag{19}$$

exists, where K is a constant. Such an integral will be termed a K-integral.

If $\alpha\neq 0$ then from equations (17)

$$\dot{\mathbf{n}}\cdot\mathbf{n}\times\dot{\mathbf{r}} = 0 \tag{20}$$

which implies that the three vectors $\mathbf{n}$, $\dot{\mathbf{n}}$ and $\dot{\mathbf{r}}$ are coplanar.

However from the rolling condition (3), on scalar multiplication by $\mathbf{n}$, we find that

$$\mathbf{n}\cdot\dot{\mathbf{r}} = 0. \tag{21}$$

Thus $\dot{\mathbf{r}}$ is perpendicular to $\mathbf{n}$ and, by (8), $\dot{\mathbf{n}}$ is perpendicular to $\mathbf{n}$, hence by equation (20) $\dot{\mathbf{r}}$ must be parallel to $\dot{\mathbf{n}}$, unless $\dot{\mathbf{n}}$ is identically zero, that is

$$\dot{\mathbf{r}} = -b\dot{\mathbf{n}}. \tag{22}$$

If b is taken to be a constant then

$$\mathbf{r} = -b\mathbf{n} + \mathbf{r}_0, \tag{23}$$

where $\mathbf{r}_0$ is a constant and the fixed surface s is a sphere of radius b and centre $\mathbf{r}_0$.

If $\dot{\mathbf{n}}$ is identically zero then the fixed surface s is restricted to be in the form of a plane of any inclination.

Considering equation (16) we see that when $\dot{\mathbf{n}}$ is identically zero, β is arbitrary but that otherwise either β is zero or $\mathbf{k}\cdot\dot{}$ is zero. However we must reject the second possibility since it would imply a restricted motion on the surface determined by equation (22). Hence, except in the case of a plane, β must vanish when α is nonzero. Thus, for motion on a plane or a sphere, a spin integral

$$\mathbf{n}\cdot\boldsymbol{\Omega} = N \tag{24}$$

exists, where N is constant. Such an integral will be termed an N-integral and we have shown that except in the case of a plane an N-integral cannot exist simultaneously with a K-integral.

The generalization of the form of the spin integral (10) to the case where α,

β are scalar functions of the vector $\mathbf{r}$ can be investigated in a similar manner but leads to no further surfaces for which spin integrals exist. It does however lead to one further integral in the case of motion on a sphere in the form

$$\left\{ma^2(\mathbf{r}\cdot\mathbf{k})\mathbf{n} + b(C + ma^2)\mathbf{k}\right\}\cdot\dot{\mathbf{\Omega}} = \text{constant}.$$

This may be easily verified and is an analogue of an angular momentum integral.

On the solution of equations (18) and (22). It is conceivable that the conditions (18) and (22) which distinguish those surfaces for which K- and N-integrals exist have solutions additional to those listed. We shall show that while equation (18) does in fact lead to new surfaces, equation (22) has only spherical solutions.

For equation (18) we consider separately the case in which $\mathbf{n}$ is perpendicular to $\mathbf{k}$ and the constant c is zero. Thus if the surface satisfying the condition has the equation

$$F(x, y, z) = 0,$$

where the z-axis is parallel to $\mathbf{k}$, then the normal is parallel to the gradient $\nabla F = (F_x, F_y, F_z)$; and when $c=0$ the condition (18) implies that $F_z=0$. Thus the surface is a cylinder with its axis vertical and cross-section $F(x, y)=0$.

If $\mathbf{n}$ is not perpendicular to $\mathbf{k}$, and $0<c\leqq 1$, we adopt the Monge form

$$z = f(x, y)$$

for the equation of the surface; then the condition (18) may be written

$$p^2 + q^2 = 1/c^2 - 1, \tag{25}$$

where $p=f_x$ and $q=f_y$. The case where c is equal to unity leads to plane surfaces and will be disregarded. Thus, defining

$$\mu^2 = 1/c^2 - 1,$$

the equation (25) has the complete integral

$$f(x, y) = \mu(x \sin\alpha + y\cos\alpha) + A,$$

where α and A are arbitrary constants. The equation has no singular integrals and the general integral for the surface is the result of eliminating α between the equations

$$(26)\qquad \begin{aligned} z - \mu(x\sin\alpha + y\cos\alpha) - g(\alpha) &= 0 \\ -\mu(x\cos\alpha - y\sin\alpha) - g'(\alpha) &= 0, \end{aligned}$$

where the function $g(\alpha)$ is arbitrary. In particular, if $g(\alpha)$ is identically zero, elimination of α leads to the equation

$$z^2 = \mu^2(x^2 + y^2) \tag{27}$$

which is the equation of a right circular cone.

Again, if $g(\alpha) = -h\alpha$, where h is a constant and we define

$$r^2 = x^2 + y^2$$
$$\theta = \tan^{-1}(y/x)$$

then elimination of α leads to the surface

$$z = \left\{ \sqrt{(\mu^2 r^2 - h^2)} - h \cos^{-1}(h/\mu r) \right\} + h\theta, \tag{28}$$

which is a helicoid.

The totality of surfaces possessing K-integrals comprises planes, cylinders with vertical axes and those surfaces included in the general integral (26).

With regard to equation (22), it is superficially possible for the multiplier b to be a function of $\mathbf{r}$ and t. Since s is a *fixed* surface, a condition on s should not involve the time t. It is sufficient, therefore, to consider the case in which

$$b = b(\mathbf{r}).$$

We will adopt surface parameters u and v, denoting partial derivatives with respect to u and v by suffixes 1 and 2 respectively. In this notation equation (22) becomes

$$\mathbf{r}_1 du + \mathbf{r}_2 dv = -b(\mathbf{n}_1 du + \mathbf{n}_2 dv)$$

or, since u, v are independent

$$\begin{aligned} \mathbf{r}_1 &= -b\mathbf{n}_1 \\ \mathbf{r}_2 &= -b\mathbf{n}_2, \end{aligned} \tag{29}$$

where b is to be considered as a function of u and v.

From equations (29), by partial differentiation,

$$\mathbf{r}_{12} = -b_2\mathbf{n}_1 - b\mathbf{n}_{12}$$
$$\mathbf{r}_{21} = -b_1\mathbf{n}_2 - b\mathbf{n}_{21}$$

whence on subtraction

$$b_1\mathbf{n}_2 = b_2\mathbf{n}_1. \tag{30}$$

Since $\mathbf{r}_1$, $\mathbf{r}_2$ cannot be parallel unless the parameter system is degenerate, it follows from equations (29) that the vectors $\mathbf{n}_1$, $\mathbf{n}_2$ are not parallel. Hence, from equation (30),

$$b_1 = b_2 = 0 \tag{31}$$

and this property holds, moreover, at all points of the surface. Thus b is necessarily a constant on the surface and direct integration of equation (22) leads to the equation of a sphere. This discussion amounts to proving that a surface for which every point is an umbilic is necessarily a sphere.

Example. On the motion of a homogeneous sphere of radius a rolling on the inside of a circular cylinder of radius b, its axis being vertical.

We take the origin O on the axis of the cylinder, with $\mathbf{n}$ as the inward drawn unit normal. Then, with the preceding notation, the position vector of the point of contact may be written in the form

$$\mathbf{r} = z\mathbf{k} - b\mathbf{n}$$

whence the equations of motion become

$$m\{\ddot{z}\mathbf{k} - (b - a)\ddot{\mathbf{n}}\} = \mathbf{R} - mg\mathbf{k} \tag{32}$$

$$C\dot{\mathbf{\Omega}} = -(a\mathbf{n}) \times \mathbf{R} \tag{33}$$

and the rolling condition is

$$\dot{z}\mathbf{k} - (b - a)\dot{\mathbf{n}} - a\mathbf{\Omega} \times \mathbf{n} = \mathbf{O}. \tag{34}$$

On eliminating $\mathbf{R}$ between equations (32) and (33) we find that

$$C\dot{\mathbf{\Omega}} = -ma\mathbf{n} \times \{g\mathbf{k} + \ddot{z}\mathbf{k} - (b - a)\ddot{\mathbf{n}}\} \tag{35}$$

whence substitution from the rolling condition, (34), gives

$$C\dot{\mathbf{\Omega}} = -ma\mathbf{n} \times \{g\mathbf{k} + a\dot{\mathbf{\Omega}} \times \mathbf{n} + a\mathbf{\Omega} \times \dot{\mathbf{n}}\}. \tag{36}$$

Since $\mathbf{k} \cdot \mathbf{n} = 0$ and $\mathbf{n} \cdot \dot{\mathbf{n}} = 0$, scalar multiplication of equation (36) by $\mathbf{k}$ gives

$$(C + ma^2)\dot{\mathbf{\Omega}} \cdot \mathbf{k} = 0 \tag{37}$$

whence

$$\mathbf{k} \cdot \mathbf{\Omega} = K, \tag{38}$$

where K is a constant, the expected K-integral.

Multiplying equation (34) vectorially by $\mathbf{k}$ twice in succession, and making use of equation (38), we obtain

$$-(b - a)\mathbf{k} \times \dot{\mathbf{n}} + aK\mathbf{n} = \mathbf{O}$$

and

$$-(b - a)\dot{\mathbf{n}} - aK\mathbf{k} \times \mathbf{n} = \mathbf{O},$$

or

$$\dot{\mathbf{n}} = \omega\mathbf{k} \times \mathbf{n} \quad \text{where} \quad \omega = -\frac{aK}{b - a}\cdot \tag{39}$$

Thus $\mathbf{n}$ rotates about the vertical with constant angular speed ω. This result is fundamental in the description of the motion and is a direct consequence of the existence of the K-integral.

Vectorial multiplication of equation (34) by $\mathbf{n}$ gives

$$\dot{z}\mathbf{n} \times \mathbf{k} - (b - a)\mathbf{n} \times \dot{\mathbf{n}} - a\mathbf{\Omega} + as\mathbf{n} = \mathbf{O}, \tag{40}$$

where $s=\mathbf{n}\cdot\mathbf{\Omega}$ and by the preceding discussion cannot be constant. Hence, substituting in equation (35) the value of $\mathbf{\Omega}$ given by equation (40), we find

$$C\{\ddot{z}\mathbf{n}\times\mathbf{k}+\dot{z}\dot{\mathbf{n}}\times\mathbf{k}-(b-a)\mathbf{n}\times\ddot{\mathbf{n}}+a\dot{s}\mathbf{n}+as\dot{\mathbf{n}}\}$$

$$=-ma^2\mathbf{n}\times\{g\mathbf{k}+\ddot{z}\mathbf{k}-(b-a)\ddot{\mathbf{n}}\},$$

or since $\dot{\mathbf{n}}=\omega\mathbf{k}\times\mathbf{n}$, $\ddot{\mathbf{n}}=-\omega^2\mathbf{n}$,

$$C\{(\ddot{z}-as\omega)\mathbf{n}\times\mathbf{k}+(\omega\dot{z}+a\dot{s})\mathbf{n}\}=-ma^2(g+\ddot{z})\mathbf{n}\times\mathbf{k}. \tag{41}$$

Since $\mathbf{n}$ and $\mathbf{n}\times\mathbf{k}$ are perpendicular, (41) yields two scalar equations:

$$\omega\dot{z}+a\dot{s}=0, \qquad C(\ddot{z}-as\omega)=-ma^2(g+\ddot{z}).$$

From the first $as=as_0-\omega z$, where s_0 is a constant. Substituting into the second, we obtain

$$(C+ma^2)\ddot{z}+C\omega^2 z=C\omega as_0-ma^2 g. \tag{42}$$

Thus the motion of the centre of the sphere is determined by equations (39) and (42) and its angular velocity is given by substitution in equation (40).

References

1. H. W. Unthank, Math. Gaz., 53 (1957) 335.
2. E. A. Milne, Vectorial Mechanics, Methuen, London, 1948.

THE LATTICE OF EQUATIONAL CLASSES OF ALGEBRAS WITH ONE UNARY OPERATION

EUGENE JACOBS AND ROBERT SCHWABAUER
State University of South Dakota

The notion of the lattice of equational classes of algebras of a given similarity type was considered by Professor Hugo Ribeiro in 1960 in seminars at the University of Nebraska. It is his insights which underlie our discussion here. We are also indebted to Professor Paul Rygg for his invaluable help and encouragement.

We define this notion and give a complete description of $\mathcal{L}$, the lattice of equational classes of algebras with one unary operation. We believe this paper ties together some isolated results, such as the fact that the atoms of $\mathcal{L}$ are $(x^1=y^1)$ and $(x=x^1)$ (since the atoms are the equationally complete classes) [3]. Perhaps our results will be useful in examining the more complex lattice based on algebras with one binary operation.

The lattice operations, $\cap$, $+$, discussed below, apply to lattices of equational classes of algebras of any given similarity type. But, since our attention is restricted to $\mathfrak{L}$, all algebras referred to here are those with one unary operation. Such an algebra is a set A with a mapping of A into A. We let x^1 denote the element into which x is mapped. Also, $x^0=x$, $(x^n)^1=x^{n+1}$, where $n=0, 1, 2, \cdots$. It is easily shown by induction that $(x^p)^q=x^{p+q}$.

If all the elements of an algebra satisfy a particular equation or each of a particular set of equations, then the algebra is a member of an equational class. For fixed c and d, where $c, d=0, 1, 2, \cdots$, $(x^c=x^{c+d})$ is the equational class of exactly those algebras $A=\langle A^1,\rangle$ such that, for each element a of A, $a^c=a^{c+d}$.

There is only one other form for equational classes of algebras, $(x^c=y^{c+d})$, and again, if $c, d=0, 1, 2, \cdots$ are fixed, then $(x^c=y^{c+d})$ is the class of all algebras, and only those, all of whose elements satisfy $x^c=y^{c+d}$. We say that $(x^c=x^{c+d})$ and $(x^c=y^{c+d})$ are *determined* by $x^c=x^{c+d}$ and $x^c=y^{c+d}$, respectively.

The notion of equational class goes back at least to Birkhoff [2] and was discussed by Tarski [4].

With certain operations the set of all equational classes of algebras with one unary operation forms the lattice $\mathfrak{L}$. If K and L are equational classes then $K\cap L$, the ordinary set intersection of K and L (clearly an equational class), is one such operation. Ordinary set union, $\cup$, does not in general yield an equational class. For example, we shall see below that $S=(x=x^2)\cup(x=x^3)$ is not an equational class.

The other operation, denoted by $K+L$, is the smallest equational class that contains $K\cup L$. That is, if M is an equational class, then

$$(1) \qquad K+L\subseteq M \quad \text{whenever} \quad K\cup L\subseteq M.$$

Such an M always exists since $(x=x)$ contains every equational class. $K+L$ is unique since it is the equational class determined by the set of all equations that hold for each algebra of $K\cup L$.

Referring again to the above example, we note that $(x=x^2)+(x=x^3)\equiv(x=x^6)$ (by Theorem (d_2)) contains $A_6=\langle\{1, 2, 3, 4, 5, 6\},^1\rangle$, where $1^1=2$, $2^1=3, \cdots, 6^1=1$, while S does not contain A_6.

The following propositions are useful in considering the properties of $\mathfrak{L}$.

(2) Substitution Principle: Let $p, q=0, 1, 2, \cdots$. Then $(x^c=x^{c+d})\subseteq(x^{c+p}=x^{c+d+p})$. Also, $(x^c=y^{c+d})\subseteq(x^{c+p}=y^{c+d+q})$ and $(x^c=y^{c+d})\subseteq(x^c=x^{c+d+p})$.

(3) Lemma (proved by induction): $(x^c=x^{c+d})\subseteq(x^c=x^{c+ed})$, where $e=1, 2, \cdots$.

This follows from (2) and from the fact that $(x^p)^q=x^{p+q}$.

We do not consider equational classes of the forms $(x^d=x^d)$ and $(x^c=y^{c+d})$, where $d\neq 0$, since $(x^d=x^d)$ like $(x=x)$ contains every algebra. And $(x^c=y^{c+d})\equiv(x^c=y^c)$ a fact which follows from (2) and from $(x^p)^q=x^{p+q}$.

Our main result is the following:

THEOREM.

(a) $$(x = x) \cap K \equiv K; \qquad (x = x) + K \equiv (x = x).$$

(b₁) $$(x^r = y^r) \cap (x^s = y^s) \equiv (x^{\min(r,s)} = y^{\min(r,s)}).$$

(b₂) $$(x^r = y^r) + (x^s = y^s) \equiv (x^{\max(r,s)} = y^{\max(r,s)}).$$

(c₁) $$(x^r = y^r) \cap (x^m = x^{m+h}) \equiv (x^{\min(r,m)} = y^{\min(r,m)}).$$

(c₂) $$(x^r = y^r) + (x^m = x^{m+h}) \equiv (x^{\max(r,m)} = x^{\max(r,m)+h}).$$

(d₁) $$(x^m = x^{m+h}) \cap (x^n = x^{n+i}) \equiv (x^{\min(m,n)} = x^{\min(m,n)+(h,i)}).$$

(d₂) $$(x^m = x^{m+h}) + (x^n = x^{n+i}) \equiv (x^{\max(m,n)} = x^{\max(m,n)+[h,i]}).$$

Here $r, s, m, n = 0, 1, 2, \cdots$; $h, i = 1, 2, \cdots$; (h, i) is the greatest common divisor and $[h, i]$ the least common multiple of h and i.

Proof. Part (a) is obvious, since $K \subseteq (x = x)$ for any K. Part (b_1) is obtained by trivial applications of (2). We now give a proof of (c_1) and the difficult part of the proof of (d_1).

This proof of (c_1) follows a suggestion of the referee. On the one hand from (2) we see that $(x^{\min(m,r)}) = y^{\min(m,r)}) \subseteq (x^m = x^{m+h})$ and $(x^{\min(m,r)} = y^{\min(m,r)}) \subseteq (x^r = y^r)$. Hence the right member of (c_1) is contained in the left.

On the other hand there exists an $e = 0, 1, 2, \cdots$ such that $m + eh \geqq r$. Then by (2) and (3) $(x^r = y^r) \equiv (x^r = y^{m+eh}) \subseteq (x^r = x^{m+eh})$ and $(x^m = x^{m+h}) \subseteq (x^m = x^{m+eh})$. So

$$(x^r = y^r) \cap (x^m = x^{m+h}) \subseteq (x^m = x^{m+eh} = x^r = y^r) \equiv (x^{\min(m,r)} = y^{\min(m,r)}).$$

As for (d_1), to show that

(i) $(x^m = x^{m+h}) \cap (x^n = x^{n+i}) \subseteq (x^{\min(m,n)} = x^{\min(m,n)+(h,i)})$ we can without loss of generality assume that $n \geqq m$ and $m + k = n$, $k = 0, 1, 2, \cdots$. We then examine

(ii) $(x = x^h) \cap (x^k = x^{k+i}) \subseteq (x = x^{(h,i)})$

and note the following results of applications of (2) and (3):

(iii) $(x = x^h) \subseteq (x^{(h,i)} = x^{(h,i)+h})$;

(iv) $(x = x^h) \subseteq (x^{h+(h,i)} = x^{h+(h,i)+eh})$,

where e is chosen so that $h + (h, i) + eh \geqq k$;

(v) $(x^k = x^{k+i}) \subseteq (x^{h+(h,i)+eh} = x^{h+(h,i)+eh+fi}) \equiv (x^{h+(h,i)+eh} = x^{R(h,i)+fi})$,

where $f = 1, 2, \cdots$, $h = a(h, i)$ and $R = a + 1 + ea$; and

(vi) $(x = x^h) \subseteq (x = x^{gh})$,

where $g = 1, 2, \cdots$.

It is well known that there exist f and g such that $gh - fi = R(h, i)$. Hence

$$(x = x^h) \cap (x^k = x^{k+i}) \subseteq (x = x^{gh} = x^{R(h,i)+fi} = x^{h+(h,i)+eh} = x^{h+(h,i)} = x^{(h,i)}).$$

This establishes (ii) and then (i) follows on replacing x by x^m in (iii) and the following equations.

This completes our discussion of intersection.

To prove (b_2) we can without loss of generality let $\max(r, s) = r$. Then

$$\begin{aligned}(x^r = y^r) + (x^s = y^s) &\equiv (x^r = y^r) + [(x^r = y^r) \cap (x^s = y^s)] \\ &\equiv (x^r = y^r)\end{aligned}$$

by (b_1) and the Absorption Axiom of lattices. Similar remarks establish (c_2) in case $\max(m, r) = m$.

We consider the following lemmas before considering proofs of (c_2) where $\max(m, r) = r$ and (d_2). Let $c, d, g = 0, 1, 2, \cdots, e, f = 1, 2, \cdots$.

LEMMA (i): *If* $(x^c = y^c) \equiv (x^d = y^d)$, *then* $c = d$.
LEMMA (ii): *If* $(x^c = x^{c+e}) \equiv (x^c = x^{c+f})$, *then* $e = f$.
LEMMA (iii): *If* $(x^c = x^{c+e}) \equiv (x^g = x^{g+f})$, *then* $c = g$.

We prove these lemmas by considering sequences of algebras. For example, for Lemma (ii) we consider $\{A_n\} \equiv \{\langle\{1, 2, \cdots, n\}, {}^1\rangle\}$, where $n = 1, 2, \cdots$ and $1^1 = 2$, $2^1 = 3, \cdots, n^1 = 1$. We then note $(x^c = x^{c+e})$ contains A_e but not $A_{e+1}, A_{e+2}, \cdots$. If $e \neq f$, therefore, $(x^c = x^{c+e}) \not\equiv (x^c = x^{c+f})$.

For Lemmas (i) and (iii) we consider the sequence $\{B_{n-1}\} \equiv \{\langle\{1, 2, \cdots, n\}, {}^1\rangle\}$, where $1^1 = 1$ but for $n = 2, 3, \cdots, n^1 = n - 1$. For Lemma (i) the procedure is as sketched for Lemma (ii). For Lemma (iii) we note that for $e = 1, 2, \cdots, (x^c = x^{c+e})$ contains B_c but not $B_{c+1}, B_{c+2}, \cdots$.

LEMMA (iv): *Let* K_1, K_2, K_3 *be equational classes. If* (1) $K_1, K_2 \subseteq K_3$ *and* (2) *for any equational class* K $K_3 \subseteq K$ *whenever* $K_1, K_2 \subseteq K$, *then* $K_1 + K_2 \equiv K_3$.

Lemma (iv) is clear since $K_1 + K_2 \subseteq K_3$ if $K_1 \cup K_2 \subseteq K_3$ by (1) and $K_3 \subseteq K_1 + K_2$ if we let $K = K_1 + K_2$. This Lemma provides the frame of the proofs for the remaining case of (c_2) and for (d_2).

For (c_2) we let $K_1 \equiv (x^r = y^r)$, $K_2 \equiv (x^m = x^{m+h})$, $K_3 \equiv (x^r = x^{r+h})$. For ($d_2$) we let $K_1 \equiv (x^m = x^{m+h})$, $K_2 \equiv (x^n = x^{n+i})$, $K_3 \equiv (x^{\max(m,n)} = x^{\max(m,n)+[h,i]})$. In both cases we show that $K_1, K_2 \subseteq K_3$ by applications of (2) and (3).

Both proofs are trivial unless K (of Lemma (iv)) is of the form $(x^c = x^{c+d})$. For if $K = (x = x)$ the proofs fall out immediately. Moreover, there is no e such that $K \equiv (x^e = y^e)$. The latter point follows (in a proof by contradiction) from (c_1) and from the fact of set theory that for sets X, Y

$$X \subseteq Y \quad \text{if and only if } X \cap Y \equiv X. \tag{4}$$

Here $X \equiv K_2$, $Y \equiv (x^e = y^e)$.

For (c_2) we note the following:

(i) $\min(r, c) = r$ by the hypothesis of (2) of Lemma (iv), by (c_1), by (4) where $X \equiv (x^r = y^r)$, $Y = (x^c = x^{c+d})$, and by Lemma (i).

(ii) $(x^r = x^{r+h}) \subseteq (x^c = x^{c+h})$ by (i) and (2), the Substitution Principle.

(iii) $(x^m = x^{m+h}) \cap (x^c = x^{c+d}) \equiv (x^m = x^{m+(h,d)})$ by (d_1) and the fact that $\mathfrak{D}$ $\min(m, c) = m$.

(iv) $(x^m = x^{m+(h,d)}) \equiv (x^m = x^{m+h})$ by (iii) and (4).

(v) $(h, d) = h$ by (iv) and Lemma (ii).

(vi) $(x^c = x^{c+h}) \subseteq (x^c = x^{c+d})$.

So by (ii) and (vi) $K_3 \subseteq K$. This part of (c_2), then, is established by Lemma (iv).

Our procedure for establishing (d_2) is similar, though here Lemma (iii) is used as well as the fact that if h and i are divisors of d then $[h, i]$ is a divisor of d.

$\mathfrak{L}$ is distributive. This can be shown by using the following facts, where a, b and c are integers:

(i) $\max(a, \min(b, c)) = \min(\max(a, b), \max(a, c))$,

(ii) $[a, (b, c)] = ([a, b], [a, c])$.

References

1. G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloquium Publications, vol. 25, rev. ed., New York, 1948.

2. ———, On the structure of abstract algebras, Proc. Cambridge Philos. Soc., 31 (1935) 443–454.

3. J. Kalicki and D. Scott, Equational completeness of abstract algebras, Akademie von Wetenschappen Proc. Ser. A, 58 (1954) 650–659.

4. A. Tarski, Contributions to the theory of models, Indag. Math., 16 (1954) 572–588, and 17 (1955) 56–64.

ON THE STRUCTURE OF PRE-p-RINGS

ALEXANDER ABIAN* AND WILLIAM A. McWORTER, Ohio State University

Let p be a prime. A p-ring is an associative ring B in which

$$x^p = x \quad \text{and} \quad px = 0, \qquad \text{for every } x \text{ in } B. \tag{1}$$

It is known [1] that an associative ring B is a p-ring if and only if B is isomorphic to a subdirect sum of fields of order p.

Obviously, in a p-ring B, the equality

$$xy^p = x^p y, \tag{2}$$

for every x and y in B is valid. It is *not true*, however, that every associative and commutative ring whose characteristic is p and in which (2) is valid is a p-ring.

Observing that in general, in a ring, *associativity*, *commutativity*, *having characteristic p* and *validity of* (2) are mutually independent conditions, we introduce the following

DEFINITION 1. *An associative and commutative ring R whose characteristic is p is called a pre-p-ring if $xy^p = x^p y$, for every x and y in R.*

The purpose of this paper is to prove the following main theorem concerning the structure of a pre-p-ring.

* Formerly, Smbat Abian.

THEOREM. *Every pre-p-ring R is a direct sum of a p-ring and a nil ring.*

More precisely, if R is a pre-p-ring, then

$$R = B \oplus N, \tag{3}$$

where B is a p-ring and N is the radical of R, given respectively by:

$$B = \{x \mid x^p = x\} \qquad x \text{ in } R \tag{4}$$

and

$$N = \{x \mid x^m = 0\} \qquad x \text{ in } R \text{ and } m = 1, 2, 3, \cdots. \tag{5}$$

The proof of the Theorem is based on the following lemmas:

LEMMA 1. *Let R be a pre-p-ring. If h and k are integers such that $h \geqq p+2$ and $k-h=t(p-1)$, $t=1, 2, 3, \cdots$ then*

$$x^h = x^k, \qquad \textit{for every } x \textit{ in } R. \tag{6}$$

Proof. Since R is a pre-p-ring, $xy^p = x^p y$ for every x in R. Substituting x^2 for y in the above equality, we obtain: $x^{p+2} = x^{2p+1}$. Multiplying both sides of the last equality by $x^{u(p-1)}$ for each $u = 1, 2, 3, \cdots$, we obtain a sequence of equalities which imply

$$x^{p+2} = x^{2p+1} = x^{3p} = \cdots = x^{p+2+t(p-1)},$$

from which (6) follows:

LEMMA 2. *Let R be a pre-p-ring and B and N as given by* (4) *and* (5). *Then every r in R has a unique representation*

$$r = b + n, \qquad b \textit{ in } B \textit{ and } n \textit{ in } N. \tag{7}$$

Proof. For every r in R, let

$$b = r^{p^2+p-1} \quad \text{and} \quad n = r - r^{p^2+p-1}.$$

Since $p^2+p-1 \geqq p+2$ and $p^3+p^2-p-(p^2+p-1) = (p^2+p-1)(p-1)$, it follows from Lemma 1 that:

$$b = r^{p^2+p-1} = r^{p^3+p^2-p} = (r^{p^2+p-1})^p = b^p.$$

Thus, for every r in R, b is in B.

Moreover, since $p^2 \geqq p+2$ and $p^4+p^3-p^2-p^2 = p^2(p+2)(p-1)$, it is a consequence of Lemma 1 that:

$$n^{p^2} = (r - r^{p^2+p-1})^{p^2} = r^{p^2} - r^{p^4+p^3-p^2} = 0.$$

Thus, for every r in R, n is in N.

Clearly, B and N as given by (4) and (5) are subrings of R. Also, it is obvious that no element z of R, except 0, is such that $z^p = z$ and $z^m = 0$ for some

$m=1, 2, 3, \cdots$. Therefore, in view of the commutativity of R, representation (7) is unique.

LEMMA 3. *Let R be a pre-p-ring and B and N as given by* (4) *and* (5). *Then $bn=0$, for every b in B and every n in N.*

Proof. In view of (2) and (4) we have $bn=b^pn=n^pb=n^pb^p=n^{p^2}b$. Since $n^m=0$ for some positive integer m it again follows, by Lemma 1, that $n^{p+2}=0$. On the other hand, $p^2 \geqq p+2$ and therefore $n^{p^2}=0$. Thus, $bn=0$, as desired.

LEMMA 4. *Let R be a pre-p-ring and R as given by* (4). *Then B is an ideal of R.*

Proof. The fact that B is a subring of R, as pointed out above, is obvious. Now, let r be in R and b be in B. By Lemma 2, $r=b'+n'$, b' in B and n' in N. By Lemma 3,

$$rb = (b' + n')b = b'b$$

and, since B is a subring of R, $b'b$ is in B, and consequently rb is in B, for every r in R and every b in B. Thus, B is an ideal of R.

Proof of the theorem. By Lemma 4, B is an ideal of R and by (4), B is a p-ring. By (5), N is the radical of R and hence N is an ideal of R. Moreover, N is a nil ring. The fact that $R=B\oplus N$ then follows from Lemmas 2 and 3. Thus, the theorem is proved.

For $p=2$, a p-ring is a Boolean ring. Moreover, in every Boolean ring B_2

$$xy(x + y) = 0 \tag{8}$$

for every x and y in B_2.

Again, it is not true that every associative and commutative ring whose characteristic is 2 and in which (8) is valid is a Boolean ring.

Motivated by Definition 1, we introduce

DEFINITION 2. *An associative and commutative ring R_2 whose characteristic is* 2 *is called a pre-Boolean ring if $xy(x+y)=0$, for every x and y in R_2.*

In view of the above Theorem, we have:

COROLLARY. *Every pre-Boolean ring is a direct sum of a Boolean ring and a nil ring.*

More precisely, if R_2 is a pre-Boolean ring, then

$$R_2 = B_2 \oplus N,$$

where B_2 is a Boolean ideal (the set of all idempotent elements of R_2) of R_2 and N is the radical (the set of all nilpotent elements of R_2) of R_2.

Reference

1. N. H. McCoy and D. Montgomery, A representation of generalized Boolean rings, Duke Math. J., 3 (1937) 455–459.

ON SEPARATION AND PROXIMITY SPACES

WILLIAM J. PERVIN, Pennsylvania State University

1. Introduction. It is generally stated that the first abstracting of the uniform topology by axiomatizing the proximity relation A *is close to* B as a binary relation on subsets of a set X was done by Efremovich [6] in 1952 when he introduced proximity spaces. In this paper we will consider to what extent this axiomatization was inherent in the axiomatization of separation spaces by Wallace [15] in 1941. We will also consider the relationship between this proximity relation, the subordination relations of Aleksandrov [1] and Freudenthal [7], and the syntopogenic relations of Császár [5].

2. Separation spaces. At the same time, Krishna Murti [8], Szymanski [13], and Wallace [15], independently used the notion of separation of sets as the primitive notion of topology. We shall follow the axiomatization of Wallace since his axioms translate into the notions of proximity most directly.

Wallace defined separation spaces by use of a binary relation $A|B$ between subsets A and B of X. In order to be able to compare his axioms directly with those for proximities, we will actually give the axioms in terms of its negation δ which we will define by setting $A\ \delta\ B$ iff $A|B$ is false. Wallace's axioms are, for all A, $B \subset X$:

(P.1) $\quad \emptyset\ \bar{\delta}\ A$

(P.2) $\quad A\ \delta\ B \Rightarrow B\ \delta\ A$

(P.3) $\quad A \cap B \neq \emptyset \Rightarrow A\ \delta\ B$

(P.4) $\quad A \supset A^*$ *and* $A^*\ \delta\ B \Rightarrow A\ \delta\ B$

(P.5) $\quad (A_1 \cup A_2)\ \delta\ B \Rightarrow A_1\ \delta\ B$ *or* $A_2\ \delta\ B$

(P.6) $\quad x\ \delta\ y \Rightarrow x = y$

(C) $\quad x\ \delta\ \{p: p\ \delta\ A\} \Rightarrow x\ \delta\ A$

(S) $\quad A\ \delta\ B \Rightarrow x\ \delta\ B$ *for some* $x \in A$ *or* $A\ \delta\ y$ *for some* $y \in B$.

Wallace's theorem characterizing separation spaces states that a relation δ, satisfying the first seven axioms, gives rise to a closure operator c defined by setting $cA = \{p: p\ \delta\ A\}$ which satisfies the Kuratowski closure axioms (including the T_1-axiom). Conversely, if c is the closure operator for a T_1-space and we let $A\ \delta\ B$ iff $(A \cap cB) \cup (cA \cap B) \neq \emptyset$, then all eight of the above axioms hold and $cA = \{p: p\ \delta\ A\}$. It is, then, the Hausdorff-Lennes separation condition that Wallace has axiomatized. We note that axiom (C) characterizes the closure operator in the space while axiom (S) specifies the type of separation to be that of Hausdorff and Lennes.

There are, of course, other equivalent forms in which these axioms could be stated. If in axiom (P. 5) we use mutual implication, then axiom (P. 4) is unnecessary. If in axiom (P. 6) we use mutual implication, then axiom (P. 3) is

unnecessary. In view of axiom (P. 4), axiom (P. 1) could also be written $\emptyset\ \bar{\delta}\ X$. With these changes in mind, the usual axioms for a proximity relation (see [10] or [12]) are (P. 1) through (P. 6) plus the "separation" axiom:

(P) $$A\ \bar{\delta}\ B \Rightarrow A\ \bar{\delta}\ C \text{ and } X - C\ \bar{\delta}\ B \text{ for some } C \subset X.$$

It should be noted that if we try to define a proximity relation by means of the Hausdorff-Lennes separation condition as done above, by setting $A\ \delta\ B$ iff $(A \cap cB) \cup (cA \cap B) \neq \emptyset$, the axiom (P) may not hold (even in a metric space) even though the other six axioms will hold. For example, in the case of the real numbers with the usual topology, if $A = \{x : x < 0\}$ and $B = \{x : x > 0\}$, then there is no set C with the desired properties. In this sense, proximity spaces are certainly distinct from the separation spaces of Wallace. The motivation for proximity spaces is, of course, the relation obtained in a metric space by setting $A\ \delta\ B$ iff $d(A, B) = 0$.

For Tychonoff spaces (i.e., completely regular T_1-spaces) it is known [10] that one may obtain a proximity space by setting $A\ \delta\ B$ iff $c^*A \cap c^*B \neq \emptyset$, where c^* is the closure operator in some fixed compactification X^* of the space X. Conversely, every proximity space gives rise to a Tychonoff space if we let $cA = \{p : p\ \delta\ A\}$. Comparing Wallace's axioms for separation with those for proximity, we see that it is the axiom (P) which implies the complete regularity of the topological space.

Again one might attempt to obtain a proximity relation in a general topological space by setting $A\ \delta\ B$ iff $cA \cap cB \neq \emptyset$ using the above motivation. All of the axioms (P. 1) through (P. 6) are again satisfied if X is a T_1-space but axiom (P) need not be satisfied. Indeed, it is easy to verify that axiom (P) is equivalent to the normality of the space X.

It is interesting to note that there are other possible binary relations between the subsets of a set which could serve as a measure of the "proximity" of the two sets. For example, we might set $A\ \delta\ B$ iff X is connected between A and B, that is, iff there does not exist any set E which is both open and closed and such that $A \subset E$ while $E \cap B = \emptyset$ (see [11]). It is well known (see [9] p. 89) that this relation satisfies axioms (P. 1) through (P. 5). Axiom (P. 6) will also be satisfied if X is *nulle part connexe* (see [9] p. 94). Furthermore, in a T_1-space, axiom (C) is also satisfied. Of course this relation cannot be useful for general topological spaces since in a connected space X, $A\ \delta\ B$ for all nonempty subsets A and B of X. The most natural use of this relation would be in the case of zero-dimensional spaces.

3. Syntopogenic spaces. Császár [5] has recently defined the notion of a syntopogenic space which generalizes the notions of topological, proximity, and uniform spaces. Since we will be concerned only with topological and proximity spaces, we need consider only the simple syntopogenic structures, called topogenic structures. The axioms for a topogenic structure are given in terms of a relation $A < B$ between subsets A and B of a set X. Following the numbering

system used in [5], the axioms for a symmetric, T_1, topogenic order are:

(0.1) $\varnothing < \varnothing$ *and* $X < X$

(Sym) $A < B \Rightarrow X - B < X - A$

(0.2) $A < B \Rightarrow A \subset B$

(0.3) $A \subset A^* < B^* \subset B \Rightarrow A < B$

(Q) $A < B$ *and* $A^* < B^* \Rightarrow A \cap A^* < B \cap B^*$ *and* $A \cup A^* < B \cup B^*$

(T_1) $x \neq y \Rightarrow x < X - y$

(7.9) $A < B \Rightarrow A < C < B$ *for some* $C \subset X$.

In order to compare these axioms with those for proximity spaces, we may restate them in terms of the dual relation δ defined by setting $A\ \delta\ B$ iff $A < X - B$ is false. It is easy to verify that the duals of the above axioms are equivalent to axioms (P. 1) through (P. 6) and (P) for proximity spaces. Császár's results include the fact that there is a one-to-one correspondence between proximity spaces and these symmetric, T_1, topogenic structures. Furthermore, there is a one-to-one correspondence between topological spaces and non-symmetric topogenic structures (ones whose dual relations may not satisfy (P. 2)) which have the additional property of being perfect; that is, are such that $x < B$ for all $x \in A$ implies that $A < B$. In terms of the dual relation δ, we have

(S′) $$A\ \delta\ B \Rightarrow x\ \delta\ B \text{ for some } x \in A.$$

We note that (S′) is not the same as (S) but again the axiom was anticipated by Wallace. The actual definition given by Császár is $A < B$ iff $A \subset G \subset B$ for some open set $G \subset X$. This is equivalent to the requirement that A is a subset of the interior of B. The dual relation $A\ \delta\ B$ is equivalent to the condition $A \cap cB \neq \varnothing$. Unfortunately, this relation is inherently unsymmetric and so $A\ \delta\ B$ will not in general imply $B\ \delta\ A$, even if the topological space is a Tychonoff space in which, therefore, a proximity could be introduced.

4. Subordination. A topogenic structure is very similar to the notion of subordination or strong inclusion used by Freudenthal [7] and Aleksandrov [1] in an attempt to find all bicompact extensions of a space. It is interesting to note that a concept similar to this was also introduced by Wallace [16] in 1942, who points out that Terasaka [14] used a similar notion in 1938.

Following Aleksandrov and Ponomarev [3] where the concept of subordination of closed sets F in open sets G is axiomatized, we may list the following subordination axioms in terms of a relation $<$ between subsets of a set X:

(K.1) $\varnothing < \varnothing$

(K.2) $F < G \Rightarrow X - G < X - F$

(K.3) $F < G \Rightarrow F \subset G$

(K.4) $F \subset F^* < G^* \subset G \Rightarrow F < G$

(K.5) $F < G$ *and* $F^* < G^* \Rightarrow F \cup F^* < G \cup G^*$

(K.6) $x \in G \Rightarrow x < G$

(K.7) $F < G \Rightarrow F < G^*$ *and* $cG^* < G$ *for some (open)* $G^* \subset X$.

It is easy to verify that the duals of the above axioms are equivalent to axioms (P. 1) through (P. 5) for proximity spaces and the following axioms:

(P.6′) $x \, \delta \, F \Rightarrow x \in F$

(P′) $F_1 \, \bar{\delta} \, F_2 \Rightarrow F_1 \, \bar{\delta} \, F$ *and* $c(X - F) \, \bar{\delta} \, F_2$ *for some (closed)* $F \subset X$,

when all are stated as relations between closed subsets of X. This relation can be extended to a relation δ^* between arbitrary subsets of X by setting $A \, \delta^* \, B$ iff $cA \, \delta \, cB$.

It is well known that a topological space in which one can introduce a subordination is Tychonoff and in any Tychonoff space one can introduce a subordination by setting $F<G$ iff F and $X-G$ can be functionally separated ($F_1 \, \delta \, F_2$ iff F_1 and F_2 cannot be functionally separated). A discussion of the relationship between subordinations and bicompact extensions of spaces appears in [2] and additional results have been obtained more recently in [4].

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References

1. P. S. Aleksandrov, Bicompact extensions of topological spaces, Mat. Sb., 5 (47) (1939) 403–424.

2. ———, Some results in the theory of topological spaces obtained within the last twenty-five years, Uspeki Mat. Nauk, 15 (1960) 23–83.

3. P. S. Aleksandrov and V. I. Ponomarev, On bicompact extensions of topological spaces, Dokl. Akad. Nauk SSSR, 121 (1958) 575–578.

4. ———, On completely regular spaces and their bicompact extensions, Moscow Univ. Vestnik, Ser. I, No. 2, (1962) 37–43.

5. Á. Császár, Fondements de la topologie générale, Akademiai Kiado, Budapest, 1960.

6. V. A. Efremovich, The geometry of proximity, Mat. Sb., 31, (73) (1952) 189–200.

7. H. Freudenthal, Neuaufbau der Endentheorie, Ann. of Math., 43 (1942) 261–279.

8. S. B. Krishna Murti, A set of axioms for topological algebra, J. Indian Math. Soc., 4 (1940) 116–119.

9. K. Kuratowski, Topologie II, Warszawa, Monografie Matematyczne, 21 (1961).

10. Z. P. Mamuzić, Uvod u opštu topologiju, Beograd, Matematička Biblioteka, 17 (1960).

11. S. Mazurkiewicz, Sur la théorie des ensembles, C. R. Acad. Sci. Paris, 151 (1910) 296–298.

12. S. G. Mrówka, Axiomatic characterization of the family of all clusters in a proximity space, Fund. Math., 48 (1960) 123–126.

13. P. Szymanski, La notion des ensembles séparés comme terme primitif de la topologie, Mathematica, Timisoara, 17 (1941) 65–84.

14. H. Terasaka, Über die Darstellungen der Verbände, Proc. Imp. Acad., Japan, 14 (1938) 306–311.

15. A. D. Wallace, Separation spaces, Ann. of Math., 43 (1941) 687–697.

16. ———, Separation spaces II, Anais Acad., Brasileira Ciencias, 14 (1942) 203–206.

UNIFORM DISTRIBUTION modulo m OF MONOMIALS

BURKE ZANE, Fresno State College

Consider an infinite sequence of integers $A=\{a_i\}$. For any integers j and $m>1$ define $A(n, j, m)$ to be the number of terms among $a_1, a_2, \cdots, a_n$ that satisfy the congruence $a_i \equiv j \pmod{m}$. Following Niven [1], we say that the sequence A is *uniformly distributed modulo m* in case

$$\lim_{n\to\infty} \frac{1}{n} A(n, j, m) = \frac{1}{m}$$

for each $j=1, 2, \cdots, m$. In the event that A is uniformly distributed modulo m we will indicate this fact by the abbreviation A is u.d. (mod m).

If f is an arbitrary polynomial with integral coefficients, then we may ask whether the infinite sequence of integers $S(f)=\{f(i)\}$ is u.d. (mod m) or not. Since $f(x+m)\equiv f(x) \pmod{m}$, an equivalent question is whether the integers $f(1), f(2), \cdots, f(m)$ constitute a complete residue system modulo m or not. This paper completely answers any such inquiries for the case where the polynomial is, in fact, a monomial.

Consider the monomial ax^k. It is obvious that if $k=1$, the $S(ax^k)$ is u.d. (mod m) iff $(a, m)=1$. Equally obvious is the observation that if $k>1$, then $S(ax^k)$ is u.d. (mod 2) iff $(a, 2)=1$. The following theorem is not quite so obvious:

THEOREM 1. *Let p be an odd prime and let $k>1$ determine the integer K by the conditions that $k=p^sK$ and $(p, K)=1$. Then $S(ax^k)$ is u.d. (mod p) iff (a, p) $=(K, p-1)=1$.*

Proof. Suppose that $S(ax^k)$ is u.d. (mod p). Clearly $(a, p)=1$. According to Fermat's Theorem, if $0<x<p$, then

$$x^k \equiv x^{p^sK} \equiv x^K \pmod{p}.$$

Also, according to another well-known result [2, p. 49], if $(b, p)=1$, then $x^K\equiv b \pmod{p}$ has $(K, p-1)$ or no solutions depending on whether

$$b^{(p-1)/(K,p-1)} \equiv 1 \pmod{p}$$

or not respectively. Since

$$1^{(p-1)/(K,p-1)} \equiv 1 \pmod{p}$$

we know that the congruence $x^K\equiv 1 \pmod{p}$ has $(K, p-1)$ incongruent solutions modulo p, and hence, that $ax^k\equiv a \pmod{p}$ has $(K, p-1)$ incongruent solutions modulo p. But we are assuming that $S(ax^k)$ is u.d. (mod p) so ax^k $\equiv a \pmod{p}$ must have exactly one solution modulo p. Therefore $(K, p-1)=1$.

Now suppose that $(a, p)=(K, p-1)=1$. Then since

$$b^{(p-1)/(K,p-1)} \equiv b^{p-1} \equiv 1 \pmod{p}$$

for each $b=1, 2, \cdots, p-1$, we know that to each such b there corresponds a

unique x such that $0<x<p$ and $x^k \equiv b \pmod{p}$. It follows that $S(ax^k)$ is u.d. (mod p).

COROLLARY 1. *There are infinitely many primes p such that $S(ax^k)$ is not* u.d. (mod p).

Proof. Let q be any prime which divides k. Consider the arithmetic progression

$$1+q, \quad 1+2q, \quad 1+3q, \cdots, \quad 1+mq, \cdots.$$

According to Dirichlet's Theorem, this progression contains an infinite number of primes. Let p be any such prime which satisfies $p>k$. Then $p=1+mq$ for some positive integer m and hence q is a divisor of $p-1$. It follows that $(k, p-1) >1$ and hence, by Theorem 1, that $S(ax^k)$ is not u.d. (mod p). Obviously there are infinitely many primes of the required type.

COROLLARY 2. *If $k>1$ is odd, then there exist infinitely many primes p such that $S(ax^k)$ is* u.d. (mod p).

Proof. We are assuming that $(2, k)=1$ so by Dirichlet's Theorem the arithmetic progression

$$2+k, \quad 2+2k, \quad 2+3k, \cdots, \quad 2+mk, \cdots$$

contains an infinite number of primes. Let $p=2+mk$ be any such prime which satisfies $p>a$. If d is a divisor of $p-1=1+mk$ and if d is also a divisor of k, then d must be a divisor of 1. It follows that $(k, p-1)=1$ and hence, by Theorem 1, that $S(ax^k)$ is u.d. (mod p). Clearly there are infinitely many primes of the required type.

Niven [1] showed that if a sequence A is u.d. (mod m) and if d is any divisor of m, then A is u.d. (mod d). He also gave an example to show that even if A happens to be u.d. (mod a) and u.d. (mod b) where $(a, b)=1$, A still need not be u.d. (mod ab). A different result holds for polynomials.

LEMMA. *If f is a polynomial with integral coefficients such that $S(f)$ is* u.d. (mod a) *and* u.d. (mod b), *where $(a, b)=1$, then $S(f)$ is* u.d. (mod ab).

Proof. Suppose that $f(x)\equiv f(y) \pmod{ab}$, where $1\leqq x,\ y\leqq ab$. Then $f(x) \equiv f(y) \pmod{a}$, and since $S(f)$ is u.d. (mod a), it follows that $x\equiv y \pmod{a}$. Similarly, we can conclude that $x\equiv y \pmod{b}$. Hence $x\equiv y \pmod{ab}$ since $(a, b)=1$. Therefore $x=y$. It follows that $f(1), f(2), \cdots, f(ab)$ are incongruent modulo ab and so $S(f)$ is u.d. (mod ab) as required.

THEOREM 2. *The sequence $S(ax^k)$ is* u.d. (mod m) iff *m is square free and $S(ax^k)$ is* u.d. (mod p) *for each prime divisor p of m.*

Proof. If we suppose that m is square free and that $S(ax^k)$ is u.d. (mod p) for each prime divisor p of m, then repeated application of the preceding lemma leads us to the conclusion that $S(ax^k)$ is u.d. (mod m).

Now suppose that $S(ax^k)$ is u.d. (mod m) and let p be any prime divisor of m. We know that $S(ax^k)$ is u.d. (mod p) by Niven's result. If we suppose that p^2 divides m, then $S(ax^k)$ is u.d. (mod p^2) by the same result and it follows that the congruence $ax^k \equiv p \pmod{p^2}$ has a solution x. Since p^2 is a divisor of $ax^k - p$, we know that p is a divisor of ax^k. If p is a divisor of x then p^2 is a divisor of x^k since $k > 1$. But then p^2 is a divisor of p which is impossible. It must follow that p is a divisor of a. Hence $(a, p) = p$, and we see by Theorem 1 that p cannot be an odd prime. But we have also seen that $S(ax^k)$ is u.d. (mod 2) iff $(a, 2) = 1$, so p cannot be the prime 2 either. The impossibility of this result forces us to conclude that p^2 cannot divide m.

THEOREM 3. *The sequence* $S(ax^{2k})$ *is* u.d. (mod m) *iff* $m = 2$ *and* $(a, 2) = 1$.

Proof. If we assume that $m = 2$ and $(a, 2) = 1$, then the result that $S(ax^{2k})$ is u.d. (mod 2) is immediate.

Suppose that $S(ax^{2k})$ is u.d. (mod m). From Theorem 2 we know that m is square free. Suppose that p is an odd prime divisor of m. If we define K by the conditions that $2k = p^s K$ and $(K, p) = 1$, then 2 is a divisor of K. Hence 2 is a divisor of $(K, p-1)$ so, by Theorem 1, we know that $S(ax^{2k})$ is not u.d. (mod p). On the other hand Theorem 2 tells us that $S(ax^{2k})$ is u.d. (mod p). The impossibility of this result forces us to conclude that no odd prime p is a divisor of m. Since $m > 1$, we conclude that $m = 2$. Clearly $(a, 2) = 1$, and the proof is complete.

While Theorems 2 and 3 completely dispose of any questions which might arise concerning the sequence $S(ax^k)$, the same questions remain open for arbitrary polynomials. Only partial results have been attained so far. For example, it is easily shown that no second degree polynomial is uniformly distributed modulo m for any $m > 2$.

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References

1. I. Niven, Uniform distribution of sequences of integers, Trans. Amer. Math. Soc., 98 (1961) 52–61.

2. I. Niven and H. S. Zuckerman, An introduction to the theory of numbers, Wiley, New York, 1960.

QUERY

Let f be a map on a ring R to another ring. What are reasonable sufficient conditions that the kernel of f be a subring of R? Address *all* replies to A. Wilansky, Department of Mathematics, Lehigh University, Bethlehem, Pa.

SEQUENCES OF MASS DISTRIBUTIONS ON THE UNIT CIRCLE WHICH TEND TO A UNIFORM DISTRIBUTION

P. T. BATEMAN, University of Illinois

1. Introduction. In a recent paper [4] L. Mirsky made the following remarks (in a slightly different notation):

"Let $\{d_n\}$ be a sequence of positive integers tending to infinity. Suppose that, for any n, D_n is a set of d_n complex numbers of unit modulus. If

$$0 < \beta - \alpha \leqq 1, \tag{1}$$

denote by $N_n(\alpha, \beta)$ the number of numbers in D_n whose arguments satisfy the inequalities $2\pi\alpha < \arg z \leqq 2\pi\beta$; and suppose that, whenever (1) is valid, we have

$$\lim_{n\to\infty} \frac{N_n(\alpha, \beta)}{d_n} = \beta - \alpha. \tag{2}$$

Finally, we write

$$T_n = \max_{D\subseteq D_n} \left| \sum_{\xi \in D} \xi \right|. \tag{3}$$

We may then conjecture that

$$\lim_{n\to\infty} \frac{T_n}{d_n} = \frac{1}{\pi}.\text{"} \tag{4}$$

The purpose of this paper is to prove the above conjecture of Mirsky. In fact we prove it in the following quantitative form. Let Δ_n be the discrepancy (cf. [1], pp. 61–62) of the set of arguments of the points in D_n modulo 2π, that is, let

$$\Delta_n = \sup_{\alpha<\beta\leqq\alpha+1} \left| \frac{N_n(\alpha, \beta)}{d_n} - (\beta - \alpha) \right|. \tag{5}$$

Then we shall prove in Section 2 that

$$\left| \frac{T_n}{d_n} - \frac{1}{\pi} \right| \leqq 2\Delta_n. \tag{6}$$

If (2) holds for every pair α, β satisfying (1), it is easy to prove (see Section 3 below) that $\lim_{n\to\infty} \Delta_n = 0$. Thus Mirsky's conjecture (4) will follow from (6).

The main purpose of Mirsky's paper was to consider the case where the set D_n consists of the $\phi(n)$ primitive nth roots of unity. In that case he showed that

$$T_n = \frac{1}{\pi}\phi(n) + O(d(n)), \tag{7}$$

where $d(n)$ is the number of divisors of n. In Section 4 we show that (7) may be derived from (6) and also consider some similar special cases.

Actually an inequality analogous to (6) holds for any mass distribution on the unit circle, not just for a discrete one, provided the quantities involved are defined appropriately. Thus (6) is contained in the following more general result.

THEOREM 1. *Suppose we have a mass distribution N on the unit circle. If $0<\beta-\alpha\leqq 1$, denote by $N(\alpha, \beta)$ the amount of mass on that part of the unit circle satisfying the inequalities $2\pi\alpha<\arg z\leqq 2\pi\beta$. Write*

$$T = \sup_{V\subseteq N}\left| \int_0^1 e^{2\pi it}dV(0, t) \right|,$$

where the notation indicates that the supremum is taken over all subdistributions V of N, that is, over all mass distributions V such that

$$V(\alpha, \beta) \leqq N(\alpha, \beta)$$

for every pair α, β satisfying $0<\beta-\alpha\leqq 1$. Put $d=N(0, 1)$ and

$$\Delta = \sup_{\alpha<\beta\leqq\alpha+1}\left| \frac{N(\alpha, \beta)}{d} - (\beta - \alpha) \right|.$$

Then

$$\left| \frac{T}{d} - \frac{1}{\pi} \right| \leqq 2\Delta.$$

The first draft of this paper contained a direct proof of the result just stated. However, I. J. Schoenberg, who was kind enough to read this first draft, pointed out that the arguments used could be adapted to prove much more. Thus we shall derive Theorem 1 from the following still more general result, in which the unit circle is replaced by an arbitrary closed rectifiable curve and a comparison is made between any two mass distributions on the curve. For convenience, we normalize the mass distributions so that the total mass is unity.

THEOREM 2. *Suppose we have a closed rectifiable curve C given by the parametric equation $z=\phi(t)$, $0\leqq t\leqq 1$. Suppose we consider two mass distributions N_1 and N_2 on C each of total mass one. If $0<\beta-\alpha\leqq 1$ and i is 1 or 2, denote by $N_i(\alpha, \beta)$ the amount of mass from N_i lying on that part of C given by the parameter values t satisfying $\alpha<t\leqq\beta$. Write*

$$T_i = \sup_{V\subseteq N_i}\left| \int_0^1 \phi(t)dV(0, t) \right| \qquad (i = 1, 2),$$

where the notation indicates that the supremum is taken over all subdistributions V of N_i as in Theorem 1. Put

$$\Delta = \sup_{\alpha<\beta\leqq\alpha+1} | N_1(\alpha, \beta) - N_2(\alpha, \beta) | .$$

Let $L=\sup_\lambda L(\lambda)$, *where* $L(\lambda)$ *is the total variation on the unit interval of the function* ϕ_λ *defined by*

$$\phi_\lambda(t) = \max\{0, \Re(e^{-2\pi i\lambda}\phi(t))\}.$$

Then $|T_1-T_2| \leqq L\Delta$.

The terminology used in the first sentence of Theorem 2 is intended to mean that ϕ is a complex-valued function on the reals which is continuous, has period one, and is of bounded variation on $(0, 1]$. (Needless to say, this has nothing to do with the Euler function occurring in (7).) Note that Theorem 2 is a purely geometrical result in the sense that the four quantities occurring in its conclusion, namely T_1, T_2, L, and Δ, are unchanged by reparametrization of the curve C.

In case the curve C is starlike with respect to the origin and one of the two mass distributions on C is the uniform distribution with respect to the parameter t, then Theorem 2 takes the form of the following corollary. (We use the term star-like to mean that there is one and only one point of C on each ray from the origin.) Note that this corollary involves a specific parametrization of the curve C.

COROLLARY. *Suppose we have a closed rectifiable curve C which is given by the parametric equation* $z=\phi(t)$ *and which is starlike with respect to the origin. Suppose we have a mass distribution N on C of total mass one and suppose*

$$T = \sup_{V\subseteq N}\left|\int_0^1 \phi(t)dV(0,t)\right|,$$

where the notation is used as in Theorem 1. *Put*

$$U = \sup_{\alpha<\beta\leqq\alpha+1}\left|\int_\alpha^\beta \phi(t)dt\right|$$

and

$$\Delta = \sup_{\alpha<\beta\leqq\alpha+1}|N(\alpha,\beta)-(\beta-\alpha)|.$$

Let L be defined as in Theorem 2. *Then* $|T-U| \leqq L\Delta$.

Theorem 1 follows immediately from the above corollary, since in the case $\phi(t)=e^{2\pi it}$ we have $L=2$. In fact, whenever C is convex and symmetrical about the origin, it is easy to see that L is the length of the longest chord of C.

As the proof in the next section will show, the result of Theorem 2 would remain valid if we were to replace Δ by

$$\Delta^* = \frac{1}{2}\sup_{0<\beta\leqq 1}\{N_1(0,\beta)-N_2(0,\beta)\} - \frac{1}{2}\inf_{0<\beta\leqq 1}\{N_1(0,\beta)-N_2(0,\beta)\}.$$

A similar remark could be made concerning (6), Theorem 1, and the Corollary

to Theorem 2. Since $\Delta^* \leqq \Delta \leqq 2\Delta^*$, we lose at most a factor of two in our conclusion by stating it in terms of Δ. Note that Δ^*, like Δ, is independent of the starting point on the curve C.

2. Proof of Theorem 2 and its corollary.

LEMMA (cf. [3]). *Suppose μ_1 and μ_2 are real-valued functions on the interval $[0, 1]$ such that $\mu_1(0) = \mu_2(0)$ and $\mu_1(1) = \mu_2(1)$. Suppose w is a complex-valued function of bounded variation on $[0, 1]$ such that $\int_0^1 w(t)d\mu_1(t)$ and $\int_0^1 w(t)d\mu_2(t)$ exist as Riemann-Stieltjes integrals and $w(0) = w(1)$. Then*

$$\left| \int_0^1 w(t)d\mu_1(t) - \int_0^1 w(t)d\mu_2(t) \right| \leqq DW,$$

where

$$D = \frac{1}{2} \sup_{0 \leqq t \leqq 1} \{\mu_1(t) - \mu_2(t)\} - \frac{1}{2} \inf_{0 \leqq t \leqq 1} \{\mu_1(t) - \mu_2(t)\}$$

and W is the total variation of w in the interval $[0, 1]$.

Proof. Let

$$M = \frac{1}{2} \sup_{0 \leqq t \leqq 1} \{\mu_1(t) - \mu_2(t)\} + \frac{1}{2} \inf_{0 \leqq t \leqq 1} \{\mu_1(t) - \mu_2(t)\}.$$

Then by the formula for integration by parts

$$\begin{aligned} \int_0^1 w(t)d\{\mu_1(t) - \mu_2(t)\} &= -\int_0^1 \{\mu_1(t) - \mu_2(t)\} dw(t) \\ &= -\int_0^1 \{\mu_1(t) - \mu_2(t) - M\} dw(t). \end{aligned}$$

Since $-D \leqq \mu_1(t) - \mu_2(t) - M \leqq D$, the result of the lemma follows.

We now turn to the proof of Theorem 2. For real λ let us put

$$T_i(\lambda) = \int_0^1 \phi_\lambda(t) dN_i(0, t) \qquad (i = 1, 2). \tag{8}$$

Note that if N_i is a discrete mass distribution, then we can interpret $T_i(\lambda)$ as follows. We take the sum of the vectors from the origin to the points in the mass distribution N_i which lie in the half-plane

$$2\pi(\lambda - \tfrac{1}{4}) < \arg z < 2\pi(\lambda + \tfrac{1}{4}); \tag{9}$$

then $T_i(\lambda)$ is the length of the projection of this sum on the ray $\arg z = 2\pi\lambda$.

We first show that

$$T_i = \sup_\lambda T_i(\lambda) \qquad (i = 1, 2). \tag{10}$$

In fact, if $V \subseteq N_i$ and $2\pi\lambda = \arg \int_0^1 \phi(t)dV(0, t)$, we have

$$\left| \int_0^1 \phi(t)dV(0,t) \right| = \mathcal{R}\left\{e^{-2\pi i\lambda} \int_0^1 \phi(t)dV(0,t)\right\}$$
$$\leqq \int_0^1 \phi_\lambda(t)dV(0,t)$$
$$\leqq \int_0^1 \phi_\lambda(t)dN_i(0,t) = T_i(\lambda).$$

Thus $T_i \leqq \sup T_i(\lambda)$. On the other hand, for each λ a legitimate choice for V is the mass distribution obtained from N_i by discarding all the mass lying outside the half-plane (9), that is, all the mass corresponding to values of the parameter t for which $\phi_\lambda(t) = 0$. Hence with this choice of V we have

$$T_i \geqq \mathcal{R}\left\{e^{-2\pi i\lambda} \int_0^1 \phi(t)dV(0,t)\right\} = \int_0^1 \phi_\lambda(t)dN_i(0,t) = T_i(\lambda),$$

so that $T_i \geqq \sup T_i(\lambda)$. Thus (10) is proved.

It is not difficult to show that

$$| T_i(\lambda + \eta) - T_i(\lambda) | \leqq K | \eta | \qquad (i = 1, 2),$$

where K is the circumference of the smallest circle with center at the origin which contains C. Accordingly $T_1(\lambda)$ and $T_2(\lambda)$ are continuous functions on the real numbers modulo one and so the supremum in (10) is actually attained. This fact is not essential, however, for what follows.

We now apply the lemma with

$$\mu_1(t) = N_1(0, t), \qquad \mu_2(t) = N_2(0, t), \qquad w(t) = \phi_\lambda(t).$$

In view of (8) we obtain

$$| T_1(\lambda) - T_2(\lambda) | = \left| \int_0^1 \phi_\lambda(t)dN_1(0,t) - \int_0^1 \phi_\lambda(t)dN_2(0,t) \right|$$
$$\leqq L(\lambda)\Delta^* \leqq L\Delta^* \leqq L\Delta$$

or

$$T_2(\lambda) - L\Delta \leqq T_1(\lambda) \leqq T_2(\lambda) + L\Delta.$$

Using (10) we find

$$T_2 - L\Delta \leqq T_1 \leqq T_2 + L\Delta,$$

so that Theorem 2 is proved.

To prove the Corollary to Theorem 2 note that, if C is starlike with respect to the origin and N_2 is the uniform distribution with respect to t, then the values

of t for which $\phi_\lambda(t)>0$ form an open interval modulo one and so by (10)

$$T_2 = \sup_\lambda \int_0^1 \phi_\lambda(t)dt \leqq \sup_{\alpha<\beta\leqq\alpha+1} \left| \int_\alpha^\beta \phi(t)dt \right| \leqq T_2.$$

Thus

$$T_2 = \sup_{\alpha<\beta\leqq\alpha+1} \left| \int_\alpha^\beta \phi(t)dt \right| = U.$$

Dropping the subscripts on N_1 and T_1, we have the assertion of the Corollary.

3. Uniform distribution. The conjecture of Mirsky quoted in the introduction is included in the following more comprehensive result on sequences of mass distributions on the unit circle which tend to a uniform distribution.

THEOREM 3. *Suppose that for each positive integer n we have a mass distribution N_n on the unit circle. Write*

$$d_n = \int_0^1 dN_n(0, t), \qquad T_n = \sup_{V\leq N_n} \left| \int_0^1 e^{2\pi i t} dV(0, t) \right|,$$

where we use the notation in the same way as in Theorem 1. *Suppose* (2) *holds for every pair α, β satisfying* (1). *Then* (4) *holds.*

Proof. In view of Theorem 1 we need only prove that $\lim_{n\to\infty}\Delta_n=0$, where Δ_n is defined by (5). But this follows from (2) by a familiar compactness argument which proceeds on the following lines (cf. the first footnote on p. 67 of [1]). For any given positive integer k there is a positive integer n_k such that, if $n\geqq n_k$, we have

$$\left| \frac{1}{d_n} N_n\left(\frac{a}{k}, \frac{b}{k}\right) - \frac{b-a}{k} \right| < \frac{1}{k} \qquad \begin{array}{l}(a = 0, 1, \cdots, k;\\ b = a, a+1, \cdots, a+k).\end{array} \tag{11}$$

Since $N_n(\alpha, \beta)$ is monotone in both α and β as long as $0<\beta-\alpha\leqq 1$, it is easy to infer from (11) that, if $n\geqq n_k$, $0\leqq\alpha\leqq 1$, and $\alpha<\beta\leqq\alpha+1$, then we have

$$\left| \frac{N_n(\alpha, \beta)}{d_n} - (\beta-\alpha) \right| < \frac{3}{k}\cdot$$

Thus $\Delta_n\leqq 3/k$ if $n\geqq n_k$. Since k is an arbitrary positive integer, $\lim_{n\to\infty}\Delta_n=0$ and Theorem 2 is proved.

4. Special examples. We now apply the inequality (6) to three particular sequences of sets D_n. In each case T_n is understood to be defined by (3) and Δ_n by (5).

For our first example, let D_n be the set of the n nth roots of unity. Then $\Delta_n=1/n$ and so (6) gives

$$|T_n - n/\pi| \leqq 2.$$

Actually in this case we have the explicit formulas

$$T_n = \csc\frac{\pi}{n} \qquad \text{if } n \text{ is even,}$$

$$T_n = \frac{1}{2}\csc\frac{\pi}{2n} \qquad \text{if } n \text{ is odd and } n > 1.$$

Thus $T_n = n/\pi + O(1/n)$, so that (6) does not give the best possible result in this trivial case.

For our second example, let D_n be the set of the $\phi(n)$ primitive nth roots of unity, which is the case considered by Mirsky. (Here ϕ denotes the Euler function, of course.) We first estimate Δ_n. In this case, if $\alpha < \beta \leqq \alpha + 1$, then $N_n(\alpha, \beta)$ is the number of positive integers m such that $(m, n) = 1$ and $\alpha < m/n \leqq \beta$. Thus in this case

$$\begin{aligned} N_n(\alpha, \beta) &= \sum_{\alpha n < m \leq \beta n} \sum_{d \mid (m,n)} \mu(d) \\ &= \sum_{d \mid n} \mu(d) \sum_{m \equiv 0 \pmod{d},\ \alpha n < m \leq \beta n} 1, \end{aligned}$$

so that

$$(12) \quad |N_n(\alpha,\beta) - (\beta - \alpha)\phi(n)| = \left| \sum_{d \mid n} \mu(d) \left\{ \left[\frac{\beta n}{d}\right] - \left[\frac{\alpha n}{d}\right] - \frac{(\beta - \alpha)n}{d} \right\} \right| < d(n).$$

Taking the supremum over all pairs α, β with $0 < \beta - \alpha \leqq 1$, we obtain

$$\phi(n)\Delta_n \leqq d(n).$$

Thus (6) gives in this case

$$(13) \qquad |T_n - \phi(n)/\pi| \leqq 2d(n),$$

which is merely a version of (7) with an explicit value of the constant implied by the O-symbol.

For our final example, let D_n be the set of the $\Phi(n)$ roots of unity of order not exceeding n, where

$$\Phi(n) = \sum_{k=1}^{n} \phi(k).$$

To obtain an estimate of the discrepancy in this case we need only sum the result (12) obtained in the previous case. We then obtain for the present case

$$|N_n(\alpha,\beta) - (\beta - \alpha)\Phi(n)| < \sum_{k=1}^{n} d(k) = \sum_{k=1}^{n} [n/k] \leqq n(\log n + 1),$$

which is a result given in [5]. Taking the supremum over all pairs α, β with $0<\beta-\alpha\leqq 1$, we obtain

$$\Phi(n)\Delta_n \leqq n(\log n + 1).$$

Thus (6) gives

$$| T_n - \Phi(n)/\pi | \leqq 2n(\log n + 1). \tag{14}$$

Since $\Phi(n)=3\pi^{-2}n^2+O(n \log n)$ (see [2], Section 18.5), we have in this case

$$T_n = 3\pi^{-3}n^2 + O(n \log n).$$

Actually, if we were to use the remark in the last paragraph of the introduction, we could dispense with the factor 2 on the right-hand side of (13) and (14).

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References

1. J. W. S. Cassels, An introduction to diophantine approximation, Cambridge, 1957.
2. G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, 4th ed. Oxford, 1960.
3. J. F. Koksma, Een algemeene steeling uit de theorie der gelijkmatige verdeeling modulo 1, Mathematica (Zutphen) B, 11 (1942) 7–11.
4. L. Mirsky, A note on cyclotomic polynomials, this MONTHLY, 69 (1962) 772–775.
5. E. H. Neville, The structure of Farey series, Proc. London Math. Soc., (2) 51 (1949) 132–144.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables 46, Florida

A THEOREM CONCERNING REARRANGEMENTS

FREDERIC T. METCALF, U. S. Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland

Suppose the sequences of real numbers, $\{a_k\}_1^n$ and $\{b_k\}_1^n$, satisfy

$$0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n; \qquad 0 \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_n. \tag{1}$$

Then a well-known result (see, e.g., [1] p. 261) states that

$$\sum_{k=1}^{n} a_k b_{n-k+1} \leqq \sum_{k=1}^{n} a_k b_{\sigma k} \leqq \sum_{k=1}^{n} a_k b_k, \tag{2}$$

where σ denotes any permutation of the integers $1, 2, \cdots, n$. (It should be mentioned that, although stated here for nonnegative numbers, this result is valid for any two sequences of real numbers.)

Let m_σ and M_σ denote, respectively, the smallest and largest terms appearing in the sum $\sum_{k=1}^{n} a_k b_{\sigma k}$. Then $m_\sigma \leqq a_k b_{\sigma k} \leqq M_\sigma$ for all $k=1, 2, \cdots, n$. It is the purpose of this note to establish that m_σ attains its *maximum* value (over all permutations σ) for the same permutation for which M_σ attains its *minimum* value; and that this permutation is the one which gives the sum on the left of (2). In other words, the sum $\sum_{k=1}^{n} a_k b_{\sigma k}$ is a minimum when the distance between the largest and smallest of the individual terms is at a minimum; and this minimum distance occurs when $b_{\sigma k} = b_{n-k+1}$ for $k=1, 2, \cdots, n$. It is easy to see that m_σ attains its minimum value (namely, $a_1 b_1$) and M_σ its maximum value (namely, $a_n b_n$) for the permutation which gives the sum on the right of (2). These results are stated precisely in the following theorem.

THEOREM. *Let the sequences of nonnegative real numbers, $\{a_k\}_1^n$ and $\{b_k\}_1^n$, be arranged in nondecreasing order, as in* (1). *Then*

$$\max_\sigma m_\sigma = \max_\sigma \min_{1 \leqq k \leqq n} a_k b_{\sigma k} = \min_{1 \leqq k \leqq n} a_k b_{n-k+1}$$

and

$$\min_\sigma M_\sigma = \min_\sigma \max_{1 \leqq k \leqq n} a_k b_{\sigma k} = \max_{1 \leqq k \leqq n} a_k b_{n-k+1};$$

while

$$\min_\sigma m_\sigma = \min_\sigma \min_{1 \leqq k \leqq n} a_k b_{\sigma k} = \min_{1 \leqq k \leqq n} a_k b_k = a_1 b_1$$

and

$$\max_\sigma M_\sigma = \max_\sigma \max_{1 \leqq k \leqq n} a_k b_{\sigma k} = \max_{1 \leqq k \leqq n} a_k b_k = a_n b_n.$$

Proof. The bounds $a_1 b_1$ and $a_n b_n$ are clear. Consider then the case of the lower bound for M_σ. The proof is by contradiction.

Label the $n!$ permutations of the integers $\{1, 2, \cdots, n\}$ as σ_i $(i=1, 2, \cdots, n!)$. Suppose that σ_1 is the permutation taking k into $n-k+1$ for $k=1, 2, \cdots, n$, i.e., σ_1 is the permutation which occurs in the theorem. For each permutation σ_i $(i=1, 2, \cdots, n!)$ there exists an index k_i such that

$$a_{k_i} b_{\sigma_i k_i} = \max_{1 \leqq k \leqq n} a_k b_{\sigma_i k}.$$

(The choice may be ambiguous, that is, there may be several possible choices for each k_i, but, e.g., the smallest such index may be chosen in order to remove the ambiguity.) Then, in this notation, the theorem asserts that

$$a_{k_1}b_{n-k_1+1} \leqq a_{k_i}b_{\sigma_i k_i} \tag{3}$$

for all $i=1, 2, \cdots, n!$. If $a_{k_1}b_{n-k_1+1}=0$, then (3) is trivial. Thus, it may be assumed that $a_{k_1}b_{n-k_1+1}>0$, without loss (this implies, in particular, that $a_{k_1}>0$, which will be used below). Suppose, contrary to what one wishes to prove, that (3) does not hold for all $i=1, 2, \cdots, n!$. Then there is an integer j, with $1<j\leqq n!$, such that $a_{k_j}b_{\sigma_j k_j} < a_{k_1}b_{n-k_1+1}$. Hence,

$$a_k b_{\sigma_j k} < a_{k_1}b_{n-k_1+1}$$

for all $k=1, 2, \cdots, n$. However, $a_{k_1}\leqq a_k$ for all $k\geqq k_1$; and thus,

$$a_{k_1}b_{\sigma_j k} < a_{k_1}b_{n-k_1+1}$$

or (since, as was shown above, a_{k_1} may be taken as positive)

$$b_{\sigma_j k} < b_{n-k_1+1}$$

whenever $k\geqq k_1$. Due to the ordering of the b's it follows that

$$\sigma_j k < n - k_1 + 1 \tag{4}$$

for $k=k_1, k_1+1, \cdots, n$. But the $n-k_1+1$ distinct positive integers,

$$\sigma_j k_1, \qquad \sigma_j(k_1 + 1), \cdots, \sigma_j n,$$

cannot all satisfy (4) since this inequality can hold for *at most* only $n-k_1$ positive integers. Thus, as was desired, a contradiction has been established, and the stated result must be valid. It is of interest to notice that this last argument is very much akin to an argument one encounters in number theory, namely, the Dirichlet "box principle": if $n+1$ objects are placed in n boxes, then at least two of the objects must reside in the same box.

In order to establish the upper bound for m_σ, the same procedure which was utilized above could be followed. However, a more direct attack is to use the formula (valid if $0<a_1b_1$)

$$\max_{\sigma} \min_{1\leqq k\leqq n} a_k b_{\sigma k} = \frac{1}{\min_{\sigma} \max_{1\leqq k\leqq n} (1/a_k b_{\sigma k})}. \tag{5}$$

If some of the a's or b's are zero, then $m_\sigma=0$ for all σ. Thus, it is safe to assume that $0<a_1b_1$. By applying the previous result to the two sequences, $\{1/a_{n-k+1}\}_1^n$ and $\{1/b_{n-k+1}\}_1^n$, one sees that equation (5) yields the desired upper bound on m_σ immediately.

Reference

1. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, 1934.

THE SEQUENCE ka^n+1 COMPOSITE FOR ALL n

ROBERT BOWEN, Fairfield, California

In [1] it was shown that if k belongs to certain arithmetic progressions, $k\cdot 2^n+1$ is composite for all n. A direct generalization to ka^n+1 is trivial if a is odd and is very easy (see Case 1 below) if $a-1$ is not a power of 2. But if we insist that k be even, the question becomes interesting.

A finite set of primes $q_1, \cdots, q_m$ is said to *cover* the sequence ka^n+1, $n\geqq 0$, if each term in the sequence is divisible by at least one of these primes. We will call the cover *perfect* if no term is divisible by two of these primes.

When $a-1$ is not a power of 2, we can cover ka^n+1 (for certain even values of k) with a single prime q (Case 1). When $a=2^s+1$, we can cover with two odd primes q_1 and q_2 if and only if $(a+1)/2=2^{s-1}+1$ has two prime factors (Case 2). Thus, as in [1], the question of whether the Fermat numbers $F_m=2^{2^m+1}$ are prime or composite enters into our problem in an interesting way. These exceptions, where $(a+1)/2$ is 9 or is a Fermat number, are treated (for $a>9$) in Case 3, where we use three prime divisors of a^4-1 to cover. For $a=9$ we cover with the four odd prime divisors of 9^8-1 and for $a=5$, the five odd prime divisors of $5^{16}-1$. Finally, for $a=3$ there is no similar method for covering $k\cdot 3^n+1$, but we obtain a perfect cover by using a different approach.

LEMMA 1. *The only solution of* $2^x=q^y-1$ *with* $y>1$ *is* $x=3$.

Proof. If $y=2t$ then $2^x=(q^t+1)(q^t-1)$. Each factor on the right is a power of two and their difference is two. Hence $q^t+1=4$ and $x=3$.

Suppose y odd. Then $q-1$ divides $q^y-1=2^x$ so $q-1=2^n$. Substituting, $2^x=(2^n+1)^y-1=2^{yn}+\cdots+\binom{y}{2}2^{2n}+y\cdot 2^n$ which is a contradiction (mod 2^{2n}) if $y>1$.

LEMMA 2. *The only solution of* $2^{2x-1}+2^x+1=5^y$ *is* $x=1$.

Proof. We have $(2^{2x-1}+2^x+1)(2^{2x-1}-2^x+1)=2^{4x-2}+1$. Now 2 is a primitive root (mod 5^y), so $2^{4x-2}\equiv -1$ (mod 5^y) implies that $4x-2\equiv 2\cdot 5^{y-1}$ (mod $4\cdot 5^{y-1}$). Hence $2x-1\geqq 5^{y-1}$ and $5(2x-1)\geqq 5^y=2^{2x-1}+2^x+1$. But this is false for $x>2$.

THEOREM. *For any a there exists an arithmetic progression* $Q=\{cx+b, x\geqq 1\}$ *such that* ka^n+1 *is odd and composite for all* $k\in Q$ *and all* $n\geqq 0$.

Proof. Case 1. $a\neq 2^s+1$. Let q be an odd prime divisor of $a-1$. Set $b=q-1$ and $c=2q$. Then $ka^n+1=(cx+b)a^n+1=(2qx+q-1)a^n+1$ is odd, is greater than q, and is divisible by q.

Case 2. $a=2^s+1$, where $s\neq 2^m+1$, $s>5$. Let $s-1=pt$, where $p>1$ is odd. Then 2^t+1 divides $2^{pt}+1=2^{s-1}+1=(a+1)/2$. Hence $2^{s-1}+1$ is composite but is not a prime power by Lemma 1 since $s-1>3$. Thus $(a+1)/2$ has two distinct prime divisors q_1 and q_2. Let b' be the least positive solution of $b'\equiv 1$ (mod $2q_1$) and $b'\equiv -1$ (mod q_2). Set $b=b'+q_1q_2$ and $c=2q_1q_2$. Then q_1 divides $ka^{2r+1}+1$ $=(2q_1q_2x+b)a^{2r+1}+1$ and q_2 divides $ka^{2r}+1$.

The argument shows that we may also include a in Case 2 whenever $(a+1)/2 = 2^{2^m}+1$ is a composite Fermat number.

Case 3. $a=2^s+1$, $s=4$ or $s=2^m+1$, $m>1$. Let q_1 be a prime divisor of $(a+1)/2$. Then q_1 divides a^2-1 and not a^2+1. Since $a-1=2^s\equiv 1$ or 2 (mod 5), 5 divides $(a^2+1)/2=2^{2s-1}+2^s+1$. By Lemma 2, $(a^2+1)/2$ has another prime divisor q_2. Let b' be the least positive solution of $b'\equiv 1 \pmod{2q_1q_2}$ and $b\equiv 4 \pmod 5$. Set $b=b'+5q_1q_2$ and $c=10q_1q_2$. Then q_1 divides $ka^{2r+1}+1$, q_2 divides $ka^{4r+2}+1$, and 5 divides $ka^{4r}+1$.

There remain only $a=2$, 3, 5, or 9.

For $a=9$, let $b'\equiv 1 \pmod{2\cdot 5\cdot 41\cdot 17}$ and $b'\equiv -1 \pmod{193}$. Set $c=(9^8-1)/2^5$ and $b=(c/2)+b'$. Then 5 divides $k\cdot 9^{2r+1}+1$, 41 divides $k\cdot 9^{4r+2}+1$, 17 divides $k\cdot 9^{8r+4}+1$ and 193 divides $k\cdot 9^{8r}+1$.

For $a=5$, let $b'\equiv 1 \pmod{2\cdot 3\cdot 13\cdot 313\cdot 17}$ and $b'\equiv -1 \pmod{11489}$. Set $c=(5^{16}-1)/2^5$ and $b=(c/2)+b'$.

For $a=2$ use the prime divisors of $2^{64}-1$. See [1] or [2].

For $a=3$, it seems doubtful that there is a cover using only odd prime divisors of $3^{2^m}-1$. Notice that any such cover is perfect, i.e., each term of the sequence $k\cdot a^n+1$ is divisible by one and only one of the covering primes. Each cover described above is perfect. To get a perfect cover for $a=3$, use the following prime divisors of $3^{432}-1$: 13, 7, 73, 6481, 97, 577, 757, 19, 37, 109, 433 and 8209. If b is suitably chosen, these divide $ka^{ir+j}+1$ where $(i, j)=(3, 0)$, $(6, 1)$, $(12, 4)$, $(24, 10)$, $(48, 22)$, $(48, 46)$, $(9, 2)$, $(18, 5)$, $(18, 14)$, $(27, 8)$, $(27, 17)$, and $(27, 26)$. As before, c equals twice the product of the covering primes.

The author is indebted to the referee for many suggestions.

References

1. W. Sierpinski, Sur un problème concernant les nombres $k\cdot 2^n+1$, Elem. Math, 15 (1960) 73–4.

2. Problem 4995, this Monthly, 70 (1963) 101–2.

SODDY'S CIRCLES AND THE DE LONGCHAMPS POINT OF A TRIANGLE

A. Vandeghen, Liège, Belgium

The Euler line of a triangle ABC contains six notable points: the orthocenter H, the nine-point center N, the centroid G, the circumcenter O, the de Longchamps point Z, and the circumcenter of the triangle formed by the tangents to the circumcircle of ABC at A, B, C [2, pp. 104, 242, 299]. The present note introduces a new property of the de Longchamps point [1, p. 370] which, being the symmetric of H with respect to O, is the external center of similitude of the circumcircle with Steiner's circle of center N and radius $3R/2$. (This is the circle within which a circle of radius $R/2$ rolls in the description of the 3-cusped hypocycloid which is the envelope of Simson lines [3, p. 115].)

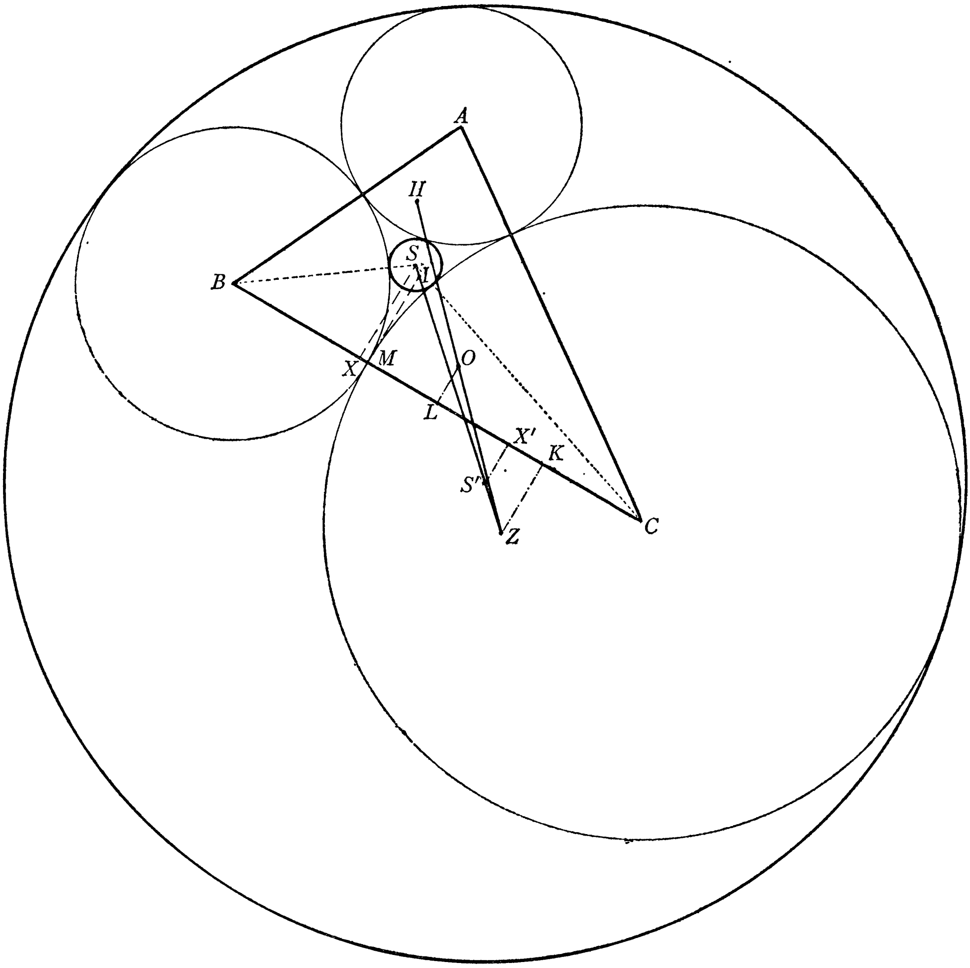

FIG. 1

In the usual notation, circles centered at the vertices of any triangle ABC touch one another if their radii are $s-a$, $s-b$, $s-c$ or s, $s-c$, $s-b$ or $s-c$, s, $s-a$ or $s-b$, $s-a$, s. The first case [3, pp. 13–16] is illustrated in Figure 1. In each of the remaining cases, one circle encloses the other two. For each system of three circles we can find two circles that touch them all. Let us call these the *Soddy circles* of the triangle, and denote their centers and radii by

$$S, \quad S'; \quad S_a, \quad S_a'; \quad S_b, \quad S_b'; \quad S_c, \quad S_c',$$
$$\sigma, \quad \sigma'; \quad \sigma_a, \quad \sigma_a'; \quad \sigma_b, \quad \sigma_b'; \quad \sigma_c, \quad \sigma_c',$$

with $\sigma<\sigma'$, and so on. We shall find that the four pairs of centers lie on the four lines IZ, I_aZ, I_bZ, I_cZ, where I, I_a, I_b, I_c are the incenter and excenters while Z is the de Longchamps point.

When $4R+r<2s$, the first pair of Soddy circles are arranged in the manner of Figure 1. Let L be the midpoint of BC, and X, X', M, K the projections on BC of S, S', I, Z. It is easy to obtain the relations

$$XL = \frac{SC^2 - SB^2}{2BC} = \frac{b-c}{2}\left(\frac{2\sigma}{a}+1\right),$$

$$LX' = \frac{S'B^2 - S'C^2}{2BC} = \frac{b-c}{2}\left(\frac{2\sigma'}{a}-1\right),$$

$$ML = \frac{b-c}{2}, \qquad LK = \frac{b^2-c^2}{2a},$$

whence

$$XM = \frac{b-c}{a}\sigma, \qquad MX' = \frac{b-c}{a}\sigma', \qquad \frac{XM}{MX'} = \frac{\sigma}{\sigma'},$$

$$XK = \frac{b-c}{a}(\sigma+s), \qquad KX' = \frac{b-c}{a}(\sigma'-s), \qquad \frac{XK}{KX'} = \frac{\sigma+s}{\sigma'-s},$$

$$\frac{XX'}{MK} = \frac{\sigma'+\sigma}{s}.$$

Thus I is the internal center of similitude of the Soddy circles, S and S' are on the line IZ, Z divides SS' in the ratio $(\sigma+s):(\sigma'-s)$, and

$$\frac{SS'}{IZ} = \frac{\sigma'+\sigma}{s}.$$

It is easy to calculate that $IZ^2=(4R+r)^2-3s^2$, and thus to obtain an expression for SS'. Incidentally, this proves that

$$4R + r \geqq \sqrt{3}\, s.$$

(According to a letter from O. Bottema to H. S. M. Coxeter, G. R. Veldkamp has observed that the external center of similitude is the Gergonne point, where AM meets the analogous cevians from B and C.)

When $4R+r>2s$, similar calculations show that I is the external center of similitude, Z divides SS' externally in the ratio $(\sigma+s):(\sigma'+s)$, and

$$\frac{S'S}{IZ} = \frac{\sigma'-\sigma}{s}.$$

For the three remaining pairs of Soddy circles we must use the excenters instead of the incenter. We find, for instance, that I_a is the external center of similitude of the Soddy circles with centers S_a and S_a'; these centers are on the line I_aZ, and Z divides S_aS_a' internally in the ratio $(s-a-\sigma):(\sigma'-s+a)$.

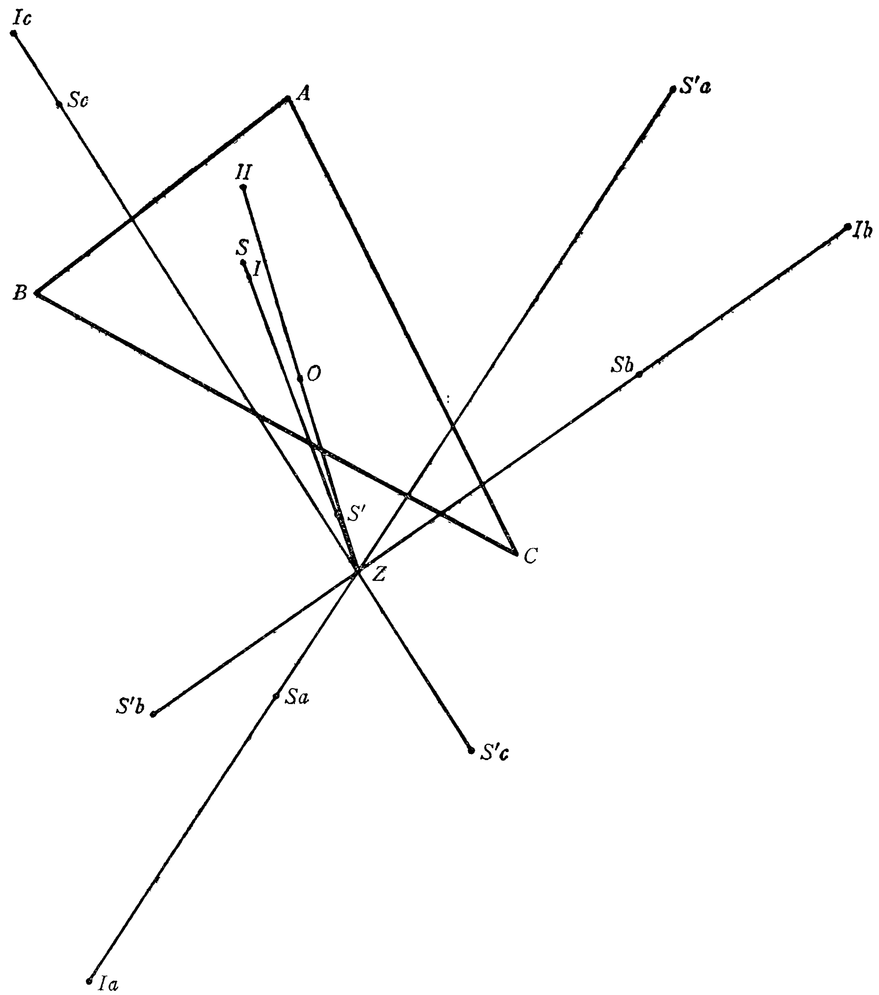

FIG. 2

Figure 2 shows the Euler line HO and the four "Soddy lines" SS', S_aS_a', S_bS_b', S_cS_c', all passing through Z. To avoid complicating the figure, we have omitted the lines OI, OI_a, OI_b, OI_c, although these are of some interest because they are the Euler lines of the four triangles formed by the points of contact of BC, CA, AB with the incircle and the three excircles.

References

1. N. A. Court, On the De Longchamps circle of the triangle, this MONTHLY, 33 (1926) 368–375.

2. ———, College geometry, 2nd ed., Barnes and Noble, New York, 1952.

3. H. S. M. Coxeter, Introduction to geometry, Wiley, New York, 1961.

YET ANOTHER PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

R. P. BOAS, JR., Northwestern University

The following proof of the fundamental theorem of algebra by contour integration is similar to Ankeny's [1], but is simpler because it uses integration around the unit circle (which is usually the first application of contour integration) instead of integration along the real axis; thus there is no need to discuss the asymptotic behavior of any integrals.

Let $P(z)$ be a nonconstant polynomial; we are to show that $P(z)=0$ for some z. We may suppose $P(z)$ real for real z. (Indeed, otherwise let $\overline{P}(z)$ be the polynomial whose coefficients are the conjugates of those of $P(z)$ and consider $P(z)\overline{P}(z)$.) Suppose then that $P(z)$ is real for real z and is never 0; we deduce a contradiction. Since $P(z)$ does not either vanish or change sign for real z, we have

$$\int_0^{2\pi} \frac{d\theta}{P(2\cos\theta)} \neq 0. \tag{1}$$

But this integral is equal to the contour integral

$$\frac{1}{i}\int_{|z|=1} \frac{dz}{zP(z+z^{-1})} = \frac{1}{i}\int_{|z|=1} \frac{z^{n-1}dz}{Q(z)}, \tag{2}$$

where $Q(z)=z^nP(z+z^{-1})$ is a polynomial. For $z\neq 0$, $Q(z)\neq 0$; in addition, if a_n is the leading coefficient in $P(z)$, we have $Q(0)=a_n\neq 0$. Since $Q(z)$ is never zero, the integrand in (2) is analytic and hence the integral is zero by Cauchy's theorem, contradicting (1).

Reference

1. N. C. Ankeny, One more proof of the fundamental theorem of algebra, this MONHTLY, 54 (1947) 464.

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

WHAT! ANOTHER NOTE JUST ON THE FUNDAMENTAL THEOREM OF ALGEBRA?

R. M. REDHEFFER, University of California, Los Angeles

1. Introduction. Throughout this discussion $P(z)$ is a nonconstant polynomial with complex coefficients. According to the fundamental theorem of algebra, $P(z)=0$ has a complex root. Some proofs, using separation properties of the plane, have loopholes hard to close. In other proofs the only topology needed is the fact that the circle $|z|\leq r$ is compact. These are the ones discussed here.

REMARK 12. *Remark* 8 *holds with e^s replaced by* $1/s$, *where s is real and positive.*

Reference

1. R. M. Redheffer, The fundamental theorem of algebra, this MONTHLY, (1957) 582–585.

A NOTE ON THE OPERATION OF MULTIPLICATION IN THE COMPLEX PLANE

D. J. HANSEN, North Carolina State College

In a first course in complex analysis, one frequently asks the following question while discussing the structure of the complex number system: Does there exist more than one way to define multiplication of ordered pairs of real numbers with the stipulation that

(i) $(x, y)=(u, v)$ if and only if $x=u$ and $y=v$,

(ii) $(x, y)\oplus(u, v)=(x+u, y+v)$,

(iii) the axioms for a field be fulfilled,

(iv) the classical definition for the modulus of an ordered pair be preserved along with the statement that $|(x, y)\odot(u, v)|=|(x, y)|\cdot|(u, v)|$, and

(v) $(c, 0)\odot(x, y)=(cx, cy)$?

The answer to the above question, as the theorem stated below indicates, is in the negative.

It is of interest to note that the hypothesis in the following theorem has been weakened by replacing the notion of a field with that of a ring in which $(c, 0)\odot(x, y)=(x, y)\odot(c, 0)$.

THEOREM. *Let S denote the collection of all ordered pairs (x, y) of real numbers with equality defined as in* (*i*) *and suppose that the binary operations $\oplus$ and $\odot$, defined on S, satisfy the axioms for a ring with $(x, y)\oplus(u, v)=(x+u, y+v)$. If $|(x, y)|=\sqrt{(x^2+y^2)}$, $|(x, y)\odot(u, v)|=|(x, y)|\cdot|(u, v)|$, and $(c, 0)\odot(x, y)=(cx, cy)=(x, y)\odot(c, 0)$ then $(x, y)\odot(u, v)=(ux-vy, uy+xv)$.*

Proof. First, one observes that

$$\begin{aligned}(0, y)\odot(0, v) &= [(y, 0)\odot(0, 1)]\odot[(v, 0)\odot(0, 1)] = (yv, 0)\odot[(0, 1)\odot(0,1)]\\ &= (yv, 0)\odot(m, n),\end{aligned}$$

where $(m, n)=(0, 1)\odot(0, 1)$. Thus

$$\begin{aligned}(x, y)\odot(u, v) &= [(x, 0)\oplus(0, y)]\odot(u, v) = [(x, 0)\odot(u, v)]\oplus[(0, y)\odot(u, v)]\\ &= (xu, xv)\oplus\{(0, y)\odot[(u, 0)\oplus(0, v)]\}\\ &= (xu, xv+uy)\oplus[(0, y)\odot(0, v)]\\ &= (xu, xv+uy)\oplus(yvm, yvn)\\ &= (xu+yvm, xv+uy+yvn).\end{aligned}$$

From the definition and product property for the modulus of an ordered pair,

it is also readily verified that

(1) $$m^2 + n^2 = 1, \text{ and}$$

(2) $$(x^2 + y^2)(u^2 + v^2) = (xu + yvm)^2 + (xv + uy + yvn)^2.$$

By expanding (2) and rearranging the terms, one is allowed to conclude that

(3) $$(1 - m^2 - n^2)y^2v^2 = 2(m + 1)xyuv + 2n(uvy^2 + xyv^2).$$

Consequently, from (1) and (3), we obtain

$$2(m + 1)xyuv + 2n(uvy^2 + xyv^2) = 0$$

for all real numbers x, y, u and v. Hence, $m = -1$ and $n = 0$. Therefore, $(0, 1) \odot (0, 1) = (-1, 0)$ and $(x, y) \odot (u, v) = (xu - yv, xv + uy)$.

ON APPROXIMATION OF SLOWLY CONVERGENT SERIES

O. E. STANAITIS, St. Olaf College

1. Introduction. The standard method of evaluating infinite series is to evaluate suitable partial sums S_k and estimate the error R_k involved in the corresponding remainder. In case of slow convergence, however, for example, with

(1) $$S = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

the upper estimate of the remainder

$$R_k = \frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} + \cdots < \frac{1}{k}$$

indicates that by the standard method it is necessary to add a million terms in order to secure 6 decimal places. In this case the method is not feasible. In many cases the problem can be solved by transforming the series into a more rapidly convergent series. Euler's, Kummer's, and Markoff's transformations are ingenious methods in this field. Kummer's and Markoff's transformations of (1) yield, respectively, (cf. [1]) the rapidly convergent series

(2) $$\sum_{n=1}^{\infty} \frac{1}{n^2} = S_p + p! \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)(n+2)\cdots(n+p)}, \qquad p = 1, 2, 3, \cdots$$

(3) $$\sum_{n=1}^{\infty} \frac{1}{n^2} = 3 \sum_{n=1}^{\infty} \frac{[(n-1)!]^2}{(2n)!}.$$

In case (2), addition of 17 terms (for $p = 9$), and in case (3), addition of 9 terms yield accuracy to 6 decimal places.

The success of Kummer's and Markoff's transformations depends for the most part, however, on special artifices, and Euler's transformation does not

always lead to a more rapidly convergent series. A case in point is Abel's series

$$S = \sum_{n=1}^{\infty} \frac{1}{(n+1)[\log(n+1)]^2}, \tag{4}$$

where Euler's transformation leads to a more slowly convergent series. Since applicable artifices for Kummer's and Markoff's transformations are not known, the above series cannot be evaluated by these transformations.

Another way of evaluating slowly convergent series is the method of upper and lower estimates of the remainder. In case (1) we have

$$\frac{1}{k+1} < R_k < \frac{1}{k}, \qquad \frac{1}{k+1} + S_k < S < S_k + \frac{1}{k}\cdot$$

That is, for an approximation to 6 decimal places one needs to add only 1000 terms, still too large a number for practical purposes. The power of this method has been considerably underestimated. Knopp observes [1, p. 260]: "But in special examples this method of upper and lower estimates of the remainder may lead to a satisfactory result. These cases are, however, so rare, that they do not come into account for practical purposes. Greater importance attaches to methods for transformation of slowly convergent into rapidly convergent series, because they admit of a far wider range of applications."

The purpose of this short note is to show, first, that the method of upper and lower estimates of the remainder can easily be refined; secondly, that the improved method admits of a wide range of applications, and, for practical purposes, is handier than the transformations.

2. Method of upper and lower estimates of the remainder. Let us consider the series

$$S = \sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{k} f(x) + \sum_{x=k+1}^{\infty} f(x) = S_k + R_k,$$

where $f(x)$ is a positive, continuous, and monotone decreasing function for $x \geqq 1$. The familiar Cauchy's integral test inequality applied to the remainder of a convergent series yields the first rough upper and lower estimate

$$\int_{k+1}^{\infty} f(x)dx < R_k < \int_{k+1}^{\infty} f(x)dx + f(k+1). \tag{5}$$

Under the additional assumption that $f''(x)$ is a positive monotonically decreasing function for $x \geqq 1$, by using trapezoids instead of rectangles, one can obtain a much sharper estimate for the remainder [2, p. 236]

$$\tfrac{1}{2}f(k+1) + \int_{k+1}^{\infty} f(x)dx - \frac{f'(k+2)}{12} < R_k < \tfrac{1}{2}f(k+1) + \int_{k+1}^{\infty} f(x)dx - \frac{f'(k)}{12}\cdot \tag{6}$$

It follows from (6) that by adding 31 terms of the series (1) and 3 additional

terms one obtains an approximation with an error less than 10^{-6}.

Note that alternating series can be treated in the same manner by combining pairs of terms to produce a new series of positive terms.

3. Generalization. If the derivatives of $f(x)$ are continuous the method can be extended to arbitrary convergent series. The familiar Euler's summation formula [1, p. 518] applied to the remainder yields

$$
\begin{aligned}
R_k = \int_{k+1}^{\infty} f(x)dx + \tfrac{1}{2}f(k+1) - \frac{B_2}{2!}f'(k+1) - \frac{B_4}{4!}f'''(k+1) - \cdots \\
- \frac{B_{2v}}{(2v)!}f^{(2v-1)}(k+1) + \int_{k+1}^{\infty} P_{2v+1}(x)f^{(2v+1)}(x)dx,
\end{aligned} \tag{7}
$$

where

$$
B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = \frac{1}{30}, \quad B_{10} = \frac{5}{66},
$$

$$
B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \cdots
$$

are Bernoulli's numbers and

$$
P_{2v+1}(x) = (-1)^{v-1} \sum_{n=1}^{\infty} \frac{2 \sin 2\pi n x}{(2\pi n)^{2v+1}} \cdot
$$

From (7) we obtain the estimate for the remainder

$$
\begin{aligned}
\Bigg| R_k - \int_{k+1}^{\infty} f(x)dx - \tfrac{1}{2}f(k+1) + \frac{B_2}{2!}f'(k+1) + \frac{B_4}{4!}f'''(k+1) + \cdots \\
+ \frac{B_{2v}}{(2v)!}f^{(2v-1)}(k+1) \Bigg| \\
\leqq \left| \int_{k+1}^{\infty} P_{2v+1}(x)f^{(2v+1)}(x)dx \right|.
\end{aligned} \tag{8}
$$

By suitable choice of k and v the error

$$
r_k = \left| \int_{k+1}^{\infty} P_{2v+1}(x)f^{(2v+1)}(x)dx \right| \tag{9}
$$

can be made arbitrarily small. If we assume further that $|f^{(2v+1)}(x)|$ is integrable, then from the familiar relation

$$
\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} < \sum_{n=1}^{\infty} \frac{1}{n^{2p}} = (-1)^p \frac{B_{2p}}{2} \frac{(2\pi)^{2p}}{(2p)!}
$$

we obtain the first rough estimate of

$$(10)\qquad r_k \leqq \int_{k+1}^{\infty} |P_{2v+1}(x)|\,|f^{(2v+1)}(x)|\,dx < \frac{B_{2v}}{2\pi(2v)!}\int_{k+1}^{\infty} |f^{(2v+1)}(x)|\,dx.$$

The result of this section we state as a theorem.

THEOREM. *If* $S=\sum_{x=1}^{\infty} f(x)$ *is a convergent series, where* $f(x)$, *defined for* $x \geqq 1$, *is* $2v+1$ *times continuously differentiable, then the error* r_k *of the approximation*

$$(11)\qquad \begin{aligned} S \approx S_k &+ \int_{k+1}^{\infty} f(x)dx + \tfrac{1}{2}f(k+1) - \frac{B_2}{2!}f'(k+1) - \frac{B_4}{4!}f'''(k+1) - \cdots \\ &- \frac{B_{2v}}{(2v)!}f^{(2v-1)}(k+1) \end{aligned}$$

is determined by (9).

4. Examples. 1. For $f(x)=1/x^2$ and $v=3$ from (10) follows

$$r_k < \frac{7}{(2\pi)42(k+1)^8}\,.$$

Hence, for $k=4$ we obtain an approximation of (1)

$$(12)\qquad S \approx 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5} + \frac{1}{2}\,\frac{1}{5^2} + \frac{1}{6}\,\frac{1}{5^3} - \frac{1}{30}\,\frac{1}{5^5} + \frac{1}{42}\,\frac{1}{5^7}$$

with an error less than 10^{-7}.

For more refined scientific needs one can increase the accuracy considerably by putting $k=9$ and $v=4$ in (11), that is, adding 9 terms of series (1) and 6 additional terms. In this case the approximation of (1) is

$$S \approx S_9 + \frac{1}{10} + \frac{1}{2}\,\frac{1}{10^2} + \frac{1}{6}\,\frac{1}{10^3} - \frac{1}{30}\,\frac{1}{10^5} + \frac{1}{42}\,\frac{1}{10^7} - \frac{1}{30}\,\frac{1}{10^9}$$

with an error less than 10^{-11}. If the three terms $B_{10}10^{-11}+B_{12}10^{-13}+B_{14}10^{-15}$ were added to the displayed series the sum would have an error less than 10^{-16}, giving $\pi^2/6=1.6449340668482323\cdots$. Incidentally, the latter approximation shows that greater accuracy does not necessarily require more effort if a suitable k yields easily calculated terms.

2. For $f(x)=1/(x+1)[\log(x+1)]^2$ and $v=3$, (10) yields $r_k<\frac{1}{42}\,\frac{1}{6}f^{(6)}(k+1)$. For $k=8$ we obtain the following approximation of (4)

$$S \approx S_8 + \frac{1}{\log 10} + \frac{1}{2}\,\frac{1}{10(\log 10)^2} - \frac{1}{12}f'(10) + \frac{1}{30}\,\frac{1}{4!}f'''(10) - \frac{1}{42}\,\frac{1}{6!}f^{(5)}(10)$$

with an error less than 10^{-7}.

5. Remarks. If $f(x)$ is positive and possesses continuous derivatives tending monotonely to 0 as $x \to \infty$ one can easily obtain a more accurate estimate for r_k. Note, first, that all derivatives $f(x)^{(2v+1)}$, $v=0, 1, 2, \cdots$, are negative; secondly, Bernoulli's numbers have alternating signs. If r_k in (9) is less than

$$\left| \frac{B_{2v}}{(2v)!} f_{(k+1)}^{(2v-1)} \right| \qquad v = 1, 2, \cdots, p$$

then (11), from the fourth term on, is a finite alternating series and the error is less than

$$\left| \frac{B_{2v+2}}{(2v+2)!} f_{(k+1)}^{(2v+1)} \right| .$$

If the series alternates it is easy to obtain more accurate estimates of the error if one first combines terms in pairs as in section 2.

Comparison of (5), (6), and (11) shows that each consecutive approximation contains terms of the preceding one. It is apparent that for practical success of the approximation easy evaluation of $\int_{k+1}^{\infty} f(x)dx$ is essential.

Approximation (6) can be considered as a special case of (11) for $v=1$, but the estimate of the error in (6), $[f'(k+2)-f'(k)]/12$, is larger than in (10). In case (1) the approximation (6) requires addition of 31 terms of (1), whereas (10) achieves the same accuracy (10^{-6}) by addition of 16 terms. Approximation (11) compares favorably in cases (2) and (3). Transformations (2) and (3) require addition of 17 and 9 terms respectively (accuracy 10^{-6}), whereas in (12) addition of 9 terms yields an approximation with an error less than 10^{-7}. We recall, first, that transformations are obtained by using special artifices; secondly, that transformation of a series requires additional effort; thirdly, that if artifices are unknown a series cannot be evaluated by a transformation (case 4); thus it follows that, for practical purposes, the method of upper and lower estimates of the remainder is superior.

References

1. Konrad Knopp, Theory and Applications of Infinite Series, 2nd English ed., Stechert-Hafner, New York, 1948, pp. 240–247; 259–267; 520–527.

2. J. M. H. Olmsted, Advanced Calculus, Appleton-Century-Crofts, New York, 1961, p. 408.

A MODIFIED MACLAURIN INTEGRAL TEST

LOUIS C. BARRETT, South Dakota School of Mines and Technology and Naval Ordnance Test Station, China Lake, California

In its most elementary form, Maclaurin's integral test is used to examine series, such as $\sum_{j=1}^{\infty} f(j)$ for convergence or divergence, when the continuous function $f(x)$ ultimately becomes and remains positive and monotone decreasing. Many extensions of this theorem have been given [**1, 2**].

This note provides a modification of the test which may be applied to series of the type $\sum_{j=1}^{\infty} f(\lambda_j)$ when $f(x)$ has the aforesaid properties and λ_j is the jth element of a strictly monotone increasing sequence. More precisely, our test reads:

THEOREM. *Suppose that*
(i) $\lambda_1<\lambda_2<\lambda_3< \cdots <\lambda_j< \cdots$,
(ii) $\lim_{j\to\infty} \lambda_j=\infty$,
(iii) *there exists* $M>0$ *such that* $\lambda_{j+1}-\lambda_j\geqq M$, $(\leqq M)$, $(j=1, 2, \cdots)$,
(iv) $f\in C$ $(\lambda_1\leqq x<\infty)$,
(v) f *is monotone decreasing and* $f(x)\geqq 0$ $(a\leqq x<\infty)$,
(vi) $\lim_{R\to\infty} \int_a^R f(x)dx=A$, $(=\infty)$;
then $\sum_{j=1}^{\infty} f(\lambda_j)<\infty$, $(=\infty)$.

Proof. Let J be a fixed positive integer such that $\lambda_J\geqq a$. Hypothesis (iv) ensures the convergence or divergence of $\sum_{j=1}^{\infty} f(\lambda_j)$, according as $\sum_{j=J}^{\infty} f(\lambda_j)$ converges or diverges. From (i) and (v)

$$f(\lambda_{j+1}) \leqq f(x) \leqq f(\lambda_j); \qquad \lambda_j \leqq x \leqq \lambda_{j+1}, \qquad j \geqq J;$$

hence, by integration,

$$f(\lambda_{j+1})(\lambda_{j+1}-\lambda_j) \leqq \int_{\lambda_j}^{\lambda_{j+1}} f(x)dx \leqq f(\lambda_j)(\lambda_{j+1}-\lambda_j); \qquad j \geqq J.$$

For every positive integer $N>J$ we find, after repeatedly adding respective members of these inequalities,

$$\sum_{j=J+1}^{N+1} f(\lambda_j)(\lambda_j-\lambda_{j-1}) \leqq \int_{\lambda_J}^{\lambda_{N+1}} f(x)dx \leqq \sum_{j=J}^{N} f(\lambda_j)(\lambda_{j+1}-\lambda_j). \tag{1}$$

Now suppose that (iii) involves $\lambda_{j+1}-\lambda_j\geqq M$ (in which case the first two hypotheses become redundant), and that (vi) involves the finite number A. Then, from (1) and the positiveness of $f(x)$,

$$\sum_{j=J}^{N+1} f(\lambda_j) \leqq f(\lambda_J) + \frac{1}{M}\int_{\lambda_J}^{\lambda_{N+1}} f(x)dx,$$

and therefore $\sum_{i=J}^{\infty} f(\lambda_j) \leqq f(\lambda_J)+A/M<\infty$.

If the alternative hypotheses in (iii) and (vi) apply, then from (1),

$$\frac{1}{M}\int_{\lambda_J}^{\lambda_{N+1}} f(x)dx \leqq \sum_{j=J}^{N} f(\lambda_j).$$

Letting N, and perforce λ_{N+1}, become infinite there follows

$$\infty = \sum_{j=J}^{\infty} f(\lambda_j).$$

This completes the proof of the theorem.

The familiar Maclaurin integral integral test occurs as a special case of this theorem when $\lambda_j = j$.

It will next be shown that the theorem may be utilized even though (i) and (iii) fail to hold.

Example 1. The sequence $\{\lambda_j\}$ whose odd and even numbered elements are defined, respectively, by

$$\lambda_j = \begin{bmatrix} \lambda_{2k+1} = 1 + k^2 - (\cos \pi k)/(1 + k^2); & k = 0, 1, 2, \cdots \\ \lambda_{2k} = 1 + k^2; & k = 1, 2, 3, \cdots \end{bmatrix}$$

satisfies (ii) but is not monotone. Furthermore, the differences $\lambda_{2k+1} - \lambda_{2k}$ tend to zero. On the other hand, the two subsequences $\{\lambda_{2k+1}\}$ and $\{\lambda_{2k}\}$ both meet the condition $\lambda_{j+1} - \lambda_j \geqq M$ of (iii). Consequently, $\sum_{k=0}^{\infty} f(\lambda_{2k+1})$, $\sum_{k=1}^{\infty} f(\lambda_{2k})$, and hence $\sum_{j=1}^{\infty} f(\lambda_j)$, are convergent series provided $f(x)$ satisfies (iv), (v), and the convergent instance of (vi). Moreover, it is now evident that, under these same conditions on $f(x)$, the series $\sum_{j=1}^{\infty} f(\lambda_j)$ is convergent whenever $\{\lambda_j\}$ is the union of a finite number of sequences each having elements that satisfy the first requirement in (iii).

Example 2. Our modified test affords a particularly convenient means of establishing the convergence of certain series which arise in connection with boundary value problems.

Suppose, for instance, that a formal solution of the boundary value problem describing the temperature distribution in an infinite cylinder has been obtained by separation of variables. To establish the validity of the result it becomes necessary to decide whether or not series such as

$$\sum_{j=1}^{\infty} \lambda_j^n e^{-\alpha\lambda_j^2} \tag{2}$$

converge, where $\alpha > 0$, n is a positive integer, and λ_j is the jth characteristic number of the problem (cf. [3]).

In a problem such as this it will usually be impossible to express λ_j exactly in terms of j; it is known, however, that $\lambda_{j+1} - \lambda_j$ will in general approach some positive constant as j increases (cf. [4]). This constant is π/c in our present example, where c is the radius of the cylinder. It was with these considerations in mind that we formulated hypothesis (iii) of our theorem.

The function

$$f(x) = x^n e^{-\alpha x^2}$$

is continuous and positive when $x > 0$, and is monotone decreasing when $x > \sqrt{(n/2\alpha)}$. Using repeated integration by parts, it may be verified that

$$\lim_{R\to\infty} \int_{\sqrt{n/2\alpha}}^{R} x^n e^{-\alpha x^2} dx$$

is finite; thus (2) converges.

Under hypotheses (i), (ii), and (iii), (with $\lambda_{j+1}-\lambda_j \leqq M$), similar extensions of the results in [1] and [2] may be formulated.

References

1. T. J. I' A. Bromwich, The relation between the convergence of series and of integrals, Proc. London Math. Soc., 6 (1908) 327–338.

2. G. H. Hardy, Theorems connected with Maclaurin's Test for the convergence of series, Proc. London Math. Soc., 9 (1910) 126–144.

3. Ruel V. Churchill, Fourier series and boundary value problems, McGraw-Hill, New York, 1941, pp. 165–167.

4. E. L. Ince, Ordinary differential equations, Dover, New York, 1926, pp. 270–273.

A NECESSARY AND SUFFICIENT CONDITION FOR RIEMANN INTEGRATION

JAMES A. LAVITA, New York University

A well-known theorem developed in the theory of Lebesgue integration is one which relates the existence of the Riemann integral to certain continuity conditions. The theorem (cf. [2]) is: *Let $f(x)$ be a bounded function on a finite interval I. Then $f(x)$ is Riemann integrable if and only if it is continuous a.e.* It is our intention to show that there is an apparently weaker theorem of this type, namely:

THEOREM. *Let $f(x)$ be a bounded function on a finite interval I. Then $f(x)$ is Riemann integrable if and only if it has a left limit a.e.*

We say this theorem is apparently weaker because it is exactly equivalent. That is, we will show that a function $f(x)$ has a left limit a.e. if and only if it is continuous a.e. This problem came to my attention indirectly from Dr. L. Levine.

We begin by proving a lemma about sets of points on the real line, which involves the concept of left limit point (*l.l.p.*). We say x_0 is an *l.l.p.* of the set X if and only if for all $\delta>0$, $X\cap(x_0-\delta, x_0)\neq\emptyset$.

LEMMA. *Let T be a bounded, uncountable set of real numbers. Then there exists an l.l.p. of T.*

Proof. Let X be the set of all real x such that there is at most a countable number of points of T to the right of x. Such points exist, for an upper bound of T is such a point. X has a lower bound, for any lower bound of T is also a lower bound of X. Thus

$$x_0 = \inf_{x\in X} x \tag{1}$$

exists as a finite number. It is easy to see that $x_0\in X$. For, let $x_n\in X$ and $x_n\searrow x_0$.

Then,

$$\{t \mid t \in T, t > x_0\} = \bigcup_{1}^{\infty} \{t \mid t \in T, t > x_n\} \tag{2}$$

but each member on the right is at most countable so that the left member is at most countable. Thus $x_0 \in X$.

Now this implies that there is an uncountable number of points of T which are less than x_0.

We claim that x_0 is the required *l.l.p.* For, if it were not, there would exist a $\delta > 0$ such that

$$T \cap (x_0 - \delta, x_0) = \varnothing. \tag{3}$$

But then $y_0 = x_0 - \delta \in X$, contradicting the definition of x_0. Thus x_0 is an *l.l.p.*

Remark. One should note that the argument hinges on the cardinality of the sets considered. In other words, if we replace uncountable by countable and countable by finite, the argument fails since we wouldn't necessarily have $x_0 \in X$.

We now introduce some definitions and notation.

Notation. On the real line we define a sphere about x_0.

$$S(x_0; \delta) = \{x \mid x \in (x_0 - \delta, x_0 + \delta)\}. \tag{4}$$

Now if $f(x)$ is a real valued function, we define

$$\liminf_{x = x_0} f(x) = \sup_{\delta > 0} \inf \{f(x) \mid x \in S(x_0; \delta)\}, \tag{5}$$

and

$$\limsup_{x = x_0} f(x) = \inf_{\delta > 0} \sup \{f(x) \mid x \in S(x_0; \delta)\}. \tag{6}$$

Now we define $J(x)$ given $f(x)$ defined on the whole real line.

$$J(x) = \min \left[1, \left| \liminf_{y = x} f(y) - \limsup_{y = x} f(y) \right| \right] \tag{7}$$

A look at $J(x)$ shows that this function, in some sense, classifies the "jump" of $f(x)$ at any point of the set S. Thus, the next thing we will do is classify the discontinuities of $f(x)$ on interval I by breaking I into other sets according to the "jump" at a point. Of course, this holds for any set $S \subset I$, also.

We define sets J_n

$$J_n = \left\{ x \mid x \in I, J(x) \geqq \frac{1}{n} \right\}. \tag{8}$$

We now need and prove several observations contained in the following lemmas.

LEMMA I. *Let J be the set of points where $J(x)>0$. If J is uncountable, then there exists an n such that J_n is uncountable.*

LEMMA II. *If $f(x)$ is a function and if x_0 is an l.l.p. of some J_n, then the left limit of $f(x)$ at $x=x_0$ does not exist.*

Proof. Obvious, with the remark that arbitrarily close to any point of J_n, and so by hypothesis to x_0, there are two values of $f(x)$ nearly $1/n$ apart.

Now we have developed the machinery necessary to prove our main lemma.

LEMMA. *Let $f(x)$ be a function defined on $-\infty<x<+\infty$ and let $f(x)$ have a left limit at every point of some closed set S. Then $f(x)$ has at most a countable number of discontinuities in S.*

Proof. With no loss in generality, we may assume that S is bounded, for it is the countable union of bounded closed sets $S\cap[n, n+1]$.

Now we proceed by contradiction. Assume there are an uncountable number of discontinuities in S. It follows from the previous lemmas that at some point of S the function has no left limit, for the set of discontinuities is precisely the set of points where $J(x)>0$.

We can now prove our main theorem.

THEOREM I. *Let $f(x)$ be a function defined on the real line. Then $f(x)$ has a left limit a.e. if and only if $f(x)$ is continuous a.e.*

Proof. If $f(x)$ is continuous a.e., it obviously has a left limit a.e.

Now assume that $f(x)$ has a left limit a.e. Let D be the set of points where $f(x)$ has no left limit, and E_n be a decreasing sequence of open sets, i.e., $E_n \supset E_{n+1}$, with measure $m(E_n)<\epsilon_n$ where $\epsilon_n \searrow 0$ and such that $E_n \supset D$.

Now for each E_n, the complement E_n' is a closed set on which $f(x)$ has a left limit everywhere. By the previous lemma $f(x)$ has at most a countable number of discontinuities in E_n'. Thus $\bigcup_1^\infty E_n'$ contains at most a countable number of discontinuities of $f(x)$. But,

$$L = \left(\bigcup_1^\infty E_n' \right)' = \bigcap_1^\infty E_n \supset D, \tag{9}$$

and $L=\bigcap_1^\infty E_n$ has measure zero. The set F of discontinuities in $\bigcup_1^\infty E_n'$ is countable and so has measure zero. Therefore, the measure of the total set of discontinuities of $f(x)$ is also zero, and our theorem is proved.

The step from here to the theorem asserted at the beginning of this paper is now obvious.

Final remark. If in the main lemma we let S be the whole real line we obtain a further result. If $f(x)$ has a left limit at every point on $-\infty<x<+\infty$ then $f(x)$ has only a countable number of discontinuities. This is a more powerful result of the type one finds for example in [1], where similar results are obtained only for functions of bounded variation or functions possessing both a left and a right limit at every point.

References

1. F. Hausdorff, Set theory (Trans. by J. R. Aumann), Chelsea, N. Y., 1957, 284.
2. I. P. Natanson, Theory of function of a real variable (Trans. by Leo F. Boron), Ungar, N. Y., 1955, 132.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

GOALS FOR SCHOOL MATHEMATICS

Summary of the Report of the Cambridge Conference on School Mathematics

A conference in Cambridge, Massachusetts, sponsored by the National Science Foundation and administered by Educational Services Inc., was held June 18 to July 12, 1963, to discuss the future of mathematics curricula. The main purpose was to reconsider the structure of mathematics education, and to sketch a rough outline of a possible new framework for the primary and secondary school. Some twenty-five mathematicians and users of mathematics, from university or industry, attended the conference. The fields represented included algebra, geometry, topology, analysis, statistics, applied mathematics, physics, and chemistry.

It was agreed from the outset that, in setting goals for mathematics curricula, the conference would have to defer consideration of the serious and closely related problem of teacher training until its first task was completed. The conference also took account of the possibility that there may be intrinsic limitations on the ability of young children to handle mathematical ideas; it felt, however, that the boundaries of these limitations, if they exist, are not well defined, and there is as yet little evidence concerning the degree to which they can be changed by the teaching process. Recognizing then that its work was necessarily of a tentative nature, the conference turned to its main objective, the curriculum from K through 12.

The conference found itself essentially in complete agreement on the mathematical aims of the elementary school.

Through the introduction of the number line, the child would be started immediately on the whole real number system, including negatives. To be sure, at first he would have formal names only for integers and the simplest rational numbers, but all of his work would keep him aware of the existence of other numbers, and the fact that they too have sums, products, etc. By this wedding of arithmetic and geometry at the pre-mathematical level, the intuition of the

child would be developed and exploited, and the significance of the arithmetical operations enriched. Moreover, the child provided with these complementary viewpoints, would have a very good chance to understand the essential nature of mathematics and its relationship to the "real" world.

The order properties of the real number system would be studied from the beginning, and would be used in inequalities, approximation, and order of magnitude estimates.

The use of Cartesian coordinates ("crossed" number lines) would begin almost as soon as the number line itself. Moreover, we agree with Freudenthal and other pioneers, that an early development of the child's spatial intuition is essential. Study of the standard shapes in two and three dimensions would continue concurrently, and would include discussion of their symmetries.

The notions of function and set are to be used throughout; of course, set theory and formal logic should not be emphasized as such, but the child should be able to build his early mathematical experience into his habitual language. Informal algebra should be taken up along with the arithmetic operations.

The Conference agreed that reasonable proficiency in arithmetic computation and algebraic manipulation is essential to the student of mathematics. But this is not an argument in favor of a curriculum devoted primarily to computation with contrived numbers through the whole of grammar school. Long pages of addition and multiplication problems add nothing to a student's understanding of the processes involved; nor do they teach him *when* to add or multiply. At best, they improve the computational speed of a student who understands how to do the algorithms (an objective that by itself had little appeal to the members of the Conference); at worst, they dissipate or destroy the interest that a good student has in the subject. Entirely adequate practice in computation can be built into problems that, on their own merits, genuinely attract the student's interest.

Because of both its intuitive appeal and its basic importance, there should be an introduction to the elementary ideas of probability and statistical judgment, accompanied by concrete experimentation with random processes.

The concern for motivation, applications, and the interplay between mathematics and the physical world, is a constant theme of the conference report. This is constrained by the limited experience with science in the elementary school. Geometry itself, however, offers a rich area within which the students can explore the relation between physical objects and their idealized mathematical abstractions. As the student's experience deepens, it will be possible to introduce more sophisticated models.

Having studied arithmetic and geometry, mostly informally, in the elementary school, the student will be prepared for a sound treatment of geometry and the algebra of polynomials, beginning in the seventh grade. The mathematics curriculum for the secondary school can therefore go much farther than it commonly does at present. The program of a student who elected mathematics each year will, at the end of the twelfth year, have contained a closely-

knit presentation of calculus, linear algebra, and probability, involving a brief introduction to other mathematical topics.

The conference did not reach any substantial agreement as to the order of presentation or the specific content for this program. Indeed the multitude of sound proposals suggest that there is certain to be no unique optimal solution. Two arrangements of the material proposed for the secondary level were developed in some detail.

The conference also arrived at other recommendations which dealt more with methods of presentation than with specific mathematical content. It was felt, for example, that it was desirable to adopt the "spiral" approach, in which every new topic is introduced under low pressure and is then reconsidered repeatedly, each time with more sophistication, and each time showing more of its interconnections with the rest of the subject. The result should be a sort of guided tour of mathematics. This approach has many important advantages. In the first place, the basic unity of the subject is automatically stressed. Moreover, in the upper grades, this approach implies that the student will be exposed concurrently to a mixture of intuitive "pre-mathematics" and rigorous mathematics. Provided that the distinction is made clear to the student, this will give a much more honest picture of what mathematics is, an organism continuously growing through the interaction of intuition and logical analysis, rather than a static structure walled about by sterile rigor.

A second aspect of the same precept led to the suggestion that topics receive multiple motivation. During the pre-mathematical stage of some topics, it may be wise to give several different informal presentations, each leading up to the desired goal (e.g. the rules for multiplication of negatives), rather than to leave students with the feeling that there is only one correct road. Ideally, this should help to convey to the student the important fact that mathematics is something one *does*, not something that one absorbs passively. One would hope to strengthen the impression that a mathematical idea appeared first as the solution to some problem by some person. The problems thus become a matter of importance equal to or even greater than that of the textual material itself. It was therefore felt that the design of imaginative problem sequences involving combinations of routine techniques and "discovery" procedures was a matter of the greatest importance in curricular development.

There is much that must be done before the ideas in this proposal can be implemented. Some of the suggestions in the report are already being tried in some of the current educational experiments, either piecemeal or as part of some more extensive program. There must be many further experiments, however, to determine just what is possible, and at what age levels. Texts and supplementary materials will have to be written. Unquestionably, the most difficult problem lies in the training of teachers, upon whom the success or failure of curricular reform ultimately rests. Nor is this an isolated problem, for the pressure to advance our mathematical goals is being felt at all levels of the profession, and one facet of the problem cannot be solved in isolation from the others. If the

proposals formulated by the Conference are to become a reality within the foreseeable future, it is necessary that the entire mathematical community devote considerable attention to the training of teachers at all levels.

The steering committee for this study consisted of E. G. Begle, J. S. Bruner, A. M. Gleason, M. Kac, W. T. Martin (Chairman), E. E. Moise, Mina Rees, P. Suppes, S. White, and S. S. Wilks. The Conference was organized and administered by Educational Services Inc., Watertown, Massachusetts, under a grant from the National Science Foundation.

The following participated in the conference: M. Auslander, E. G. Begle, R. C. Buck, G. F. Carrier, J. Cole, R. B. Davis, R. P. Dilworth, B. Friedman, H. L. Frisch, A. M. Gleason, P. J. Hilton, J. L. Hodges, S. Koenig, G. C. Lin, E. L. Lomon, E. E. Moise, F. Mosteller, H. O. Pollak M. Rees, M. M. Schiffer, G. Springer, P. Suppes, A. H. Taub, S. S. Wilks, J. R. Zacharias.

This summary of the full report was prepared by a subcommittee consisting of R. C. Buck, P. J. Hilton and H. O. Pollak.

The full report became available as a 100-page booklet late in November at a price of $1.00 postpaid for single copies from Houghton Mifflin Company, 2 Park Street, Boston 7, Massachusetts. Orders should be addressed to Houghton Mifflin Company.

MANPOWER PROBLEMS IN TEACHING

The Conference on Manpower Problems in the Training of Mathematicians (April, 1963), sponsored by the Conference Board of the Mathematical Sciences and supported by the National Science Foundation, included a panel on education of teachers. The recommendations of the panel and excerpts from the remarks addressed to the Conference by Frank B. Allen, president of NCTM, follow.

Recommendations of Panel 6 (A. S. Galbraith, Julius Hlavaty, H. Vernon Price). To maintain and increase the trend of students towards graduate study, instruction for undergraduate and high school students must be strengthened and improved.

The problem of the improvement of instruction at the undergraduate level is of special urgency because there are in general no agreed criteria for the selection, training, certification and supervision of in-service training of teachers at that level.

We recommend that large and well-staffed departments which nevertheless make use of graduate students and teaching fellows should accept the responsibility for the training and supervision of these assistant teachers. Some of the techniques that might be utilized are:

1. Departmental seminars on teaching problems.
2. In-service and summer institutes in content and methods of presentation.
3. Large group teaching by outstanding teachers with several trainees assigned to follow the lectures. The trainees should then be given teaching responsibilities with small sub-sets of the large—not merely problem-solving, paper-grading and question-answering responsibilities.
4. Departmental encouragement of the use of enrichment materials to inspire and maintain individual study and investigation by capable students.

5. Continuing study of the over-all curriculum of the department—in consultation and cooperation with such bodies as CUPM and with other institutions.

6. Professors, instructors and fellows should seek to present to their students classical as well as evolving applications of mathematics.

We recommend that smaller institutions and high schools should actively be working on the improvement of instruction by utilizing some of the following techniques:

1. Establishing a close relation with a nearby large institution. Periodic seminars of smaller departments with representatives of the large institution should be held.

2. Extension of the visiting lectureship program to provide lectures to staffs and students by outstanding teachers from nearby institutions and from industry and government.

3. Carrying on the activities suggested in the first section of this report.

The accelerating growth of college enrollment indicates the need for a great expansion of teaching staffs. This increase is indicated not only by the growing number of majors in mathematics, but also by the needs of large numbers of students who are not going on into graduate work but who are going on to careers in mathematics at various technical levels and in various applied fields.

On a short range scale this might be met in part by establishing relations with nonacademically employed mathematicians and scientists and using them in a part-time teaching commitment. This might indicate a partial subsidy by the industries concerned or some income tax allowance to them.

Remarks by Frank B. Allen. Since this conference is focused on the need for manpower at the graduate level in mathematics, any consideration of an alleged shortage of qualified mathematics teachers at the secondary level may seem to be peripheral to our major area of concern. This may be true in an immediate sense. In the long range view, I think that it is not true. I venture to suggest that, in the long run, our supply of able graduate students is largely determined by the quality of mathematics instruction provided in college preparatory courses. I am sure that many of you will agree that the enthusiastic well-prepared teacher of high school mathematics is a powerful influence in persuading mathematically gifted students to continue their studies. We must have an adequate supply of such teachers if we are to keep our academic pipelines full of worthy candidates for high degrees.

According to the U. S. Registry maintained jointly by the National Council of Teachers of Mathematics and the National Science Teachers Association, there are about 80,000 persons who teach mathematics 50% or more of their time in grades 7–12. It is estimated that about 8% of these teachers must be replaced each year due to retirements, career changes, and other attritions. Thus, about 6400 teachers are needed for replacement alone. Inference based on facts provided by the U. S. Department of Health, Education, and Welfare

indicate that enrollments in grades 9–12 will increase at an annual rate of about 4% a year between now and 1980. At present, this means that we need about 3200 teachers to provide for increased enrollments. Thus for the next twenty years, requirements for new high school mathematics teachers must be set at about 10,000 per year. Indeed, this figure will rise gradually throughout that period.

Having looked at the demand for new mathematics teachers in grades 9–12, let us now inquire about the supply. The principal supply of new teachers is obtained from newly certified college graduates. According to a recent report on teacher supply and demand published by the National Education Association there were about 7000 newly certified high school mathematics teachers produced in 1962. Thus we already have a shortage of about 3000 teachers in high school mathematics even if we assumed that all newly certified personnel actually entered the teaching profession. This is not the case. According to the same study, only about three-quarters of these newly certified persons entered the teaching profession and there is no way of knowing how many of them were assigned outside the field of mathematics. Moreover, we know that many of these certified teachers would not be qualified according to the higher standards of qualifications which are currently being applied by high schools that are trying to improve their programs in mathematics. More specifically, we know that many certified teachers would not be qualified according to the criteria contained in the recommendations of the Panel on Teacher Training of the Committee on the Undergraduate Program of the Mathematical Association of America. For example, the panel recommends that prospective teachers of elementary algebra and geometry should have a very strong minor in mathematics and that prospective teachers of high school mathematics beyond the elements of algebra and geometry, should complete a very strong major in mathematics, and a minor in some field in which a substantial amount of mathematics is used. These two levels of competence are referred to in the Panel's report as levels two and three. In their recommendations for level four for teachers of the elements of calculus, linear algebra, probability and so forth, the Panel's criteria call for a master's degree in mathematics with at least two-thirds of the courses being in mathematics. Anyone who has interviewed newly certified teachers is well aware of the fact that many of them do not qualify on the basis of the criteria established by the Panel on Teacher Training.

There is mounting evidence to indicate that the high schools are just beginning to do a better job. Thousands of high school seniors are taking courses in calculus. In 1961, 9% of MIT freshmen entered with advanced credit for the first semester of their calculus courses, and 11% more got credit for a full year. Twenty per cent of Cal Tech freshmen were able to skip the first half of the calculus course, which is a far more advanced course than it was six years ago. All this is a result of advanced placement courses in high school. It is well known, moreover, that the introduction of such courses benefits not only the elite groups but tends to raise the level of instruction for all students. Together with the

rapid introduction of the improved programs referred to above, this will enable the high schools to do a better job for students of all ability levels. It will also enable the colleges to discard their remedial courses and to get on with the business of teaching more advanced mathematics.

ABSTRACT OF REPORT ON MATHEMATICS IN THE CATHOLIC HIGH SCHOOL

BROTHER EDWARD DANIEL, C.F.X., St. Xavier High School, Louisville

The report on "Mathematics in the Catholic High School" was prepared by an Advisory Committee on Mathematics appointed by the Secondary School Department of the National Catholic Educational Association. The fourteen-member committee, which included representation from all regions of the United States, was appointed in the fall of 1961 and submitted its report in June 1963.

The introductory section of the report surveys recent developments in secondary school mathematics in the United States. The committee found that mathematics programs in Catholic high schools reflect the uncertainty, the tension between the conflicting claims of "modern" and "traditional" mathematics that characterizes generally the state of mathematics in the schools of the nation. Catholic school administrators and teachers expressed a particular need for leadership and guidance in revising existing mathematics curricula. The major portion of the report consists of recommendations for this purpose.

The committee goes on record "as strongly supporting the movement for a thorough modernization of the secondary school mathematics program." This revision should provide for: the earliest possible introduction of mathematically significant facts and concepts; enrichment through contact with a variety of mathematical ideas; emphasis on the axiomatic approach to mathematics in each area studied; training in precision of statement and mathematical rigor appropriate to the maturity of the student; increasingly higher standards of achievement. The report suggests that helpful assistance in achieving these objectives can be found in the *Report of the Commission on Mathematics* of the College Entrance Examination Board.

The report declares that a critical shortage of well-trained mathematics teachers exists in Catholic high schools and urges that "all levels of the Catholic educational system . . . assign the highest priority to the preparation and recruitment of well-trained mathematics teachers . . . and make realistic provisions for upgrading the mathematical background of teachers now in service." The *Recommendations of the Mathematical Association of America for the Training of Teachers of Mathematics* should be considered a basic minimum of preparation.

The report concludes with some recommended procedures for schools to follow in introducing new programs. These include the development of an over-all plan, careful advanced preparation of teachers, and extensive use of consultants during the preparation and introduction of the new materials.

MATHEMATICS FOR ELEMENTARY TEACHERS

RALPH CROUCH AND GEORGE BALDWIN, New Mexico State University Project

A course content project in mathematics for elementary teachers, with support from the National Science Foundation, began June 1, 1962. The summer months of 1962 were spent in writing a preliminary version of the first two-thirds of the text. The fall of 1962 was spent in finishing the first version, and the revision started in the spring of 1963 and continued through September 1963. It is a text for future elementary teachers designed to be used in a two three-semester hour sequence.

The first text was tried out during the school year 1962–63 in the following institutions: New Mexico State University; Michigan State University Oakland; Michigan State University; University of Southwestern Louisiana; Pennsylvania State University. A meeting of the teachers of the materials was held under the auspices of CUPM in February 1963. Comments of the teachers were included in the revised edition. Two thousand copies of the first half of the revised edition were published in September, 1963. To this date, November 1, 800 copies have been sold at cost ($2.50 per copy). The second half of the text will be published and ready for use by February 1, 1964. It will be for sale at cost ($2.00 per copy).

NATIONAL SCIENCE SEMINARS

A series of seminars for high school students and teachers called National Science Seminars were sponsored by the New Mexico Academy of Science in conjunction with the National Science Fair in Albuquerque, New Mexico, in May, 1963. Only the most competent students and teachers were selected for the seminars by local school systems. Their participation in the seminars was made possible through the cooperation of the Junior and Senior Academies of Science. A number of the seminars were on mathematics. Some of the mathematicians participating in the seminars, with their seminar topics, were: Paul B. Bailey, Applied Mathematics Division, Sandia Corporation: mathematical analysis. E. F. Beckenbach, Department of Mathematics, UCLA: matrix algebra and algebraic and geometric inequalities. Julius R. Blum, Department of Mathematics, University of New Mexico: mathematical models and elementary probability theory. Edmund D. Cashwell, Theoretical Division, Los Alamos Scientific Laboratory of the University of California: application of Monte Carlo techniques to problems in physics and unique factorization in mathematics. H. S. M. Coxeter, Department of Mathematics, University of Toronto: geometrical transformations and map-coloring. John H. Giese, Chief of the Computing Laboratory, U. S. Army Ballistic Research Laboratories, Aberdeen Proving Ground: numerical solution of partial differential equations. H. J. Greenberg, Assistant Director, Mathematical Sciences Department, IBM Research, Yorktown Heights, New York: pattern recognition. Preston C. Hammer, De-

partment of Mathematics, University of California, San Diego: role and nature of mathematics, convex figures, and topology. Franz E. Hohn, Department of Mathematics, University of Illinois: modern application of Boolean algebra, and career opportunities in mathematics. Donald R. Morrison, Supervisor, Computer and Numerical Analysis Division, Sandia Corporation: iterative processes. Joseph A. Shatz, Staff Member, Computer Numerical Analysis Division, Sandia Corporation: computability. Joachim F. Weyl, Chief Scientist of the Office of Naval Research, Washington: modern applications of mathematics. G. Milton Wing, Applied Mathematics Division, Sandia Corporation: difference equations and some of their applications. Leo Zippin, Professor of Mathematics, Queens College: topology.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine
COLLABORATING EDITOR: C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1661. *Proposed by K. S. Williams, University of Toronto*

A student thought that the formula for differentiating a product was

$$d\{u(x)v(x)\}/dx = d\{u(x)\}/dx\cdot d\{v(x)\}/dx.$$

He used this formula with $u(x)=(2-x)^{-2}$ and $v(x)=x^2$ and obtained the correct result! Find a general class of functions $u(x)$ and $v(x)$ satisfying the above formula.

E 1662. *Proposed by C. A. Nicol, University of South Carolina*

Prove that if p is the smallest prime divisor of a perfect number n, then n has at least p distinct prime divisors.

E 1663. *Proposed by H. D. Ruderman, Hunter College High School*

What is the maximum number of regions into which n spheres can partition space?

E 1664. *Proposed by Barry Wolk, Cornell University*

It is easily shown that $n+2$ hyperplanes in general position in Euclidean n-space determine $n+2$ (closed) simplices. Show that each simplex is contained in the union of the others.

E 1665. *Proposed by Azriel Rosenfeld, Yeshiva University*

Construct a commutative ring in which the square of every element is zero but not every product is zero. Prove that such a ring must have at least eight elements.

E 1666. *Proposed by Arthur Engel, Stuttgart, Germany*

Let $P_1P_2 \cdots P_n$ be any planar polygon and let G_i, $i=1, \cdots, n$, be the centroid of triangle $P_iP_{i+1}P_{i+2}$, where $P_{n+i}=P_i$. Is the polygon determined if the G_i are given?

E 1667. *Proposed by G. J. Minty, University of Michigan*

Suppose that the surface of a sphere is divided into triangular "countries," where *triangular* means that each country touches exactly three others. A vertex of the graph formed by the boundary lines of the countries is called *even* or *odd* according as an even or an odd number of boundary lines run into it. Is there such a triangulation having exactly two odd vertices in which these vertices are adjacent?

E 1668. *Proposed by D. G Wilson, IBM Corporation, Bethesda, Maryland*

Farmer Jones has a cart with square wheels. However, it suits his needs since with it he is able to travel the washboard road without any bumping. Assuming no slipping of the wheels, describe the washboard road.

E 1669. *Proposed by Ralph Greenberg, University of Pennsylvania*

Let $\phi_k(n)=\phi\{\phi_{k-1}(n)\}$ and $\phi_0(n)=\phi(n)=$ Euler's ϕ-function. Show that for any k, $\phi_k(n)>1$ for all sufficiently large n.

E 1670. *Proposed by Tai-ichi Kitamura, Ibaraki University, Japan*

If $\mathfrak{D}$ is the differential operator xd/dx, prove that $e^{\mathfrak{D}}P(x)=P(ex)$, where $P(x)$ is any polynomial in x.

SOLUTIONS

The Slopes of the Sides of an Equilateral Triangle

E 1581 [1963, 437]. *Proposed by Erwin Just, Bronx Community College*

If m_1, m_2, m_3 are finite nonzero slopes of the sides of an equilateral triangle, prove that:

(a)
$$m_1m_2 + m_2m_3 + m_3m_1 = -3,$$

(b)
$$\sum_{i=1}^{3} m_i \sum_{i=1}^{3} 1/m_i = 9.$$

I. *Solutions by M. T. L. Bizley, London, England.* (A) If m represents any one of m_1, m_2, m_3, then the angles between the positive direction of the x-axis

and the three sides are $\tan^{-1} m$, $120°+\tan^{-1} m$, and $240°+\tan^{-1} m$. Hence if we write, for brevity, $p=m_1m_2m_3$, we have

$$p = m[(m - \sqrt{3})/(1 + m\sqrt{3})][(m + \sqrt{3})/(1 - m\sqrt{3})] = m(m^2 - 3)/(1 - 3m^2).$$

Therefore m_1, m_2, m_3 are the roots of the cubic

$$m^3 + 3pm^2 - 3m - p = 0.$$

Hence immediately

$$m_1m_2 + m_2m_3 + m_3m_1 = -3,$$

which is part (a) of the problem, and

$$\sum_{i=1}^{3} m_i \sum_{i=1}^{3} 1/m_i = (-3p)(-3/p) = 9,$$

which is part (b) of the problem.

(B) Since the tangents of the angles between successive pairs of sides are all equal (in a given cyclic order),

$$(m_2 - m_3)/(1 + m_2m_3) = (m_3 - m_1)/(1 + m_3m_1) = (m_1 - m_2)/(1 + m_1m_2) = \lambda,$$

say, where $\lambda=\sqrt{3}$ or $-\sqrt{3}$. Therefore $m_2-m_3=\lambda(1+m_2m_3)$, with two similar equations. Adding these equations yields (a) at once. Multiplying the equations respectively by m_1, m_2, m_3 and adding yields

$$0 = m_1 + m_2 + m_3 - 3m_1m_2m_3,$$

from which (b) follows at once, using (a).

II. *Solution by D. C. Kay, Michigan State University.* Let m_i $(i=0, 1, \cdots, n-1)$ be the finite nonzero slopes of the sides P_iP_{i+1} $(i=0, 1, \cdots, n-1$, reducing modulo $n)$ of any regular n-gon taken in counterclockwise order. Using the formula for the angle from P_iP_{i+1} to $P_{i-1}P_i$, we have, for $i=1, 2, \cdots, n$,

$$\tan(P_{i-1}P_iP_{i+1}) = (m_{i-1} - m_i)/(1 + m_{i-1}m_i).$$

But $\measuredangle P_{i-1}P_iP_{i+1}=\alpha$, say, for each i, and we obtain the two sets of equations

(1) $$m_{i-1}m_i = (m_{i-1} - m_i)\cot\alpha - 1, \qquad i = 1, 2, \cdots, n,$$

(2) $$(m_{i-1}m_i)^{-1} = (m_i^{-1} - m_{i-1}^{-1})\cot\alpha - 1, \qquad i = 1, 2, \cdots, n.$$

Adding the equations in (1) and in (2) we obtain

(3) $$\sum_{i=1}^{n} m_{i-1}m_i = -n,$$

(4) $$\sum_{i=1}^{n} (m_{i-1}m_i)^{-1} = -n,$$

where, in the special case $n=3$, we can use (3) to express (4) in the form

$$(4') \qquad \sum_{i=0}^{2} m_i \sum_{i=0}^{2} m_i^{-1} = 9.$$

Also solved by J. C. Abad, R. J. Addison, A. N. Aheart, Marc Aronson, K. F. Bailie, Merrill Barnebey, E. R. Barnes, Suzanne Bedford, A. Behr, E. D. Bender, T. E. Black, Jr., D. A. Blaeuer, W. R. Boland, D. A. Breault, M. H. Brodsky, R. E. Brown, F. P. Callahan, Jr., Leonard Carlitz, R. L. Carmichael, D. I. A. Cohen, Martin Cohen, R. J. Cormier, Frank Dapkus, H. J. de St. Germain, Gus DiAntonio, J. F. Dillon, J. R. Fall, Roy Feinman, Stephen Fisk, C. M. Frye, José Gallego-Diaz, Anton Glaser, Michael Goldberg, Ralph Greenberg, S. H. Greene, Cornelius Groenewoud, J. D. Haggard, F. C. Hall, W. J. Halm, Jerry Harpster and Terry Hinrich (jointly), Ned Harrell, Mark Hayamizy, S. Heller, W. R. Hutcherson, Howard Jacobowitz, R. A. Jacobson, J. E. Jean, Jr., Diane M. Johnson, Roman Kaluzniacki, Geoffrey Kandall, J. H. Kaplan, Joel Kugelmass, George Kurata, G. J. Kurowski, Harry Langman, Stark Lash, Lawrence Lessner, L. P. Lewis, D. L. Linfield and Esther A. Linfield (jointly), Nicholas Macri, Coline M. Makepeace, C. F. Marion, D. C. B. Marsh, Stephen Montague, M. G. Murdeshwar, J. P. Muskat, Jack Nebb, P. R. Nolan, F. D. Parker, Fritz Parmenter, Robert Patenaude, C. B. A. Peck, R. R. Perez, D. J. Peterson, Stanton Philipp, J. P. Phillips, Rodger Poore, B. E. Rhoades, Henry Ricardo, L. A. Ringenberg, P. A. Rognlie, J. S. Scandale, Jr., Perry Scheinok, E. M. Scheuer, Lawrence Schulman, R. Sibson, Jr., D. L. Silverman, C. S. Smith, Eric Sturley, R. L. Syverson, Tom Tarzian, Elaine Tatham, P. D. Thomas, Rory Thompson, W. Toalson, Simon Vatriquant, Gary Venter, William Wernick, H. R. Wier, Ron Wilder, Hazel S. Wilson, K. L. Yocom, and the proposer.

Several solvers pointed out that from (a) one immediately obtains

$$(c) \qquad 1/m_2m_3 + 1/m_3m_1 + 1/m_1m_2 = -3,$$

since $1/m_1$, $1/m_2$, $1/m_3$ are also slopes of the sides of an equilateral triangle. The product of relations (a) and (c) then yields relation (b). Carlitz showed that relations (a) and (c) constitute necessary and sufficient conditions for a triangle with sides of slopes m_1, m_2, m_3 to be equilateral. Some solvers employed the trigonometric identity

$$\tan\theta + \tan(\theta + \pi/3) + \tan(\theta + 2\pi/3) = -3\tan\theta\tan(\theta + \pi/3)\tan(\theta + 2\pi/3).$$

Integration Over the Unit Sphere

E 1582 [1963, 437]. *Proposed by Harley Flanders, Purdue University*

Prove that

$$\iint x^6 d\sigma = 15\iint x^2y^2z^2 d\sigma,$$

where the integrations are taken over the unit sphere centered at the origin with respect to the area element $d\sigma$.

Solution by L. A. Ringenberg, Eastern Illinois University. Using spherical coordinates and Wallis's formula,

$$\iint x^6 d\sigma = 8\int_0^{\pi/2}\int_0^{\pi/2} \sin^6\phi\cos^6\theta\sin\phi\, d\theta d\phi = 4\pi/7,$$

and

$$15\iint x^2y^2z^2d\sigma = 120\int_0^{\pi/2}\int_0^{\pi/2}\sin^5\phi\cos^2\phi\cos^2\theta\sin^2\theta d\theta d\phi = 4\pi/7.$$

Also solved by B. W. Banks, Merrill Barnebey, George Bergman, J. P. Brazy, M. H. Brodsky, R. L. Carmichael, S. H. Greene, Cornelius Groenewoud, Emil Grosswald, Kit Hanes, Geoffrey Kandall, M. S. Klamkin, Harry Langman, T. J. Lee, Coline M. Makepeace, D. C. B. Marsh, C. B. A. Peck, Perry Scheinok, F. C. Smith, Rory Thompson, J. A. Tierney, Simon Vatriquant, and the proposer.

Bergman established the more general result: If $d\sigma$ denotes the surface element of the unit sphere in d-dimensional space, and $k_1, \cdots, k_d$ are nonnegative integers with $k=k_1+\cdots+k_d$, then

$$\int\cdots\int x_1^{2k}d\sigma = \frac{k_1!\cdots k_d!}{k!}\frac{(2k)!}{(2k_1)!\cdots(2k_d)!}\int\cdots\int x_1^{2k_1}\cdots x_d^{2k_d}d\sigma.$$

The given problem is the case where $d=3$, $k_1=k_2=k_3=1$.

Klamkin pointed out that, by use of the Dirichlet integral, the general integral $\iint x^ay^bz^cd\sigma$ can be evaluated, and in particular, if $a+b+c=\alpha+\beta+\gamma$,

$$\frac{\iint x^{2a}y^{2b}z^{2c}d\sigma}{\iint x^{2\alpha}y^{2\beta}z^{2\gamma}d\sigma} = \frac{(2a)!(2b)!(2c)!}{a!b!c!}\frac{\alpha!\beta!\gamma!}{(2\alpha)!(2\beta)!(2\gamma)!}.$$

The given problem is the case where $a=3$, $b=c=0$, $\alpha=\beta=\gamma=1$.

Application of a Formula of Ramanujan

E 1583 [1963, 437]. *Proposed by C. D. Zimmerman, Southern Missionary College*

Is $\sum_{x=0}^{m-1} m^x/x! \geqq e^{m-1}$ for all positive integral m?

I. *Solution by Stanton Philipp, Long Beach, Calif.* Yes. By Taylor's theorem

$$\sum_{x=0}^{m-1} m^x/x! = e^m - [e^m/(m-1)!]\int_0^m e^{-t}t^{m-1}dt.$$

Hence it will suffice to establish that

$$\int_0^m e^{-t}t^{m-1}dt \leqq (m-1)!(1-1/e) \tag{1}$$

for all positive integral m. This will be done by induction.

(a) Relation (1) holds, with equality, if $m=1$.

(b) Suppose relation (1) holds for $m=k$. Integrating the left member by parts and multiplying both members of the last mentioned inequality by m

shows that

$$k^k e^{-k} + \int_0^k e^{-t} t^k dt \leqq k!(1 - 1/e).$$

But

$$\int_k^{k+1} e^{-t} t^k dt \leqq \max_{k \leqq t \leqq k+1} e^{-t} t^k = e^{-k} k^k.$$

Therefore

$$\int_0^{k+1} e^{-t} t^k dt = \int_k^{k+1} e^{-t} t^k dt + \int_0^k e^{-t} t^k dt \leqq k^k e^{-k} + \int_0^k e^{-t} t^k dt \leqq k!(1 - 1/e).$$

The proof is now complete.

II. *Solution by J. H. Foster, University of Notre Dame.* A formula of Ramanujan (G. Szegö, *Jour. London Math. Soc.*, 3 (1928) 225–32) states that

$$e^n/2 = \sum_{k=0}^{n-1} n^k/k! + (n^n/n!)\theta_n, \qquad 1/3 < \theta_n < 1/2.$$

We have

$$\sum_{x=0}^{m-1} m^x/x! - e^{m-1} = e^m/2 - e^{m-1} - (m^m/m!)\theta_m$$

$$\geqq e^m[1/2 - 1/e - (m^m/2m!)e^{-m}] = e^m f(m), \text{ say.}$$

Since $(m^m/m!)e^{-m}$ is a decreasing sequence $(m=1, 2, \cdots)$, $f(m)$ is increasing, and since $f(m) \geqq 0$ for $m=3$, we get

$$\sum_{x=0}^{m-1} m^x/x! \geqq e^{m-1}$$

for $m=3, 4, \cdots$. This gives us the desired result, since the inequality is obviously true for $m=1$ and $m=2$.

Also solved by K. R. Bailie, Martin Cohen, Joseph Gayda, Emil Grosswald, J. E. Hafstrom, and A. E. Livingston.

Rectangular Rugs in a Square Room

E 1584 [1963, 438]. *Proposed by D. J. Newman, Yeshiva University*

What are all the sizes of rugs which will fit on a given square floor? In other words, what is the condition on a and b which insures that a rectangle with sides a and b can be contained wholly within the unit square?

Solution by D. L. Silverman, Beverly Hills, Calif. The rug will obviously fit if $\max(a, b) \leqq 1$. If, however, $\max(a, b) > 1$ and the rug fits, it will still fit if moved to a position symmetric with a diagonal of the square, that is, when $a+b \leqq \sqrt{2}$. These alternative conditions can be summed up in the single inequality

$$\min[\max(a, b), (a + b)/\sqrt{2}] \leqq 1,$$

or, using the identities

$$\begin{matrix}\max \\ \min\end{matrix}(x, y) = (x + y \pm |x - y|)/2,$$

$$(1 + \sqrt{2})(a + b) + |a - b| - |(1 - \sqrt{2})(a + b) + |a - b|| \leqq 4.$$

Also solved by J. C. Abad, Merrill Barnebey, M. J. Behr, Walter Bluger, Judy Caspino and Charles Conlin (jointly), D. I. A. Cohen, R. J. Cormier, Frank Dapkus, D. L. Deever, J. F. Dillon, Roy Feinman, Stephen Fisk, Michael Goldberg, Ralph Greenberg, S. H. Greene, R. A. Jacobson, J. E. Jean, Jr., Joel Kugelmass, Robert Maas, C. F. Marion, D. C. B. Marsh, C. B. A. Peck, Stanton Philipp, R. D. Spitz, Eric Sturley, Rory Thompson, K. L. Yocom, and the proposer.

Some solvers considered the problem where cutting the rug and then piecing it together is allowed.

Nonattacking Knights on a Chessboard

E 1585 [1963, 438]. *Proposed by Irving Newman, Monroe, North Carolina*

What is the maximum number of knights which can be placed on a chessboard in such a way that no knight attacks any other?

I. *Solution by Robert Patenaude, Humboldt State College, Calif.* Considering a smaller board of 2×4 squares, it is seen that a knight on any square attacks exactly one other square. Hence no more than 4 knights can be placed on this board, and it follows that no more than 32 can be placed on an 8×8 board. This number is shown to be possible by covering the squares of one color with 32 knights.

II. *Solution by Ralph Greenberg, University of Pennsylvania.* Since a knight moves from a black square to a white square, or vice versa, we may obviously fill all the black squares. Hence the maximum number is not less than 32. Since a knight can tour the chessboard touching every square exactly once, at most 32 knights can exist peacefully on the board, for along such a path only alternate squares may be occupied.

III. *Solution by the proposer.* The maximum number is 32. This can be obtained by placing the knights on either the 32 black or the 32 white squares. Suppose the black squares are chosen. Then each white square is threatened by at least two knights. Therefore, the transfer of a knight from a black to a white square would cause a removal of at least one other knight and would reduce the total number of knights.

Also solved by J. C. Abad, H. L. Abbott and M. G. Murdeshwar (jointly), P. H. Arregotti, Charles Bacon, E. D. Bender, Walter Bluger, D. A. Breault, W. E. Buker, John Carreia, Judy Caspino and Charles Conlin (jointly), Larry Castelli, Allan Chuck, D. I. A. Cohen, Martin Cohen, R. J. Cormier, Frank Dapkus, D. L. Deever, J. F. Dillon, J. R. Fall, J. A. Faucher, T. M. Feder, Sanford Fleezer, E. T. Frankel, Michael Goldberg, Bill Heidrick, K. D. Herr, J. A. H. Hunter, Erwin Just, Roman Kaluzniacki, M. S. Klamkin, Harry Langman, Douglas Lind, Robert Maas, C. F. Marion, D. C. B. Marsh, Stephan Montague, D. A. Moran, P. N. Nagara, G. O. Perez, Stanton Philipp, W. R. Scott, D. L. Silverman, Eric Sturley, Rory Thompson, P. O. Wood, Jr., and the proposer.

Many of these solutions showed that 32 nonattacking knights can be placed on a chessboard, but failed to show that this is the maximum possible number. The problem, with a solution but no proof, can be found in W. W. R. Ball, *Mathematical Recreations and Essays* (1926), p. 171, and in Henry Dudeney, *Amusements in Mathematics*, p. 96. It can be shown that for an $n\times n$ chessboard, $n>2$, the maximum number of nonattacking knights is $[2n^2+1-(-1)^n]/4$; for a 2×2 board the number is 4. For other generalizations see Francis Scheid, Some packing problems, this MONTHLY, Mar. 1960, pp. 231–5. The answer to the corresponding problem for queens is 8, for rooks is 8, and for bishops is 14. Perez showed that if the problem is varied by stating that no knight is to attack any knight of the other color, then 36 knights, 18 of each color, can exist peacefully on the board.

Six Circles Equal to the Circumcircle of a Triangle

E 1586 [1963, 438]. *Proposed by J. F. Darling, Woodstown, New Jersey*

In a triangle with circumradius R reflect the circumcenter in the sides. From these points and the vertices describe circles of radius R. Letter the outer intersections of consecutive pairs of these six circles A, B, C, D, E, F, and the corresponding inner intersections A', B', C', D', E', F'. Prove that triangles ACE and DFB, and likewise triangles $A'C'E'$ and $D'F'B'$, are equilateral, have equal and parallel corresponding sides, and are in perspective from the nine-point center of the given triangle.

Solution by the proposer. Choose the circumcircle as the circle $|z|=1$ of the complex plane and let t_1, t_2, t_3 denote the turns representing the vertices, taken in counterclockwise order, of the given triangle. Then the reflection of O in the side t_1t_2 is t_1+t_2, etc. Each unknown vertex is then the third vertex of an equilateral triangle of which we know the other two vertices. Using the criterion that $a+\omega b+\omega^2c=0$, where $\omega=e^{2\pi i/3}$, for an equilateral triangle with counterclockwise vertices having affixes a, b, c, we find for the counterclockwise vertices of the two large triangles

$$t_1-\omega t_2,\qquad t_2-\omega t_3,\qquad t_3-\omega t_1,$$

and

$$t_1-\omega^2 t_3,\qquad t_2-\omega^2 t_1,\qquad t_3-\omega^2 t_2.$$

Now

$$t_1-\omega t_2+\omega(t_2-\omega t_3)+\omega^2(t_3-\omega t_1)=0$$

and

$$t_1 - \omega^2 t_3 + \omega(t_2 - \omega^2 t_1) + \omega^2(t_3 - \omega^2 t_2) = 0,$$

and the two triangles are equilateral. For the midpoints of the joins of corresponding opposite vertices we have

$$(t_1 - \omega t_2 + t_3 - \omega^2 t_2)/2 = (t_1 + t_2 + t_3)/2,$$
$$(t_2 - \omega t_3 + t_1 - \omega^2 t_3)/2 = (t_1 + t_2 + t_3)/2,$$
$$(t_3 - \omega t_1 + t_2 - \omega^2 t_1)/2 = (t_1 + t_2 + t_3)/2,$$

which proves that the triangles have equal and parallel corresponding sides, and are in perspective from the nine-point center of the given triangle. One may give a similar treatment for the two smaller triangles.

A Restricted Semigroup

E 1587 [1963, 438]. *Proposed by T. C. Brown, Washington University*

Let S be the semigroup of words in two generators a, b subject to the relation $w^3 = w$ for all w in S. Show that

$$(aba^2b)^2 = (ab^2ab)^2 = (ab^2a^2b)^2 = (ab)^2.$$

I. *Solution by Lawrence Lessner, San Diego State College.* We have

$$\begin{aligned}(aba^2b)^2 &= [(ab)a(ab)]^2 = (ab)a(ab)^2a(ab)\\ &= ab(ab)^2a(ab)^2a(ab)^2(ab) = ab(ab)^2ab = (ab)^2.\end{aligned}$$

Also

$$\begin{aligned}(ab^2ab)^2 &= [(ab)b(ab)]^2 = (ab)b(ab)^2b(ab)\\ &= ab(ab)^2b(ab)^2b(ab)^2ab = ab(ab)^2ab = (ab)^2.\end{aligned}$$

Finally

$$\begin{aligned}(ab^2a^2b)^2 &= [ab(ba)ab]^2 = ab(ba)(ab)^2ba(ab)\\ &= ab(ab)^2ba(ab)^2ba(ab)^2ab\\ &= aba^2(ab)^2ba(ab)^2ba(ab)^2ab\\ &= aba^2(ab)^2ba(ab)^2ba(ab)^2b^2ab\\ &= aba[a(ab)^2b][a(ab)^2b][a(ab)^2b]bab\\ &= aba[a(ab)^2b]bab = aba^2(ab)^2b^2ab\\ &= ab(ab)^2ab = (ab)^2.\end{aligned}$$

II. *Solution by the proposer.* The subsemigroup $abSab$ is a group, with identity $(ab)^2$ (see Green and Rees, On semigroups in which $x^r = x$, *Proc. Cambr. Phil. Soc.*, 48 (1952) 35–40).

Also solved by K. R. Bailie, K. Balachandran, L. P. Bush, R. A. Cislo, Robert Cohen, S. Heller, R. R. Korfhage, and S. J. Ryan.

Some of these solutions involved unjustified assumptions.

Curves with Circular Orthoptics

E 1588 [1963, 438]. *Proposed by M. S. Klamkin, The State University of New York at Buffalo*

An ellipse has the property that the sum of the moments of inertia of its area about two orthogonal tangents is constant. Does this property characterize the ellipse?

Solution by the proposer. It follows from the parallel-axis transfer theorem that in order for this property to hold for a given curve, its orthoptic curve (locus of intersection of orthogonal tangents) must be a circle whose center coincides with the centroid of the area enclosed by the given curve. Another curve having this property (see R. C. Yates, *A Handbook on Curves and Their Properties*, J. W. Edwards, Ann Arbor, 1947) is the deltoid

$$x = a(2\cos t + \cos 2t),$$
$$y = a(2\sin t - \sin 2t),$$

which is a 3-cusped hypocycloid whose orthoptic is the inscribed circle.

Also solved by Michael Goldberg, who gave an envelope construction for all such curves.

A Criterion for a Triangle to be Isosceles

E 1589 [1963, 438]. *Proposed by W. J. Blundon, Memorial University of Newfoundland*

Find necessary and sufficient conditions for a triangle of inradius r, circumradius R, and semiperimeter s to be isosceles.

Solution by D. C. B. Marsh, Colorado School of Mines. If we denote the sides of a triangle by a, b, c, we have the easily verified relations

$$a + b + c = 2s, \qquad ab + bc + ca = r^2 + s^2 + 4rR, \qquad abc = 4rRs.$$

Thus, a, b, c are the roots of the cubic equation

$$x^3 - 2sx^2 + (r^2 + s^2 + 4rR)x - 4rRs = 0.$$

The triangle will be isosceles if and only if two roots of this cubic are real and equal; this will be the case if and only if the cubic's discriminant is zero. The discriminant is

$$4r^2[4R(R - 2r)^3 - (s^2 + r^2 - 10rR - 2R^2)^2],$$

whence a necessary and sufficient condition on r, R, s that the triangle be

isosceles is that

$$4R(R-2r)^3 = (s^2+r^2-10rR-2R^2)^2.$$

Also solved by Walter Bluger, J. P. Brazy, Leonard Carlitz, D. I. A. Cohen, Roy Feinman, José Gallego-Diaz, Michael Goldberg, Hazel S. Wilson, and the proposer.

A Series Involving the Legendre Polynomials

E 1590 [1963, 438]. *Proposed by James Nearing and J. L. Pietenpol, Columbia University*

Sum, for all x and y, the series

$$\sum_{n=0}^{\infty} P_n(x)y^n/n!,$$

where $P_n(x)$ is the nth order Legendre polynomial.

I. *Solution by A. E. Livingston, University of Alberta.* The formula

$$e^{xy}J_0(y\sqrt{(1-x^2)}) = \sum_{n=0}^{\infty} P_n(x)y^n/n!,$$

in which $J_0(z)$ is the Bessel function of the first kind of order zero, appears on p. 165 of E. D. Rainville, *Special Functions*, Macmillan, New York (1960), with the following comment: "The relation (above) was being used at the beginning of this century. We have not been able to determine when or by whom it was first discovered."

The quoted formula is obtained by substituting

$$\sum_{k=0}^{[n/2]} n!(x^2-1)^k x^{n-2k}/[2^{2k}(k!)^2(n-2k)!]$$

for $P_n(x)$, inverting the order of summation in the resulting iterated series, and making the change of summing index $n=\nu+2k$.

II. *Solution by the proposers.* Denote the sum by $f(y)$. Then, by using the identity

$$(2n+1)xP_n(x) = nP_{n-1}(x)+(n+1)P_{n+1}(x),$$

one can find a differential equation satisfied by $f(y)$. It is

$$yf''+(1-2xy)f'+(y-x)f=0.$$

At $x=+1$ and $x=-1$, $f(y)=e^y$ and $f(y)=e^{-y}$ respectively, which suggests the substitution

$$f(y) = e^{xy}g(y).$$

Then $g(y)$ satisfies the equation

$$g'' + (1/y)g' + (1 - x^2)g = 0,$$

which is a Bessel equation, and the result is

$$\sum_{n=0}^{\infty} P_n(x)y^n/n! = e^{xy}J_0(y\sqrt{(1 - x^2)}).$$

Also solved by J. L. Brown, Jr., Donald Childs, A. E. Danese, J. A. Faucher, Stephen Fisk, Ralph Greenberg, Emil Grosswald, S. Heller, V. E. Hoggatt, Jr., M. S. Klamkin, M. G. Murdeshwar, W. J. Pervin, Stanton Philipp, Perry Scheinok, and F. C. Smith.

Several solvers referred to Eq. (40), p. 182 of Higher Transcendental Functions, Vol. II (Bateman Manuscript Project).

ADVANCED PROBLEMS AND SOLUTIONS

Edited by E. P. Starke, Bloomfield College

Collaborating Editors: L. Carlitz, Duke University; H. S. M. Coxeter, University of Toronto; and A. Wilansky, Lehigh University.

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editors. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5171. *Proposed by P. T. Bateman and L. A. Rubel, University of Illinois*

Suppose that f is a real-valued function belonging to $L_1(-\infty, +\infty)$. Put $H(t) = \sum' tf(kt)$, where $\sum'$ indicates that the summation is over all nonzero integers k.

(1) Prove that $\sum' |f(kt)| < \infty$ for almost all positive t.

(2) Prove that

$$\operatorname{ess}\liminf_{t\to 0+} H(t) \leqq \int_{-\infty}^{\infty} f(x)dx \leqq \operatorname{ess}\limsup_{t\to 0+} H(t).$$

(3) Construct an f for which

$$\operatorname{ess}\liminf_{t\to 0+} H(t) \neq \operatorname{ess}\limsup_{t\to 0+} H(t).$$

5172. *Proposed by Kwangil Koh, University of North Carolina*

An associative ring R, not necessarily commutative, is said to be an integral domain if it has no zero divisor. It is known that any commutative integral domain can be imbedded in a field [1], and there is an integral domain which cannot be imbedded in a skew field [2]. Prove that an integral domain can be imbedded in a primitive ring (a ring with faithful irreducible right module [3]).

References

1. Birkhoff and MacLane, *A Survey of Modern Algebra*, New York, 1947.
2. A. Malcev, *On the immersion of an algebraic ring into a field*, Math. Ann., 113 (1936), 689–691.
3. Nathan Jacobson, *Structure of rings*, Amer. Math. Soc., Colloquium Publication, 37, 1956.

5173. *Proposed by James E. Potter and Robert Fitzgerald, Massachusetts Institute of Technology*

Given that $v(t)$ is an n-dimensional vector valued function, that $P(t)$ is a one parameter family of $n\times n$ positive definite matrices, that μ is a positive measure on the real line, and that $v(t)$ and $P^{-1}(t)$ are in $L^1(\mu)$, show that

$$\int v^*(t)P(t)v(t)\mu(dt) \geqq \int v^*(t)\mu(dt)\left(\int P^{-1}(t)\mu(dt)\right)^{-1}\int v(t)\mu(dt),$$

where * denotes conjugate transpose.

5174. *Proposed by Ranko Bojanic, University of Notre Dame*

In A. Zygmund's *Trigonometric Series*, v. 1, Ch. V, a positive and continuous function $l(x)$ defined for $x\geqq x_0$ is called a slowly varying function if for every $\delta>0$ there exists $x_\delta\geqq x_0$ such that

$$x^\delta l(x) \text{ increases and } x^{-\delta}l(x) \text{ decreases if } x \geqq x_\delta. \tag{1}$$

Prove that a positive and continuous function $l(x)$ is slowly varying in the above sense if and only if

$$-\infty < D_+l(x) \leqq D^+l(x) < +\infty \qquad (x \geqq x_1) \tag{2}$$

$$xD_+l(x) = o(l(x)) \quad\text{and}\quad xD^+l(x) = o(l(x)) \qquad (x\to\infty).$$

Here $D_+l(x)$ and $D^+l(x)$ are the lower and the upper right hand derivatives of $l(x)$ respectively.

5175. *Proposed by Ranko Bojanic, University of Notre Dame*

Let f be continuous for all $x\geqq 0$ and

$$\eta(x) = \sup_{a\geqq 0}\left(\int_x^{x+a} f(t)dt\right).$$

Then $\eta(x)\to 0(x\to\infty)$ if and only if there exists a Riemann integrable function g such that $f(x)\leqq g(x)$ for all x and $\int_0^\infty g(t)dt$ exists.

5176. *Proposed by Orrin Frink, Pennsylvania State University*

Let there be given a square array of elements of a division ring (e.g. quaternions). Prove that if the rows of the array are left linearly dependent, then the columns are right linearly dependent.

5177. *Proposed by P. R. Vein, Leatherhead, Surrey, England*

Find $q(x)$ such that the integral

$$u = -\int_{-1}^{1} q(x)\log\left\{x^2 - 2x\cosh b(r^2 - \sinh^2 b)^{1/2} + r^2\right\}dx$$

is independent of r in the interval $\sinh b\leqq r\leqq\cosh b$.

5178. *Proposed by Robert Breusch, Amherst College*

Find A_i $(i=0, 1, 2, \cdots)$ if for every nonnegative integer n,

$$\sum_{i=0}^{n}\binom{2n-2i}{n-i}A_i = \binom{2n+1}{n} \qquad \left[\text{with}\binom{0}{0}=1\right].$$

5179. *Proposed by David Carlson, Oregon State University*

It is well known that, for any square matrix A with complex elements, there is a unique decomposition $A=B+C$, where $B=(A+A^*)/2$ is Hermitian (and has all roots along the real axis) and $C=(A-A^*)/2$ is skew-Hermitian (and has all roots along the imaginary axis). Given any two distinct lines through the origin in the complex plane, prove an analogous result for a unique decomposition of A into two "Hermitian-like" matrices, each with its roots along one of the two given lines.

5180. *Proposed by A. E. Livingston, University of Alberta*

Find $\lim_{x\to 1-0}(1-x)\sum_{n=0}^{\infty} a_n x^n$ if each set $\{a_{kN}, a_{kN+1}, \cdots, a_{(k+1)N-1}\}$ is a permutation of the set $\{a_0, a_1, \cdots, a_{N-1}\}$ for $k=1, 2, 3, \cdots$, where N is a fixed positive integer.

SOLUTIONS

Length of a Graph

5023 [1962, 317; 1963, 766]. *Proposed by D. J. Newman, Yeshiva University*

Let $P(z)$ be a nonconstant polynomial and for each positive number t define $L(t)$ to be the length of the curve $|P(z)|=t$. Prove that $L(t)$ is an increasing function.

Solution by I. N. Baker, Imperial College, London, England. That the conjecture is false is shown by the polynomials $P(z)=z^n-1$.

In the case $t=1$ the curve $|P|=1$ consists of n loops given by the polar equation $r^n=2\cos n\theta$, $r>0$. Each loop joins the origin to a point on the circumference $r=2^{1/n}>1$, so that

$$L(1) > 2n. \tag{1}$$

For $t>1$ the equation $|P|=t$ may be written

$$r^{2n} - (2\cos n\theta)r^n + (1-t^2) = 0, \tag{2}$$

where r^n is the positive root of (2) viz.

$$r^n = \cos n\theta + (\cos^2 n\theta + t^2 - 1)^{1/2},$$

whence

$$t - 1 \leqq r^n \leqq t + 1, \tag{3}$$

since $x+(x^2+t^2-1)^{1/2}$ is an increasing function of x. Thus the curve has an equation $r=r(\theta)$, where r is a single-valued function of θ satisfying (2) and (3). Differentiating (2) yields

$$\frac{dr}{d\theta} = \frac{-r\sin n\theta}{r^n - \cos n\theta}. \tag{4}$$

If $t>2$, (3) and (4) give $|dr/d\theta| \leqq (t+1)^{1/n}(t-2)^{-1}$. If s is the length of arc measured along $|P|=t$ we have

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2 \leqq \frac{(t+1)^{2/n}}{(t-2)^2} + (t+1)^{2/n}.$$

Thus

$$L(t) = \int_0^{2\pi}\left(\frac{ds}{d\theta}\right)d\theta \leqq 2\pi(t+1)^{1/n}\{1+(t-2)^{-2}\},\ t>2.$$

For any fixed $t>2$ this number is uniformly bounded and it is easy to choose n so that $L(1)>L(t)$; e.g., if $t=3$ we have for all n, $L(3)\leqq 2\pi\cdot 4^{1/n}\cdot 2<16\pi$, while if $n>26>8\pi$ we have from (1), $L(1)>2n>16\pi>L(3)$.

Also solved by L. A. Karlovitz and J. A. Voytuk and by Rainer Schulz and V. E. Hoggatt.

"Angles of Inclination" in Correlation Theory

5076 [1963, 215]. *Proposed by Hans Zassenhaus, University of Notre Dame*

In the correlation theory of multiple events one can define a set of correlation coefficients in the form of cosines of the angles of inclination between two linear subspaces S_1, S_2 of Euclidean n-space. Let q_j be the dimension of S_j, $q_1 \leqq q_2$ and let the row vectors of the $q_j \times n$ matrix A_j form a basis of S_j. Give a definition of

the angles of inclination and show that the characteristic roots of the matrix

$$f(A_1, A_2) = A_1 A_2^T (A_2 A_2^T)^{-1} A_2 A_1^T (A_1 A_1^T)^{-1}$$

are the squares of the cosines of the angles of inclination.

Solution by the proposer. The transition to another basis of S_j corresponds to premultiplication of A_j by a nonsingular matrix P_j of degree q_j: $A_j \to P_j A_j$. A simple computation shows that

$$F(P_1 A_1, P_2 A_2) = P_1 F(A_1, A_2) P_1^{-1}$$

so that the characteristic roots remain unchanged. Moreover, $F(A_1, A_2)$ remains unchanged under the transition $A_j \to A_j R$ if R is an orthogonal matrix of degree n corresponding to another choice of Cartesian coordinates of the E_n. For $A_j A_j^T$, being a symmetric positive matrix, there exists a suitable nonsingular matrix P_j of degree q_j such that $P_j A_j A_j^T P_j^T = B_j B_j^T = I_{q_j}$ where $B_j = P_j A_j$ corresponds to an orthonormal basis of S_j. For $S = (B_1 B_2^T)(B_1 B_2^T)^T$, being symmetric nonnegative, there is an orthogonal matrix Q_1 of degree q_1 such that

$$Q_1 X Q_1^{-1} = (c_i^2 \delta_{ik}) = (C_1 B_2^T)(C_1 B_2^T)^T \qquad (c_1 \geqq c_2 \geqq \cdots \geqq c_{q_1} \geqq 0,$$

where $C_1 = Q_1 B_1$ corresponds to another orthonormal basis of S_1. Hence $1 \geqq c_1$, $c_i = \cos \phi_i$, $0 \leqq \phi_1 \leqq \phi_2 \leqq \cdots \leqq \phi_{q_1} \leqq \frac{1}{2}\pi$; the row vectors of $C_1 B_2^T$ are mutually orthogonal, hence there is an orthogonal matrix Q_2 of degree q_2 such that $C_1 B_2^T Q_2^T = C_1 C_2^T = (\cos \phi_i \delta_{ik})$, where $C_2 = Q_2 B_2$ corresponds to another orthonormal basis of S_2. This shows that the set of the angles $\phi_1, \phi_2, \cdots, \phi_{q_1}$ may be defined as a normalized set of angles of inclination between S_1 and S_2. The invariance of this set follows from the factorization

$$\det (tI_{q_1} - f(C_1, C_2)) = \prod_{i=1}^{q_i} (t - (\cos \phi_i)^2) = \det (tI_{q_1} - f(A_1, A_2)).$$

Partition of a Domain

5077 [1963, 216]. *Proposed by Hewitt Kenyon, George Washington University*

If f is a function and S is a set included in the domain of f, denote by S' the map $f(S)$ of S by f. If f has no fixed points, show that the domain of f can be partitioned into three pairwise disjoint sets A, B, C, none of which intersects its map: $A \cap B = B \cap C = C \cap A = A \cap A' = B \cap B' = C \cap C' = \varnothing$.

Solution by I. N. Baker, Imperial College, London, England. Assume that for sets S and X we have $S \subset X$ and f: $S \to X$, where f has no fixed points. Subject to the condition $f(x) \neq x$ extend f in any manner to a mapping f: $X \to X$, which agrees with the original in the latter's range (and hence on S). Then for every positive integer n the nth iterate f_n: $X \to X$ exists. Introduce the equivalence

relation R for points of X: xRy if and only if there exist positive integers m and n such that $f_m(x)=f_n(y)$.

Suppose that an equivalence class K contains a periodic point x_1, i.e. a point such that $f_p(x_1)=x_1$ for some integer $p>0$; then $x_1, x_2=f(x_1), \cdots, x_k=f(x_{k-1}), \cdots, x_p$ form a cycle of p periodic points ($x_1=f(x_p)$) and these are the only periodic points in K.

We now define the sets A, B, C:

In each equivalence class K choose an element x_1, taking a periodic x_1 if K contains one; assign all $f_n(x_1)$ to A, B or C as follows: x_1 is in A and

(i) if x_1 is not periodic, $f_n(x_1)$ is in A if n is even, in B if n is odd;

(ii) if x_1 belongs to a cycle $x_1, x_2, \cdots, x_p$ with p even, then do as in (i), i.e. $x_1, x_3, \cdots, x_{p-1}\in A$; $x_2, x_4, \cdots, x_p\in B$;

(iii) if x_1 belongs to a cycle $x_1, x_2, \cdots, x_p$ with p odd, then take $x_1, x_3, \cdots, x_{p-2}\in A$; $x_2, x_4, \cdots, x_{p-1}\in B$, $x_p\in C$.

For any other $y\in K$ there is a least positive m such that $f_m(y)=f_n(x_1)$ for some $n>0$. If $f_n(x_1)\in A$ or C, take $y\in A$ if m even, $y\in B$ if m odd; if $f_n(x_1)\in B$, take $y\in B$ if m even, $y\in A$ if m odd.

These definitions partition X onto three disjoint sets A, B, C such that $A'=f(A)\subset B$, $B'=f(B)\subset A\cup C$, $C'=f(C)\subset A$. Thus $A\cap A'=B\cap B'=C\cap C'=\emptyset$. Replacing A, B, C by $A\cap S$, $B\cap S$, $C\cap S$ we get a similar partition of S.

We note that C can be chosen empty if there are no periods of odd length, e.g. if there are no periodic points at all. Thus if $X=E^1$, $f=x+1$, then we can take $A=\bigcup_{-\infty}^{\infty} [2n, 2n+1)$, $B=X-A$, $C=\emptyset$.

Also solved by W. H. Bonney, Robert Bowen, D. Ž. Djoković, D. P. Giesy, R. D. Johnson, W. M. Lambert, Jr., Sibe Mardešić, George Senge, D. L. Silverman, W. C. Waterhouse, Carroll Webber, Jr., Oswald Wyler, Alexander Zabrodzky, and the proposer.

Prime Ideal with Infinite Depth

5079 [1963, 216]. *Proposed by Fred Suvorov, Princeton University*

In *Commutative Algebra*, v. I, p. 241, Zariski and Samuel assert that a prime ideal in a noetherian ring need not have finite depth: the depth of a prime ideal is the upper bound (if it exists) of the lengths of strictly ascending chains of prime ideals starting with the given one. Find an example of a prime ideal in a noetherian ring with infinite depth.

Note by Veselin Perić, Sarajevo, Yugoslavia. Such an example is given by Krull, W. in *Dimensionstheorie in Stellenringen*, J. Reine Angew. Math., 179 (1938), p. 222. The same reference is noted in Northcott, D. G., *Ideal Theory*, p. 106.

Squares in the Fibonacci Series

5080 [1963, 216]. *Proposed by A. P. Rollett, Crediton, England*

In the Fibonacci series ($F_1=1$, $F_2=1$, $F_{n+1}=F_n+F_{n-1}$) the first, second and twelfth terms are squares. Are there any others?

Solution by Oswald Wyler, University of New Mexico. There are no other squares. Because of its length, the proof is subdivided into seven steps.

(1) *Relations between the F_n.* We put $\omega = \frac{1}{2}(\sqrt{5}+1)$, so that $\omega^2 = \omega + 1$. It follows by induction that $\omega^n = F_n\omega + F_{n-1}$ for all positive integers n (with $F_0 = 0$). Using this and the laws of exponents, we obtain

$$\begin{aligned} F_{m+n} &= F_mF_n + F_mF_{n-1} + F_{m-1}F_n, \\ F_{kn} &= kF_n(F_{n-1})^{k-1} + (F_n)^2P_k(F_n, F_{n-1}), \qquad (1.1) \\ F_{kn-1} &= (F_{n-1})^k + (F_n)^2Q_k(F_n, F_{n-1}), \end{aligned}$$

for any positive integers m, n, k, where P_k and Q_k are homogeneous polynomials with integral coefficients. We have in particular

$$F_{2n} = F_n(2F_{n-1} + F_n), \qquad F_{2n-1} = (F_{n-1})^2 + (F_n)^2. \tag{1.2}$$

Putting $\bar{\omega} = \frac{1}{2}(1-\sqrt{5})$ and multiplying ω^n by $\bar{\omega}^n = F_n\bar{\omega} + F_{n-1}$, we have

$$(F_{n-1})^2 + F_nF_{n-1} - (F_n)^2 = (-1)^n, \tag{1.3}$$

for any positive integer n.

(2) *On prime divisors.* It follows from (1.3) that F_n and F_{n-1} are relatively prime. By this and (1.1), any prime factor p of F_n which is not a factor of k occurs in the factorizations of F_n and of F_{kn} with the same exponent.

If $p \mid F_n$, then $p \mid F_{m+n}$ if and only if $p \mid F_m$ by (1.1), and it follows that $p \mid F_n$ for an integer n if and only if n is a multiple of the least integer m such that $p \mid F_m$. In particular, $2 \mid F_n$ if and only if $3 \mid n$, and $3 \mid F_n$ if and only if $4 \mid n$.

(3) A LEMMA. *Let p be prime and let m be the least integer such that $p \mid F_m$. If m is even, then $p \not\equiv 13$ or 17 (mod 20); and if $p \equiv 3$ or 7 (mod 20), then $F_{m-1} \equiv -1$ (mod p).*

Proof: Let $m = 2m'$, $F_{m'} \equiv a$ (mod p), $F_{m'-1} \equiv b$ (mod p). By (1.2) $a + 2b \equiv 0$ (mod p), and $F_{m-1} \equiv a^2 + b^2$ (mod p). Thus $F_{m-1} \equiv 5b^2$(mod p). By (1.3), $(F_{m-1})^2 \equiv 1$ (mod p), so that $F_{m-1} \equiv 5b^2 \equiv \pm 1$ (mod p). If $p \equiv 13$ or $p \equiv 17$ (mod 20), then 1 and -1 are quadratic residues mod p, and 5 is a quadratic nonresidue, so this case cannot occur. If $p \equiv 3$ or $p \equiv 7$ (mod 20), then 5 and -1 are quadratic nonresidues so that we must have $F_{m-1} \equiv -1$ (mod p).

We remark without proof that the cases $p \equiv 11$ or 19 (mod 20) are also excluded if m is a multiple of 4.

(4) *F_n for powers of 2.* Using (1.2), one obtains the following values mod 20 for F_n and F_{n-1} if $n = 2^m$:

m	2	3	4	5	6
F_{n-1}	2	13	10	9	2
F_n	3	1	7	9	3

For $m \geqq 6$, the table repeats itself with period 4. (For convenience we will sometimes put $F(x)$ for F_x.) Now $F(2^m) = F(2^{m-1})\ G_m$, where $F(2^{m-1})$ and $G_m = 2F(2^{m-1}-1) + F(2^{m-1})$ are relatively prime. If p is a prime factor of G_m, then

$n=2^m$ is the least integer n such that $p|F_n$. Thus $p\not\equiv 13$ and $p\not\equiv 17$ (mod 20) by (3). On the other hand, our table shows that $G_m\equiv 7$ (mod 20) for all $m>2$, and $G_2=3$. Thus not all prime factors of G_m can be $\equiv \pm 1$ (mod 5); and G_m, for $m>2$, has at least one prime factor p such that $p\equiv 3$ or $p\equiv 7$ (mod 20). Then $F(2^m-1)\equiv -1$ (mod p) by (3), and -1 is not a quadratic residue mod p.

(5) *First case: n odd.* If $n\geqq 3$ is odd, then we can write n in the form $n=k2^m\pm 1$, with k odd and $m\geqq 2$. By (4), there is a prime factor p of $F(2^m)$ such that $F(2^m\pm 1)\equiv -1$ (mod p) and that -1 is not a quadratic residue mod p. By (1.1), we have $F_n=F(k2^m\pm 1)\equiv(-1)^k\equiv -1$ (mod p), and it follows that F_n cannot be a square.

(6) *Second case: $n\neq 2^k3^l$.* Let us call a prime number p a good prime factor of F_n if $p\geqq 3$ and p occurs in the factorization of F_n with an odd exponent. It follows from (2) that p is a good prime factor of F_{2n} and of F_{3n} if p is a good prime factor of F_n. If $n=m2^k3^l$, with $m>1$ prime to 6, then F_m is not a square by (5), and not a multiple of 2 or of 3 by (2). Thus F_m has a good prime factor. It follows from this and the preceding remark that F_n also has a good prime factor. Thus F_n is not a square.

(7) *Third case: $n=2^k3^l$.* Since $F_8=21$ and $F_9=34$ have good prime factors, F_n has a good prime factor if $k\geqq 3$ or $l\geqq 2$, by (6). This leaves six cases open, viz. $F_1=1$, $F_2=1$, $F_3=2$, $F_4=3$, $F_6=8$, $F_{12}=144$. Of these, only F_1, F_2 and F_{12} are squares.

The problem is mentioned in Ogilvy, Tomorrow's Math., 1962, p. 100.

A Partial Difference Equation

5081 [1963, 216]. *Proposed by D. S. Adorno, California Institute of Technology*

a. Determine the solution to the partial difference equation with indicated boundary conditions: for all $i=0, 1, \cdots, n+2$; $n=0, 1, 2, \cdots$; $0<\alpha<1$,

$$A_{i,n+1} = A_{i,n} - \alpha^{n+1}A_{i-1,n}, \quad A_{-1,n} = 0, \quad A_{0,n} = 1, \quad A_{n+2,n} = 0.$$

b. Determine $\lim_{n\to\infty}A_{i,n}$.

Solution by L. Carlitz, Duke University. Put $f_n(x)=\sum_{i=0}^{n+1} A_{i,n}\, x^i$. Then

$$f_{n+1}(x) = \sum_{i=0}^{n+2} (A_{i,n} - \alpha^{n+1}A_{i-1,n})x^i = f_n(x) - \alpha^{n+1}xf_n(x),$$

so that $f_{n+1}(x)=(1-\alpha^{n+1}x)f_n(x)$. Take $f_0(x)=1+A_{1,0}x$. ($A_{1,0}$ is apparently arbitrary). It follows that

$$f_n(x) = (1 + A_{1,0}x)\prod_{r=1}^{n}(1 - \alpha^r x).$$

Now it is well known that

$$\prod_{r=1}^{n}(1 - \alpha^r x) = \sum_{r=0}^{n}(-1)^r\alpha^{r(r+1)/2}\begin{bmatrix} n \\ r \end{bmatrix}x^r,$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(1-\alpha^n)(1-\alpha^{n-1})\cdots(1-\alpha^{n-r+1})}{(1-\alpha)(1-\alpha^2)\cdots(1-\alpha^r)}, \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1.$$

It follows that

$$A_{r,n} = (-1)^r\alpha^{r(r+1)/2}\begin{bmatrix} n \\ r \end{bmatrix} + (-1)^{r-1}\alpha^{r(r-1)/2}\begin{bmatrix} n \\ r-1 \end{bmatrix} A_{1,0}.$$

In particular, if we take $A_{1,0} = -1$ this reduces to

$$A_{r,n} = (-1)^r\alpha^{r(r-1)/2}\begin{bmatrix} n+1 \\ r \end{bmatrix}.$$

Moreover

$$\lim_{n\to\infty} A_{r,n} = \frac{(-1)^r\alpha^{r(r+1)/2}}{(1-\alpha)\cdots(1-\alpha^r)} + \frac{(-1)^{r-1}\alpha^{r(r-1)/2}A_{1,0}}{(1-\alpha)\cdots(1-\alpha^{r-1})}$$

and in particular when $A_{1,0} = -1$

$$\lim_{n\to\infty} A_{r,n} = \frac{(-1)^r\alpha^{r(r-1)/2}}{(1-\alpha)\cdots(1-\alpha^r)}.$$

Angle Preserving Map

5084 [1963, 335]. *Proposed by Robert Spira, University of California, Berkeley*

Find a one-to-one continuous function on the unit disk into the plane which is angle preserving, yet not analytic. (If no such function exists, then we have a simple geometric characterization of analytic functions. If such a function does exist, then from it we will obtain a clue as to a further geometric restriction necessary to characterize the idea of analytic function.)

Comment by George Bergman, Harvard University. In requiring that a function be angle preserving, we presume that it sends smooth curves—those curves between which we can measure angles—into smooth curves. To find a good analytic characterization of such functions seems to be a more fundamental problem, which probably has to be solved before the one given.

An example of a function which sends the set of smooth curves on the unit disk onto itself, but is not differentiable is $(\rho, \theta)\to(\sqrt{\rho}, \theta)$.

For differentiable functions, it is easy to show that angle preserving implies analytic.

A Fallacious Deduction

5085 [1963, 335]. *Proposed by E. R. Gentile, Universidad del Sur, Argentina*

Let G be any abelian group and $G^2 = G\oplus G$ the direct sum of two copies of G. G^2 admits in a natural way structures of left and right modules over the ring

Z_2 of 2×2 matrices over the integers Z. We recall that for every abelian group H the mapping ϕ: $G^2 \otimes_{Z_2} H^2 \to G \otimes_Z H$ defined by $\phi((g_1 \oplus g_2) \otimes_{Z_2} (h_1 \oplus h_2)) = (g_1 + g_2) \otimes_Z (h_1 + h_2)$ is a canonical isomorphism.

This permits an incorrect proof of the statement: *If G^2 is isomorphic to H^2, then G is isomorphic to H*, viz.

$$G \cong G \otimes_Z Z \cong G^2 \otimes_{Z_2} Z^2 \cong H^2 \otimes_{Z_2} Z^2 \cong H \otimes_Z Z \cong H.$$

Where is the error?

Solution by W. C. Waterhouse, Harvard University. It has been implicitly assumed that the group isomorphism between G^2 and H^2 is also an isomorphism of their Z_2-module structures.

Also solved by George Bergman, W. H. Bonney, Veselin Peric, J. J. Zeltmacher, Jr., and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, Carleton College and E. P. VANCE, Oberlin College

The Calculus of Variations. By N. I. Akhiezer. Translated by Aline H. Frink. Blaisdell, New York, 1962. 247 pp. \$7.50.

Calculus of Variations. By L. E. Elsgolc. Addison-Wesley, Reading, Mass., 1962. 178 pp. \$4.50.

These two volumes are translations of texts first published in Russia in the 1950's. That by Akhiezer is informal in style, but careful and clear in its treatment. It is well suited for use by any reader with a reasonable mathematical background. In Chapter 1, the Euler equations are derived for the simplest problem for curves in $n+1$ dimensions. The existence and differentiability of families of extremals are treated, and canonical variables are introduced. The last section of the chapter treats the parametric problem in the plane. Chapter 2 takes up the theory of fields and sufficient conditions for a minimum, followed by a brief consideration of the necessary conditions of Weierstrass, Legendre, and Jacobi. Chapter 3 derives the first necessary conditions for an extremum in the following cases: problems with variable end points in the plane, problems with discontinuous integrand, double integrals, simple integrals involving derivatives of the nth order, isoperimetric problems, and simple cases of the problem of Lagrange. Chapter 4 gives proofs of the existence of an absolute minimum for both nonparametric and parametric problems in the plane. The method of Ritz for finding a minimizing sequence is included. An appendix of 78 pages contains exercises, worked examples, and sections on the variational prin-

ciples of mechanics, the Dirichlet principle, double integrals (including Haar's lemma), the Bernstein theorem on analyticity of extremals, and the Sturm-Liouville theory for second order differential equations.

The book by Elsgolc professes to be designed primarily for engineers. Chapters 1 to 4 treat most of the topics included in Chapters 1 to 3 of Akhiezer, but in somewhat more cursory fashion. The treatment of fields in space of more than two dimensions is restricted to a short paragraph in fine print in Chapter 3, in which no mention is made of the need to restrict the field to be a Mayer field if it is to be used in proving a sufficiency theorem. Chapter 5 gives a brief consideration of direct methods, but includes no existence proofs. There are many exercises and many worked examples. The reviewer believes that the book could be made much more enlightening for its users without increasing its size. The style is informal and discursive, but tends to leave some of the proofs rather hazy. Many of the results are stated without proofs.

L. M. GRAVES, Illinois Institute of Technology

The Theory of Lebesgue Measure and Integration. By S. Hartman and J. Mikusinski. Vol. 15, International Series of Monographs on Pure and Applied Mathematics. Translated from the Polish edition by Leo F. Boron. Pergamon Press, 1961. 176 pp. $6.50.

This short book presents a clear and self-contained exposition of what may be called the "classical" Lebesgue theory. By this is meant that the development is confined entirely to the real line (except for a brief indication of the extension of the theory to higher-dimensional Euclidean spaces), and no reference is made to more general types of measure spaces.

The first seven chapters cover what may be called the basic theory: measure of sets (beginning with the theorem on the structure of open subsets of the real line and then employing open coverings), preservation of measurability under countable unions and intersections, measurable and integrable functions, termwise integration of sequences of integrable functions, almost-everywhere differentiability of functions of bounded variation, absolutely continuous functions and their representation as integrals of their derivatives. Chapter 8 consists of a brief and clear treatment of the L^p-spaces (including their completeness) while Chapter 9 contains the elements of the L^2-theory (orthogonal expansions, best mean-square approximation, Riesz-Fischer theorem, and Parseval relation) and Chapter 10 extends these chapters to complex-valued functions. The next two chapters define measure and the integral in E^n and include the theorems of Fubini and Tonelli. The final chapter presents a brief and interesting treatment of Riemann-Stieltjes integration, concluding with the important theorem that when $f(x)$ is continuous and $g(x)$ is absolutely continuous, the Stieltjes integral $\int f dg$ is equal to the Lebesgue integral $\int f g' dx$.

The coverage of the first ten chapters is quite similar to that of Chapters 10, 11, 12 of Titchmarsh's "Theory of Functions," and the styles of exposition are somewhat similar. The book under review impressed the reviewer as being a

little more relaxed in its pace, and hence more suitable as a text or for individual study. On the other hand, the complete absence of problems is a serious defect, and the instructor or student who wishes to use this book would be well-advised to secure problems from other sources. A new edition with an abundant supply of problems would be a welcome addition to the textbook literature on measure and integration.

The translation reads smoothly, except for a few places where the original word order or idiomatic structure appears to have been followed.

BERNARD EPSTEIN, University of New Mexico

Industrial Dynamics. By Jay W. Forrester, MIT Press and Wiley, New York, 1961. 464 pp. $18.00.

One of the current views about industrial organizations is that they are information-feedback systems. In its most general form, such a system includes all the men, machines, and materials that comprise a working organization as well as the intangible factors of information flow, policy formation, decision making, decision enforcing, etc.

In *Industrial Dynamics,* Jay Forrester describes a computer language (DYNAMO) suitable for simulating such organizations in considerable detail. Numerous applications to organizations are presented in the book. A simulation of a firm can be formally described as a finite set of simultaneous difference equations. These are usually of such mathematical complexity, however, that a solution in "closed form," even if it could be found, would be so complicated as to be worthless. Hence a useful form of computer output, which plots the levels of various variables through time, is adopted in the book as the best form in which to present "answers."

As might be expected, the difference equations so obtained have a natural period of vibration, and Forrester asserts that so does the real-life system being simulated. The latter conclusion rests, however, upon the verification of actual similarities between the model and the real system. Such validation of the model seems to the reviewer to be the weakest step in the theory, since the only verification checks seem to depend upon casual questioning of the management of the firm. The answers to such questions emphasize the "normal" operating areas of the company, yet in a simulation run it is very easy to drive the company into an abnormal mode of operation for which the answers may no longer be valid. The other drawback of the theory is that it cannot (as yet) handle interactions between two or more firms and hence is not able to study oligopoly theory.

Nevertheless the book under review will interest readers who like their mathematics in the "raw" or pre-formal stage. Electronic computers are presently being used by many people for simulation purposes and much intuitive knowledge is being gained about the virtues and drawbacks of the method. The time for formal mathematical study of simulation techniques will soon be at hand if, indeed, it is not already here.

G. L. THOMPSON, Carnegie Institute of Technology

Symbols, Signals and Noise: The Nature and Process of Communication. By J. R. Pierce. Harper & Row, New York, 1961. xi+305 pp. $6.50.

This book, another in the Harper Modern Science Series, is intended for the general reader who knows no mathematics but is willing to work. Beginning with explanations of such notions as *variable*, *logarithm*, and *nonconstructive proof*, the author sets forth Shannon's theory of information, noise, signals, and coding, and discusses various attempts to apply information theory in physics, psychology, and art.

The exposition, on the whole, is careful, well developed, and remarkably clear. The author distinguishes the various motivations for Shannon's theory and gives a good indication of the types of mathematical argument used in it. In these respects Pierce's book is outstanding among the existing popularizations of information theory.

On the other hand, the book could all too easily mislead the general reader in regard to studies of communication outside of electrical engineering. Many inquiries and theories are ignored or dismissed on the ground that they are still immature and—by innuendo—not worthwhile or not interesting. There is a chapter on "Language and Meaning," e.g., but nowhere any reference to the work of Carnap, Bar-Hillel, and Kemeny, or Osgood, or Marschak. There is no bibliography, not even any suggestions for further reading.

There are quite a few typographical errors and "slips of the pen." On page 226 the date for Turing should be 1950.

DAVID HARRAH, University of California, Riverside

Elements of Probability and Statistics. By Frank L. Wolf. McGraw-Hill, New York, 1962. 322 pp. $7.50.

This book provides greater exactness in its presentation than most elementary texts. The reviewer did not find any statement which would have to be contradicted in more advanced courses in probability and statistics—something one cannot say about many elementary books. The only formal prerequisite is high school algebra, but as usual considerable maturity in mathematical thinking is required.

The book is divided into fourteen chapters: 1. Empirical frequency distribution; 2. Sets and set operations; 3. Notational and computational devices; 4. Averages; 5. Measures of dispersion; 6. Probability; 7. Discrete probability distributions; 8. Applications of discrete distributions; 9. Continuous probability distributions; 10. Normal distributions; 11. Chi-square distributions; 12. F distributions; 13. Student's distributions; 14. Bivariate distributions. There follows an appendix of eight tables, including as a special feature the cumulative binomial.

An innovation worthy of notice is the use of the plural of "distribution"; this is clearly justified. In the definition of the sample variance, the author's use of the sample size as the divisor is advantageous on account of the analogy with the population variance. Another special feature of the book is the statement of Chebyshev's theorem for samples. This seems to be a fine idea.

The aim of the author is to provide material for the general education of mathematics majors and a basic course for social and physical sciences. The reviewer has reservations about whether this goal is in fact achieved. The statistical ideas discussed do not go beyond the extent of most elementary "cook books" in statistics. There is no mention of modern statistical decision theory. The Neyman-Pearson fundamental lemma is not discussed except for an example of Neyman structure test in a problem. Type I and II errors are defined, but they are not linked to the notion of hypothesis tested and alternative hypothesis. On the contrary, the author states, "In the general case, it is not always clear which of the two hypotheses should be called the null hypothesis." Then he defines Type I and II error as usual. Game theory is introduced through problems only. Non-parametric tests are not discussed with the exception of the chi-square test of goodness of fit.

ESTHER SEIDEN, Michigan State University

A Brief Introduction to Theta Functions. By Richard Bellman. Holt, Rinehart and Winston (Athena Series), New York, 1961. x+78 pp. $2.50.

Dedicated "To devotees of analytic number theory," this book is written by one of today's most versatile research mathematicians. There are seventy sections, some only a few lines long. A partial listing may give some idea of the far-flung contents: § 2: The Four Types of Theta Functions. §§ 4, 9: The Transformation Formula for $\theta_3(z, t)$. § 10: Numerical Application. § 11: Modular Functions and Eisenstein Series. § 13: The Heat Equation. § 17: The [Laplace]-Transformed Transformation Formula. § 24: The Riemann Zeta Function. § 26: The Riemann Hypothesis. § 27: The Poisson Summation Formula and the Zeta Function. § 29: Gaussian Sums. § 30: Polya's Derivation. § 32: A Fundamental Infinite Product [for θ_4]. § 42: Mock Theta Functions. § 47: Landen's Formula. § 50: Functional Equations. § 52: Entire Solutions. § 61: Multidimensional Theta Functions. § 69: The Modular Transformation.

These sections are grouped, to quote from the Introduction, about " . . . three principal results in the theory of elliptic functions . . . as a stage upon which to parade some of the general factota of analysis, and as an excuse to discuss some intimately related results of great mathematical elegance. Our aim is to indicate the applicability and versatility of analytic techniques that should be part of the hope chest of every young mathematician." The first forty-one pages present various proofs, and applications, of the transformation formula for $\theta_3(z, t)$. The next sixteen pages develop the infinite product expansion of $\theta_4(z, t)$ and related material. The final portion centers around the theta functions as entire solutions of certain simple functional equations. Interrelations abound; the final portion, for example, contains yet another proof of the fruitful transformation formula for θ_3 (with one of the very few, minor misprints occurring in its restatement on p. 61).

It is difficult to take the title of the book at face value. A prior acquaintance with, say, the chapters on Elliptic Functions and on the Theta Functions in

Whittaker and Watson's *A Course of Modern Analysis* would seem to constitute the necessary and sufficient background for the general reader to derive inspiration, rather than desperation, from Bellman's enthusiastic collection of gems. His over-generous use of "elegant" and similar adjectives cannot take the place of such essential background and substantial motivation. Much more helpful are the many Comments and References at the end of most sections. To a reader who is well prepared, or willing to follow up the main references actively, Bellman's short monograph could be an exciting tour guide to a subject he refers to, in his Foreword, as "the fairyland of mathematics."

FRITZ STEINHARDT, The City College of New York

Introduction to Electromagnetic Fields and Waves. By D. Corson and P. Lorrain. Freeman, San Francisco, 1962. 552 pp. $12.00.

This text at the advanced undergraduate level might well have been subtitled *Maxwell for the Senior*. The authors' attitude towards their subject is expressed clearly by the following quotation from their preface: " . . . The discussion has been kept as systematic and as thorough as possible, and hazy 'physical' arguments have been avoided. We have also stressed the internal logic of the subject, and we have clearly stated all assumptions. . . . " The reader would infer correctly from this statement that the authors intend to treat electromagnetic theory as a branch of applied mathematics, in the tradition of Maxwell's treatise. Teachers and students who prefer this approach, and these will be many, will welcome this book as a solution to a difficult pedagogical problem.

The principal topics discussed follow the classical tradition: electro- and magnetostatics, low frequency induction, electromagnetic waves in free space, wave guide theory, and radiation from antennas and from moving charges. The full apparatus of Maxwell's equations has been used in nearly half of the book. Vector methods are used throughout. It is evident that the authors have selected their material and mode of presentation with great care, so that students with only modest attainments in physics and mathematics need seldom falter. The book would seem to be particularly well adapted to a course in applied physics for engineering students, and for review study by engineers and applied physicists.

The reviewer would like to express a mild dissent, however, from the authors' dogged insistence on the phenomenological character of electromagnetic theory. It cannot be concealed forever, even from engineering students, that the emphatic statement of the generality of the Maxwell field equations which is made on page 305 is subject to qualification. Remarkable as the macroscopic theory is, it has definite limitations. More particularly, the authors are as well aware as anyone else that the theory of radiation from moving charges is of questionable validity, to say nothing of its failure to explain the photon structure of light. It is regrettable that these limitations on the theory were not made more explicit.

E. L. HILL, University of Minnesota

Handbook of Automation, Computation and Control. Vol. 3.—Systems and Components. Edited by E. M. Grabbe, S. Ramo, and D. E. Woolridge. Wiley, New York, 1961. xxi+1172 pp. \$19.75.

This handbook is a compendium of the state of the art in industrial process control and closely related areas, and contains little of interest to mathematicians as such. The twenty-eight separately authored sections are naturally rather uneven, but, in general, they seem to be intended as a survey of their subject matter for a reader who is working in a not too closely related field of engineering, and should be quite useful for this purpose. They range from surveys of commercially available components and systems to a rather nice intuitive treatment of semiconductor theory. References, given at the end of each chapter, range in number from 3 to 171, again indicative of the unevenness of the collection.

J. D. RUTLEDGE, Thomas J. Watson Research Center, IBM

Elements of Linear Spaces. By A. R. Amir-Moéz and A. L. Fass. Pergamon Press, New York, Oxford, London, Paris, 1962. ix+149 pp. \$5.50.

The approach of this text is strongly geometrical. (For example, linear dependence is first defined in terms of collinearity and coplanarity.) The first sixty pages present the elementary theory of vectors, linear transformations, matrices, determinants, characteristic equations, and quadratic forms, all in the context of real Euclidean space of dimension $\leqq 3$.

In the next fifty-five pages or so, the earlier ideas are generalized to complex n-space, quadric surfaces are studied in detail, and a number of classical applications to geometry are developed. Then about thirty pages are devoted to the abstract theory of linear algebras, and to "some of the more accessible recent results on singular values . . . ." Finally, a four page appendix reviews solid geometry.

The authors, in their preface, affirm their belief that "students and instructors are intelligent and would like to supply details of proof or technique in many places." But they also say that they are aiming at an "elementary undergraduate course to bridge the gap between freshman mathematics and modern abstract algebra." It seems to me that the extent to which they have gone in implementing the first statement has vitiated the second. Furthermore, their scanting has produced take-it-or-leave-it definitions ("The product AB of two transformations A and B is defined by the relation $(AB)V=B(AV) \cdots$"); patchy proofs (the uniqueness of the adjoint of a transformation and even the associativity of matrix multiplication are tacitly assumed and used without explicit mention); and unreasonable demands on the student (the hard core of the theory of determinants is airily left as a problem for the student).

I found only a very few typographical errors (all trivial), and practically no mathematical sins of *commission.*

I. H. ROSE, Hunter College

Theory of Markov Processes. By E. B. Dynkin. Translated from the Russian by D. E. Brown. Prentice-Hall, Englewood Cliffs, N. J., 1961. x+210 pp. $8.95.

By definition a Markov process is a stochastic process without aftereffects, or a non-hereditary stochastic process. Originally the impetus for studying such processes came from a number of physical problems, which were essentially diffusion processes. More applications have arisen recently, and discussions of them can be found in "Elements of the Theory of Markov Processes and their Applications" by Bharucha-Reid. At the same time the subject has been investigated by mathematicians in a rigorous and abstract fashion. Many of the intimate connections between Markov processes and the theory of differential equations, semigroups and spectral decomposition of operators have been investigated.

The book under review deals exclusively with the abstract theory of Markov processes, by an author who has himself made many important contributions to the subject in recent years. It represents the first of two volumes on the subject. The approach of the author is not intended to appeal to the novice. A thorough understanding of measure theory and some knowledge of Markov processes are presupposed; they can be found in the books on Probability Theory by Feller or Gnedenko. The book lays a complete and rigorous foundation. No applications are mentioned. The second volume, promised for the future, will deal with the connection between Markov processes and the theory of semigroups of linear operators.

HARRY HOCHSTADT, Polytechnic Institute of Brooklyn

A FORTRAN Primer. By Elliott I. Organick. Addison-Wesley, Reading, Mass., 1963. 186 pp. $3.95.

This is a pedagogically well written introductory text about a number of (but not all) versions of FORTRAN. The first part is about (a) a quick look at digital computers, (b) use of computers, (c) approach to computer use with FORTRAN, and (d) some highlights of the FORTRAN language. The second part contains (a) arithmetic expressions and substitution statements, (b) conditional control, (c) unconditional transfer and other control statements, (d) input-output, (e) the comment, dimension and equivalence declarations, (f) defining and calling internal and external functions, and (g) preparation of punch card program deck. Many worked examples and problems with solutions appear at the end of the book. Part three contains the complete solutions of seven problems. The differences in the solutions for the various machines are clearly brought out. The fourth part contains a good discussion of the newly issued FORTRAN IV, pointing out the advantages over FORTRAN II. This part also contains a few minor misprints, and in the version of FORTRAN IV actually released there are some changes not contained in the text.

I. FARKAS, University of Toronto

A Guide to ALGOL Programming. By Daniel D. McCracken. Wiley, New York, 1962. 106 pp. $3.95.

This is an excellent text both for beginners and experienced programmers who would like to familiarize themselves with the ALGOL language. The first chapter is an introduction to computers, algorithms and ALGOL. The second chapter deals with numbers, variables and expressions, the third with program organization, if-statements and Boolean variables, the fourth chapter with the for-statement, the fifth with indexed variables, the sixth with switches and blocks, the seventh with procedures, and the eighth with input and output. Every chapter contains a great number of worked examples and problems to be solved, for half of which there are complete answers at the end of the book. After careful studying of the text and solving the given problems, a student will have a good knowledge of ALGOL. Without doubt this is the best text that has appeared on ALGOL.

I. FARKAS, University of Toronto

Geometric Transformations. By I. M. Yaglom. Random House (New Mathematical Library 8), New York, 1962. vii+133 pp. $1.95.

This book does brilliantly all of the following:

1. It introduces groups of Euclidean transformations into the study of Plane Geometry.

2. It is at the level of a bright high school student who has had a term or a year of Plane Geometry. Thus, it does not overwhelm the reader with extraneous notions and it is always simple rather than rigorous when there is a nontrivial choice. (Maybe this level is suitable for the college student too.)

3. It is a natural continuation of Plane Geometry. All of its exercises can be understood by a high school student, but the solutions are best understood by using transformations. After reading the book it will be hard, for example, to see a midpoint of a segment without executing a half turn.

4. It does much "pure" transformation theory (and all geometrically—a matrix is never mentioned). Any isometry is proved to be one of the type introduced in the text, and various composition rules are stated and proved.

The format of the book encourages active participation by the reader at all times. The first half contains the short, basic text and many problems, all interesting and mostly hard. The second half contains solutions in full as well as further comments.

The only error observed was the interchange of the diagrams on page 26. The only objection imaginable by this reviewer is the choice of chapter headings, since symmetry should connote more than an improper motion. But this should not stop any high school teacher from showing this book immediately to all of his bright students.

MELVIN HAUSNER, New York University

Optimization Techniques with Applications to Aerospace Systems. Vol. 5 of Mathematics in Science and Engineering, A Series of Monographs and Textbooks. Edited by G. Leitmann. Academic Press, New York, 1962. xiii+453 pp. $16.00.

According to the foreword, the purpose of this volume is to assist those confronted by problems of systems optimization and optimal control in the choice of optimization technique. This is done by assembling ten chapters by ten authors dealing with various methods for solving such problems. Four additional chapters deal with applications. The following list of chapter titles indicates the contents: 1. Theory of maxima and minima, 2. Direct methods, 3. Extremization of linear integrals by Green's theorem, 4. The calculus of variations in applied aerodynamics and flight mechanics, 5. Variational problems with bounded control variables, 6. Methods of gradients, 7. Pontryagin maximum principle, 8. On the determination of optimal trajectories via dynamic programming, 9. Computational considerations for some deterministic and adaptive control processes, 10. General imbedding theory, 11. Impulsive transfer between elliptical orbits, 12. The optimum spacing of corrective thrusts in interplanetary navigation, 13. Propulsive efficiency of rockets, and 14. Some topics in nuclear rocket optimization.

As the title implies, the book is directed at engineers and applied mathematicians rather than at the pure mathematician. In view of the current revival of interest in the calculus of variations and its numerous applications to modern systems engineering, this volume should prove to be a useful addition to the literature on the subject.

E. K. Blum, Wesleyan University

NEWS AND NOTICES

Edited by Raoul Hailpern, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Associate Secretary, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor M. Gweneth Humphreys, Randolph-Macon Woman's College, represented the Association at the meetings of the American Council on Education held at the Mayflower Hotel in Washington, D. C. on October 2 through October 4, 1963.

Professor H. E. Woodward, Jr., Texas Technological College, represented the Association at the inauguration of Dr. Roy C. McClung as President of Wayland Baptist College on October 8, 1963.

University of Akron: Associate Professor J. M. Egar, Ball State Teachers College, has been appointed Associate Professor; Dr. Rodney Angotti, University of Pittsburgh, has been appointed Assistant Professor; Assistant Professor W. H. Beyer has been promoted to Associate Professor; Mr. Leonard Sweet has been promoted to Assistant

Professor; Professor Margaret E. Mauch retired June 1963 but will remain for the 1963–64 academic year.

University of Alaska: Professor Philip Van Veldhuizen, Sacramento State College, has been appointed Professor; Dr. R. E. Carr has been appointed Acting Head of the Department of Mathematics.

Arizona State University: Visiting Associate Professor E. E. Grace, University of Georgia, has been appointed Professor; Dr. Alvin Swimmer, St. Mary's College, and Mr. J. D. Bedient, University of Colorado, have been appointed Assistant Professors; Associate Professor Abraham Sinkov, University of Maryland, has been appointed Visiting Professor; Assistant Professor L. T. Smith has been promoted to Associate Professor.

California State Polytechnic College: Messrs. J. R. Gilbert, Lockheed Aircraft, Vandenberg Air Force Base, California, and C. T. Haskell, University of Arizona, have been appointed Assistant Professors; Associate Professor C. J. Hanks has been promoted to Professor; Assistant Professors M. L. Clinnick, K. G. Fuller and G. R. Mach have been promoted to Associate Professors.

University of California, Davis: Drs. G. D. Chakerian, California Institute of Technology, and E. J. Tully, University of California, Los Angeles, have been appointed Lecturers.

University of Colorado: Professor W. J. LeVeque, University of Michigan, has been appointed Visiting Professor; Dr. Richard Roth, University of California, Berkeley, and Dr. W. B. Jones, National Bureau of Standards, Boulder, Colorado, have been appointed Acting Assistant Professors; Assistant Professor John Hodges has been promoted to Associate Professor.

Colorado State University: Associate Professor F. M. Stein has been promoted to Professor; Assistant Professor E. R. Deal has been promoted to Associate Professor.

Drexel Institute of Technology: Assistant Professor Judith Richman, SUNY at Buffalo, has been appointed Assistant Professor; Assistant Professor J. H. Staib has been promoted to Associate Professor; Mr. R. E. Russell has been promoted to Assistant Professor; Professor F. H. M. Williams retired June 15, 1963 with the title of Professor Emeritus.

Fort Hays Kansas State College: Associate Professor Wilmont Toalson has been promoted to Professor; Miss Ellen Veed has been promoted to Assistant Professor.

Long Beach State College: Messrs. H. D. Eylar, University of Washington, and E. M. Stone, University of Wisconsin, have been appointed Assistant Professors.

Northwestern State College of Louisiana: Mr. W. C. Pine, Marshall Space Flight Center, Huntsville, Alabama, has been appointed Assistant Professor; Associate Professor S. W. Shelton has been promoted to Professor; Mr. B. R. Waldron has been promoted to Assistant Professor.

Pennsylvania State University: Dr. Paromita Chowla, University of Colorado, has been appointed Assistant Professor; Mr. Frank Kocher has been promoted to Assistant Professor.

San Antonio College: Mr. E. H. Sullivan has been appointed Chairman of the Department of Mathematics; Dr. P. R. Culwell has been appointed Dean of the College.

San Diego State College: Dr. Albert Romano, National Academy of Sciences, Washington, D. C., has been appointed Assistant Professor; Associate Professor L. J. Warren has been promoted to Professor; Assistant Professors C. R. Burton, E. I. Deaton, and R. L. Van de Wetering have been promoted to Associate Professors.

Seton Hall University: Professor L. M. Rauch, University of San Diego, has been appointed Professor; Professor M. G. Ossesia, Slippery Rock State College, has been appointed Associate Professor.

University of the South: Assistant Professors S. E. Puckette and S. A. McLeod have been promoted to Associate Professors.

Southern Illinois University, Carbondale: Mr. Robert Silber, Marshall Space Flight Center, Huntsville, Alabama, has been appointed Lecturer; Assistant Professor M. R. Kenner has been promoted to Associate Professor.

Southern Illinois University, Edwardsville: Professor A. O. Lindstrum, Knox College, has been appointed Professor; Dr. Orville Goering, IBM, Poughkeepsie, New York, has been appointed Assistant Professor; Assistant Professor C. C. Oursler has been promoted to Associate Professor.

Tuskegee Institute: Mr. C. W. Dyche, Menlo College, has been appointed Assistant Professor; Assistant Professor A. J. Scavella has been appointed Acting Head of the Department of Mathematics.

Vanderbilt University: Dr. D. J. Rodabaugh, Illinois Institute of Technology, has been appointed Assistant Professor; Professor J. A. Hyden retired June 30, 1963 with the title of Professor Emeritus.

Western Washington State College: Drs. W. B. Laffer, Ohio State University, and J. R. Reay, University of Washington, have been appointed Assistant Professors; Assistant Professor J. E. McFarland has been promoted to Associate Professor.

University of Wisconsin, Milwaukee: Assistant Professor R. H. Moore, University of Wisconsin, Madison, has been appointed Assistant Professor; Dr. G. G. Walter has been promoted to Assistant Professor.

Worcester Polytechnic Institute: Mr. B. C. McQuarrie has been promoted to Assistant Professor; Professor Edward Brown retired June 1963.

Mr. Edward Anders, Northeast Louisiana State College, has been promoted to Assistant Professor.

Rev. Matthew Audibert, OFM, St. Francis College, has been promoted to Associate Professor and appointed Chairman of the Division of Natural Sciences and Mathematics.

Professor R. W. Bagley, University of Alabama, has been appointed Associate Professor at the University of Miami.

Assistant Professor H. G. Bergmann, City College of New York, has been promoted to Associate Professor.

Dr. C. H. Boll, Aerospace Corporation, Los Angeles, California, has been appointed Associate Professor at Southern Methodist University.

Professor G. M. Ewing, University of Oklahoma, has been promoted to Research Professor of Mathematics.

Assistant Professor A. N. Feldzamen, University of Wisconsin, is on leave during the academic year 1963–64 to serve as the Executive Director of the Committee on Educational Media of the MAA.

Professor D. T. Finkbeiner, Kenyon College, has been appointed Visiting Professor at the University of Western Australia during 1964.

Assistant Professor William Forman, Brooklyn College, has been promoted to Associate Professor.

Professor J. S. Georges, Mundelein College, has been appointed Associate Professor at DePaul University.

Mr. R. M. Gordon, Arthur D. Little, Inc., Cambridge, Massachusetts, has accepted a position as Director of Data Processing at The Northrop Corporation; he has been appointed Site Director in the Nortronics Division.

Miss Camilla Hayden resigned from the faculty at the University of Toledo, effective June, 1963.

Assistant Professor Robert Kalechofsky, New York State University, Long Island Center, has been appointed Assistant Professor at Union College.

Associate Professor J. S. Klein, Wilson College, has been appointed Associate Professor at Monmouth College.

Associate Professor E. P. Merkes, Marquette University, has been appointed Associate Professor at the University of Cincinnati.

Dr. J. W. Moeller, Bell Telephone Laboratories, Whippany, New Jersey, has been appointed Assistant Professor at Case Institute of Technology.

Mrs. Dalia K. Motzkin, Philco Corporation, Willow Grove, Pennsylvania, has accepted a position as a Senior Engineer in advanced development at the Great Valley Laboratories of the Burroughs Corporation, Paoli, Pennsylvania.

Assistant Professor T. A. Newton, Washington State University, has been promoted to Associate Professor.

Associate Professor C. J. Oberist, U. S. Merchant Marine Academy, has been promoted to Professor.

Associate Professor S. C. Saxena, Atlanta University, has been appointed Associate Professor at Northern Illinois University.

Dr. J. L. Sieber, Pennsylvania State University, has been appointed Associate Professor at Shippensburg State College.

Assistant Professor W. E. Smith, on leave from the University of Colorado, has been appointed Visiting Assistant Professor of Biostatistics, School of Public Health, University of California, Los Angeles.

Assistant Professor Raymond Smullyan, Yeshiva University, has been promoted to Associate Professor.

Professor J. R. K. Stauffer, University of Rhode Island, retired January 31, 1963.

Dr. S. A. Stone, on leave from Bradford Durfee College of Technology, has been appointed Educational Planner for the Southeastern Massachusetts Technical Institute.

Mr. Alexander Weiner, Hofstra University, has been promoted to Assistant Professor.

Mrs. Nell G. Whipkey, Youngstown University, has been promoted to Assistant Professor.

Assistant Professor E. M. J. Wright, Washington University, has been appointed Lecturer at San Fernando Valley State College.

Dr. P. R. Young, Massachusetts Institute of Technology, has been appointed Assistant Professor at Reed College.

Assistant Professor R. C. Davis, University of Akron, died on June 7, 1963. He was a member of the Association for 15 years.

Visiting Professor C. A. Garabedian, University of Rhode Island, died on June 12, 1963. He was a member of the Association for 47 years.

Dr. H. V. Gosling, Kingston, Ontario, Canada, died on October 6, 1963. He was a member of the Association for 10 years.

Professor Takashi Terami, College of St. Thomas, died on July 3, 1963. He was a member of the Association for 17 years.

Professor J. I. Tracy, Yale University, died on October 7, 1963. He was a charter member of the Association.

THE UNIVERSITY OF WISCONSIN-MILWAUKEE

The University of Wisconsin-Milwaukee has just been authorized to begin a Ph.D. program in Mathematics, starting in September 1964. Although thesis work may be done in a number of subjects, the program will emphasize classical analysis and applied mathematics. A number of major appointments are now being made along these lines. Inquiries should be addressed to Professor Morris Marden.

SUMMER 1964 RESEARCH PARTICIPATION

College teachers of mathematics who wish to spend the summer (10 weeks) of 1964 doing research in mathematics or in computer science are encouraged to apply for NSF Research Participation Awards at the University of Oklahoma. Stipends vary from $750 to $1600, plus travel.

Participants will devote full time to research of their own choice using the excellent facilities at the University of Oklahoma. Each participant will enroll in at least one advanced course or seminar in mathematics or computer science (June 8–August 2) and will also be expected to contribute two lectures to a research seminar. Special informal seminars can be arranged for small groups with common interests. Preference will be given to applicants who have a project sufficiently well in mind to outline it in their application.

Academic Year Extensions of $2000 each will also be made available by NSF to support the work of several participants at their home institution during the following year—$1500 will be for direct support of the participant's research and $500 for strengthening the mathematical program of the participant's home institution by improving the library in his field of research, or other mathematical activity. These extensions can be made to holders of either the predoctoral or postdoctoral awards.

If you are interested, write to Professor Richard V. Andree, Mathematics Service Committee, University of Oklahoma, Norman, Oklahoma.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the Mathematical Association of America was held on Saturday, October 12, 1963, at North Central High School, Indianapolis. One hundred and eleven persons attended, of whom 49 were members of the Association. Chairman Harley Flanders of Purdue University presided at both morning and afternoon sessions. The meeting consisted of a symposium on the subject *Mathematics Education Abroad* and an invited hour address. Symposium discussion centered around three lectures as follows:

1. *On some aspects of Soviet mathematics education*, by Professor Izaak Wirszup, University of Chicago.
2. *The mathematical tripos and mathematical education in Great Britain*, by Professor Daniel Pedoe, Purdue University.
3. *Mathematical training in Indian universities*, by Professor Pesi R. Masani, Indiana University.

The invited hour address by Professor Timothy O'Meara of the University of Notre Dame was entitled *The Celebrated Theorem of Hasse-Minkowski in Number Theory.*

The meeting opened with a short welcoming address by Mr. Milo Eiche, Principal of North Central High School, whose mathematics staff, under the chairmanship of Mr. Allan Weinheimer, was responsible for local arrangements for the meeting.

At a short business meeting the Section voted to continue the practice of holding two meetings a year but to discontinue holding its fall meeting in conjunction with the Indiana Academy of Science.

P. T. Mielke, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fifth Summer Meeting, University of Massachusetts, Amherst, August 24–26, 1964.

Forty-eighth Annual Meeting, Denver-Hilton Hotel, Denver, Colorado, January 28–30, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pa., May 2, 1964.

ILLINOIS, Bradley University, Peoria, May 8–9, 1964.

INDIANA, Butler University, Indianapolis, May 2, 1964.

IOWA, Luther College, Decorah, April 17–18, 1964.

KANSAS, Kansas State University, Manhattan, April 18, 1964.

KENTUCKY, University of Kentucky, Lexington, Spring 1964.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Pace College, New York, April 11, 1964.

MICHIGAN, Michigan State University, East Lansing, March 28, 1964.

MINNESOTA, College of St. Thomas, St. Paul, May 9, 1964.

MISSOURI, University of Missouri, Columbia, April 18, 1964.

NEBRASKA, University of Nebraska, Lincoln, May 2, 1964.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 7, 1964.

NORTHEASTERN, Worcester Polytechnic Institute, Worcester, Mass., November 28, 1964.

NORTHERN CALIFORNIA

OHIO, University of Akron, May 9, 1964.

OKLAHOMA, East Central State College, Ada, Oklahoma, April 10–11, 1964.

PACIFIC NORTHWEST, Washington State University, Pullman, Washington, June 19, 1964.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 21, 1964.

ROCKY MOUNTAIN, Colorado College, Colorado Springs, May 1–2, 1964.

SOUTHEASTERN, The Citadel, Charleston, South Carolina, March 20–21, 1964.

SOUTHERN CALIFORNIA, San Fernando Valley State College, Northridge, March 14, 1964.

SOUTHWESTERN, New Mexico State University, University Park, March 1964.

TEXAS, Texas Technological College, Lubbock, April 10–11, 1964.

UPPER NEW YORK STATE, New York State Education Department, Albany, May 16, 1964.

WISCONSIN, Wisconsin State College, Whitewater, May 2, 1964.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN MATHEMATICAL SOCIETY, Reno, Nevada, April 18, 1964.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Maine, Orono, June 22–26, 1964.

ASSOCIATION FOR COMPUTING MACHINERY, Philadelphia, August 25–28, 1964.

CALIFORNIA MATHEMATICS COUNCIL, Northern Section, Sacramento State College, April 4, 1964.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Detroit, November 26–28, 1964.

INSTITUTE OF MATHEMATICAL STATISTICS, Berne, Switzerland, September 14–16, 1964.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Miami Beach, Florida, April 22–25, 1964.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Queen Elizabeth Hotel, Montreal, May 27–29, 1964.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, The Hotel Shoreham, Washington, D. C., May 11–14, 1964.

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FREDERICK A. FICKEN, *Editor*

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ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

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RECENT DEVELOPMENTS IN MATHEMATICS

JEAN DIEUDONNÉ, Institut des Hautes Etudes Scientifiques, Paris

About a year ago, I rather lightheartedly accepted the invitation to give this talk, [at a conference to dedicate Van Vleck Hall at the University of Wisconsin] but, as the time drew near, I began to realize the pitfalls ahead of me, and my recklessness. When so many different topics are touched upon, for every statement of mine there will necessarily be in the audience someone who knows the subject much better than I do, and who therefore will with good reason take exception to my superficial remarks. But I am afraid I will be the target of far more stringent criticism for the selection I have had to make: it is quite clear that *some* choice was imperative in such a short review, and I had therefore to decide on what was important and what was not; no objective criterion being available, I must admit that I have merely followed my own tastes. I tried, however, not to imitate certain of our colleagues, who are so entranced by the beauty of their tiny nook in some highly specialized field, that for them it is the only and unique important thing in the whole world; and to counterbalance the danger of subjectivity, I have taken into account the opinions I have heard expressed by some of the best mathematicians of our time. I confess that this is not a very democratic procedure, but I am afraid I don't believe very much in democracy, at least in scientific matters. To make things clearer, I have selected, as prominent landmarks of present-day mathematics, the solutions of some outstanding problems bequeathed to us by previous generations of mathematicians. I readily admit that there may be some point in claiming, as some will do, that more ingenuity can be spent in unraveling, say, the structure of some fancy nonassociative algebra, than in solving Hilbert's fifth problem or disproving Burnside's conjecture; but I am not very receptive to such arguments, and on these matters I will rather follow C. L. Siegel or A. Weil.

Mathematics progresses essentially in two different ways. The mathematicians whom I might call the tacticians pounce head on at a problem, using only old and well-tested tools, and they merely rely on their cleverness to give some new twist to traditional arguments, and thus reach the solution which had eluded previous attempts. The strategists, on the other hand, will never be satisfied until the concepts involved in a problem have been so thoroughly analyzed, and their connections put in such a clear light, that the final solution almost appears as a triviality; but of course this may demand lengthy and tedious developments of seemingly unrelated very general theories, which some people will deem out of proportion with the initial question.

I believe, however, that both approaches are essential to the well-being of mathematics. Excessive reliance on individual prowess, without a corresponding renewal of methods and outlook, may well end in sterility, through intensive concentration on tiny aspects of a theory, unduly magnified; and on the other hand, an exclusive lover of generality will too often lose sight of the proper motivations, and indulge in endless churning out of more and more empty theo-

ries. In the better men the two tendencies fortunately blend into a harmoniously balanced and fruitful combination, of which Hilbert is perhaps the perfect example. Let it therefore be quite clear that in following (somewhat loosely) the preceding classification, I do not intend in the least to assign higher value to one type of results over the other.

As a final preliminary matter, I had to decide what I would mean by "recent" developments. I think that in order to get a proper background we should go back approximately to the end of World War II. Future historians may well ponder over the curious fact that the end of both World Wars was accompanied by a remarkable outburst of scientific creativity, and look for sociological or psychological explanations of that phenomenon, but I think no one will doubt its reality; and one of my purposes is to point out that, although much of the mathematics done after 1945 was a natural continuation of previous work, a sizable part has consisted in radically new departures which, in my opinion, herald another era in mathematics.

Let us first turn to recent results which could have been obtained 30 or 50 years ago. Among the most notable are Roth's definitive version of the Thue-Siegel approximation theorem in the theory of numbers, and the splendid work of J. Thompson on finite groups, especially his proof (in collaboration with Feit) of the long standing conjecture of Burnside that all finite simple noncommutative groups are of even order. I understand that much of the proof of that theorem could have been found by Frobenius or Schur in the early 1900's. It is also probable (from the preliminary announcements) that the example by which Novikov has disproved another famous Burnside conjecture on finitely generated groups does not require any more modern machinery. Finally, in that same category I could put much of the startling revival which has taken place in algebraic topology and brought back some of its original geometric flavor; but it is hardly separable from other developments of a quite different origin in that field and I prefer to postpone this to a later section of my talk.

After these exciting examples of the impact of fresh imagination on old problems, there comes a somewhat different and much larger category, which I would venture to describe as the well-earned fruits of a tremendous labor, done during the period immediately preceding the one we are considering, and which aimed at reshaping mathematics according to modern standards and forging efficient tools for the new generations. I would not, however, be suspected of implying that no great effort was needed to wield those tools with success, and the examples which I am about to give would immediately dismiss such a silly notion; but it seems clear to me that these beautiful results could not possibly have been obtained, nor even sometimes formulated, without the fundamental concepts of modern algebra, topology, and topological vector spaces, as they were laid down between 1920 and 1940.

The typical example which immediately comes to mind is the solution in 1951 by Gleason and Montgomery-Zippin of Hilbert's fifth problem, where Haar measure and F. Riesz's characterization of finite dimensional vector spaces

come as the "deus ex machina" to clinch the argument. Another application of Haar measure, perhaps less well-known but to my mind even more remarkable, is the recent work of Tamagawa on "adelic groups": developing earlier ideas of A. Weil, Iwasawa and Tate, it gives an extraordinarily simple and striking formulation, in the terminology of locally compact groups and measure theory, to the main theorems of the classical theory of algebraic numbers, including Siegel's monumental work on quadratic forms; and as usual this new formulation immediately opens the way to unsuspected generalizations. I would not leave this topic without mentioning the closely related and no less remarkable results obtained in 1961 by A. Borel and Harish-Chandra on discrete subgroups of semi-simple Lie groups, which at last clarify and unify the various "finiteness" theorems of the "arithmetic theory of forms," going back to Hermite and Jordan, by formulating them in their proper setting.

The phenomenal growth of the theory of partial differential equations, during the last 10 years, can also be taken as an excellent example of the impact of the general theory of topological vector spaces on classical analysis. Here the catalyst undoubtedly was the theory of distributions, although much of the technique is of earlier origin. Schwartz's theory itself had had many forerunners, and indeed it may best be compared to what we call the "invention of Calculus": it is quite clear that long before Newton and Leibniz, practically all prominent mathematicians of Europe around 1650 could solve most of the problems where elementary calculus is now used; but instead of having a ready made tool universally applicable, they had to resort to *ad hoc* considerations in each instance. Similarly, most of the problems which belong to the theory of distributions had been considered and essentially solved before Schwartz, but no one had succeeded in building up a formalism which would dispense of special arguments in each particular case. This was made possible by the theory of topological vector spaces, and although many other approaches to distribution theory have since been proposed, none offers, in my opinion, the flexibility and power of the original description of Schwartz. The applications of these new ideas, and in particular the extended range offered to the convolution product and the Fourier transformation, were not long in making themselves felt; I need only mention here the work of Gårding, Hörmander, Malgrange, Ehrenpreis, Lojasiewicz, Calderon and many others, which has taught us so much on the general properties of linear partial differential equations, especially on existence and uniqueness problems, now essentially solved for systems of arbitrarily high order with constant coefficients; I don't think anybody would seriously claim that these results could have reached the same scope and generality, had it not been for the new basic ideas in functional analysis.

The same may be said of the theory of group representations in infinite dimensional spaces, which has been so brilliantly developed in the 1950's, after the pioneering work of von Neumann and the Gelfand school had built up the necessary algebraico-topological machinery. Time prevents me from giving any detail here, but I think that what may rightly be regarded as the climax of these

efforts, the long (and still unfinished) series of papers by Harish-Chandra on representations of Lie groups can hardly be matched by any contemporary mathematical work in depth and originality, linking in a grandiose synthesis (in which distribution theory is an essential tool) Lie algebras, harmonic analysis and partial differential equations.

The next item in my second category is the algebraic geometry of what I may call the Weil-Zariski period, roughly 1945–1955; the main problem here was to build up from scratch the algebraic geometry over *arbitrary* fields (in particular fields of characteristic $\neq 0$), not out of a mere desire for greater generality, but because this had become imperative for a better understanding of diophantine analysis. The challenging difficulty was to find a substitute for the deep geometric insight of earlier generations, in particular the brilliant Italian school, which would work in this vastly expanded context, and at the same time rest on less shaky foundations. Earlier attempts, notably by van der Waerden, did not quite reach that goal, the chief trouble lying in the theory of intersections, which was finally mastered by A. Weil in his difficult "Foundations"; whereas in a different direction, Zariski was patiently exploring new algebraic and topological concepts which, as we now see it, lay down the groundwork for still better things to come. Here again I cannot give any detail on the immediate consequences of their work (in their own papers and those of their followers Rosenlicht, Matsusaka, Lang, Nagata, Chow, Igusa, Néron, etc.). The most conspicuous success, of course, was the famous proof by A. Weil in 1948 of the so-called "Riemann hypothesis for curves over a finite field." Another great conquest of the new algebraic geometry was the development of the theory of algebraic groups, practically nonexistent before 1945, which reached complete maturity in less than 15 years. It was due essentially to the work of 3 men: A. Weil, who single-handed created the general theory of abelian varieties over fields of arbitrary characteristic, and A. Borel and C. Chevalley, who did the same for linear algebraic groups, bringing their theory to the same high level of perfection which had been achieved by E. Cartan and H. Weyl 30 years earlier for semi-simple Lie groups over the real or complex field. I can give here, unfortunately, only a fleeting mention of a closely related work, the remarkable Tohoku paper of Chevalley in 1955, which for the first time, and in the most unsuspected way, established a bridge between Lie theory and the theory of finite simple groups, thus opening a rapidly expanding new field where already many beautiful results have been obtained, and many more undoubtedly still lie ahead of us.

At this point, to remain faithful to my program, I should speak of the developments of "analytic geometry" and differential geometry between 1945 and 1953, since they also were a natural continuation of the earlier fundamental work of Oka and H. Cartan on the one hand, E. Cartan, Whitney, Chern, Pontrjagin, etc. on the other. But this would be highly artificial, as there is here no such sharp difference in outlook as between the two periods of algebraic

geometry, and they will much better be tackled as a whole in the third section of this talk, to which I now proceed.

It is certainly too early to pass final judgments in this matter, but I would readily venture to predict that the main fact about our time which will be emphasized by future historians of mathematics is the extraordinary upheaval which has taken place in and around what was earlier called algebraic topology. To evaluate the magnitude of this transformation, just recall that of the two most lively entries in each present-day issue of Mathematical Reviews, homological algebra and differential topology, one was totally nonexistent 20 years ago, and the other was practically limited to De Rham and Hodge's theorems. Hardly a year now passes without bringing the solution of some famous old problem which seemed tantalizingly out of reach: the Hauptvermutung has been disproved (Mazur-Milnor); the Poincaré conjecture is now a theorem except in dimensions 3 and 4 (Smale-Stallings); we know that spheres may have several distinct differential structures and we are even able to compute their number in many cases (Milnor-Smale), whereas on the other hand there exist topological manifolds with no differential structure at all (Kervaire); the exact number of linearly independent vector fields on a sphere is now entirely determined (J. Adams); so are the parallelizable spheres and those which may admit a complex structure; more and more refined criteria are now available for problems of embeddings, of unknotting of manifolds, of extensions of homeomorphisms, etc., etc. It should be stressed, of course, that these results rest primarily on fundamental concepts developed before 1945, such as fiber spaces, homotopy groups, characteristic classes, Whitney's work on differentiable manifolds and the Morse theory of the calculus of variations in the large. As already said, much of the recent progress lies in a better and deeper use of these tools, of which Thom's famous work on cobordism, Bott and Smale's remarkable applications of the Morse theory, and the quite recent work of Milnor on microbundles are shining examples. This is the new geometrical trend in differential topology which I mentioned earlier; it was preceded by the deep results in 3-dimensional topology by men like Moise, Bing, and Papakyriakopoulos, and it has been followed by a similar renewal in combinatorial topology, with the work of J. H. C. Whitehead (a pioneer in this field), Mazur, Stallings, Morton Brown and Zeeman among others. A little earlier, around 1950, the algebraic aspects of topology had also been considerably enriched by the introduction of a whole panoply of new tools, such as Steenrod's reduced powers, Bockstein operators, Postnikov systems, Eilenberg-MacLane spaces, loop spaces, and Whitehead products, to name only a few.

All this, however, is only one half of the story, and in spite of its remarkable successes, it is not to me the most impressive half. The new methods and concepts which I have just mentioned are topological in nature, and were introduced to solve topological problems. What was totally unexpected in 1940 is that the methods of algebraic topology could be bodily transplanted to a host

of mathematical situations in algebra and analysis. I refer of course to homological algebra, which is now in the process of invading the whole of mathematics, just as group theory and linear algebra did 50 years ago.

Homological algebra, as you know, started in fact as a kind of glorified linear algebra, by introducing concepts such as the Ext and Tor functors, which in a way measure the manner in which modules over general rings misbehave when compared to the nice vector spaces of classical linear algebra; and the similarity with homology groups, which tell us how much a complex deviates from being acyclic, is now commonplace. But it took some time to realize that similarity and express it in mathematical terms, and it was only after the discovery, in the early 1940's, of the cohomology exact sequence (by several topologists working independently), the introduction of resolutions by H. Hopf, the definition of the Ext functor by Eilenberg-MacLane in 1942, of the Tor functor by H. Cartan in 1948, that finally H. Cartan and Eilenberg welded these scattered results into a general theory, and had the courage to write the first book on it in 1955. It is a measure of the success and vitality of this theory that the book already should be completely rewritten to take care of a lot of new tools introduced since then in homological algebra, notably the very recent Grothendieck groups and rings; above all, of course, it should now deal with the general concept of *abelian category*, of which the category of modules is only a particular case.

It seems likely, however, that the impact of these new ideas on other parts of mathematics would have been less spectacular, if the introduction of sheaves by J. Leray, at the very same moment, had not enormously increased their scope, by giving for the first time a workable mathematical formalism for the intuitive concept of "variation" of structures. Leray himself had in mind chiefly applications to the topology of manifolds, and in particular fibre bundles, doing away with the cumbersome triangulations of former methods; and it may be recalled here that he also created for the same purpose the concept of spectral sequence, one of the most powerful tools of homological algebra. That he was on the right track was immediately evident from the results he and H. Cartan were able to derive in homology theory, followed in a short time by Serre's breakthrough in the theory of homotopy groups, A. Borel's thesis on the cohomology of Lie groups and a little later J. Adams' deep work on the relations between homology and homotopy. But within a few years the tremendous flexibility and versatility of these new tools had revolutionized other fields as well. First came "analytic geometry," where H. Cartan and Serre, working in close collaboration, and followed in quick succession by Kodaira-Spencer, Hirzebruch, Grauert, and Remmert, found in the concept of coherent sheaf a wonderfully well adapted means of expressing the Cartan-Oka results in a simple and powerful formulation. This led rapidly to the solution of some key problems of that fast expanding branch of mathematics, such as E. Levi's conjecture concerning the characterization of domains of holomorphy and the imbedding problem for analytic manifolds, to name only two questions which had remained unsolved for a long time. These results were already having exciting consequences in classical al-

gebraic geometry, such as the cohomological characterization of algebraic varieties by Kodaira and the now famous generalization of the Riemann-Roch theorem by Hirzebruch. Then, in his celebrated FAC paper of 1955, Serre discovered that the same methods which worked so well in the analytic case could be adapted to a seemingly much more algebraic and "abstract" situation, namely algebraic geometry over an arbitrary field, by using the Zariski topology to carry over the whole geometric machinery of "ringed spaces" to algebraic varieties as defined by A. Weil. This, as you know, has been pushed farther ahead with enormous energy by A. Grothendieck, ushering in the present era of algebraic geometry, with such a wealth of new concepts, methods and problems that several generations of mathematicians may well find the job of their lives in exploring all the fascinating possibilities of this vast and still largely uncharted territory. After only a few years, this new approach has already yielded such substantial dividends as Serre's proof of the Severi conjecture. Grothendieck's generalization of the Riemann-Roch-Hirzebruch formula, the resolution of singularities (in the case of characteristic 0) by Hironaka, the solution by Mumford of the problem of "moduli" for algebraic curves over an arbitrary field, and very recently some decisive progress towards the Weil conjectures by Grothendieck and M. Artin. It should be stressed that much of this was made possible by another breakthrough, where Serre again was the outstanding pioneer, the application of homological algebra (and in particular the introduction of the concept of *flatness*) to the theory of local rings, crowning 20 years of achievements in that field by Krull, Chevalley, Zariski, Samuel, Nagata, I. Cohen and M. Auslander-Buchsbaum, and making available its powerful results to the Grothendieck theory.

There are still other important applications of homological algebra, for instance Tate's formulation of class-field theory in cohomological terms, immediately following the pioneering work of Hochschild, Nakayama and A. Weil in that direction, and which is now blossoming forth in the very promising theory of Galois cohomology in the hands of Tate himself, Lang, Serre, A. Borel, M. Kneser, and M. Lazard to name only a few. In a totally different direction, the great novelties last year in the theory of linear partial differential equations were Malgrange's success in giving a very neat and useful formulation of some key results in terms of Ext functors, and applications of the Grothendieck groups to elliptic equations by Atiyah-Singer. All the experience of the last 10 years suggests that we may expect important consequences of their ideas, and that we may look forward with confidence to other conquests of homological algebra in areas of mathematics still untouched by it.

I would like to conclude with some general remarks. The first one, and the one I would like to emphasize most, is that, despite the tremendous surge and somewhat bewildering diversity of unpredictable developments during the last 20 years, mathematics is now more *unified* than it ever was. Of course, striking examples of the deep kinship of various parts of mathematics have not been unknown to classical times, from the use of Dirichlet's series in number theory to

Riemann's introduction of topology in function theory. But we now have reached a point where it is practically impossible to apply to a large part of contemporary mathematical literature any one of the old labels of "algebra," "analysis" or "geometry." Algebraic geometry and "analytic geometry" already behave like identical twins, any advance in one being almost invariably matched by the corresponding theorem in the other within a short time; on the other hand, the merger of commutative algebra and algebraic geometry is all but complete, and the theory of algebraic numbers is confidently expected to fall in line within a few years. Some of the most remarkable theorems derive from the successful confrontation of two seemingly unrelated theories: it was by translating the "Riemann hypothesis" for curves over a finite field in purely geometric terms that A. Weil realized that what was needed was the "abstract" counterpart of Lefschetz's fixed point formula in topology, finally succeeded in forging in that way his famous proof, and was further led to the formulation of his conjectures, the proof of which is expected to give us at last general methods of attack in diophantine analysis. Still more revealing is the history of the "Riemann-Roch-Hirzebruch-Grothendieck" theorem. Around 1950, Kodaira understood that the Riemann-Roch theorem for classical algebraic varieties of dimension 2 and 3 could be formulated as an *equality* between topological invariants of the variety instead of an inequality as with Zariski and the Italian geometers; but he still lacked the machinery to extend these results to higher dimensional cases. A little later, he realized that the topological invariants he needed were linked to the properties of vector bundles over complex manifolds (Chern classes, Pontrjagin classes); as soon as Serre and Cartan started using coherent sheaves in that theory, Kodaira saw that this gave him one of his essential tools and in a few months he had (in partial collaboration with Spencer, and independently of Serre and Cartan) obtained far reaching results in that theory, using in particular theorems from the theory of elliptic partial differential equations. In 1952, Hirzebruch became interested in the problem; by ingenious algebraic devices, he linked the Chern classes of vector bundles on algebraic varieties to earlier invariants introduced in algebraic geometry by Eger and Todd, and a little later Serre was able to guess what the formulation of the general Riemann-Roch theorem should be. But a proof was still lacking, and it was obtained only in 1954 by Hirzebruch *via* the use of yet another set of ideas, this time Thom's cobordism theory, which had just appeared and gave just the information on Pontrjagin classes needed to fill in the gap. Such a bewildering maze of arguments was not very satisfactory, and Hirzebruch himself was aware that simpler and better proofs could probably be obtained. This was done in 1957 by Grothendieck, who kept the essential algebraic and homological ideas of his predecessors, but could dispense with all the machinery coming from harmonic forms or differential topology; this enabled him, not only to extend the formula to abstract algebraic geometry, but also to show that it was a special case of a still more general and simpler one. To do that, however, he had in particular to introduce new tools in homological algebra, what are now called the Grothen-

dieck groups and rings, whose latent potentialities immediately attracted attention. The first step in that direction has been made by Hirzebruch and Atiyah, in an extraordinary combination where this time the new ingredient is, of all things, Bott's periodicity theorem for the homotopy groups of simple Lie groups. This has enabled them to give an extension of the Riemann-Roch theorem to differentiable manifolds; it has also provided Adams with the necessary tools for his solution of the problem of vector fields on spheres. Still more surprising is the way in which Atiyah has used these concepts in the theory of representations of finite groups, and H. Bass is now exploring with success the application of similar methods to projective modules and linear groups over Dedekind rings. This is where we stand today, and we certainly are still very far from the end of the story; but I hope this is enough to show you how complex and fruitful the interaction now is of mathematical ideas coming from every conceivable quarter.

It is sometimes feared (even by young graduates) that these powerful tendencies towards complete unification of the various branches of mathematics may in the end be self-defeating, through sheer mental impossibility to get a firm and competent grasp of so many different ideas and theories at the same time. Fortunately it seems that, as it has been the case in similar "Sturm und Drang" periods in the history of our science, the bewildering diversity of the new concepts naturally produces by reaction a search for a simplification of the unwieldy mess. This time it seems that our salvation will come from the new concepts of categories and functors, introduced in the early 1940's by Eilenberg and MacLane; they have already demonstrated their versatility and usefulness in the work of men like Eckmann, Hilton, Kan and Grothendieck, and many younger mathematicians are now engaged in this work of concentration and simplification, which on a new level repeats the story of algebra and topology of 40 years ago. Of course, as always, the price to be paid is in more "abstraction"; but it is now a well-established phenomenon that what is highly abstract for a generation of mathematicians is just commonplace for the next one, and the cries of anguish one still hears from time to time usually come from older men visibly afraid of being unable to catch up with the younger set. One tends, however, to be a little more impatient towards this manifestation of human frailty than, say, 30 years ago and the traditional jokes about "hard" and "soft" mathematics have now become a little stale. It is of course very easy, from the heap of axiomatic trash dumped every year by would-be mathematicians on the unhappy public, to select some particularly nonsensical paper and exhibit it as the typical product of modern mathematics; I leave it to you to judge whether such an attitude is compatible with even a minimum of intellectual honesty and whether those who indulge in it should not have the decency to remain silent until they have made the effort of getting more accurate information.

As a final remark, I would like to stress how little recent history has been willing to conform to the pious platitudes of the prophets of doom, who regularly warn us of the dire consequences mathematics is bound to incur by cut-

ting itself off from the applications to other sciences. I do not intend to say that close contact with other fields, such as theoretical physics, is not beneficial to all parties concerned; but it is perfectly clear that of all the striking progress I have been talking about, not a single one, with the possible exception of distribution theory, had anything to do with physical applications; and even in the theory of partial differential equations, the emphasis is now much more on "internal" and structural problems than on questions having a direct physical significance. Even if mathematics were to be forcibly separated from all other channels of human endeavour, there would remain food for centuries of thought in the big problems we still have to solve within our own science.

Indeed our wealth is now so great that even a cursory inventory of it could not possibly be made within the span of a single talk, and I woefully realize how much valuable work I had to leave reluctantly aside in such fields as complex multiplication (A. Weil, Shimura, Taniyama), nonlinear partial differential equations, infinite Lie groups (Chern, Kuranishi, Sternberg), Riemann surfaces (Teichmüller, Ahlfors, L. Bers), potential theory (Deny, Beurling, Hunt, Choquet), harmonic analysis (Kahane, Katznelson, Rudin, Helson, Malliavin and many others), to say nothing of mathematical logic or probability theory, where my ignorance prevents me from entering at all. I hope at least to have given you some faint idea of the tremendous progress accomplished during the last 20 years; no other comparable period of our history has been so rich in new ideas and results, and we have every reason to be confident that the future will ever more fulfill Hilbert's motto: "We must know and we shall know."

BANACH ALGEBRAS AND THEIR APPLICATIONS

B. R. GELBAUM, University of Minnesota

0. This article discusses Banach algebras that are related to the special examples I–III below. The aim is to indicate on the one hand the extent of the classes typified by these examples and on the other hand the generality of techniques and results that lead to well-known classical theorems.

The bibliography consists of books and expository papers on the subject. Detailed references of a more technical nature will be found in these sources.

1. The examples in question are:

I. $C([0,1]) \equiv \{f(x): f(x)$ continuous, complex-valued on $[0,1]\}$;

II. $L_1(\mathbf{R}) \equiv \{f(x): f(x)$ complex-valued and Lebesgue-integrable on $\mathbf{R} \equiv (-\infty, \infty)\}$;

III. $\mathcal{E}(\mathbf{C}^m) \equiv \{T$: T a linear transformation (endomorphism) of the vector space $\mathbf{C}^m$, of all m-tuples of complex numbers, into itself$\}$.

As sets of functions with ranges in $\mathbf{C}$-modules ($\mathbf{C}$ resp. $\mathbf{C}^m$), I–III have naturally associated notions of addition and multiplication by complex scalars.

Ring multiplication in I is defined by pointwise multiplication, in III by endomorphism products. Although II is a function space, its (ring) multiplication is not that of the pointwise product of functions (the pointwise product of two functions in $L_1(\mathbf{R})$ need not be in $L_1(\mathbf{R})$!). Having its origins in the classical group algebras of finite groups, $L_1(\mathbf{R})$ has *convolution* as its (ring) multiplication: if $f, g \in L_1(\mathbf{R})$, then their product $h = f * g$ is given by the formula

$$h(x) = \int_{-\infty}^{\infty} f(y)g(-y+x)dy.$$

Hence I–III have reasonable algebraic structures. Their topologies are drawn from metrics based on invariant norms:

I. dist $(f, g) \equiv \sup_{x\in[0,1]} |f(x)-g(x)| \equiv \|f-g\|_\infty$;

II. dist $(f, g) \equiv \int_{-\infty}^{\infty} |f(x)-g(x)|dx \equiv \|f-g\|_1$;

III. dist $(T, S) \equiv \sup_{\|x\|\leq 1} \|(T-S)x\| \equiv \|T-S\|$;

where in III, for $x = (x_1, \cdots, x_m) \in \mathbf{C}^m$, $\|x\|^2 \equiv \sum_{i=1}^m |x_i|^2$.

The metric defined in II is degenerate ($\|f-g\|_1 = 0$ does *not* imply $f = g$), but if functions that are equal almost everywhere are identified, the metric loses its degeneracy.

In I–III, the metric spaces are complete (Cauchy sequences converge).

In each of I–III, there is a natural *involution* (an automorphism of period two):

I*. $f^*(x) \equiv \overline{f(x)}$;

II*. $f^*(x) \equiv \overline{f(-x)}$;

III*. T^*;

where in III*, T^* is the adjoint linear transformation arising from T, (matrically T^* is the conjugate transpose) and is uniquely defined by the requirement

$$(Tx, y) = (x, T^*y)$$

for all $x, y \in \mathbf{C}^m$. (For $u = (u_1, \cdots, u_m)$, $v = (v_1, \cdots, v_m) \in \mathbf{C}^m$, $(u, v) \equiv \sum_{i=1}^m u_i\bar{v}_i$. Note: $\|u\| = \sqrt{(u, u)}$.)

The algebra, topology and involution are related in the following way: If $\| \cdots \|$ stands for any of the functions $\|\cdots\|_\infty$, $\|\cdots\|_1$, $\|\cdots\|$ of I, II, III, if a, b are a pair of elements from any one of I, II, III, if λ is a complex number, and if * stands for any of the three involutions, then

$$\begin{aligned} \|a+b\| &\leq \|a\| + \|b\| \\ \|\lambda a\| &= |\lambda| \, \|a\| \\ \|ab\| &\leq \|a\|\|b\| \\ \|a^*\| &= \|a\|, \end{aligned}$$

where ab stands for the product of a and b as defined on the relevant space. Consequently $a+b$ and ab are continuous functions of the pair a, b, λa is a continuous function of the pair λ, a, and involution is continuous.

The distillate of all the above may be described as follows: A complete normed algebra A over **C**. (We shall discuss involutions later.) Such an algebra A is called a Banach algebra. As algebras, I and II are commutative; III is not commutative unless $m=1$.

Of course, I–III are prototypes of generalizations that are themselves special cases of Banach algebras. If X is a locally compact Hausdorff space, G a locally compact group, $\mathcal{H}$ a Hilbert space, then one may define:

I′. $C_\infty(X) \equiv \{f(x): f(x)$ continuous complex-valued and vanishing at ∞ on $X\}$;

II′. $L_1(G) \equiv \{f(x): f(x)$ complex-valued and Haar-integrable on $G\}$;

III′. $\mathcal{E}(\mathcal{H}) \equiv \{T: T$ a bounded linear transformation (endomorphism) of $\mathcal{H}$ into itself$\}$.

For I′, "vanishing at ∞" means for $f(x)$ that for each $\epsilon>0$, $\{x: |f(x)| \geqq \epsilon\}$ is compact; for II′ "Haar-integrable" means that $f(x)$ is measurable with respect to the σ-ring **K** generated by the compact sets of G and that with respect to the (essentially unique) group-invariant (translation-invariant) Haar-measure defined on the sets of **K**, $f(x)$ is absolutely integrable; for III′, "bounded" means for T that

$$\sup_{\|x\| \leqq 1} \|T(x)\| \equiv \|T\| < \infty.$$

Convolution in $L_1(G)$ is defined by the formula:

$$f*g(x) = \int_G f(y)g(y^{-1}x)dy.$$

All other concepts are carried over in direct analogy from I–III to I–III′. III′ always has an identity; I′ has an identity ($f(x) \equiv 1$) iff X is compact; II′ has an identity iff G is discrete (then $f(e)=1$, $f(x)=0$, $x \neq e$, where e is the group identity, is the identity for convolution).

2. If a Banach algebra A is *commutative*, and if M is a modular maximal ideal (i.e. A/M has an identity) then A/M *qua* Banach algebra is isometrically isomorphic to **C** because (Gelfand-Mazur) it is a normed division algebra over **C**. Hence each $a \in A$ is mapped into the complex number a/M denoted by $\hat{a}(M)$. As M varies over the set $\mathfrak{M}$ of all modular maximal ideals of A, $\hat{a}(M)$ assumes a set of complex values and $\hat{a}(M)$ is thus regarded as a function on $\mathfrak{M}$. $\mathfrak{M}$ is then given the weakest (weak*) topology relative to which all such functions $\hat{a}(M)$ are continuous. The association $a \rightarrow \hat{a}(M)$ between elements a of A and functions $\hat{a}(M)$ is an algebraic **C**-homomorphism. $\mathfrak{M}$ in its weak* topology is a locally compact Hausdorff space and each $\hat{a}(M)$ is in $C_\infty(\mathfrak{M})$. Finally $\|\hat{a}\|_\infty \leqq \|a\|$.

The association $a \rightarrow \hat{a}(M)$ is one-one under special circumstances and when these prevail, A is called semisimple. Even then, the collection $\hat{A} \equiv \{\hat{a}(M) \mid a \in A\}$ may be a proper subset of $C_\infty(\mathfrak{M})$. The situation is clarified by the discussion of involutions.

If there is an involution * defined in A and subject to

1. $a^{**}=a$
2. $(\alpha a)^*=\bar{\alpha}a^*,\quad \alpha\in\mathbf{C}$
3. $(ab)^*=b^*a^*$
4. $(a+b)^*=a^*+b^*$
5. $\|a\|^2=\|aa^*\|$

then, when A is commutative, the association $a\to\hat{a}(M)$ is a one-one isometric epimorphism: $\|a\|=\|\hat{a}\|_\infty$. In this case $\hat{a}^*(M)=\overline{\hat{a}(M)}$. (More generally, if, for each $a\in A$, there is a $b\in A$ such that $\hat{b}(M)=\overline{\hat{a}(M)}$, then $\hat{A}$ is dense in $C_\infty(\mathfrak{M})$.) It is a simple matter to verify that $1-4$ are satisfied in I, I′, II, II′, III, III′, and 5 in I, I′, III, III′.

3. When G is an abelian locally compact group and $A=L_1(G)$ (II′) then for each $M\in\mathfrak{M}$ and $f\notin M$ the formula $\hat{f}_x(M)/\hat{f}(M)$, where $(f_x(y)=f(xy))$ defines a function $\alpha_M(x)$ independent of f (so long as $f\notin M$) and α_M satisfies:

(i) $\alpha_M(xy)=\alpha_M(x)\cdot\alpha_M(y)$;
(ii) $|\alpha_M(x)|=1$;
(iii) $\alpha_M(x)$ is continuous in the pair x, M.

In other words, each M gives rise to a *character* α_M mapping G continuously and homomorphically into $\mathbf{T}\equiv\{z\colon |z|\equiv 1\}$, viewed as a group relative to multiplication. The association $M\to\alpha_M$ is one-one and each character α of G is so engendered. This means there is a one-one natural map between $\mathfrak{M}$ and $\hat{G}$, the *character group* of G. Finally if we identify $\mathfrak{M}$ and $\hat{G}$ in this case, then (writing $\alpha_M(x)=(\alpha,x)$) we find

$$\hat{f}(\alpha)=\int_G f(x)\overline{(\alpha,x)}dx.$$

$\hat{G}$, endowed with the locally compact topology inherited from $\mathfrak{M}$, is actually a topological *group*. If $\alpha,\beta\in G$, the product $\alpha\beta$ and the inverse α^{-1} are defined by

$$(\alpha\beta,x)=(\alpha,x)(\beta,x)$$
$$(\alpha^{-1},x)=\overline{(\alpha,x)}=(\alpha,x)^{-1}$$

for $x\in G$. The product $\alpha\beta$ is continuous in the pair α, β and the inverse α^{-1} is continuous in α.

The following table shows the situation for the elementary and basic groups of the subject ($\mathbf{Z}\equiv$the set of integers):

Group (G)	Typical element	Character Group ($\hat{G}$)	Typical element	Mapping $G\to\mathbf{T}$
$\mathbf{R}$	x	$\mathbf{R}$	t	$x\to e^{2\pi itx}$
$\mathbf{Z}$	n	$\mathbf{T}$	$z=e^{2\pi i\theta}$	$n\to z^n=e^{2\pi in\theta}$
$\mathbf{T}$	$z=e^{2\pi i\theta}$	$\mathbf{Z}$	n	$z\to z^n=e^{2\pi in\theta}$

In all the cases above $\hat{\hat{G}}=G$, and this relationship is valid for an arbitrary locally compact abelian group G (see A below).

Clearly the classical theories of Fourier series and Fourier integrals for functions of finitely many real (or complex) variables are thus subsumed by the general theory of $L_1(G)$.

If we set

$$L_2(G) \equiv \left\{f(x): f(x) \text{ complex-valued and Haar-measurable, } \int_G |f(x)|^2 dx < \infty\right\}$$

(an example of Hilbert space), even the "L_2-theory" may be covered.

For example, the classical Plancherel theorem now reads:

If $f \in L_1(G) \cap L_2(G)$ then $\hat{f} \in L_2(\hat{G})$ and

$$\|f\|_2^2 = \int_G |f(x)|^2 dx = \int_G |\hat{f}(\alpha)|^2 d\alpha = \|\hat{f}\|_2^2.$$

This isometry carrying the dense subset $L_1(G) \cap L_2(G)$ of $L_2(G)$ onto a subset of $L_2(\hat{G})$ can be extended to all $L_2(G)$. Its image is then all $L_2(\hat{G})$. In this sense the L_2-Fourier-transform exists and is invertible.

A special case of the Plancherel theorem for $G = \mathbf{T}$, $\hat{G} = \mathbf{Z}$ reads:

$$\sum_{n=-\infty}^{\infty} |\hat{f}_n|^2 = \int_0^{2\pi} |f(x)|^2 d(x/2\pi) \qquad (\textit{Parseval's relation}),$$

for the Fourier coefficients $\{\hat{f}_n\}$ *of* f, $\hat{f}_n = \int_0^{2\pi} f(x) e^{-inx} d(x/2\pi)$.

Other consequences of this development are:

A. THE PONTRJAGIN DUALITY THEOREM: $\hat{\hat{G}} = G$; *G is discrete iff $\hat{G}$ is compact; G is neither discrete nor compact iff $\hat{G}$ is neither discrete nor compact.*

B. THE INVERSION FORMULA: *There is a directed set $\{u\}$ of functions in $L_1(G)$ such that for any $f \in L_1(G)$*

$$\left\| f - \int_G \hat{f}(\alpha) \hat{u}(\alpha)(\alpha, x) d\alpha \right\|_1 \to 0$$

on the directed set $\{u\}$. This means that $f \to \hat{f}$ is a one-one correspondence.

C. THE GENERALIZED BOCHNER THEOREM: *If $p(x)$ is a positive definite function on G*

$$\left(\int_G \int_G p(x-y) f(x) \overline{f(y)} dx dy \geqq 0 \textit{ for every continuous } f \textit{ with compact support}\right)$$

then there is a measure μ defined on the σ-ring generated by the compact sets of $\hat{G}$ such that

$$p(x) = \int_{\hat{G}} (\alpha, x) d\mu(\alpha).$$

D. THE RIEMANN-LEBESGUE THEOREM: *The Fourier transform $\hat{f}$ of a function f in $L_1(G)$ (G locally compact abelian) vanishes at infinity. This follows from the general fact that if a belongs to a commutative Banach algebra then $\hat{a}$ belongs to $C_\infty(\mathfrak{M})$.*

E. WIENER'S TAUBERIAN THEOREM: *If $f \in L_1(G)$ (G locally compact abelian) and if $\hat{f}$ never vanishes, then any ideal containing f is dense in $L_1(G)$. This is a consequence of the statement that every closed proper ideal of $L_1(G)$ is contained in some modular maximal ideal.*

4. The classical problem of reducing an $n \times n$ Hermitian matrix $M=(m_{ij})$, where $m_{ij}=\bar{m}_{ji}$, or an $n \times n$ normal matrix $N=(n_{ij})$, where $NN^* \triangleq N^*N$, to diagonal form leads to the following solution: There are n complex numbers $\{\lambda_i\}$ (not necessarily distinct and in the Hermitian case real) and n orthonormal basis vectors $\{e_i\}$ for $\mathbf{C}^n$ such that

$$Te_i = \lambda_i e_i,$$

where T is the operator corresponding either to M or to N.

If we order the complex plane $\mathbf{C}$ by: $x_1+iy_1=z_1<z_2=x_2+iy_2$ iff $x_1 \leqq x_2$, $y_1<y_2$, we may for each complex number z define a projection operator E_z by the prescription: If $v \in \mathbf{C}^n$ is written in terms of the basis $\{e_i\}$

$$v = \sum_{i=1}^{n} \alpha_i e_i \qquad (\alpha_i = (v, e_i))$$

then

$$E_z v = \sum_{\lambda_i < z} \alpha_i e_i.$$

This readily implies that $E_z = E_z^* = E_z^2$ for all z. Furthermore,

$$Tv = \sum_{i=1}^{n} \alpha_i T e_i = \sum_{i=1}^{n} \alpha_i \lambda_i e_i.$$

If T is Hermitian and the λ_i are real, we may assume their numbering is such that $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ and the formula above becomes

$$Tv = \sum_{i=1}^{n} \lambda_i (E_{\lambda_i} - E_{\lambda_{i-1}}) v,$$

which we write in the suggestive notation

$$Tv = \int_{\mathbf{R}} \lambda dE_\lambda v \quad \text{or} \quad T = \int_{\mathbf{R}} \lambda dE_\lambda.$$

A similar discussion for the case of where T is normal yields

$$T = \int_{\mathbf{C}} z dE_z,$$

and this formula includes the preceding one as a special case.

We note that for any polynomial P

$$P(T) = \sum_{i=1}^{n} \alpha_i P(\lambda_i) e_i$$

or

$$P(T) = \int_{\mathbf{C}} P(z) dE_z.$$

This formalism can be extended to functions more general than polynomials and provides the basis for a functional calculus of (finite-dimensional) operators.

It is essential to note that the ring of polynomials in T is thereby made isomorphic to a ring of complex-valued functions on the set $\{\lambda_i\}$, the *spectrum* of T. It is appropriate then to regard the diagonalization problem for T as the problem of finding a function-ring isomorphic to the ring of polynomials for T. This formulation is extremely useful in dealing with operators in Hilbert space where the "matrix-basis" approach is unyielding. The theory of Banach algebras provides a handy device for constructing the desired isomorphism.

If T is a bounded Hermitian or normal operator on a Hilbert space ($T \in \mathcal{E}(\mathcal{H})$ and $T = T^*$ or $TT^* = T^*T$) then the smallest closed algebra $\mathcal{A}$ containing all polynomials in I (the identity), T, and T^* is a commutative Banach algebra with an involution * satisfying the requirements 1–5 above. Thus $\mathcal{A}$ is isometrically isomorphic to $C_\infty(\mathfrak{M})$. The association $C_\infty(\mathfrak{M}) \to \mathcal{A}$ sends each $f \in C_\infty(\mathfrak{M})$ into an operator $T_f \in \mathcal{A}: f \to T_f$. Via the use of the Daniell approach to (Lebesgue) integrals, this isometric isomorphism can be extended to $\mathcal{B}_b(\mathfrak{M})$ (the class of bounded Baire functions, i.e. the class of bounded functions in $\mathcal{B}(\mathfrak{M})$, the smallest monotone class containing $C_\infty(\mathfrak{M})$): $\mathcal{B}_b(\mathfrak{M}) \to \mathcal{A}_b \supset \mathcal{A}$, where $\mathcal{A}_b$ is a super algebra of $\mathcal{A}$ in $\mathcal{E}(\mathcal{H})$. Now we order the complex plane $\mathbf{C}$ as we did above. For each z, let $K_z = \{M \mid \hat{T}(M) \prec z\}$ and let E_z be the operator corresponding to χ_{K_z} = the characteristic function of $K_z (\chi_{K_z} \in \mathcal{B}_b(\mathfrak{M}))$. Since $\mathcal{B}_b(\mathfrak{M}) \to \mathcal{A}_b$ is an algebraic and isometric association, obviously $E_z = E_z^* = E_z^2$, i.e., E_z is a projection operator for each z. It is also clear that

$$\hat{T}(M) = \int_{\mathbf{C}} z d\chi_{K_z}(M),$$

$$T = \int_{\mathbf{C}} z dE_z.$$

This is the celebrated spectral theorem for normal operators. We see that it permits definition of $f(T)$ at least for bounded measurable functions:

$$f(T) = \int_{\mathbf{C}} f(z) dE_z$$

and thus yields a functional calculus of operators. The set of complex numbers $\{\hat{T}(M)\}$ is homeomorphic to $\mathfrak{M}$ and is the *spectrum* of T.

In another direction, if $x \to U_x$ is a continuous unitary representation (each U_x is a unitary operator, i.e., $(U_x w, U_x z) = (w, z)$ for any pair w, z of elements in $\mathcal{H}$), of a locally compact abelian group G into $\mathcal{E}(\mathcal{H})$, then the representation may be extended to $L_1(G)$ by the formula

$$f \to T_f = \int_G f(x) U_x dx,$$

where $(T_f h_1, h_2) = \int_G f(x)(U_x h_1, h_2) dx$. Since there is a one-one algebraic isomorphism between functions f in $L_1(G)$ and their Gelfand-Fourier transforms $\hat{f}$ in $C_\infty(\hat{G}) = C_\infty(\mathfrak{M})$, the above may be viewed as a correspondence

$$\hat{f} \to f \to T_f$$

which may, via Daniell integral techniques, be extended to a correspondence for bounded Baire functions on $\hat{G}$. Again characteristic functions of sets and projections correspond and at last we arrive at Stone's theorem

$$U_x = \int_{\hat{G}} (x, \alpha) dE_\alpha,$$

where E_α is a spectral family of projections.

There is a theorem of von Neumann that reads as follows:

If $\{A_n\}$ is a sequence of bounded normal operators that commute in pairs, there is a single family of projections $\{E_t\}$ and a family of continuous complex-valued functions $\{a_n(t)\}$ such that

$$A_n = \int_0^1 a_n(t) dE_t, \qquad n = 1, 2, \cdots.$$

Setting $A = \int_0^1 t dE_t$, we see that $A_n = a_n(A)$, i.e., that there is a single operator A such that each A_n is a function of A.

Banach algebras permit a neat proof of von Neumann's theorem. If $\mathfrak{A}$ is the closed algebra generated by I, A_n, A_m^*, n, $m = 1, 2, \cdots$, then $\mathfrak{A}$ is a commutative Banach algebra isomorphic to $C_\infty(\mathfrak{M})$. Since $\mathfrak{A}$ is separable (a countable dense set consists of the polynomials in I, A_n, A_m^* with rational coefficients) and $\mathfrak{A}$ has an identity, $\mathfrak{M}$ is a compact metric space and hence the continuous image of the Cantor set D: ϕ: $D \to \mathfrak{M}$. For each $t \in [0, 1]$, let $\chi_t(M)$ be the characteristic function of the set $\phi([0, t] \cap D)$ and let E_t be the corresponding projection in $\mathfrak{A}_b \subset \mathcal{E}(\mathcal{H})$. For each $t \in D$, and $B \in A$, let $b(t) = \hat{B}(\phi(t))$ and extend $b(t)$ to $[0, 1]$ by linearity over the intervals deleted in constructing D. Then, as direct computation shows,

$$B = \int_0^1 b(t) dE_t$$

and in particular

$$A_n = \int_0^1 a_n(t)dE_t,$$

where $a_n(t)$ is the linear extension of $A_n(\phi(t))$. (This proof will appear in a forthcoming publication of the author.)

Finally, we mention an isolated but very high peak in the theory. It is the following theorem of Gelfand and Naimark: If A is an arbitrary Banach algebra, commutative or not, with an involution for which 1–5 and

6. *For all* a, $(-aa^*)$ *is not an identity modulo any modular maximal ideal.* (*Equivalently if* A *has an identity*, e, *then for all* a, $(e+aa^*)^{-1}$ *exists.*)

obtain, e.g., III or III′, then A is isomorphic to a closed subalgebra of $\mathcal{E}(\mathcal{H})$ for some $\mathcal{H}$.

If one regards the representation of a *commutative* Banach algebra A as a subalgebra of $C_\infty(\mathfrak{M})$ to be a useful reduction from the abstract to the concrete, one may similarly regard the Gelfand-Naimark theorem as providing a "concrete" version of an abstract noncommutative Banach algebra. The isolation of the Gelfand-Naimark result typifies the poverty in noncommutative theory as contrasted with the wealth in commutative theory.

The general methodology of abstract mathematics is exemplified here. One begins by considering concrete examples of topological algebras. Then a reasonable and minimal set of properties of these algebras is chosen as the axiomatic basis for a class of "Banach" algebras. Finally, one seeks through investigation of the abstract form, to find the complete class of concrete forms that the abstract form represents. The discussion above gives some indication of the extent to which this program has been carried out.

The foregoing paper gives the substance of an invited address at the meeting of the Minnesota Section of the Association in November, 1962.

References

1. E. Hewitt, A survey of abstract harmonic analysis in *Some aspects of analysis and probability*, Wiley, New York, 1958.

2. E. Hewitt and K. A. Ross, Abstract harmonic analysis, in vol. 1, *Structure of topological groups, Integration theory*, Academic Press, Inc., New York, 1963.

3. I. Kaplansky, Functional analysis, in *Some aspects of analysis and probability*, Wiley, New York, 1958.

4. L. Loomis, Introduction to abstract harmonic analysis, Van Nostrand, New York, 1953.

5. G. Mackey, Functions on locally compact groups, Bull. Amer. Math. Soc., 56 (1950) 385–412.

6. M. Naimark, Normed rings (translated by Leo Boron), P. Noordhoff, Groningen, 1959.

7. C. E. Rickart, General theory of Banach algebras, Van Nostrand, Princeton, 1960.

8. W. Rudin, Fourier analysis on groups, Interscience Publishers, New York, 1962.

A PARTIAL DIFFERENTIAL EQUATION AND PARALLEL PLANE CURVES

A. W. GOODMAN, University of Kentucky

1. Introduction. A certain problem on the characterization of analytic functions leads quite naturally to the differential equation

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = 1. \tag{1}$$

For the present we devote our attention to the study of the solutions of (1) and postpone to the following paper the application to analytic functions.

It is easy to find solutions of (1) using Charpit's method [3] and in fact several text books use equation (1) as an illustrative example (see [1] p. 31; and [2] p. 18). Although this method gives the "complete" solution of (1), it seems to me that it falls far short of describing *all* of the solutions of (1). In section 2 we find all solutions ϕ for which the second order partial derivatives exist and are continuous in a given (open and connected) region $\mathcal{R}$. In section 3 we discuss the possibility of relaxing the condition on the partial derivatives of ϕ. Finally we remark that the solution of (1) suggests an interesting problem on parallel plane curves, and section 4 is devoted to some remarks on this problem.

2. The solution of the differential equation. We will see that if $\phi(x, y)$ is a solution of (1) then the surface

$$z = \phi(x, y) \tag{2}$$

is a certain ruled surface (called a distance surface) generated by a set of lines each of which makes an angle $\pi/4$ with the xy-plane. Further if all of these lines actually meet the plane then the intersection is a smooth curve $\mathcal{C}$ such that the projection of each ruling on the xy-plane is normal to $\mathcal{C}$. If the region $\mathcal{R}$ is not too complicated, the surface (2) can be described by saying that $\phi(x, y)$ is the directed distance of (x, y) from the curve $\mathcal{C}$, positive on one side of $\mathcal{C}$ and negative on the other side.

In the general case, unfortunately, the region $\mathcal{R}$ and the surface over $\mathcal{R}$ may be so complicated that neither of the above tentative descriptions of the surface is quite accurate. Before giving a precise definition of a distance surface, and an accurate statement of our results, it will be helpful to look at several examples.

Henceforth we let $\mathcal{C}$ be the set on which $\phi(x, y) = 0$, and we call this set the *base set* for the surface. If $\mathcal{C}$ is the straight line $ax+by+c=0$, with $a^2+b^2=1$, then the plane $z=ax+by+c$ is a distance surface with this base set. If $\mathcal{C}$ is the circle $(x-h)^2+(y-k)^2=r^2$ and $\mathcal{R}$ is any region that does not contain the center (h, k) then either of the cones $z=\pm(\{\sqrt{(x-h)^2+(y-k)^2}\}-r)$ is a distance surface with the circle for its base set. It is easy to see that for these surfaces the corresponding $\phi(x, y)$ is a solution of (1).

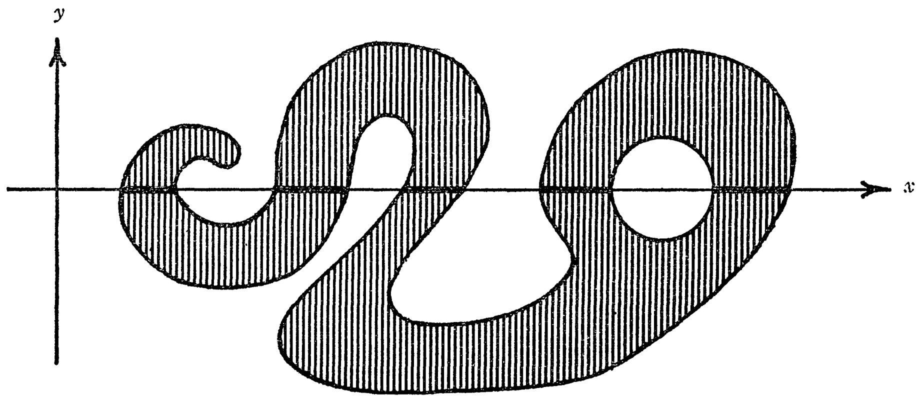

FIG. 1

Consider next the surface $z=y$ defined over the region shown in Fig. 1. Clearly $\phi(x, y)=y$ is a solution of (1) in $\mathcal{R}$, but the base set (shown heavily shaded on the x-axis) has five components. Further, it is not true that each of the lines $z=y$, $x=x_0$, on the surface meets the x-axis. Of course in this special case we can enlarge both the region $\mathcal{R}$ and the surface, and then this defect will disappear.

As a final example, let $\mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2\cup I$, where $\mathcal{R}_1$ is the semiring region defined by inequalities

$$1 < x^2 + y^2 < 2, \qquad x < 0, \tag{3}$$

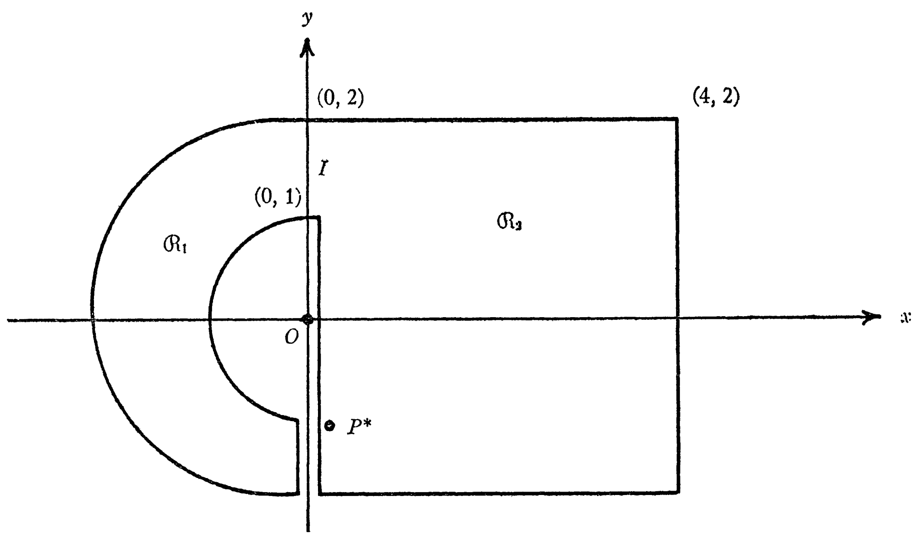

FIG. 2

$\mathcal{R}_2$ is the rectangle defined by the inequalities

$$(4) \qquad -2 < y < 2, \qquad 0 < x < 4,$$

and I is the interval $x=0$, $1<y<2$. The region is shown schematically in Figure 2. Let the surface be defined over $\mathcal{R}$ by

$$(5) \qquad z = \phi(x, y) = \begin{cases} \sqrt{\{x^2+y^2\}} - 1, & (x, y) \text{ in } \mathcal{R}_1, \\ y - 1, & (x, y) \text{ in } \mathcal{R}_2 \cup I. \end{cases}$$

Clearly this $\phi(x, y)$ solves (1) in $\mathcal{R}$. The base set $\mathcal{C}$ consists of the semicircle $x^2+y^2=1$, $x<0$, and the line $y=1$, $0 \leqq x < 4$. The surface can be generated by drawing through each point of $\mathcal{C}$ a suitable line segment that makes an angle of $\pi/4$ with the xy-plane. But observe that $\phi(x, y)$ is not always the directed distance between (x, y) and $\mathcal{C}$. For example, if $\epsilon>0$ the point $P^*(\epsilon, -1)$ has distance ϵ from $\mathcal{C}$, but $\phi(P^*)=-2$. These examples force upon us the following complicated definition.

DEFINITION 1. *A surface $z=\phi(x, y)$ is called a distance surface over a plane region $\mathcal{R}$ if at each point of $\mathcal{R}$ it is a local distance surface. A surface is a local distance surface at a point P if there is a circular neighborhood $\mathfrak{N}$ of P, a smooth curve $\mathcal{C}$ that divides $\mathfrak{N}$ into two subregions, and a constant c, such that $d=\phi(x, y)-c$ is the directed distance of (x, y) from $\mathcal{C}$, for each point of $\mathfrak{N}$.*

Clearly each of the examples cited so far is a distance surface. We can now state:

THEOREM 1. *Let ϕ_{xx}, ϕ_{xy}, and ϕ_{yy} be continuous in a region $\mathcal{R}$. Then $\phi(x, y)$ is a solution of* (1) *if and only if $z=\phi(x, y)$ is a distance surface.*

Proof. Assume that $\phi(x, y)$ is a solution of (1). Let $\mathcal{C}_0$ be the curve determined by solving the system of differential equations

$$(6) \qquad \begin{aligned} \frac{dx}{dt} &= \phi_y(x, y) \\ \frac{dy}{dt} &= -\phi_x(x, y) \end{aligned}$$

with the initial conditions $x(0)=x_0$, $y(0)=y_0$, where $(x_0, y_0)\in\mathcal{R}$. Under the hypothesis that ϕ_{xx}, ϕ_{xy}, and ϕ_{yy} are continuous in $\mathcal{R}$, this set has a unique solution in $\mathcal{R}$ [3, p. 93]. Furthermore, from the theory of differential equations, $\mathcal{C}_0$ is a simple smooth curve which may be closed (for example the circle) or it may be open, either extending infinitely in either direction or terminating at boundary points of $\mathcal{R}$ (see Figure 1). But on $\mathcal{C}_0$ the function $\phi(x, y)$ is the constant $\phi(x_0, y_0)$, for on $\mathcal{C}_0$

$$(7) \qquad \frac{d\phi}{dt} = \phi_x \frac{dx}{dt} + \phi_y \frac{dy}{dt} = \phi_x\phi_y + \phi_y(-\phi_x) = 0.$$

We have already seen that the level sets of ϕ may have more than one component. The above considerations show that each component is a simple smooth curve. The orthogonal trajectories of these level curves are obtained as solutions of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= \phi_x(x, y) \\ \frac{dy}{dt} &= \phi_y(x, y). \end{aligned} \tag{8}$$

For any such curve, we have from (1) and (8)

$$\frac{d^2x}{dt^2} = \phi_{xx}\frac{dx}{dt} + \phi_{xy}\frac{dy}{dt} = \phi_{xx}\phi_x + \phi_{xy}\phi_y = \frac{1}{2}(2\phi_x\phi_{xx} + 2\phi_y\phi_{yx})$$

$$\frac{d^2x}{dt^2} = \frac{1}{2}\frac{\partial}{\partial x}(\phi_x^2 + \phi_y^2) = \frac{1}{2}\frac{\partial}{\partial x}(1) = 0. \tag{9}$$

Similarly $d^2y/dt^2 = 0$. The vanishing of these second derivatives shows that the orthogonal trajectories are straight lines.

Since $|\nabla\phi| = 1$, ϕ must vary with the distance along each of these straight lines, and hence $z = \phi(x, y)$ is a distance surface. I am indebted to the referee for suggesting this proof which is simpler than the one originally submitted.

We now consider the converse, namely, if $z = \phi(x, y)$ is a distance surface over a region $\mathcal{R}$ then ϕ is a solution of (1). This proposition is geometrically obvious (as was first pointed out to me by my colleague Wasley Krogdahl), but a rigorous proof based solely on geometric considerations seems to be difficult, so we give an analytic proof. Let t be a suitably selected parameter for the curve $\mathcal{C}$ (see Definition 1) such that $x = f(t)$, $y = g(t)$ are twice differentiable and $D^2 \equiv f'(t)^2 + g'(t)^2 \neq 0$. Then a curve parallel to $\mathcal{C}$ at the signed distance q from $\mathcal{C}$ will have a parameterization

$$x = f(t) - q\frac{g'(t)}{D}, \quad y = g(t) + q\frac{f'(t)}{D}, \tag{10}$$

where $q > 0$ for the parallel curves on one side of $\mathcal{C}$ and $q < 0$ for those on the other side. In order to construct the distance surface for $\mathcal{R}$ we merely replace q by z in (10). In other words, $z = \phi(x, y)$ is defined implicitly, in a sufficiently small neighborhood $\mathfrak{N}$, by the pair of equations

$$x = f(t) - z\frac{g'(t)}{D}, \quad y = g(t) + z\frac{f'(t)}{D}. \tag{11}$$

If $\mathcal{C}$ is a straight line or a circle the elimination of t in (11) leads to the surfaces already discussed. For any other curve, the detailed computation is more difficult. But we can compute z_x and z_y from (11) using the standard implicit func-

tion theorems. We write (11) in the form $F(z, t, x, y)=0$, $G(z, t, x, y)=0$. Then (see [4] p. 27)

$$\frac{\partial z}{\partial x}=\frac{\begin{vmatrix} F_2 & F_3 \\ G_2 & G_3 \end{vmatrix}}{\Delta}, \qquad \frac{\partial z}{\partial y}=\frac{\begin{vmatrix} F_2 & F_4 \\ G_2 & G_4 \end{vmatrix}}{\Delta}, \qquad \text{where} \quad \Delta=\begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix}. \tag{12}$$

In our case direct computation from (12) yields

$$F_2=\frac{zf'}{D^3}(f'g''-g'f'')-f', \qquad G_2=\frac{zg'}{D^3}(f'g''-g'f'')-g',$$

and the simpler results, $F_1=g'/D$, $G_1=-f'/D$, $F_3=1$, $G_3=0$, $F_4=0$, $G_4=1$. A little labor now gives

$$\Delta=\frac{z}{D^2}(f'g''-g'f'')-D. \tag{13}$$

From (12) we have

$$\Delta\frac{\partial z}{\partial x}=\begin{vmatrix} F_2 & F_3 \\ G_2 & G_3 \end{vmatrix}=-G_2=g'-\frac{zg'}{D^3}(f'g''-g'f'') \tag{14}$$

and

$$\Delta\frac{\partial z}{\partial y}=\begin{vmatrix} F_2 & F_4 \\ G_2 & G_4 \end{vmatrix}=F_2=-f'+\frac{zf'}{D^3}(f'g''-g'f''). \tag{15}$$

Finally a brief computation with (14) and (15) gives $(z_x)^2+(z_y)^2=1$ wherever $\Delta\neq 0$. If $\Delta=0$, equation (13) gives $z=D^3/(f'g''-g'f'')$, the radius of curvature of $\mathcal{C}$. Consequently we can always find a neighborhood of $\mathcal{C}$ in which $\Delta\neq 0$. This completes the proof of Theorem 1.

It is worth remarking that if ϕ is a distance surface it also satisfies

$$\phi_{xx}\phi_{yy}-\phi_{xy}^2=0, \tag{16}$$

the differential equation for a developable surface ([2] p. 10). For equation (1) obviously yields $\phi_x\phi_{xx}+\phi_y\phi_{yx}=0$ and $\phi_x\phi_{xy}+\phi_y\phi_{yy}=0$, and because of (1) this pair leads immediately to (16).

If the solution ϕ is also required to be harmonic then the surface is a plane. For, combining (16) with $\phi_{xx}+\phi_{yy}=0$, we obtain $-\phi_{yy}^2-\phi_{xy}^2=0$. Hence $\phi_{yy}=\phi_{xy}=0=\phi_{xx}$ and hence ϕ_x and ϕ_y are constants.

3. Existence of the second derivatives. We have made the assumption that ϕ_{xx}, ϕ_{xy}, and ϕ_{yy} are continuous in $\mathcal{R}$ in order that the systems (6) and (8) have solutions. It would have been sufficient to assume that ϕ_x and ϕ_y satisfy a Lipschitz condition. There are solutions of (1), however, for which ϕ_{xx} and ϕ_{yy} are not continuous. In fact ϕ_x and ϕ_y need not satisfy a Lipschitz condition. We

obtain such a function by putting together two planes and a cone. Let $\mathfrak{R}$ be the region obtained by deleting the third quadrant $x \leqq 0$, $y \leqq 0$ from the plane. Set

$$\phi = \begin{cases} x \text{ in the subregion } A\colon x > 0,\ y < 0, \\ \sqrt{\{x^2 + y^2\}} \text{ in the subregion } B\colon x > 0,\ y > 0, \\ y \text{ in the subregion } C\colon x < 0,\ y > 0, \end{cases}$$

and define ϕ on the positive x-axis and y-axis, by the common limit values. Clearly ϕ is a solution of (1) in each of the subregions A, B and C. It is easy to check that ϕ_x and ϕ_y have the same values on the common boundaries $\bar{A} \cap \bar{B}$ and $\bar{B} \cap \bar{C}$. Hence ϕ is a solution in $\mathfrak{R}$. But ϕ_{xx} does not exist on the positive y-axis because the limit from the right gives $\phi_{xx} = 1/y$ and the limit from the left gives $\phi_{xx} = 0$. Similarly ϕ_{yy} fails to exist on the positive x-axis, being $1/x$ from above and 0 from below.

If ϕ_x satisfies a Lipschitz condition with respect to x in $\mathfrak{R}$ then ϕ_{xx} is bounded in the subregion B. But in B $\phi_{xx} = y^2/(x^2+y^2)^{3/2}$, which is not bounded on the line $y = x$, as $y \to 0$. Hence we have a solution of (1) whose first derivatives do not satisfy a Lipschitz condition in $\mathfrak{R}$.

Of course for this example the "bad" point occurs on the boundary of $\mathfrak{R}$. We conjecture that if ϕ is a solution of (1) in a region $\mathfrak{R}$ then ϕ_x and ϕ_y satisfy a Lipschitz condition on every closed subset of $\mathfrak{R}$.

4. Parallel plane curves. Let $\mathcal{C}$ be a smooth curve in the plane region $\mathfrak{R}$ and let $\mathcal{C}(q)$ denote that portion of the parallel curve at signed distance q from $\mathcal{C}$ that also lies in $\mathfrak{R}$. We say that the family $\mathcal{C}(q)$ covers $\mathfrak{R}$ in a simple manner if each curve of the family is a simple curve and no two curves of the family intersect. Clearly if the family $\mathcal{C}(q)$ does cover $\mathfrak{R}$ in a simple manner, then the surface obtained by setting z equal to q in equation (10) gives a solution of (1). The problem is to determine nice conditions on $\mathfrak{R}$ and $\mathcal{C}$ that will assure this.

Let $\mathcal{E}$ be the evolute of $\mathcal{C}$. Then each curve $\mathcal{C}(q)$ is an involute of $\mathcal{E}$ and can be drawn mechanically by the well-known device of unwinding a thread from the evolute, with different lengths of thread for the different curves. While such an argument can not be regarded as a proof it suggests the conjecture that if the evolute of $\mathcal{C}$ lies outside a simply-connected region $\mathfrak{R}$, then the family $\mathcal{C}(q)$ covers $\mathfrak{R}$ in a simple manner. In its place we will prove the following weaker result.

THEOREM 2. *Let m be the minimum value of the radius of curvature along $\mathcal{C}$, and suppose that the length of $\mathcal{C}$ is less than $m\pi/2$. Then the region covered by the family $\mathcal{C}(q)$ with $|q| < m$ is covered in a simple manner by that family.*

Proof. Suppose that the two curves $\mathcal{C}(q_1)$ and $\mathcal{C}(q_2)$, with $m > q_2 > q_1 > 0$ meet at a point P_0. The case of a curve intersecting itself is handled by setting $q_2 = q_1$. We introduce intrinsic equations for the curve $\mathcal{C}$ in the following way: let P_0P_1 be the normal from the curve $\mathcal{C}$ that determines P_0 as a point on $\mathcal{C}(q_2)$, so that

$|P_0P_1| = q_2$. As indicated in the Figure 3 we draw a circle C_m with center at the origin and radius m. Then by a rigid motion we place the base curve so that P_1 lies at $(m, 0)$ and P_0 lies at $(m-q_2, 0)$. The circle with P_0 as center and radius q_2 lies inside C_m except for the common point at $(m, 0)$. If $\mathcal{C}$ lies outside or on the boundary of C_m, then for each $q_1 < q_2$, it is obvious that $\mathcal{C}(q_1)$ will not contain P_0. In case $q_1 = q_2$, then the curve $\mathcal{C}(q_2)$ runs through P_0 only when the corresponding point P on $\mathcal{C}$ is at P_1. Hence it suffices to show that $\mathcal{C}$ never crosses C_m.

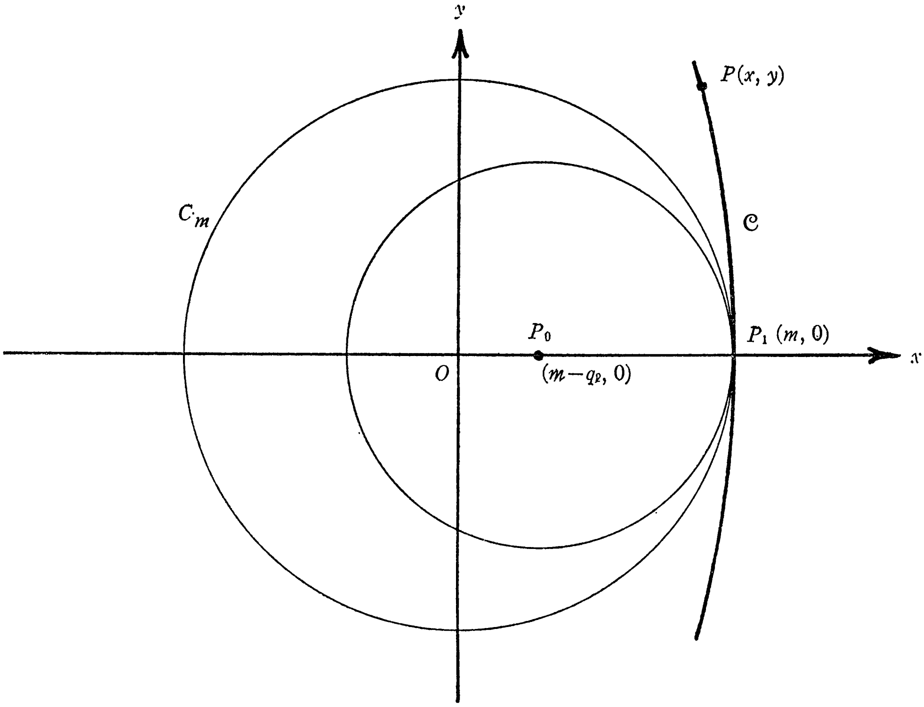

FIG. 3

We let s be the arc length on $\mathcal{C}$, measured from P_1 with the positive direction upward. The curve is clearly tangent to the two circles at P_1. Under these conditions it is easy to show that, with s as a parameter, the intrinsic equations for $\mathcal{C}$ are

$$x = m - \int_0^s \sin \delta(t) dt, \tag{17}$$

$$y = \int_0^s \cos \delta(t) dt, \tag{18}$$

where $\rho(s)$ is the radius of curvature and

$$\delta(t) = \int_0^t \frac{d\sigma}{\rho(\sigma)} \cdot \tag{19}$$

By hypothesis $|s| \leqq m\pi/2$, and we first suppose that $s \geqq 0$. Then $0 \leqq \sigma \leqq t \leqq m\pi/2$, and $\rho(\sigma) \geqq m$, and hence from (19), $\delta(t) \leqq t/m$. Under these conditions $0 \leqq \delta(t) \leqq t/m \leqq \pi/2$. Since δ is increasing we have from (17) and (19)

$$x = m - \int_0^s \sin \delta(t) dt \geqq m - \int_0^s \sin \frac{t}{m} dt = m - \left[m - m \cos \frac{s}{m} \right] = m \cos \frac{s}{m} \cdot$$

Similarly $\cos \delta(t)$ is decreasing, so from (18)

$$y = \int_0^s \cos \delta(t) dt \geqq \int_0^s \cos \frac{t}{m} dt = m \sin \frac{s}{m} \cdot$$

Then $x^2 + y^2 \geqq m^2$, and the point P lies outside or on the boundary of C_m. A similar argument can be used for $-\pi m/2 \leqq s \leqq 0$, or we may just reflect $\mathcal{C}$ about the x-axis.

Naturally the restriction on the length of $\mathcal{C}$ is too severe, but some restriction is necessary. For, referring to the figure, we see that the curve $\mathcal{C}$ might continue outward, then loop around a second circle of radius larger than m, and then return and cross C_m. It would be interesting to find the sharp upper bound for the length of $\mathcal{C}$ in Theorem 2.

After this paper was accepted for publication, my colleague Prof. Harold Robertson called my attention to an article by Z. A. Melzak, *Plane Motion with Curvature Limitations*, Journal of the Soc. for Ind. and Applied Math., vol. 9, 1961, pp. 422–432. Here Melzak, in his Lemma 2, proves a result which, if correct, would allow us to replace $m\pi/2$ by $m\pi$ in our Theorem 2. Although I am convinced that Melzak's lemma is correct, I am not certain about the proof. It seems to me that at the top of page 425, he squares an inequality in which one side is negative if $\pi/2 < s/\rho_0 < \pi$, and that at this point his proof breaks down, unless $|s/\rho_0| < \pi/2$.

References

1. B. Baule, Partielle Differentialgleichungen, S. Hirzel Verlag, Leipzig, 1944.
2. R. Courant, David Hilbert, Methods of Mathematical Physics, Vol. 2, Interscience, New York, 1962.
3. L. R. Ford, Differential Equations, McGraw-Hill, New York, 1957.
4. D. V. Widder, Advanced Calculus, 2nd ed. Prentice-Hall, Englewood Cliffs, N. J., 1961.

The unusually large audience at the visiting geometer's lecture was eventually seen to be the result of the new secretary's manner of transcribing the title, which appeared on the notice board as "Convicts, Sex, and Inner Qualities."

ON A CHARACTERIZATION OF ANALYTIC FUNCTIONS

A. W. GOODMAN, University of Kentucky

1. Introduction. In a recent paper [1] Dzyadyk proved the following theorem which gives a geometric characterization for analytic functions.

THEOREM 1. *Let* $\mathcal{R}$ *be a given region in the plane and let* $u(x, y)$, $v(x, y)$ *be continuous real valued functions with* u_x, u_y, v_x *and* v_y *continuous in* $\mathcal{R}$. *Set*

$$w = f(x + iy) = u(x, y) + iv(x, y). \tag{1}$$

Then in order that either f *or* $\bar{f}$ *be analytic in* $\mathcal{R}$ *it is necessary and sufficient that, if* $\mathcal{S}$ *is an arbitrary subregion of* $\mathcal{R}$, *then all three of the surfaces*

$$z = u(x, y), \qquad z = v(x, y), \qquad z = \sqrt{\{u^2(x, y) + v^2(x, y)\}} \tag{2}$$

have the same area over $\mathcal{S}$.

Our objective here is to examine the three surfaces defined in (2) and to show that the set may be replaced by a more general set.

$$z = u(x, y), \qquad z = v(x, y), \qquad z = \phi(u, v) \tag{3}$$

with the same conclusion. In fact we find a general form of the function $\phi(u, v)$ in (3) for which Theorem 1 with (2) replaced by (3) remains valid. To this end we recall the outline of the proof given by Dzyadyk.

Since the surface area is given by

$$S = \int\int_{\mathcal{S}} \sqrt{\{1 + z_x^2 + z_y^2\}}\, dxdy, \tag{4}$$

the condition of the theorem requires that the three integrands for the three different surfaces be the same at each point of $\mathcal{R}$. For the surfaces $z = u(x, y)$ and $z = v(x, y)$, this requires that $D_1 = D_2$, where

$$\begin{aligned} D_1 &\equiv u_x^2 + u_y^2 \\ D_2 &\equiv v_x^2 + v_y^2. \end{aligned} \tag{5}$$

It is easy to see from the Cauchy-Riemann equations that if f or $\bar{f}$ is analytic then $D_1 = D_2$.

For the surface $z = \sqrt{(u^2 + v^2)}$ we have

$$D_3 \equiv z_x^2 + z_y^2 = \frac{u^2(u_x^2 + u_y^2) + v^2(v_x^2 + v_y^2) + 2uv(u_x v_x + u_y v_y)}{u^2 + v^2} \tag{6}$$

whenever $u^2 + v^2 \neq 0$. It is easy to see that if f or $\bar{f}$ is analytic then (6) reduces to D_1. Hence the necessity of the condition is proved.

Conversely, if the three surfaces have equal areas, then $D_1 = D_2 = D_3$, and a

careful argument [1] leads either to the Cauchy-Riemann equations, or to the set $u_x = -v_y$, $u_y = v_x$.

2. The extension of the theorem. We first remark that the equality of the area of the two surfaces $z = u(x, y)$ and $z = v(x, y)$ is not sufficient. For if $u = x^2 + y^2$ and $v = x^2 - y^2$, then $D_1 = D_2$ but $x^2 + y^2 + i(x^2 - y^2)$ is neither analytic nor the conjugate of an analytic function. In fact we might take u any function and use for v the same function. Then $D_1 = D_2$ would be satisfied trivially while $u(x, y) + iu(x, y)$ is not analytic unless it is a constant. Hence, as indicated in equations (3), we seek to add a third function.

Formal computation for the surface $z = \phi(u(x, y), v(x, y))$ yields

$$D_3 \equiv \phi_x^2 + \phi_y^2 = \phi_u^2(u_x^2 + u_y^2) + \phi_v^2(v_x^2 + v_y^2) + 2\phi_u\phi_v(u_xv_x + u_yv_y). \tag{7}$$

Necessity. Suppose now that either $f = u + iv$ or $\bar{f}$ is analytic. We wish to conclude that the three surfaces always have equal area, for each subregion $\mathcal{S}$ of $\mathcal{R}$. Thus we wish to show that $D_1 = D_2 = D_3$. But if f or $\bar{f}$ is analytic, then

$$u_xv_x + u_yv_y = 0, \tag{8}$$

$$u_x^2 + u_y^2 = v_x^2 + v_y^2 \tag{9}$$

and hence (7) reduces to

$$D_3 = (\phi_u^2 + \phi_v^2)(u_x^2 + u_y^2). \tag{10}$$

Consequently if $D_1 = D_3$ we find that $\phi(u, v)$ must satisfy the partial differential equation

$$\phi_u^2 + \phi_v^2 = 1. \tag{11}$$

Now all solutions of this equation, for which ϕ_{uu}, ϕ_{uv}, and ϕ_{vv} are continuous, are known, and these are discussed fully in [2]. One solution must be $\phi = \sqrt{\{u^2 + v^2\}}$ as used by Dzyadyk. A second and simpler solution is $\phi = au + bv$ where $a^2 + b^2 = 1$. In order to give a surface different from $z = u$ and $z = v$, we require that $ab \neq 0$.

Sufficiency. Suppose conversely that u and v are such that all three of the surfaces (3) lying over any subregion $\mathcal{S}$ of $\mathcal{R}$ have the same area, and further suppose that $\phi_u^2 + \phi_z^2 = 1$. Then $D_1 = D_2 = D_3$, and equation (9) is satisfied. Using D_3 as given by (7) we can write $D_1 = D_3$ in the form

$$\begin{aligned} u_x^2 + u_y^2 &= \phi_u^2(u_x^2 + u_y^2) + \phi_v^2(u_x^2 + u_y^2) + 2\phi_u\phi_v(u_xv_x + u_yv_y) \\ &= (\phi_u^2 + \phi_v^2)(u_x^2 + u_y^2) + 2\phi_u\phi_v(u_xv_x + u_yv_y). \end{aligned}$$

Since (11) is satisfied this gives

$$\phi_u\phi_v(u_xv_x + u_yv_y) = 0. \tag{12}$$

Let us suppose for the moment that the factor $\phi_u\phi_v$ can be dropped. Then (12) gives (8). It can be proved that if u and v satisfy (8) and (9) in $\mathfrak{R}$ then either f or $\bar{f}$ is analytic in $\mathfrak{R}$. The proof is a little involved, and requires careful consideration of a number of cases, but this can be found in Dzyadyk's work [1] and so we omit it here.

The factors ϕ_u and ϕ_v are treated as follows: Let $\mathfrak{R}^*$ be the region in the uv-plane over which the function $\phi(u, v)$ is defined. Since $\phi_u^2+\phi_v^2=1$, the surface of $z=\phi(u, v)$ is a distance surface. For simplicity we assume that each level curve $\mathcal{C}$ of ϕ has only a finite number of points with normals that are horizontal or vertical. Along the horizontal normals $\phi_v=0$, and along the vertical normals $\phi_u=0$, and these are the only points for which $\phi_u\phi_v=0$. The mapping $u=u(x, y)$, $v=v(x, y)$ takes $\mathfrak{R}$ into some subset of $\mathfrak{R}^*$. Suppose that there is some open set $\mathcal{G}$ of $\mathfrak{R}$ which goes into one of the horizontal normals of $\mathcal{C}$. Then in $\mathcal{G}$, v is a constant, and $v_x=v_y=0$. Consequently, from (9), $u_x=u_y=0$, and then (8) holds. A similar argument yields (8) for any open set of $\mathfrak{R}$ that maps into a vertical normal of $\mathcal{C}$. Any other point P of $\mathfrak{R}$ which yields $\phi_u\phi_v=0$ is a limit point of points P_n at which $\phi_u\phi_v\neq 0$. But then at each P_n we have $u_xv_x+u_yv_y=0$. Consequently, by a continuity argument, this equation is also satisfied at P_0. Hence (8) is satisfied throughout $\mathfrak{R}$. We have proved the following theorem:

THEOREM 2. *Let $\mathfrak{R}$ be a given region in the plane and let $u(x, y)$, $v(x, y)$ be continuous real valued functions with u_x, u_y, v_x and v_y continuous in $\mathfrak{R}$. Let $\mathfrak{R}^*$ contain the image of $\mathfrak{R}$ under $u=u(x, y)$, $v=v(x, y)$. Finally let $\phi=\phi(u, v)$ have second order partial derivatives and satisfy the differential equation* (11) *in $\mathfrak{R}^*$, and suppose that each level curve for the surface $z=\phi(u, v)$ has only a finite number of horizontal or vertical normals. Then in order that either f or $\bar{f}$ given by* (1) *be analytic in $\mathfrak{R}$ it is necessary and sufficient that, if $\mathcal{S}$ is an arbitrary subregion of $\mathfrak{R}$, then all three of the surfaces* (3) *have the same area over $\mathcal{S}$.*

References

1. V. K. Dzyadyk, Geometrical definition of analytic functions, Uspehi Mat. Nauk, 15 (1960) number 1 (91) 191–194.

2. A. W. Goodman, A partial differential equation and parallel plane curves, this MONTHLY, the preceding article.

ON THE COMMUTATIVITY OF RINGS

RAYMOND AYOUB AND CHRISTINE AYOUB, Pennsylvania State University

1. Introduction. As a consequence of his theory of algebras, Jacobson [1] proved that an algebraic algebra, without nilpotent elements over a finite field, is commutative. As a corollary, he deduced that if in a ring R there exists an integer $n>1$ such that for every a in R, $a^n=a$, then the ring is commutative.

This result implies, in particular, the celebrated theorem of Wedderburn that any finite division ring is a field. There have been in recent years several generalizations and variations of Jacobson's result but proofs of all these commutative theorems use "transcendental" methods in the sense that they use Zorn's lemma implicitly or explicitly. Details of these theorems and their proofs may be found in Jacobson's Colloquium book on the theory of rings. Jacobson, moreover, remarks that it would be interesting to prove these theorems without recourse to transfinite methods.

It is the object of the present note to prove the theorem in its simplest form for a certain class of exponents n. In addition, in an attempt to construct a general method, several specific cases of the theorem are proved.

Let p be an odd prime and suppose that in the binary expansion of $p+1$, precisely two ones occur. Set $n=p+1$. Thus n is of the form $p+1$, where p is a prime of the form 2^k+2^m-1. Examples of such primes are 5, 11, 17, 19, etc. in which case $n=6$, 12, 18, 20, etc. The class includes in particular the Fermat primes by choosing $m=1$ and $k=2^r$. An interesting arithmetic question is whether the expression 2^k+2^m-1 represents infinitely many primes. Specifically, we prove:

THEOREM A. *Let n be of the above form. If R is an arbitrary ring in which $a^n=a$ for every a in R, then R is commutative.*

THEOREM B. *If $n=2$, 3, 4, 5, 7, and $a^n=a$ for every a in R, then R is commutative.*

2. Some preliminary theorems. Proofs of these two theorems are rather different. We prove in this section some theorems which will be used in the ensuing proofs. Most of these are known and appear in one place or another but we gather them here for easy reference.

THEOREM 2.1. *In a ring R without nilpotent elements, an idempotent element lies in the center.*

Proof. Suppose that $a^2=a$ and let x be an arbitrary element of R, then

$$\begin{aligned}(axa - ax)^2 &= axa^2xa - axa^2x - axaxa + axax\\ &= axaxa - axax - axaxa + axax\\ &= 0.\end{aligned}$$

Since the ring was assumed to have no nilpotents, it follows that $ax=axa$, and then by symmetry that $xa=axa$, and so a commutes with every element of R and hence lies in the center.

THEOREM 2.2. *Let Z be the center of the ring R. If for every a in R, a^2+a lies in the center Z, then R is commutative.*

Proof. Let x and y be arbitrary elements of the ring. Then $(x+y)^2+(x+y)$

is in Z. Expanding and noting that both x^2+x and y^2+y lie in the center, we see that $xy+yx$ lies in Z. Hence

$$x(xy + yx) = (xy + yx)x$$

or $x^2y=yx^2$. This implies that x^2 lies in the center and therefore x itself lies in the center.

THEOREM 2.3. *If a is an element of a ring R and n an integer such that $a^n=a$, then for all integers $m\geqq 0$, $a^{m(n-1)+1}=a$. Furthermore, if k and l are positive integers and $k\equiv l \pmod{n-1}$, then $a^k=a^l$.*

Proof. Using induction on m, we get:

$$a^{(m+1)(n-1)+1} = a^{m(n-1)+1}a^{n-1} = a\, a^{n-1} = a^n = a.$$

We may assume that $k\geqq l$. If $l=1$, $k\equiv 1 \pmod{n-1}$ or $k=m(n-1)+1$ with $m\geqq 0$ so that $a^k=a=a^l$. So assume that $k\geqq l>1$. Then $k-1\equiv l-1 \pmod{n-1}$ and $k-1-q(n-1)+(l-1)$ with $q\geqq 0$. Hence we have:

$$a^k = a^{k-1}\, a = a^{q(n-1)+(l-1)}\, a = a^{l-1}a^{q(n-1)+1} = a^{l-1}a = a.$$

THEOREM 2.4. *If every element of the ring R satisfies $x^n=x$, then for any element a in R and any integer k, $(k^n-k)a=0$. Thus R has finite characteristic.*

Proof. $ka=(ka)^n=k^na^n=k^na$. Hence $(k^n-k)a=0$.

THEOREM 2.5. *If every element of the ring R satisfies $x^n=x$, then the characteristic of R is square free and is a divisor of the g.c.d. of k^n-k, where k runs over all integers k.*

Proof. The second statement is clear from the previous theorem. Now let c be the characteristic of R and assume that R is not square free; then there is a prime p such that p^2 divides c. Taking $k=p$, we see that $c\mid p^n-p$. Hence $p^2\mid p(p^{n-1}-1)$ but this is clearly impossible. Thus c is square free.

THEOREM 2.6. *Let $n=e_12^r+\cdots+e_r$, with e_i equal 0 or 1 be the binary expansion of the integer n. If exactly s of the e_i are equal to 1, then there are precisely 2^s odd binomial coefficients $\binom{n}{k}$.*

Proof. The proof may be found in [2]; the theorem appears to be due to Kummer.

3. Proof of Theorem A. We assume that $n=p+1$, where p is a prime and that the binary expansion of n has exactly two ones; moreover we assume that p is odd. We further suppose that for every element a in the ring R, we have $a^n=a$. Since n is even,

$$a = a^n = (-1)^na^n = (-a)^n = -a.$$

Therefore $2a=0$ so that the ring has characteristic 2. Our object will be to prove that $a^2=a$. We have

$$a + a^2 = (a + a^2)^n$$
$$= a^n + \binom{n}{1} a^{n+1} + \cdots + \binom{n}{r} a^{n+r} + \cdots \binom{n}{n-r} a^{2n-r} + \cdots + a^{2n}.$$

Let $\binom{n}{n-r}$ and $\binom{n}{r}(r \neq 0)$ be odd and all other binomial coefficients $\binom{n}{k}(k \neq 0, n)$ even—there are exactly 4 odd coefficients by the above theorem. Since $a^n = a$, and $a^{2n} = a^2$, we conclude that $a^{n+r} = a^{2n-r}$, i.e. $a^{2r} = a^n = a$. Since $r < (n/2)$, $2r < n$, we have therefore found an integer $k < n$, such that $a^k = a$. By Theorem 2.3, it follows that $a^m = a^q$ if $m \equiv q \pmod{n-1}$. We now try to solve the congruence

$$(2r - 1)x + 1 \equiv 2 \pmod{n - 1}, \text{ i.e.}$$
$$(2r - 1)x \equiv 1 \pmod{p}.$$

Since $2r - 1 < p$, $(2r - 1, p) = 1$, and the congruence is solvable.

Since $a^n = a$, applying Theorem 2.3 we have: $a^{(2r-1)x+1} = a^2$. On the other hand, since $a^{2r} = a$ by applying Theorem 2.3 we have $a^{(2r-1)x+1} = a$. Hence $a^2 = a$, and the commutativity of R follows from Theorem 2.1 once we have verified that R contains no nonzero nilpotent elements.

Suppose that x is nilpotent and that $x^t = 0$; then for $t' \geqq t$, $x^{t'} = 0$. Choose m so that $m(n-1)+1 > t$. Then $x = x^{m(n-1)+1} = 0$. Hence any nilpotent x in R is zero.

4. Some reduction theorems.

THEOREM 4.1. *Let R be a ring in which every element satisfies $x^n = x$. Then there exist rings $R_1, \cdots, R_m$ such that $R = R_1 \dotplus \cdots \dotplus R_m$ ($\dotplus$ denotes direct sum), where each R_i has prime characteristic.*

Proof. By Theorem 2.5. R has square free characteristic c. Hence we can write $c = q_1 \cdots q_m$, where the q_i are distinct primes. Then by a well-known theorem (cf. [3] Theorem 29) R is a direct sum of rings $R_1, \cdots, R_m$ of characteristics $q_1, \cdots, q_m$ respectively.

THEOREM 4.2. *Let R be a ring in which every element satisfies $x^n = x$ and assume that R has prime characteristic q. Then $q - 1$ divides $n - 1$.*

Proof. Let t be a primitive root, mod q. Since by Theorem 2.5, $q \mid k^n - k$ for every integer k, we have $t^n \equiv t \pmod{q}$ and hence $t^{n-1} \equiv 1 \pmod{q}$ since $(t, q) = 1$. Therefore, $q - 1 \mid n - 1$.

THEOREM 4.3. *Let R be a ring of prime characteristic q and assume that every element of R satisfies $x^n = x$ and that n is minimal. Then $n - 1$ is not divisible by q.*

Proof. If $q \mid n - 1$, let $n - 1 = mq$. Then if a is any element of R, $(a^{n-1})^q = a^{n-1} = (a^m)^q$. Therefore, $(a^{n-1} - a^m)^q = a^{n-1} - a^{n-1} = 0$. But R contains no nonzero nilpotent elements; hence $a^{n-1} = a^m$ and we have: $a^{m+1} = a^n = a$. Further-

more, $m+1<n$ since if $m+1=n$, $m=n-1$ and $q=1$. Now n was assumed minimal and so we have a contradiction if $q \mid n-1$.

We will also need the following:

THEOREM 4.4 (cf. [3] Theorem 43.) *If R has prime characteristic p, and every element of R satisfies $x^p=x$, then R is commutative.*

5. Proof of Theorem B. By the theorems of the previous paragraph we need only consider rings R in which $x^n=x$ and with prime characteristic q with $q-1 \mid n-1$, $q \nmid n-1$ and $q \neq n$.

(i) $n=2$. The theorem follows from Theorem 2.1.

(ii) $n=3$. Then $n-1=2$ and $q-1=1$ or 2. Thus $q=2$ or 3, but $2 \mid n-1=2$, and $3=n$.

(iii) $n=4$. Then $n-1=3$ and $q-1=1$ or 3. Thus $q=2$. Since $(a+a^2)^2 = a^2+a^4=a^2+a$, every element of the form $a+a^2$ is idempotent. Hence if a is in R, $a+a^2$ lies in the center by Theorem 2.1. Therefore, by Theorem 2.2, R is commutative.

(iv) $n=5$. Then $n-1=4$ and $q-1=1$, 2 or 4. But $q=2$ divides $n-1=4$, and $q=5=n$. Thus we may assume that $q=3$.

Let a be any element of R. Then we have:

$$a + a^2 = (a + a^2)^5 = a^5 + 2a^6 + a^7 + a^8 + 2a^9 + a^{10},$$

from which it follows that $2a^2+a^3+a^4+2a=0$. Multiplying by a, we get:

$$2a^3 + a^4 + a + 2a^2 = 0.$$

Now we add the last two equations and obtain

$$4a^2 + 2a^4 = 0.$$

Hence $a^2=a^4$ and $a^3=a^5=a$. Thus R has characteristic 3 and every element of R satisfies $x^3=x$. By Theorem 4.4, R is commutative.

(v) $n=6$. This case follows from Theorem A.

(vi) $n=7$. Then $n-1=6$. Thus $q-1=1$, 2, 3 or 6. But $q=2$ or 3 divides $n-1=6$, and $q=7=n$.

In conclusion we remark that the special methods employed in this paper may be used to prove the theorem for other values of n but some new technique is needed to take care of all cases—for example, we could not prove the theorem for $n=8$ by use of any of the devices developed here.

References

1. N. Jacobson, Structure theory for algebraic algebras of bounded degree, Ann. of Math., 46 (1945) 695–707.

2. G. Glaisher, Quarterly Journal of Pure and Applied Mathematics, 30 (1899) 150–156.

3. Neal H. McCoy, Rings and Ideals, The Mathematical Association of America (Carus Monograph) 1948.

THE NUMBER OF PLANTED PLANE TREES WITH A GIVEN PARTITION

W. T. TUTTE, University of Waterloo, Ont., Canada

1. Introduction. A *tree* is a connected finite graph containing no polygon. In this paper we consider only trees with at least two vertices. Such a tree is said to be *planted* when one monovalent vertex is specified as the *root*. If in addition one or more other monovalent vertices are specified as *secondary roots* we say the tree is *doubly planted*. (The *valency* of a vertex is the number of edges incident with it.)

Let T be any planted or doubly planted tree. A *proper vertex* of T is any vertex which is not a root or a secondary root. Let the number of proper vertices of T of valency m be $v(m)$. Then the *partition* of T is the vector

$$V = (v(1), v(2), v(3), \cdots). \tag{1}$$

Formally V has infinitely many components, but only a finite number of them may be nonzero.

A *plane* tree is a tree which is embedded in the Euclidean plane. Two planted or doubly planted plane trees are *equivalent* if and only if each can be transformed into the other by an orientation-preserving homeomorphism of the plane onto itself which maps root onto root and proper vertices onto proper vertices. For doubly planted plane trees this implies that secondary roots are mapped onto secondary roots. In what follows we do not distinguish between equivalent planted or doubly planted plane trees. We determine the number of planted plane trees having a given partition.

A planted or doubly planted tree is *k-coloured* when to each of its proper vertices there is assigned a unique member of a given set of k colours, subject to the condition that no two adjacent vertices of the tree may have the same colour.

Let T be any k-coloured planted or doubly planted plane tree. Let the k colours be enumerated as $C_1, C_2, \cdots, C_k$. Then for each colour C_i we can define a vector

$$V_i = (v_i(1), v_i(2), v_i(3), \cdots) \tag{2}$$

whose mth component $v_i(m)$ is the number of proper vertices of T of valency m and colour C_i. We call V_i the ith *colour-partition* of T. In this paper we consider the problem of finding the number of k-coloured planted plane trees with given colour-partitions, obtaining a complete solution only in the case $k=2$.

2. Plane trees. We use the term "doubly planted plane tree" only in a restricted sense now to be explained. Let R be the root of a planted plane tree T, and let S be the vertex to which R is joined. If T is to be doubly planted we require that each secondary root shall be distinct from S and shall be joined to S by an edge. The edges joining S to secondary roots must form a consecu-

tive block in the cyclic order of incident edges at S, and this block must occur immediately after the edge SR with respect to the positive sense of rotation about S.

Let $f(x)$ be a function having derivatives of all orders with respect to x, but otherwise arbitrary. With the vector V of (1) we associate the product

$$\pi(V) = \prod_{m=1}^{\infty}\left\{\frac{1}{(m-1)!}\left(\frac{d}{dx}\right)^{m-1} f(x)\right\}^{v(m)}. \tag{3}$$

We note the law

$$\pi(V+W) = \pi(V)\pi(W).$$

Let $q(V)$ denote the number of planted plane trees whose partition is V. Similarly let $q(s, V)$ be the number of doubly planted plane trees with s secondary roots and partition V. We write

$$Q = \sum_V q(V)\pi(V), \tag{4}$$

$$Q(s) = \sum_V q(s, V)\pi(V). \tag{5}$$

We shall use these expressions only as formal series, and no question of convergence or divergence need arise.

Consider any planted or doubly planted plane tree T with root R. Let S be the vertex joined to R. It may happen that S is joined only to R and to secondary roots. If so, the valency of S is $s+1$, where s is the number of secondary roots.

In the remaining case let SS' be the edge immediately succeeding the block of edges joining S to secondary roots, or immediately succeeding SR if $s=0$, in the positive cyclic order of edges at S. We subdivide SS' into two edges SR' and $R'S'$. We can now regard T as the union of two plane trees having only the vertex R' in common. One of these trees, T_1 say, has S' as a vertex. We think of it as a planted plane tree with root R'. The second tree, T_2 say, has S as a vertex. We regard it as a doubly planted plane tree with root R and $s+1$ secondary roots, the last of which is R'. The partition of T is the sum of the partitions of T_1 and T_2.

On the other hand it is clear that any planted plane tree T_1 can be combined with any doubly planted plane tree T_2 having $s+1$ secondary roots, the root of T_1 being identified with the last secondary root of T_2, to give a unique planted or doubly planted plane tree T.

These results can be expressed as follows:

$$Q(s) = QQ(s+1) + \frac{1}{s!}\left(\frac{d}{dx}\right)^s f(x),$$

$$Q = QQ(1) + f(x).$$

The first of these equations holds for all positive integers s. Multiplying the first equation by Q^s, summing over s, and adding the second equation, we obtain

$$Q + \sum_{s=1}^{\infty} Q^s Q(s) = \sum_{s=1}^{\infty} Q^s Q(s) + \sum_{s=0}^{\infty} \frac{Q^s}{s!}\left(\frac{d}{dx}\right)^s f(x).$$

To justify this as an equation between formal series we observe that each product of powers of $f(x)$ and its derivatives can occur only a finite number of times on each side of the equation. We thus arrive at the functional equation

$$Q = f(x + Q).$$

Expanding Q by Lagrange's formula we obtain

$$\text{(6)} \qquad \begin{aligned} Q &= \sum_{n=1}^{\infty} \frac{1}{n!}\left[\left(\frac{d}{da}\right)^{n-1} f^n(x + a)\right]_{a=0} \\ Q &= \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{d}{dx}\right)^{n-1} f^n(x). \end{aligned}$$

For any vector $V = (v(1), v(2), v(3), \cdots)$ we write

$$\lambda(V) = \sum_{m=1}^{\infty} v(m), \qquad \mu(V) = \sum_{m=1}^{\infty} mv(m).$$

Let $r(V)$ be the coefficient of $\prod_{m=1}^{\infty} \{(d/dx)^{m-1}f(x)\}^{v(m)}$ in Q. Then by (3) we have

$$q(V) = r(V) \prod_{m=1}^{\infty} \{(m-1)!\}^{v(m)}.$$

By (6) $r(V) = 0$ unless there is an integer n such that

$$n - 1 = \sum_{m=1}^{\infty} (m-1)v(m) = \mu(V) - \lambda(V),$$

$$n = \sum_{m=1}^{\infty} v(m) = \lambda(V).$$

Such an integer exists if and only if

$$\text{(7)} \qquad \mu(V) = 2\lambda(V) - 1.$$

This result merely reflects the elementary property of a tree that the number of vertices, counting roots, exceeds the number of edges by 1. For if V is a partition of a planted tree, the number of vertices is $\lambda(V)+1$, and the number of edges is $\frac{1}{2}(\mu(V)+1)$.

Suppose V satisfies (7). Then by (6) $r(V)$ is the coefficient of

$$\prod_{m=1}^{\infty} \left\{\left(\frac{d}{dx}\right)^{m-1} f(x)\right\}^{v(m)} \quad \text{in } \frac{1}{\lambda(V)!}\left(\frac{d}{dx}\right)^{\lambda(V)-1} \{f^{\lambda(V)}(x)\},$$

that is

$$\frac{1}{\lambda(V)!}\cdot\frac{(\lambda(V)-1)!}{\Pi\{(m-1)!\}^{v(m)}}\cdot\frac{\lambda(V)!}{\Pi(v(m)!)}\,.$$

We deduce that

$$q(V) = \frac{(\lambda(V)-1)!}{\Pi(v(m)!)} \tag{8}$$

if V satisfies (7), and $q(V)=0$ otherwise.

3. Coloured trees. Let T be any k-coloured planted or doubly planted plane tree. If the colour of the vertex joined to the root is C_i we say that T has *basic colour* C_i.

We retain the arbitrary function f, but instead of x we use k independent variables $x_1, x_2, \cdots, x_k$. Instead of (3) we write

$$\pi(V_i) = \prod_{m=1}^{\infty}\left\{\frac{1}{(m-1)!}\left(\frac{\partial}{\partial x_i}\right)^{m-1} f(x_i)\right\}^{v_i(m)}. \tag{9}$$

Let $q_i(V_1, V_2, \cdots, V_k)$ denote the number of k-coloured planted plane trees of basic colour C_i and with colour-partitions $V_1, V_2, \cdots, V_k$. Similarly let $q_i(s; V_1, V_2, \cdots, V_k)$ be the number of doubly planted k-coloured plane trees with s secondary roots, with basic colour C_i, and with the same colour-partitions. We write

$$Q_i = \sum_{V_1,\cdots,V_k} q_i(V_1, \cdots, V_k)\prod_{j=1}^{k}\pi(V_j), \tag{10}$$

$$Q_i(s) = \sum_{V_1,\cdots,V_k} q_i(s; V_1, \cdots, V_k)\prod_{j=1}^{k}\pi(V_j), \tag{11}$$

$$J = \sum_{i=1}^{k} Q_i. \tag{12}$$

We now make use of the decomposition of a planted or doubly planted plane tree T into two simpler ones, T_1 and T_2, as explained in Section 2. We now take the trees to be k-coloured, T having basic colour C_i. Then T_2 has basic colour C_i and T_1 can have any basic colour other than C_i. We are led to the following equations:

$$Q_i(s) = (J - Q_i)Q_i(s+1) + \frac{1}{s!}\left(\frac{\partial}{\partial x_i}\right)^{s} f(x_i),$$

$$Q_i = (J - Q_i)Q_i(1) + f(x_i).$$

The first of these is valid for any positive integer s.

From these equations, by the same process of summation as is used in Section 2, we obtain the functional equations

$$Q_i = f(x_i + J - Q_i), \qquad 1 \leqq i \leqq k. \tag{13}$$

We solve these equations in the simplest non-trivial case $k=2$. Undoubtedly they could be solved for higher values of k by using the extension of Lagrange's formula to k variables, [1].

In the case $k=2$ we have

$$Q_1 = f(x_1 + Q_2), \qquad Q_2 = f(x_2 + Q_1). \tag{14}$$

Hence

$$\begin{aligned} Q_1 &= f(x_1 + f(x_2 + Q_1)) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!}\left[\left(\frac{d}{da}\right)^{n-1} f^n(x_1 + f(x_2 + a))\right]_{a=0} \\ &= \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\partial}{\partial x_2}\right)^{n-1} f^n(x_1 + f(x_2)) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\partial}{\partial x_2}\right)^{n-1} \sum_{p=0}^{\infty} \frac{f^p(x_2)}{p!}\left(\frac{\partial}{\partial x_1}\right)^{p} f^n(x_1), \end{aligned} \tag{15}$$

$$Q_1 = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{1}{n!p!}\left(\frac{\partial}{\partial x_1}\right)^{p} f^n(x_1)\left(\frac{\partial}{\partial x_2}\right)^{n-1} f^p(x_2).$$

We write

$$\lambda(V_i) = \sum_{m=1}^{\infty} v_i(m), \qquad \mu(V_i) = \sum_{m=1}^{\infty} m v_i(m).$$

Let $r(V_1, V_2)$ be the coefficient of

$$\prod_{m=1}^{\infty}\left\{\left(\frac{\partial}{\partial x_1}\right)^{m-1} f(x_1)\right\}^{v_1(m)}\left\{\left(\frac{\partial}{\partial x_2}\right)^{m-1} f(x_2)\right\}^{v_2(m)}$$

in Q_1. Then by (9) we have

$$q_1(V_1, V_2) = r(V_1, V_2)\prod_{m=1}^{\infty}\left\{(m-1)!\right\}^{v_1(m)+v_2(m)}.$$

By (15) $r(V_1, V_2)=0$ unless there are integers n and p such that

$$p = \sum_{m=1}^{\infty}(m-1)v_1(m) = \mu(V_1) - \lambda(V_1),$$

$$n = \sum_{m=1}^{\infty} v_1(m) = \lambda(V_1),$$

$$n - 1 = \sum_{m=1}^{\infty} (m - 1)v_2(m) = \mu(V_2) - \lambda(V_2),$$

$$p = \sum_{m=1}^{\infty} v_2(m) = \lambda(V_2).$$

Such integers exist if and only if

$$\mu(V_1) = \lambda(V_1) + \lambda(V_2) = \mu(V_2) + 1, \tag{16}$$

a pair of equations which expresses some elementary properties of a 2-coloured planted tree with basic colour C_1.

Suppose V_1 and V_2 satisfy (16). Then, by (15), $r(V_1, V_2)$ is the coefficient of

$$\prod_{m=1}^{\infty} \left\{\left(\frac{\partial}{\partial x_1}\right)^{m-1} f(x_1)\right\}^{v_1(m)} \left\{\left(\frac{\partial}{\partial x_2}\right)^{m-1} f(x_2)\right\}^{v_2(m)}$$

in

$$\frac{1}{\lambda(V_1)!\ \lambda(V_2)!}\left(\frac{\partial}{\partial x_1}\right)^{\lambda(V_2)} f^{\lambda(V_1)}(x_1)\left(\frac{\partial}{\partial x_2}\right)^{\lambda(V_1)-1} f^{\lambda(V_2)}(x_2).$$

Hence

$$r(V_1, V_2) = \frac{\lambda(V_2)!}{\prod \{(m-1)!\}^{v_1(m)}} \times \frac{\lambda(V_1)!}{\prod (v_1(m)!)} \times \frac{(\lambda(V_1) - 1)!}{\prod \{(m-1)!\}^{v_2(m)}} \times \frac{\lambda(V_2)!}{\prod (v_2(m)!)} \times \frac{1}{\lambda(V_1)!\ \lambda(V_2)!}.$$

We deduce that

$$q_1(V_1, V_2) = \frac{(\lambda(V_1) - 1)!\ \lambda(V_2)!}{\prod (v_1(m)!) \prod (v_2(m)!)} \tag{17}$$

if V_1 and V_2 satisfy (16), and $q_1(V_1, V_2) = 0$ otherwise.

Reference

1. I. J. Good, Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes, Proc. Cambridge Philos. Soc., 56 (1960) 367–380.

A GENERALIZATION OF THE INTEGRAL OF THE CIRCULAR COVERAGE FUNCTION

WILLIAM C. GUENTHER, University of Wyoming

1. Introduction. Let $X=(x_1, \cdots, x_n)$ be the center of a sphere S_1 of radius R in an n-dimensional coordinate system. Assume that X has an n-dimensional standard normal distribution so that its probability density function is

$$f(x_1, \cdots, x_n) = \frac{1}{(2\pi)^{n/2}} \exp\left\{-\sum_{i=1}^{n} x_i^2/2\right\}.$$

Let $X'=(x_1', \cdots, x_n')$ be a randomly selected point inside of or on a sphere S_2 of radius D centered at the origin. That is, X' is uniformly distributed within or on S_2 and has for its p.d.f.

$$g(x_1', \cdots, x_n') = \frac{1}{V}, \qquad \sum_{i=1}^{n} x_i'^2 \leqq D^2,$$

where V is the volume within S_2. We seek the probability that the sphere S_1 will contain X' and will show that it is expressible in terms of integrals of the non-central chi-square distribution.

The result lends itself to several interpretations. First, summarizing the previous paragraph, it may be regarded as the probability that a sphere S_1 of radius R, whose center has an n-dimensional standard normal distribution, will contain a point which is distributed uniformly inside of or on a sphere S_2 of radius D centered at the origin. Secondly, it is the probability that a sphere S_1 of radius R contains a randomly selected point in a sphere S_2 of radius D after S_1 reaches its destination if the center of S_1 is thrown at the center of S_2 and the point at which the center of S_1 comes to rest is spherically normally distributed. Finally, it may be interpreted as the expected fraction of a sphere S_2 of radius D lying within a sphere S_1 of radius R when S_1 reaches its destination if the center of S_1 is thrown at the center of S_2 and the point at which the center of S_1 comes to rest is spherically normally distributed.

2. Evaluation of the probability. Given that the randomly selected point has assumed a position X', the probability that it is contained in S_1 is

$$Q(x_1', \cdots, x_n') = \int_{\sum_{i=1}^{n}(x_i-x_i')^2 \leqq R^2} \cdots \int f(x_1, \cdots, x_n)dx_1 \cdots dx_n. \tag{1}$$

The probability that S_1 covers X' regardless of the position it chooses is

$$P_n(R, D) = \int_{\sum_{i=1}^{n} x_i'^2 \leqq D^2} \cdots \int Q(x_1', \cdots, x_n') \frac{1}{V} dx_1' \cdots dx_n'. \tag{2}$$

The multiple integral $Q(x_1', \cdots, x_n')$ can be considerably simplified. If we let $y_i = x_i - x_i'$, then (1) becomes

$$(3)\quad Q(x_1', \cdots, x_n') = \underset{\sum_{i=1}^n y_i^2 \leqq R^2}{\int \cdots \int} \frac{1}{(2\pi)^{n/2}} \exp\left\{-\sum_{i=1}^n (y_i + x_i')^2/2\right\} dy_1 \cdots dy_n.$$

But (3) is $Pr[\sum_{i=1}^n y_i^2 \leqq R^2] = Pr[W^2 \leqq R^2]$, where W^2 has a noncentral chi-square distribution with n degrees of freedom and noncentrality parameter $r^2 = \sum_{i=1}^n x_i'^2$. It is well known that the p.d.f. of W^2 is:

$$h(W^2; n, r^2) = \frac{1}{2}\left(\frac{W}{r}\right)^{(n-2)/2} \exp\left\{-\frac{W^2 + r^2}{2}\right\} I_{(n-2)/2}(Wr),$$

where $I_k(x)$ is the modified Bessel function of the first kind of order k. Thus (3) can be written

$$(4)\quad Q(x_1', \cdots, x_n') = \int_0^{R^2} h(W^2; n, r^2) dW^2 = H(R^2; n, r^2).$$

We can now rewrite (2) as

$$(5)\quad P_n(R, D) = \underset{\sum_{i=1}^n x_i'^2 \leqq D^2}{\int \cdots \int} H(R^2; n, r^2) \frac{1}{V} dx_1' \cdots dx_n',$$

where

$$V = \frac{\pi^{n/2} D^n}{\Gamma\left(\dfrac{n+2}{2}\right)}.$$

To simplify (5) change to spherical coordinates. Thus

$$\begin{aligned} x_1' &= r \cos \theta_1, \\ x_2' &= r \sin \theta_1 \cos \theta_2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-1}' &= r \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_{n-1}, \\ x_n' &= r \sin \theta_1 \cdots \sin \theta_{n-2} \sin \theta_{n-1}, \\ |J| &= r^{n-1} (\sin \theta_1)^{n-2} (\sin \theta_2)^{n-3} \cdots \sin \theta_{n-2}, \end{aligned}$$

so that

$$P_n(R, D) = 2^n \int_0^D \int_0^{\pi/2} \cdots \int_0^{\pi/2} H(R^2; n, r^2) \frac{|J|}{V} d\theta_1 \cdots d\theta_{n-1} dr.$$

Integrating out the θ_i's yields

$$P_n(R, D) = \int_0^D H(R^2; n, r^2) \frac{nr^{n-1}}{D^n} dr. \tag{6}$$

Next integrate (6) by parts with

$$dv = \frac{nr^{n-1}}{D^n} dr, \qquad u = H(R^2; n, r^2).$$

We need

$$\begin{aligned}
\frac{dH}{dr} &= -\frac{n-2}{2}\int_0^{R^2} \frac{1}{2r}\left(\frac{W}{r}\right)^{(n-2)/2} \exp\left\{-\frac{W^2+r^2}{2}\right\} I_{(n-2)/2}(Wr)dW^2 \\
&\quad - rH(R^2; n, r^2) + \int_0^{R^2} \frac{1}{2}\left(\frac{W}{r}\right)^{(n-2)/2} \exp\left\{-\frac{W^2+r^2}{2}\right\} I'_{(n-2)/2}(Wr)WdW^2 \\
&= \int_0^{R^2} \frac{1}{2}\left(\frac{W}{r}\right)^{(n-2)/2} \\
&\quad \cdot \exp\left\{-\frac{W^2+r^2}{2}\right\}\left[-\frac{n-2}{2r} I_{(n-2)/2}(Wr) + \frac{Wr}{r} I'_{(n-2)/2}(Wr)\right] dW^2 \qquad (7) \\
&\quad - rH(R^2; n, r^2) \\
&= \int_0^{R^2} \frac{1}{2}\left(\frac{W}{r}\right)^{(n-2)/2} \exp\left\{-\frac{W^2+r^2}{2}\right\} I_{n/2}(Wr)WdW^2 - rH(R^2; n, r^2).
\end{aligned}$$

In the first term of (7) let

$$dv = \exp\left\{-\frac{W^2+r^2}{2}\right\} dW^2, \qquad u = \frac{1}{2} \frac{W^{n/2}}{r^{(n-2)/2}} I_{n/2}(Wr).$$

Then (7) becomes

$$\frac{dH}{dr} = -\frac{R^{n/2}}{r^{(n-2)/2}} I_{n/2}(Rr) \exp\left\{-\frac{R^2+r^2}{2}\right\}.$$

From the first integration by parts we get

$$P_n(R, D) = H(R^2; n, D^2) + \frac{R^{n/2}}{D^n}\int_0^D r^{(n+2)/2} \exp\left\{-\frac{R^2+r^2}{2}\right\} I_{n/2}(Rr)dr.$$

Use parts integration once more with

$$dv = r \exp\left\{-\frac{R^2+r^2}{2}\right\} dr, \qquad u = \frac{R^{n/2}}{D^n} r^{n/2} I_{n/2}(Rr)$$

and this leads to

$$
\begin{aligned}
P_n(R, D) &= H(R^2; n, D^2) - \left(\frac{R}{D}\right)^{n/2} \exp\left\{-\frac{R^2 + D^2}{2}\right\} I_{n/2}(RD) \\
&\quad + \left(\frac{R}{D}\right)^n H(D^2; n, R^2).
\end{aligned} \tag{8}
$$

Germond [3] obtained this result for the case $n=2$ and has prepared a table. When $n=3$ (8) reduces to

$$
\begin{aligned}
P_3(R, D) &= \phi(D + R) - \phi(D - R) + \frac{R^3}{D^3}\left[\phi(D + R) - \phi(R - D)\right] \\
&\quad + \frac{1}{D^3\sqrt{2\pi}}\left[(D^2 - RD + R^2 - 1) \exp\left\{-\frac{(D + R)^2}{2}\right\}\right. \\
&\quad \left. - (D^2 + RD + R^2 - 1) \exp\left\{-\frac{(D - R)^2}{2}\right\}\right],
\end{aligned}
$$

a result found in [4] where an abbreviated table is presented. Here

$$
\phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx.
$$

3. A useful recursion formula. If we integrate (4) by parts letting

$$
dv = \frac{1}{2} \exp\left\{-\frac{W^2 + r^2}{2}\right\} dW^2, \qquad u = \left(\frac{W}{r}\right)^{(n-2)/2} I_{(n-2)/2}(Wr)
$$

then

$$
\begin{aligned}
du = \Bigg\{&\frac{n - 2}{4} \frac{W^{(n-6)/2}}{r^{(n-2)/2}} I_{(n-2)/2}(Wr) \\
&+ \left(\frac{W}{r}\right)^{(n-2)/2} \frac{r}{4W}\left[I_{n/2}(Wr) + I_{(n-4)/2}(Wr)\right]\Bigg\} dW^2.
\end{aligned}
$$

Using

$$
I_{n/2}(Wr) = I_{(n-4)/2}(Wr) - \frac{n - 2}{Wr} I_{(n-2)/2}(Wr)
$$

gives

$$
du = \frac{1}{2}\left(\frac{W}{r}\right)^{(n-4)/2} I_{(n-4)/2}(Wr) dW^2.
$$

Hence (4) becomes for $n > 2$

$$
H(R^2; n, r^2) = H(R^2; n - 2, r^2) - \left(\frac{R}{r}\right)^{(n-2)/2} \exp\left\{-\frac{R^2 + r^2}{2}\right\} I_{(n-2)/2}(Rr). \tag{9}
$$

Although the recursive relationship is not well known, it has been observed by Quenouille [12].

Applying (9) to (8) yields

$$P_n(R, D) = H(R^2; n + 2, D^2) + \left(\frac{R}{D}\right)^n H(D^2; n, R^2). \tag{10}$$

4. Numerical evaluation. To use (9) to evaluate (10) we need $H(R^2; n, D^2)$ for $n=1$ and $n=2$. If $n=1$, then

$$H(R^2; 1, D^2) = \phi(D + R) - \phi(D - R)$$

which can be computed from numerous tables. For example in [9] the integral

$$\int_{-x}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-t^2/2\}\, dt$$

is given to fifteen decimal places for

$$x = 0(.0001)1(.001)7.800 \text{ (various) } 8.285.$$

Two extensive tables are available for the case $n=2$. The Bell Aircraft Corporation [2] has tabulated $H(R^2; 2, D^2)$ to five decimal places for $R=0.01(0.01)4.59$, $D=0.00(0.01)3.00$. Marcum [7] has tabulated $1-H(R^2; 2, D^2)$ to six decimal places for $R=0.10(0.10)20.0$ over values of D by intervals of .05 as are necessary to cover the range $H(R^2; 2, D^2)=0$ to $H(R^2; 2, D^2)=1$. The most extensive table of the noncentral chi-square is apparently the one prepared by Haynam and Leone [5]. Their table gives $H(R^2; n, D^2)$ to five decimal places for all combinations of values

$$D^2 = 0(0.1)1.0(0.2)3.0(0.5)5.0(1.0)34.0$$

$$n = 1(1)30(2)50(5)100$$

$$R^2 = 0.01(0.01)0.1(0.1)1.0(0.2)3.0(.5)10.0(1.0)30.0(2)50.0(5)165.$$

Unfortunately this latter table is not readily available.

Several good approximations for the noncentral chi-square distribution are available in the literature [1], [6], [10], [11]. Using the Pearson result on (10) we have

$$P_n(R, D) \cong Pr[w_1^2 < M_1] + \left(\frac{R}{D}\right)^n Pr[w_2^2 < M_2],$$

where

$$M_1 = \frac{R^2 + [D^4/(n + 2 + 3D^2)]}{(n + 2 + 3D^2)/(n + 2 + 2D^2)}, \qquad M_2 = \frac{D^2 + [R^4/(n + 3R^2)]}{(n + 3R^2)/(n + 2R^2)},$$

and w_1^2 and w_2^2 are distributed as central chi-square with fractional degrees of freedom

$$\nu_1' = \frac{[n + 2 + 2D^2]^3}{[n + 2 + 3D^2]^2} \quad \text{and} \quad \nu_2' = \frac{[n + 2R^2]^3}{[n + 3R^2]^2}.$$

References

1. S. H. Abdel-Aty, Approximate formula for the percentage points of the probability integral of the noncentral chi-square distribution, Biometrika, 41 (1954) 538–540.
2. Bell Aircraft Corporation, Tables of circular normal probabilities, Operations Analysis Group, Report No. 02-949-106, Buffalo, New York.
3. H. H. Germond, The circular coverage function, Rand Report No. RM-330 (1950), Santa Monica, California.
4. W. C. Guenther, On the probability of capturing a randomly selected point in three dimensions, SIAM Rev., 3 (1961) 247–250.
5. G. E. Haynam and F. C. Leone, Tables of the cumulative noncentral chi-square distribution, Unpublished (1962).
6. N. L. Johnson, On an extension of the connection between Poisson and chi-square distributions, Biometrika, 46 (1959) 352–363.
7. J. I. Marcum, Table of Q functions, Rand Report No. RM-339 (1950) Santa Monica, California.
8. National Bureau of Standards, Tables of the exponential function e^x, N.B.S. Applied Mathematics Series, 14 (1951) U. S. Govt. Printing Office, Washington, D. C.
9. National Bureau of Standards, Tables of normal probability functions, N.B.S. Applied Mathematics Series, 23 (1953) U. S. Govt. Printing Office, Washington, D. C.
10. P. B. Patnaik, The noncentral chi-square and F distributions and their applications, Biometrika, 36 (1949) 206–232.
11. E. S. Pearson, Note on an approximation to the distribution of noncentral chi-square, Biometrika, 46 (1959) 364.
12. M. H. Quenouille, The evaluation of probabilities in a normal multivariate distribution with special reference to the correlation ratio, Proc. Edinburgh Math. Soc., 8 (1949) 95–100.
13. G. N. Watson, A treatise on the theory of Bessel functions, 2nd ed., (1952) Cambridge University Press.

Further items on Umbugio's bookshelf

Cf. this MONTHLY, 69 (1962) 980–981.

Insequential Analysis	History of Arcs, by Joan
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Deferential Equations	Eccentricities of Orbiting Mathematicians
Natural Logs in Public Forests	1,000,000 Random Numbers in Ascending Order

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida, 33146.

ON $n+2$ POINTS IN n-DIMENSIONAL SPACE

O. BOTTEMA, Delft, Netherlands

N. Altshiller Court has recently [1] proved the following theorem: *Given five points in space, if for each of them we construct the harmonic plane with respect to the tetrahedron determined by the remaining four, the five planes thus obtained are such that any four of them form a tetrahedron perspective to the tetrahedron determined by their four corresponding points. The center and plane of perspectivity are in each case the remaining point and the remaining plane. The constants of the five perspectivities considered are equal, their common numerical value being* 5.

At the end of the paper it is said that the proposition may, presumably, hold for $n+2$ points in an n-dimensional space. In what follows we give a proof of this conjecture.

If in n-dimensional space S_n a point P has the $m=n+1$ homogeneous projective coordinates, a_i $(i=1 \cdots m)$ with respect to the coordinate simplex T then it is well known that the harmonic $(n-1)$-space of P with respect to T has the equation

$$\sum_1^m a_i^{-1} x_i = 0.$$

Now consider $n+2$ points of S_n in general position; take m of them $A_1, A_2, \cdots, A_m$ as the vertices of a coordinate simplex and the remaining one, A_0, as the unit point. Thus for A_i $(i=1, \cdots, m)$ we have $x_i=1$, $x_j=0$ $(j\neq i)$ and for A_0: $x_j=1$ for all j. For a fixed value of k we introduce the transformation of coordinates

$$y_k = x_k, \qquad y_i = x_i - x_k, \qquad (i \neq k). \tag{1}$$

Then for A_i $(i\neq 0, i\neq k)$ all the new coordinates y are equal to the old coordinates x; for A_0 we have $y_k=1$, $y_i=0$ $(i\neq k)$ and for A_k: $y_k=1$, $y_i=-1$ $(i\neq k)$. Therefore the m points A_i $(i\neq k)$ are the vertices of the coordinate simplex for the y-coordinates, from which it follows that the harmonic space V_k of A_k with respect to this simplex has the equation

$$-y_k + \sum_{i\neq k} y_i = 0,$$

which means that the equation of this space in the x-coordinates reads

$$\sum_{i=1}^m x_i = (m+1)x_k. \tag{2}$$

We now take successively $k=1, 2, \cdots, m$, that is, we construct the harmonic space V_k of each point A_k ($k\neq 0$) with respect to the simplex determined by the remaining points A. There are m such spaces; the vertex B_k of the simplex formed by them is the point of intersection of $V_1, V_2, \cdots, V_{k-1}, V_{k+1}, \cdots, V_m$. From (2) it follows that we have for the coordinates of B_k:

$$x_i = 1 \ (i \neq k), \qquad x_k = 2. \tag{3}$$

It is now immediately seen that the line $l_k = A_0A_k$, which has the equations

$$x_i = \lambda \ (i \neq k), \qquad x_k = 1 + \lambda, \tag{4}$$

where λ is a parameter ($\lambda=0$ for A_k, $\lambda=\infty$ for A_0) passes through B_k, the corresponding parameter value being $\lambda=1$. Moreover, the harmonic space V_0 of A_0 with respect to the simplex $A_1 \cdots A_m$ has the equation $\sum_1^m x_i=0$ and the intersection C_k of l_k and V_0 is therefore given by $\lambda=-1/m$. Hence, the biratio $(A_0C_kA_kB_k)$ has the constant value $m+1=n+2$. Therefore:

Given $n+2$ points in n-dimensional space, if for each of them we construct the harmonic $(n-1)$-space, with respect to the simplex determined by the remaining points, the $n+2$ spaces thus obtained are such that any $n+1$ of them form a simplex perspective to the simplex determined by their corresponding points. The center and space of perspectivity are in each case the remaining point and the remaining space. The common value of the perspectivity constants is $n+2$.

Reference

1. N. Altshiller Court, Five points in three space, this MONTHLY, 69 (1962) 614–617.

ON A THEOREM OF MANN

WILLIAM A. MCWORTER, Ohio State University

In [1] Mann, using surprisingly elementary methods, proved the famed $\alpha+\beta$ theorem for which he received the Cole Prize for Number Theory. In [2] he applied his methods to abelian groups. His paper contains, curiously, only one result on nonabelian groups; namely,

THEOREM. *Let G be a group (written multiplicatively) and let A and B be two subsets of G such that*
(*) *not every element of G has the form ab, with a in A and b in B. Then*

$$(G) \geqq (A) + (B), \tag{1}$$

where (X) is the cardinal of the set X.

The purpose of this note is to generalize this theorem to arbitrary quasigroups.

A quasigroup $Q(\cdot)$ is a nonempty set Q on which a binary operation is defined such that the equations $a\cdot x=b$ and $y\cdot a=b$ are uniquely solvable for x

and y, for all a and b in Q. In case Q is finite, we can form the multiplication table M of Q by placing the product $x\cdot y$ in the xth row and yth column of M (x and y in Q). The fact that the equations $a\cdot x=b$ and $y\cdot a=b$ are uniquely solvable for all a and b in Q implies that M is a latin square.

We may now proceed to prove the theorem for quasigroups.

If Q is an infinite quasigroup, (1) is obvious. Hence we assume that Q is a finite quasigroup containing two subsets A and B satisfying (*). We may rearrange the rows of M so that the first (A) rows of M have entries consisting of all right multiples of elements of A, and then rearrange the columns of M so that the first (B) columns consist of all left multiples of elements of B. Thus, M will have the form

C	X
*	Y

where C is an (A) by (B) submatrix of M. Hence the entries of C consist precisely of the elements of the form $a\cdot b$, with a in A and b in B. Now, let $N(x)$ be the number of times that the element x in Q appears in C. Since x appears (A) times in C and X together, x appears $(A)-N(x)$ times in X. But x appears $(Q)-(B)$ times in X and Y together. Hence x appears $(Q)-(B)-[(A)-N(x)]$ times in Y. Certainly the number of times that x appears in Y is not negative, so we must have

$$(Q) - (A) - (B) + N(x) \geqq 0, \qquad \text{for all } x \text{ in } Q.$$

Now, since A and B satisfy (*), $N(x)=0$ for some x in Q. Hence (1) now follows.

References

1. H. B. Mann, On the fundamental theorem on sums of sets of positive integers, Ann. of Math., Ser. 2, 43 (1942) 523–527.

2. ———, On products of sets of group elements, Canad. J. Math., 4 (1952) 64–66.

ANOTHER PROOF OF LAGRANGE'S FOUR SQUARE THEOREM

JOHN D. DIXON, California Institute of Technology

The famous theorem that every positive integer can be written as the sum of four squares has had many different proofs (for example, see [1] pp. 275–303). One of the simplest and most beautiful of these proofs (and the one which appears in most recent elementary textbooks on number theory) is the one given in [2; Theorem 169]. The principal common element of all these proofs is that they use the identity of Euler on the products of sums of four squares, and that the arguments used are combinatorial. The present proof differs in both these

respects, and uses an argument (Lemma 2) which is a form of diophantine approximation.

THEOREM. *Every integer $n>0$ can be written*

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

as the sum of four square integers.

Note. It is clearly sufficient to prove the theorem in the case that no square integer >1 divides n. Therefore, in the following we assume that n is squarefree (and also greater than 1).

The proof of the theorem rests on two lemmas.

LEMMA 1. *For all squarefree integers $n>1$, there exist integers x, y such that*

$$x^2 + y^2 + 1 \equiv 0 \pmod{n}. \tag{1}$$

Note. This assertion may be false if n is not squarefree; for example, if $n=4$.

Proof. We use induction on the number of prime divisors of n. If n is a prime p, then the $[p/2]+1$ numbers x^2, $0 \leqq x \leqq p/2$ are all incongruent (mod p) since, if $0 \leqq x < z \leqq p/2$ then $p \nmid (x-z)(x+z) = x^2 - z^2$. Similarly the $[p/2]+1$ numbers $-1-y^2$ are all incongruent (mod p) for $0 \leqq y \leqq p/2$. Since these two sets together contain $2[p/2]+2>p$ numbers, two numbers must be congruent and some $x^2 \equiv -1-y^2 \pmod{p}$ as required.

If n is not prime, then $n = n_1 n_2$ with $1 < n_1$, $n_2 < n$. Therefore by the inductive hypothesis

$$x_i^2 + y_i^2 + 1 \equiv 0 \pmod{n_i} \qquad (i = 1, 2).$$

Since n is squarefree, n_1 and n_2 are relatively prime and so for some integers m_1, m_2 we have $n_1 m_1 + n_2 m_2 = 1$. Taking

$$x = n_1 m_1 x_2 + n_2 m_2 x_1, \qquad y = n_1 m_1 y_2 + n_2 m_2 y_1,$$

we get $x^2+y^2+1 \equiv 0 \pmod{n_1 n_2}$, which proves the lemma.

If x, y are chosen as in (1) then we note that for any integers s, t

$$(xs + yt)^2 + (xt - ys)^2 + s^2 + t^2 = (s^2 + t^2)(x^2 + y^2 + 1) \equiv 0 \pmod{n}. \tag{2}$$

From this we shall prove

LEMMA 2. *There exist integers y_i such that $y_1^2+y_2^2+y_3^2+y_4^2 = n$ or $2n$.*

Proof. The average density of integer lattice points (s, t) in the plane is 1 per unit area. Therefore, there exists in the (s, t)-plane some circle $\mathcal{C}$, of radius $\sqrt{n/3}$ and area $\pi n/3 > n$, which contains more than n lattice points.

Now consider the set $\mathcal{S}$ of all integral lattice points (a, b) such that

$$a \equiv xs + yt, \qquad b \equiv xt - ys \pmod{n} \tag{3}$$

for some $(s, t) \in \mathcal{C}$, where account is taken of the number of (s, t) which give rise to a given (a, b). Since there are more than n points in $\mathcal{C}$, the average density of the points of $\mathcal{S}$ in the (a, b)-plane is $> n \cdot 1/n^2 = 1/n$. Therefore, there exists in the (a, b)-plane a circle $\mathcal{C}'$ of radius $\sqrt{n/3}$ and area $\pi n/3 > n$ which contains more than $1/n \cdot n = 1$ point.

Thus for some two points (s, t), $(s', t') \in \mathcal{C}$, there are two corresponding points (a, b), $(a', b') \in \mathcal{C}'$. Write

$$y_1 = s - s', \qquad y_2 = t - t', \qquad y_3 = a - a', \qquad y_4 = b - b'$$

and note that $y_3 \equiv xy_1 + yy_2$, $y_4 \equiv xy_2 - yy_1 \pmod{n}$ from (3). Since the circles $\mathcal{C}$ and $\mathcal{C}'$ are each of diameter $2\sqrt{n/3}$, we have

$$0 < y_1^2 + y_2^2 \leqq \left(2\sqrt{\frac{n}{3}}\right)^2 = \frac{4}{3}n$$

$$0 \leqq y_3^2 + y_4^2 \leqq \frac{4}{3}n.$$

Hence

$$0 < y_1^2 + y_2^2 + y_3^2 + y_4^2 \leqq 8/3n < 3n$$

and $0 \equiv y_1^2 + y_2^2 + y_3^2 + y_4^2 \pmod{n}$ from (2). The lemma follows.

Proof of the theorem. The proof of the theorem for the case n follows immediately from Lemma 2 unless

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 2n.$$

In this case the y_i must be congruent (mod 2) in pairs. Suppose $y_1 \equiv y_2$, $y_3 \equiv y_4 \pmod{2}$. Then

$$\left(\frac{y_1 + y_2}{2}\right)^2 + \left(\frac{y_1 - y_2}{2}\right)^2 + \left(\frac{y_3 + y_4}{2}\right)^2 + \left(\frac{y_3 - y_4}{2}\right)^2 = \frac{2n}{2} = n,$$

where the terms on the left are all integers.

References

1. L. E. Dickson, History of the Theory of Numbers, v. 2, Chelsea, New York, 1952.
2. E. Landau, Elementare Zahlentheorie, Chelsea, New York 1947.

ON THE CASUS IRREDUCIBILIS

HENRY B. MANN, Ohio State University

The theorem on the casus irreducibilis is usually formulated (see [1] and [2]) in terms of irreducible cubic equations over a real number field. In this formulation it does not tell us anything about the solutions of cubic equations

over the field of all real numbers since there are no irreducible cubics over this field. More generally, the usual formulation does not make any statement about formulae for the solution of cubic equations valid over any field. The following formulation fills this gap.

THEOREM. *Let F be any field whose characteristic is different from 2 or 3. Let*

$$f(x) = x^3 + a_1x^2 + a_2x + a_3, \qquad a_i \in F, \tag{1}$$

be irreducible in F. Let D be the discriminant of f and α one of its roots. If F' is a field obtained from F by successive adjunction of irreducible radicals and if $\alpha \in F'$ then $F'(\sqrt{D})$ contains a primitive cube root of unity.

The proof is completely analogous to that given in [1] or [2], the main point being that the Galois group of f over $F(\sqrt{D})$ is the alternating group on three letters. The details may be left to the reader.

If F is any field we may always adjoin indeterminates a_1, a_2, a_3 and consider (1) as a (necessarily irreducible) polynomial over $F(a_1, a_2, a_3)$. The theorem then asserts: If there is an expression for the root α of $f(x)$ in terms of a succession of irreducible radicals $\beta_1, \cdots, \beta_s$ over $F(a_1, a_2, a_3)$ then the cube roots of unity can be expressed in terms of $\beta_1, \cdots, \beta_s$ and $\sqrt{D}$.

The restriction that the characteristic be different from 3 is obviously necessary, otherwise there are no primitive cube roots of unity. For characteristic 2 we have $\sqrt{D} \in F$. Let F be the field of rational functions modulo 2; then $F(\sqrt[3]{x}) = F(\sqrt{D}, \sqrt[3]{x})$ does not contain the cube roots of unity.

If in the theorem, however, we replace $\sqrt{D}$ by

$$\theta = \alpha_1^2\alpha_2 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_1,$$

where α_1, α_2, α_3 are the roots of (1), then the theorem and its proof will be valid also for the case of characteristic 2. If in (1) we have $a_1 = 0$, as may always be assumed, then $\theta = \frac{1}{2}(\sqrt{D} + 3a_3)$ if the characteristic of F is distinct from 2, and

$$\theta^2 + a_3\theta + a_2^3 + a_3^2 = 0$$

if F has characteristic 2.

The solution of (1) can easily be given in terms of θ. The Lagrange resolvent if $a_1 = 0$ becomes

$$(\rho, \alpha_1) = \sqrt[3]{\{(9\rho^2 - 9)a_3 + 3(\rho - \rho^2)\theta\}},$$

where ρ is a primitive cube root of unity. For characteristic 2 this reduces to

$$(\rho, \alpha_1) = \sqrt[3]{\{\theta + \rho a_3\}}.$$

The solutions of (1) can be expressed by (ρ, α_1) and (ρ^2, α_1) in the usual manner where $(\rho, \alpha_1)(\rho^2, \alpha_1) = -3p$ ($= p$ if the characteristic is 2).

References

1. Garrett Birkhoff and Saunders MacLane, A Survey of Modern Algebra, Macmillan, New York, 1953, pp. 450–451.
2. B. L. van der Waerden, Modern Algebra, Ungar Publishing Company, New York, 1949, pp. 179–180.

ON ASSIGNING AN ARBITRARY LIMIT TO A LINEARLY INDEPENDENT SEQUENCE OF VECTORS

DETLEF LAUGWITZ, Technische Hochschule, Darmstadt, Germany

It is well known that in a finite dimensional vector space a sequence (x_n) converges to a limit x_0 in a certain norm if and only if it has the same limit x_0 according to every other norm. It seems to be unknown that in an infinite dimensional space a sequence may be made convergent to an arbitrary limit by choice of a suitable norm, if only the sequence has a not very restricting property. See, for example, problem No. 132 of Nieuwe opgaven, Nieuw Archief voor Wiskunde, 1962, p. 3, which is posed as follows by N. G. de Bruijn and G. W. Veltkamp: Show by a counter-example that the following theorem is not true: *Let* $\|\ \|_1$ *and* $\|\ \|_2$ *be different norms in a linear space M. If a sequence* $x_1, x_2, \cdots$ *converges in both norms then the limits are equal.*

Before giving the general result, let us consider an instructive example. Let P be the space of all real polynomials, $x(t)=a_0+a_1t+\cdots+a_nt^n$, and define

$$\|x\|_N = \left|\sum a_j\right| + \sum_{j\neq N} |a_j|/(j+1).$$

The relations $\|x+y\|_N \leqq \|x\|_N+\|y\|_N$, $\|a\cdot x\|_N=|a|\cdot\|x\|_N$, and $\|x\|_N\geqq 0$ are easily verified. It remains to show that $\|x\|_N=0$ only if the polynomial x vanishes identically; $\|x\|_N=0$ implies

$$\text{(A)} \qquad \left|\sum a_j\right| = 0$$

and

$$\text{(B)} \qquad \sum_{j\neq N} |a_j|/(j+1) = 0.$$

Equation (B) yields $a_j=0$ for $j\neq N$, hence $a_N=0$ by (A). Thus $\|x\|_N$ is a norm.

I shall prove that the sequence $x_n=t^n$ converges to t^N, if the norm of P is defined by $\|x\|_N$. In fact,

$$\|t^N - t^n\|_N = \frac{1}{n+1},$$

which tends to 0 for $n\to\infty$.

Now let L be a real or complex vector space. A set $E\subseteq L$ is called linearly independent, if every finite subset of E is linearly independent. A subset E is called infinite dimensional, if it is not contained in a finite dimensional subspace.

The following theorem will generalize our example. Moreover, the norm yielding the result is not pathological: it may be chosen as an inner product norm.

THEOREM. *Let* (x_n) $(n=0, 1, 2, \cdots)$ *be a linearly independent sequence of a (necessarily infinite dimensional) vector space L. Then there exists an inner product norm of L such that* (x_n) $(n=1, 2, 3, \cdots)$ *converges to* x_0 *in this norm.*

Proof. The vectors $y_n=x_n-x_0$ $(n=1, 2, 3, \cdots)$ form a linearly independent sequence. Let M be the linear hull of the set $E=\{y_n\}$, such that $y\in M$ admits of a unique representation $y=\sum_{n=1}^{\infty} a_n y_n$, with $a_n=0$ for almost all n. Define in M

$$\|y\| = \left(\sum_{n=1}^{\infty} |a_n|^2/n^2\right)^{1/2}.$$

By a routine procedure, this may be extended to an inner product norm for the whole of L. The proposition follows from the relations

$$\|x_n - x_0\| = \|y_n\| = 1/n.$$

COROLLARY 1. *Let E be an infinite dimensional subset of L. Then, for an arbitrary* $x_0\in L$, *there exist a sequence* $x_n\in E$, $n=1, 2, 3, \cdots$, *and an inner product norm of L such that* x_n *converges to* x_0 *according to this norm.*

Proof. Since E is infinite dimensional, a linearly independent sequence $y_n=x_n-x_0$, $x_n\in E$, $n=1, 2, \cdots$, will exist. Apply the former reasoning.

Example. In the space of all real functions $x(t)$, $-\infty<t<+\infty$, the sequence $x_n(t)=t^n$ may be made convergent to an arbitrary (not necessarily continuous) function by the choice of a suitable inner product norm.

Since the limit of a finite dimensional sequence (if it exists) is uniquely determined, the results give rise to a number of characterizations of infinite dimensionality; one of them is:

COROLLARY 2. *A subset of a vector space L is infinite dimensional if and only if it contains a sequence which converges to different limits according to two suitably chosen norms of L.*

A NOTE ON THE GENERALIZED WILSON'S THEOREM

L. CARLITZ, Duke University

1. Let P_m denote the product of the integers $\leqq m$ and prime to p, where p is a fixed prime. The writer has proved [1] that if $p^r|m$, $r\geqq 1$, and $p>3$ then P_m satisfies the congruence

$$P_m \equiv ((p-1)!)^{m/p} \pmod{p^{r+2}}. \tag{1}$$

It is evident from $P_{np}=\prod_{h=1}^{n}\prod_{k=1}^{p-1}(hp-k)$ that

$$P_{np} = ((p-1)!)^n \prod_{h=1}^{n} \binom{hp-1}{p-1}. \tag{2}$$

A CONVERGENCE CRITERION FOR POSITIVE SERIES

JACK P. TULL, University of Adelaide, Australia, and Ohio State University;
DAVID REARICK, University of Colorado

Consider a sequence of positive terms a_n tending to zero, and for any positive x let $N(x)$ denote the number of terms $a_n \geqq x$. One of the authors proposed the problem (E1552, this MONTHLY, vol. 69, no. 10, 1962, p. 1008) of showing that if $\sum_{n=1}^{\infty} a_n$ converges, then $xN(x) \to 0$ as $x \to 0$. This latter condition is not sufficient to ensure the convergence of $\sum_{n=1}^{\infty} a_n$. For example, taking a_n to be the reciprocal of the nth prime, we have $N(x) = \pi(1/x)$, and $xN(x) = \pi(1/x)/1/x \to 0$ as $x \to 0$ whereas the series diverges. The following theorem gives a necessary and sufficient condition for the convergence of $\sum_{n=1}^{\infty} a_n$ in terms of the function $N(x)$.

THEOREM. *$\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_0^{\infty} N(x)dx$ converges, and in the event of convergence we have*

$$\sum_{n=1}^{\infty} a_n = \int_0^{\infty} N(x)dx. \tag{1}$$

The integral is improper at the lower limit only, for if b is any number larger than the largest of the a_n, then $N(x) = 0$ for $x \geqq b$. For any $\epsilon > 0$, let $A_\epsilon = \sum_{a_n \geqq \epsilon} a_n$. As $\epsilon \to 0$, A_ϵ converges to a finite limit A if and only if $\sum_{n=1}^{\infty} a_n$ converges to A. We can form the Stieltjes integral of x with respect to $-N(x)$ on $[\epsilon, b]$, and in fact,

$$A_\epsilon = \int_\epsilon^b xd(-N(x)).$$

Integration by parts yields

$$A_\epsilon = \int_\epsilon^b N(x)dx + \epsilon N(\epsilon).$$

If $\sum_{n=1}^{\infty} a_n$ converges to A, then $A_\epsilon \to A$ and $\epsilon N(\epsilon) \to 0$ as $\epsilon \to 0$ (the latter by problem E 1552), and consequently $\int_\epsilon^b N(x)dx \to A$. On the other hand, if $\sum_{n=1}^{\infty} a_n$ diverges, then $A_\epsilon \to +\infty$ as $\epsilon \to 0$, and

$$\begin{aligned} \int_\epsilon^b N(x)dx &= A_\epsilon - \epsilon N(\epsilon) = \sum_{a_n \geqq \epsilon} (a_n - \epsilon) \\ &\geqq \sum_{a_n \geqq 2\epsilon} (a_n - \epsilon) \\ &\geqq \sum_{a_n \geqq 2\epsilon} (a_n - \tfrac{1}{2}a_n) = \tfrac{1}{2}A_{2\epsilon} \to +\infty. \end{aligned}$$

This theorem is occasionally of use as a convergence test. For example, we may use it to show the divergence of $\sum_{k=2}^{\infty} r(k)/k \log k$, where $r(k)$ is the number

of representations of k as a sum of two squares, $k=m^2+n^2$. Consider the equivalent double series

$$\sum_{\substack{m=-\infty \\ m^2+n^2\neq 0}}^{+\infty} \sum_{n=-\infty}^{+\infty} \frac{1}{(m^2+n^2)\log(m^2+n^2)}. \tag{2}$$

The general term of the series (2) is greater than x if the lattice point (m, n) lies inside a circle of radius $(x \log 1/x)^{-1/2}$ about the origin, and hence $N(x)$ is not less than the number of such lattice points, which is $\pi(x \log 1/x)^{-1} + O((x \log 1/x)^{-1/2})$. Therefore $\int_0^\infty N(x)dx$ diverges, and consequently so do the series (2) and the original series.

The identity (1) provides an integral representation for functions expressed as series. For instance, if s is real and greater than 1 we may put $a_n = 1/n^s$. Then $N(x) = [x^{-1/s}]$, and we obtain for the Riemann zeta function,

$$\zeta(s) = \int_0^\infty [x^{-1/s}]dx = \frac{s}{s-1} - \int_0^1 \{x^{-1/s}\} dx,$$

where brackets denote fractional part. Upon setting $x=y^{-s}$, this becomes the known formula

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{y\}}{y^{s+1}} dy,$$

which makes sense for complex $s \neq 1$ with positive real part and hence serves as an analytic continuation of $\zeta(s)$ to the right half plane.

A BIBLIOGRAPHICAL NOTE

NATHAN ALTSHILLER COURT, University of Oklahoma

In his recent note (this MONTHLY, March, 1963, Vol. 70, pp. 294–298) G. D. Chakerian quoted "a well-known result of H. A. Schwarz" (1843–1921) that of all triangles inscribed in an acute-angled triangle the pedal triangle (of the orthocenter) is the unique one having minimal perimeter.

The renowned analyst provided an ingenious and most elegant geometrical proof of that proposition. But the property itself is of a much older vintage. In the Acta Eruditorum, 1775, p. 297, J. F. de Fuschis a Fagnano, stated the proposition and proved it using differential calculus.

Schwarz's proof of Fagnano's theorem, together with four other such proofs, and a goodly number of bibliographical references regarding that theorem, may be found in Scripta Mathematica, Vol. 17, 1951, pp. 147–150 and Vol. 18, 1952, pp. 95–96.

CORRECTION

In the paper by N. A. Khan entitled "The Characteristic of a Ring," this MONTHLY, 70 (1963) 736, the inequality $0<f<e$ on p. 736, l.16, should read $0\leq f<e$.

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Md.

A NOTE ON EXPONENTS (MOD 2^r)

L. CARLITZ, Duke University

1. It is familiar that if $r \geqq 3$, then the number 5 belongs to the exponent 2^{r-2} (mod 2^r). It is perhaps not quite so familiar how to specify the exponent to which a given odd number belongs. The following theorem answers this question for numbers of the form $4n+1$.

THEOREM 1. *Let $a \equiv 1$ (mod 4) and let 2^s be the highest power of 2 dividing $a-1$, so that $s \geqq 2$. Then if $r \geqq 3$, a belongs to the exponent 2^{r-s} (mod 2^r).*

The proof is simple. Put $a = 1 + 2^s m_0$ $(2 \nmid m_0)$. Then we have

$$a^{2^j} = 1 + 2^{s+j} m_j \qquad (2 \nmid m_j) \tag{1}$$

for $j = 0, 1, 2, \cdots$. Indeed, assuming the truth of (1) for the value j, we have

$$a^{2^{j+1}} = (1 + 2^{s+j} m_j)^2 = 1 + 2^{s+j+1} m_j (1 + 2^{s+j-1} m_j),$$

so that

$$m_{j+1} = m_j (1 + 2^{s+j-1} m_j) \equiv 1 \pmod 2.$$

For numbers of the form $4n+3$ we have

THEOREM 2. *Let $a \equiv 3$ (mod 4) and let 2^s be the highest power of 2 dividing $a+1$, so that $s \geqq 2$. Then if $r > s$, a belongs to the exponent 2^{r-s} (mod 2^r). If $r = s$, a belongs to the exponent 2.*

We may assume $r > s$. Then by Theorem 1, a^2 belongs to the exponent 2^{r-s-1} (mod 2^r). Thus $(-a)^{2^{r-s}} \equiv 1$ (mod 2^r). If we assume that

$$(-a)^{2^{r-s-1}} \equiv 1 \pmod{2^r},$$

we have a contradiction when $r > s+1$; when $r = s+1$ we get

$$-a \equiv 1 \pmod{2^r},$$

which contradicts the hypothesis concerning s.

It follows from Theorem 1 that the numbers $8n+5$ all belong to the ex-

ponent 2^{r-2} (mod 2^r); by Theorem 2 the same is true of the numbers $8n+3$. On the other hand, the numbers $16n+9$ belong to the exponent 2^{r-3} while the numbers $16n+7$ belong to the exponent 2^{r-3} for $r>3$.

2. It is of some interest to discuss the modulus $(1+i)^r$ in the Gaussian field. For basic properties of the Gaussian integers see, for example [1, Ch. 5]. In particular we recall that the integer α is odd provided it is not divisible by $1+i$; this is equivalent to the statement

$$\alpha \equiv 1 \pmod{1+i}.$$

Corresponding to Theorem 1 we have

THEOREM 3. *Let α be an odd Gaussian integer and let $(1+i)^s$ be the highest power of $1+i$ that divides $\alpha-1$. Let $3\leqq s\leqq r$. Then α belongs to the exponent $2^{[(r-s+1)/2]}$ (mod $(1+i)^r$), where $[(r-s+1)/2]$ denotes the greatest integer $\leqq(r-s+1)/2$.*

Proof. Put $\alpha=1+(1+i)^s\beta_0$, $(1+i\nmid\beta_0)$. Then we have

$$\alpha^{2^j} = 1 + (1+i)^{s+2j}\beta_j \qquad (1+i\nmid\beta_j) \tag{2}$$

for $j=0, 1, 2, \cdots$. Indeed if we assume that (2) holds for the value j, we get

$$\begin{aligned}\alpha^{2^{j+1}} &= 1 + 2(1+i)^{s+2j}\beta_j + (1+i)^{2s+4j}\beta_j^2 \\ &= 1 + (1+i)^{s+2j+2}\beta_j\{-i + (1+i)^{s+2j-2}\beta_j\},\end{aligned}$$

so that, since $s\geqq 3$,

$$\beta_{j+1} = \beta_j\{-i + (1+i)^{s+2j-2}\beta_j\} \equiv 1 \pmod{1+i}.$$

The theorem follows at once from (2).

As an example we take $\alpha=3+2i$. Then $s=3$, so that $3+2i$ belongs to the exponent 2^k (mod $(1+i)^{2k+3}$).

To treat the cases not covered by Theorem 3 we recall that if α is an arbitrary odd Gaussian integer then $\alpha\equiv\pm 1$ or $\pm i$ (mod $(1+i)^3$). We may state

THEOREM 4. *Let $(1+i)^s$ be the highest power of $1+i$ that divides $\alpha+1$ and assume $s>3$. Then if $r>s$, α belongs to the exponent*

$$2^{[(r-s+1)/2]} \tag{3}$$

(mod $(1+i)^r$). If $r=s$, α belongs to the exponent 2.

THEOREM 5. *Let $(1+i)^s$ be the highest power of $1+i$ that divides $\alpha-\epsilon$, where $\epsilon=\pm i$; assume $s\geqq 3$. Then if $r>s+2$, α belongs to the exponent (3) (mod $(1+i)^r$). If $s\leqq r\leqq s+2$, α belongs to the exponent 4.*

The proof of Theorem 4 is similar to the proof of Theorem 2 and, in much the same way, Theorem 5 is derived from Theorem 4.

As an example it follows from Theorem 4 that $5+2i$ belongs to the exponent $2^k \pmod{(1+i)^{2k+3}}$; it follows from Theorem 5 that $2+i$ belongs to the exponent $2^k \pmod{(1+i)^{2k+3}}$ provided $k \geqq 2$.

Supported in part by National Science Foundation Grant G-16485.

Reference

1. L. W. Reid, The elements of the theory of algebraic numbers, Macmillan, New York, 1910.

INDEFINITE INTEGRATION BY RESIDUES

R. P. BOAS, JR., Northwestern University

It is sometimes incorrectly asserted that only definite integrals with very special limits can be evaluated by contour integration. Neville [1] has pointed out that integrals of the form

$$\int_a^b R(x)dx \tag{1}$$

can be evaluated by contour integration when R is a rational function of the usual form and a and b are arbitrary. In a similar way one can evaluate any integral of the form

$$\int_\alpha^\beta R(\sin\theta, \cos\theta)d\theta, \qquad 0 \leqq \alpha < \beta \leqq 2\pi, \tag{2}$$

where R is a rational function without poles on $[\alpha, \beta]$. Incidentally, *any* proper integral of the form (1) can be transformed into the form (2) and so evaluated, irrespective of whether $\int^\infty R(x)dx$ converges or not.

We begin with the case $\alpha=0$, $\beta=\pi$, where the method perhaps shows to best advantage; we suppose that R is real (otherwise consider its real and imaginary parts separately). We then have to evaluate an integral of the form

$$\int_0^\pi f(e^{i\theta})d\theta, \tag{3}$$

where the rational function $f(z)$ is real-valued for $|z|=1$ and has no poles for $|z|=1$, $0 \leqq \theta \leqq \pi$.

Assume first that $f(z)$ has no poles at all on $|z|=1$. Then we replace (3) by

$$\Re \int_0^{2\pi} f(e^{i\theta})g(e^{i\theta})d\theta, \tag{4}$$

where $g(z)$ is analytic for $|z|<1$, g is dominated by an integrable function in $|z|<1$, $\Re(g)=1$ on $|z|=1$ for $0<\theta<\pi$, and $\Re(g)=0$ on the rest of $|z|=1$; here

we use the fact that $f(e^{i\theta})$ is real. We can then write the integral in (4) as

$$-i\int_{|z|=1} f(z)g(z)z^{-1}dz, \tag{5}$$

and this is 2π times the sum of the residues of $f(z)g(z)/z$ in $|z|<1$.

A suitable $g(z)$ is

$$1-(i\pi)^{-1}\log\{i(1-z)/(1+z)\}. \tag{6}$$

In fact, $w=i(1-z)/(1+z)$ maps $|z|\leqq 1$ on the half plane $\mathscr{I}(w)\geqq 0$ with the arc $0\leqq\theta\leqq\pi$ corresponding to the positive real axis; hence the principal branch of $\log w$ is real for $0<\theta<\pi$ and has imaginary part π for $\pi<\theta<2\pi$; that is, (6) has real part 1 for $0<\theta<\pi$ and real part 0 for $\pi<\theta<2\pi$.

More generally, if $f(z)$ is analytic for $z=e^{i\theta}$, $0\leqq\theta\leqq\pi$, the integral (3) is finite, even though the integral in (4) may diverge because of poles of $f(z)$ on the complementary arc of the unit circumference. Formally, it is only the imaginary part of this integral that diverges, and this is irrelevant. We may argue more precisely that (5) converges when the integration is over $|z|=\rho<1$, and its value is still 2π times the sum of the residues of $f(z)g(z)/z$ when ρ is near 1; the real part of this integral converges to (3) as $\rho\to 1$.

Hence we have the following theorem.

THEOREM. *If $f(z)$ is a rational function, real on $|z|=1$, with no poles on the arc $|z|=1$, $0\leqq\theta\leqq\pi$, then $\int_0^\pi f(e^{i\theta})d\theta$ is equal to 2π times the real part of the sum of the residues of*

$$z^{-1}f(z)\{1-(i\pi)^{-1}\log[i(1-z)/(1+z)]\}$$

inside $|z|=1$, where the logarithm has its principal value.

The general case

$$\int_\alpha^\beta f(e^{i\theta})d\theta \tag{7}$$

requires a function g such that $\mathfrak{R}(g)=1$ on $|z|=1$ for $\alpha<\theta<\beta$, $\mathfrak{R}(g)=0$ on the rest of $|z|=1$. We can take

$$g(z)=1-(i\pi)^{-1}\log\{i(1-ze^{-i\alpha})/(1-ze^{-i\beta})\},$$

and hence (7) is equal to 2π times the real part of the sum of the residues of $z^{-1}f(z)g(z)$ inside $|z|=1$.

Example. We evaluate

$$\int_0^\pi \frac{d\theta}{a+b\sin\theta}, \tag{8}$$

where a and b are real and $|b|<a$.

Here

$$f(e^{i\theta}) = \frac{2i}{2ia + b(e^{i\theta} - e^{-i\theta})}, \quad f(z) = \frac{2iz}{bz^2 + 2aiz - b},$$

and $z^{-1}f(z)g(z)$ has a single pole, at the zero r_1 of $bz^2+2aiz-b$ that is inside $|z|=1$. We have

$$ibr_1 = a - (a^2 - b^2)^{1/2}, \qquad ibr_2 = a + (a^2 - b^2)^{1/2},$$

and the residue is

$$\frac{2i}{b(r_1 - r_2)}\left\{1 - \frac{1}{i\pi}\log i\,\frac{1 - r_1}{1 + r_1}\right\}$$
$$= (a^2 - b^2)^{-1/2}\left\{1 - \frac{1}{i\pi}\log i\,\frac{ib - a + (a^2 - b^2)^{1/2}}{ib + a - (a^2 - b^2)^{1/2}}\right\}.$$

Now the quantity of which we are taking the logarithm has modulus 1, so that the logarithm is simply i times the principal angle of this quantity. This is

$$i\tan^{-1}\left\{-(a^2 - b^2)^{1/2}/b\right\},$$

where the inverse tangent is taken in $(0, \pi)$. Hence the residue is

$$(a^2 - b^2)^{-1/2}\{1 - \pi^{-1}\tan^{-1}[-(a^2 - b^2)^{1/2}/b]\}$$
$$= \pi^{-1}(a^2 - b^2)^{-1/2}\tan^{-1}\left\{(a^2 - b^2)^{1/2}/b\right\},$$

and finally the integral (8) is equal to

$$2(a^2 - b^2)^{-1/2}\tan^{-1}\left\{(a^2 - b^2)^{1/2}/b\right\},$$

with the inverse tangent in $(0, \pi)$ (not the principal value when $b<0$).

Reference

1. E. H. Neville, Indefinite integration by means of residues, Math. Student, 13 (1945) 16–25.

A DIVISIBILITY THEOREM

L. M. COURT, George Washington University

Here is a curious little result in elementary number theory (it might be called a divisibility theorem), which may merit some attention:

THEOREM 1. *Let w be a number of the form $u^{p^k}+v^{p^k}$, where u and v are integers, p is an odd prime, and k is a positive integer; if w is divisible by p, then it is automatically divisible by p^{k+1}. Either u or v or both can be negative as well as positive.*

Proof. Since

$$p^k = (p - 1)(p^{k-1} + p^{k-2} + \cdots + 1) + 1 \tag{1}$$

VECTOR IDENTITIES IN $\mathfrak{S}_3$

W. F. EBERLEIN, University of Rochester

The spin model $\mathfrak{S}_3$ of Euclidean 3-space (see [1]) is the vector space of self-adjoint linear transformations of trace 0 in a two dimensional unitary space H_2 plus the operations $A\cdot B=\frac{1}{2}(AB+BA)$ and $A\times B=(AB-BA)/2i$. (We identify a scalar c with cI, where I is the identity transformation in H_2.) Although the real significance of the spin model lies on the quantum mechanical level, it is instructive and, perhaps, amusing to exhibit the basic identities of vector algebra as reflections of the associativity of the (Clifford) algebra $B(H_2)$ of linear transformations in H_2.

The Jacobi identity is a standard property of the Lie product in any associative algebra:

$$
\begin{aligned}
&(A\times B)\times C+(B\times C)\times A+(C\times A)\times B\\
(1)\qquad &= -\tfrac{1}{4}[(AB-BA)C-C(AB-BA)+(BC-CB)A-A(BC-CB)\\
&\quad +(CA-AC)B-B(CA-AC)]=0.
\end{aligned}
$$

Now employ the quaternion identity (see [2])

$$AB=\tfrac{1}{2}(AB+BA)+i(AB-BA)/2i=A\cdot B+i(A\times B)$$

to write the product ABC in two ways:

$$
\begin{aligned}
A(BC)&=A[B\cdot C+i(B\times C)]=(B\cdot C)A+iA\cdot(B\times C)-A\times(B\times C)\\
(AB)C&=[A\cdot B+i(A\times B)]C=(A\cdot B)C+i(A\times B)\cdot C-(A\times B)\times C.
\end{aligned}
$$

Equating scalar components yields

$$(2)\qquad A\cdot(B\times C)=(A\times B)\cdot C \quad \text{(scalar triple product identity)},$$

while the vector components give

$$(B\cdot C)A-A\times(B\times C)=(A\cdot B)C-(A\times B)\times C$$

or

$$(A\times B)\times C-A\times(B\times C)=(B\cdot A)C-(B\cdot C)A.$$

Since the Jacobi identity can be rewritten as $(A\times B)\times C-A\times(B\times C)=B\times(C\times A)$, it follows that

$$(3)\qquad B\times(C\times A)=(B\cdot A)C-(B\cdot C)A \quad \text{(vector triple product identity)}.$$

The Lagrange identity $(A\times B)\cdot(C\times D)=(A\cdot C)(B\cdot D)-(A\cdot D)(B\cdot C)$ then follows in elementary fashion from (2) and (3).

References

1. W. F. Eberlein, The Spin Model of Euclidean 3-Space, this MONTHLY, 69 (1962) 587–598.
2. ———, The Geometric Theory of Quaternions, this MONTHLY, 70 (1963) 952–954.

ARC LENGTH

CASPER GOFFMAN, Purdue University

1. Let S be a metric space with metric d. By a curve C in S, we shall mean a continuous mapping f on the closed interval $I=[0, 1]$ into the space S. The standard definition of the length of the curve C is as follows:

For every partition $\pi=[t_0=0<t_1< \cdots <t_n=1]$ of I, consider the number

$$\lambda(\pi, C) = \sum_{i=1}^{n} d(f(t_{i-1}), f(t_i)),$$

and let the length of C be the number

$$L(C) = \sup \lambda(\pi, C),$$

where the supremum is taken for all partitions π, and the supremum of an unbounded set of nonnegative numbers is taken to be $+\infty$.

The norm, $\|\pi\|$, of a partition $\pi=[t_0=0<t_1< \cdots <t_n=1]$, is the number

$$\|\pi\| = \max [t_i - t_{i-1}: i = 1, \cdots, n].$$

It is well known (see [1]) that if $\{\pi_m\}$ is a sequence of partitions of I whose norms converge to zero, then $\{\lambda(\pi_m, C)\}$ converges, (in the extended sense where the limit may be $+\infty$) and that

$$L(C) = \lim_{m\to\infty} \lambda(\pi_m, C).$$

2. In general, the sequence $\{\lambda(\pi_m, C)\}$ is not monotonic. It is the purpose of this note to point out that the reason for its convergence is that it satisfies a condition which we shall call quasi-monotonicity.

A sequence $\{s_n\}$ of real numbers is called *quasi-monotonically non-decreasing* if, for every $\epsilon>0$ and every n, there is an N such that $m>N$ implies $s_m>s_n-\epsilon$.

THEOREM. *If $\{s_n\}$ is a quasi-monotonically non-decreasing sequence, it converges to a real number or to $+\infty$.*

Proof. Obviously, $\{s_n\}$ is bounded from below.

Suppose that $\{s_n\}$ is unbounded. For every M, there is an n such that $s_n>M+1$. There is an N such that $s_m>s_n-1>M$, for every $m>N$. Hence $\{s_n\}$ converges to $+\infty$.

Suppose that $\{s_n\}$ is bounded. Let $u=\sup s_n$. Let $\epsilon>0$. There is an $s_n>u-\frac{1}{2}\epsilon$. There is an N such that $m>N$ implies $s_m>s_n-\frac{1}{2}\epsilon$. Hence, $m>N$ implies $u-\epsilon<s_m\leqq u$. Thus, $\{s_n\}$ converges.

3. Let C be a curve given by a continuous mapping f on I into S, and let $\{\pi_m\}$ be a sequence of partitions of I whose norms converge to zero. Fix $\epsilon>0$ and let $\pi_n = [t_0 = 0 < t_1 < \cdots < t_k = 1]$. Let $\delta > 0$ be such that $\delta<\frac{1}{3}\min [t_i-t_{i-1}: i=1, 2, \cdots, k]$, and if $x, y\in I$ and $|x-y|<\delta$ then $d(f(x),$

$f(y)) < \epsilon/2k$. Now, let N be such that $m > N$ implies $\|\pi_m\| < \delta$. For every $m > N$, each interval $[t_{i-1}, t_i]$, $i = 1, \cdots, k$, contains points s_i, u_i, which are in π_m, and are such that $s_i < u_i$, $d(f(t_{i-1}), f(s_i)) < \epsilon/2k$, and $d(f(u_i), f(t_i)) < \epsilon/2k$. It follows that $\lambda(\pi_m, C) > \lambda(\pi_n, C) - \epsilon$. Hence, the sequence $\{\lambda(\pi_m, C)\}$ is quasi-monotonically non-decreasing, so that it converges to a real number or to $+\infty$.

4. Let $\{\sigma_m\}$ be another sequence of partitions of I whose norms converge to zero. Then the norms of the sequence

$$\sigma_1, \pi_1, \sigma_2, \pi_2, \cdots, \sigma_m, \pi_m, \cdots$$

converge to zero, so that $\lambda(\sigma_1, C), \lambda(\pi_1, C), \cdots, \lambda(\sigma_m, C), \lambda(\pi_m, C), \cdots$ converges, and so

$$\lim_{m\to\infty} \lambda(\sigma_m, C) = \lim_{m\to\infty} \lambda(\pi_m, C).$$

Now, if a sequence $\{s_n\}$ is quasi-monotonically non-decreasing, it follows that for every $k = 1, 2, \cdots$, $\lim_{m\to\infty} s_m \geqq s_n$. Let π be any partition of I, and let $\sigma_1 = \pi$ in the above sequence $\{\sigma_m\}$. Then

$$\lim_{m\to\infty} \lambda(\pi_m, C) = \lim_{m\to\infty} \lambda(\sigma_m, C) \geqq \lambda(\pi, C),$$

so that $\lim_{m\to\infty} \lambda(\pi_m, C) \geqq L(C)$. But, it is obvious that

$$\lim_{m\to\infty} \lambda(\pi_m, C) \leqq L(C).$$

This is taken from a talk on length and area given to the Pi Mu Epsilon Chapter at the Pennsylvania State University in May, 1962.

Reference

1. L. Cesari, this MONTHLY, 65 (1958) pp. 318 and 489.

EXTENSION OF A CONJECTURE

R. D. LARSSON, Mohawk Valley Community College

In a paper of mine in this MONTHLY [1], I proved the following: *If* $[p_n/p_i] = 2m_i + 1$ *for all primes* p_i, $3 \leqq p_i < \sqrt{(2p_n)}$, *then* $p_n + 2$ *is a prime.*

I stated that the hypothesis holds for $p_n = 3$, 5, 11 and 17 and fails for all other p_n equal to the first member of any prime pair, up to and including the prime 3557. I conjectured that no other prime pairs beyond 17, 19 satisfy the hypothesis.

A group of students of mine in a National Science Foundation Summer Science Training Program for High Ability Secondary School Students at Clarkson College of Technology in 1962 obtained the following extension of data substantiating this conjecture.

The conjecture holds for all prime pairs in the intervals

3–170,099; 1,000,037–1,003,109; 1,321,301–1,322,219;
2,000,081– 2,004,347; 2,792,087– 2,792,861;
3,000,131– 3,005,129; 3,146,831– 3,148,097;
4,000,037– 4,006,229; 4,727,297– 4,729,367;
5,000,111– 5,005,307; 5,216,681– 5,220,539;
6,000,101– 6,004,781; 6,135,251– 6,136,679;
7,000,127– 7,005,527; 7,233,047– 7,234,481;
8,000,051– 8,004,011; 8,496,491– 8,498,141;
9,000,377– 9,007,049; 9,572,837– 9,575,009;
10,000,139–10,006,427; 10,000,451–10,001,819.

The maximum p_i for which the truncated value of the quotient was first even was 61. It occurred only once, for $p_n = 33{,}347$. In general the hypothesis failed for a small value of p_i.

The students had the use of an IBM 1620 to aid them in their calculations. Their names were Stephen F. Levitas, Milne School, Albany, N. Y.; Jeffrey S. Rakoff, Central High School, Philadelphia, Pa.; Alan Schmerler, Bloomfield S. High School, Bloomfield, N. J.; Eugene A. Smith, West Canada Valley Central, Middleville, N. Y.; Robert H. Waldman, Valley Stream North High School, Franklin Square, N. Y.

Reference

1. R. D. Larsson, Necessary and Sufficient Conditions for Prime Pairs, this MONTHLY, 68 (1961) 549–550.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

PROGRAMMED LEARNING

The Panel on Programmed Learning of the Committee on Educational Media is investigating the possible usefulness of programming in the teaching of collegiate mathematics. Mathematicians who have used programmed materials or are engaged in projects involving the writing or testing of such materials could provide substantial help by communicating with the Panel through the central office: CEM, P.O. Box 2310, San Francisco, Calif. 94126.

REPORT ON THE SYRACUSE UNIVERSITY-WEBSTER COLLEGE MADISON PROJECT

ROBERT B. DAVIS, Syracuse University and Webster College

Various aspects of the Madison Project having been reported elsewhere (cf. part A of the bibliography), this note will discuss three matters only. One of these is a mathematical over-view of Project materials; another is the philosophy underlying the development of the Project's new experimental 9th grade algebra course; the third is the matter of psychology appropriate for "new curriculum" work.

I. In general, the mathematical content of Madison Project materials is characterized by simultaneous study of arithmetic, axiomatic algebra, matrix algebra, co-ordinate geometry, the rudimentary study of functions, and the study of certain aspects of physical science (such as empirically-determined functions, velocity and acceleration, rate of change, and graphical differentiation and integration). This occurs mainly in grades 2–8 (ages: approximately 8 through 14). Heavy emphasis is placed upon student discovery, originality, creativity, and student exploration of open-ended situations. Some materials in more tentative form are concerned with a careful ϵ, N approach to the concept of limit of a sequence, and the introduction of irrational numbers, thus far tested at the eighth-grade level (with students who have completed several years of previous study of Project materials). Mathematicians interested in Project work can best learn about it by viewing one or more of the films listed below:

Graphing an Ellipse. This film shows an actual classroom lesson. Seventh-grade children who are familiar with graphs of straight lines, circles, and parabolas, but who have not previously encountered ellipses, nor equations containing a parameter, are asked to graph

$$x^2 + 4y^2 = 25$$

and, for $0 \leqq k$,

$$x^2 + ky^2 = 25.$$

Matrices. Another actual classroom lesson, this time with a mixed group of 5th and 6th graders, all of whom have had some experience adding and multiplying matrices, and some of whom have studied associative and commutative properties, etc., for matrix operations. The main material of the lesson is new to all of the children, and consists of seeking matrices that "behave like 'zero' and 'one'."

In the course of this investigation the children inadvertently concoct matrices that reverse rows or columns, behave like "2," and so forth. They are encouraged to explore these directions further, and do so.

Complex Numbers via Matrices. A classroom lesson in which 7th grade children develop an isomorphism between rational numbers and a subset of 2×2

matrices, and then make use of this to extend the number system to include imaginary and complex numbers.

II. The Project is presently developing, by means of actual classroom trials, a ninth-grade algebra course. The content is on two levels. For children who have not previously studied Project materials, the course consists of a careful axiomatic study of algebraic identities, plus other topics generally similar to those listed above. Some of the course is informal. The study of empirically-determined functions, for example, brings experiments from the physics laboratory straight into the mathematics classroom; so does some material on measurement and the variability of data (variance, inner-quartile range, "trimmed" range, etc.). The axiomatic algebra, however, is genuinely axiomatic. So far as the Project knows, a genuinely axiomatic algebra that does not make frequent nonaxiomatic appeals to "rules for removing parentheses," or "rules for adding signed numbers," or "rules for combining like terms," or "rules for solving equations" is rarely if ever actually taught at the 9th grade level, assertions to the contrary not withstanding. Project experience thus far would indicate that this is not due to the reluctance of students to view algebra in a genuinely axiomatic light, nor is it due to the fact that an axiomatic approach necessarily lacks mathematical power; in fact, of course, it does not. The cause appears to be the ubiquity in textbooks and in teacher education programs of statements such as "the absolute value of a number is the actual value of the number, disregarding its condition or sign; absolute value is neither positive nor negative." (The inference, from the law of trichotomy, is clear.)

For students who have previously studied Project materials, a more ambitious 9th grade course deals with problems of summation and limits, as in Archimedes' methods for finding areas and volumes, etc. Finite difference methods, and proofs by mathematical induction, are also included.

III. The "new curricula" programs generally involve the collaboration of psychologists or psychiatrists. Schools of thought among psychologists are at least as diverse as among mathematicians and statisticians. A psychological point of view which the Project has found valuable and apparently appropriate is expressed in the publications listed in part B of the bibliography which follows.

References

PART A. Reports of other aspects of Madison Project activity.

Robert B. Davis, The Evolution of School Mathematics, (to appear).

———, A Modern Mathematics Program as It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere. (The Madison Project.) Report for 1963 submitted to the Commissioner of Education, U. S. Dept. of Health, Education and Welfare.

———, Math Takes a New Path, The PTA Magazine, No. 6, 57 (1963) 8–11.

———, Mathematical Thought and the Nature of Learning: The Madison Project View, Frontiers of Education. American Council on Education, (21 Audubon Avenue, New York 32, N. Y., 1963) pp. 79–83.

Robert B. Davis, Report on the Madison Project, Science Education News (a publication of the American Association for the Advancement of Science). December (1962) 15–16.

———, The Madison Project: A Brief Introduction to Materials and Activities, Madison Project.

PART B. Psychological Literature Relevant to New Curriculum Efforts.

Ruth Brecher and Edward Brecher, Gifted Children Need Freedom to Learn, Parents' Magazine. (Bergenfield, N. J.) June (1962) 44 ff.

Jerome S. Bruner, The Process of Education, Harvard University Press, Cambridge, Mass., 1963.

Nathaniel Cantor, Dynamics of Learning, Henry Stewart, Inc., New York (80 East 11th Street) 1961.

Stuart A. Courtis, Our Choice: Revolution or Destruction, Educational Leadership, No. 8, May 20 (1963) 520–522.

Earl C. Kelley, The Workshop Way of Learning, Harper, New York, 1951.

Hughes Mearns, Creative Power, Dover, New York, 1958.

A. S. Neill, Summerhill, Hart, New York, 1960.

George Polya, How to Solve It (paperback) Doubleday, New York, 1957.

Carl R. Rogers, Client-Centered Therapy, Houghton Mifflin, Boston, 1959.

Donald Snygg, A Learning Theory for Curricular Change. Chapter II of "Using Current Curriculum Developments," Association for Supervision and Curriculum Development, Washington 36, D. C. (1963) 109–115.

E. Paul Torrance, Creativity. Department of Classroom Teachers, American Educational Research Association of the National Education Association. Research Pamphlet Series "What Research Says to the Teacher," No. 28, April 1963.

Alfred North Whitehead, Aims of Education, (Mentor Books) Macmillan, New York, 1929.

THE FOURTH INTERNATIONAL MATHEMATICAL OLYMPIAD FOR STUDENTS OF EUROPEAN COMMUNIST COUNTRIES

IZAAK WIRSZUP, University of Chicago

From July 3 to July 16, 1962, the Fourth International Mathematical Olympiad [1, 3] was held in Czechoslovakia. Teams from the following seven European Communist countries participated in the competitions: Bulgaria, Czechoslovakia, East Germany, Hungary, Poland, Rumania, and the Soviet Union. The Russians were fully represented for the first time at this international contest; they had sent an incomplete team to the First Olympiad [4], and missed the Second and Third [5]. The Olympiad was conducted by the Mathematics Institute of the Czechoslovakian Academy of Sciences and by the Society of Czechoslovakian Mathematicians and Physicists as part of the Society's centennial celebration. Each team was made up of eight students from the last two grades of the secondary schools who were winners in their country's national mathematical olympiad, and was accompanied by two mathematics educators, one of whom was the team leader.

A special committee of the Czech Society, chaired by Academician J. Novák, looked over all the problems sent in beforehand by the ministries of education of the participating countries. It chose a preliminary list of 12 problems for con-

sideration and further selection by the International Committee of the Olympiad, the official adjudicating body, which comprised all team leaders. The International Committee met before the teams arrived in Czechoslovakia, and after a thorough examination and analysis of the problems and of their solutions, decided to select seven, one from each participating country. The International Committee formulated the texts of the problems, translated them into seven languages, and set a maximum point score for each problem. On the first day three problems were to be solved in four hours, and on the second day four problems in five hours. The contests were conducted in the picturesque Hluboká castle in Southern Bohemia.

The problems for the Olympiad are given below. The country that submitted the problem and the maximum score for the solution appear in parentheses.

First day of the Olympiad.

1. Find the smallest natural number n which has the following properties:

a) Its decimal representation has 6 as the last digit.

b) If the last digit, 6, is erased and placed as the first digit in front of the remaining digits, the resulting number is four times as large as the original number n. (Poland, 6 points)

2. Determine all real numbers x that satisfy the inequality

$$\sqrt{(3-x)} - \sqrt{(x+1)} > \tfrac{1}{2}. \qquad \text{(Hungary, 6 points)}$$

3. Given the cube $ABCDA'B'C'D'$ ($ABCD$ and $A'B'C'D'$ are the upper and lower faces, respectively, and $AA' \| BB' \| CC' \| DD'$). The point X moves at a constant speed along the perimeter of the square $ABCD$ in the direction $ABCDA$, and the point Y moves at the same speed along the perimeter of the square $B'C'CB$ in the direction $B'C'CBB'$. Points X and Y begin their motion at the same instant from the starting positions A and B', respectively. Determine and draw the locus of the midpoints Z of the segments XY.

(Czechoslovakia, 8 points)

Second day of the Olympiad.

4. Solve the equation

$$\cos^2 x + \cos^2 2x + \cos^2 3x = 1. \qquad \text{(Rumania, 5 points)}$$

5. On the circle K there are given three distinct points A, B, and C. Construct (using only a compass and a ruler) on the circle K a fourth point D such that a circle can be inscribed in the quadrilateral $ABCD$ that is obtained.

(Bulgaria, 7 points)

6. Given an isosceles triangle. Let r denote the radius of its circumscribed circle, and ρ the radius of its inscribed circle. Prove that the distance d between the centers of these two circles is

$$d = \sqrt{\{r(r-2\rho)\}}. \qquad \text{(East Germany, 6 points)}$$

7. Given the tetrahedron $SABC$ with the following property: there exist five spheres, each of which is tangent to the edges SA, SB, SC, AB, BC, CA, or to their extensions.

a) Prove that the tetrahedron $SABC$ is regular.

b) Prove that, conversely, for every regular tetrahedron five such spheres exist. (Soviet Union, 8 points)

The International Committee decided, for the first time, to make the grading of the solutions as uniform and coordinated as possible. The following procedure was strictly observed: There was one Czech coordinator in charge of each problem; for each country's team, all solutions to a given problem were checked by that country's two team leaders in the presence of the Czech coordinator. The solutions of the Czech participants were corrected jointly by their team leaders and by the leader of the team from whose country the given problem was suggested; the solutions to the Czech problem (problem 3) that were done by the Czech contestants were graded by the leaders of the Polish team.

Of the 56 Olympiad participants, one Soviet student, Iosif Bernstein, achieved a perfect total score of 46 points. (The same student was a first-prize winner in the First All-Russian Mathematical Olympiad of 1961. See *Matematika v shkole*, 4 (1961) p. 12.) A Hungarian student scored 45, a Russian girl scored 43, and another Hungarian student 41. These were the first-prize winners of the Olympiad. Twelve second prizes were given to students who made total scores between 40 and 34, and 15 third prizes for scores between 33 and 29. All remaining contestants received certificates of participation in the Fourth International Mathematical Olympiad.

The report by the leader of the East German team [3] states that "since the Olympiad is not a contest of countries but a contest of individuals, the International Committee undertook no ranking by country." A rating based only on the number of prizes won for each country [1], not taking into account the scores of participants who did not win prizes, is given below.

Country	First Prize 46-41 points	Second Prize 40-34 points	Third Prize 33-29 points
Hungary	2	3	2
Soviet Union	2	2	2
Rumania		3	3
Poland		1	3
Czechoslovakia		1	3
Bulgaria		1	2
East Germany		1	

A detailed analysis of the performance of all the participants on each of the problems is summarized in the following table [1]:

Distribution of Point Scores among Contestants

Problem (in parentheses is the maximum score)	8	7	6	5	4	3	2	1	0
No. 1 (6)			45	5	2	0	1	2	1
No. 2 (6)			21	11	7	9	4	2	2
No. 3 (8)	14	8	6	6	8	4	2	2	6
No. 4 (5)				34	10	3	5	2	2
No. 5 (7)		12	12	5	6	1	2	4	14
No. 6 (6)			9	8	16	4	3	3	13
No. 7 (8)	2	2	5	7	1	0	6	8	25

These results and a report on the Olympiad [3] show that in all probability too many problems were given. The contestants found themselves pressed for time, especially on the second day. Some were no longer able to concentrate to the extent required for solving the problems. It is thus expected that in the future there will be only three problems on each of two days, with four hours allowed each day.

According to a report by the leader of the East German team [3], the standards at this Olympiad and the general level of accomplishment were considerably higher than in the previous year. Concerning his own team's performance, this leader asserted that, although the team clearly could not keep up with the Hungarian, Soviet, and Rumanian teams, they did reach the level of achievement of the teams from the other countries, particularly in algebra and arithmetic. He stated that geometry was the East Germans' great weakness, showing up mainly in the construction problems. Indicative of the general standards at the Olympiad is his statement that although his students considered the principal task to be just to find a solution to a problem, at the level of these international competitions the precision and completeness of the way in which the solutions were presented proved to be equally important.

As is usually the case at these competitions, this Olympiad was given a very festive character, with a stress on the spirit of international friendship among the students and teachers from the socialist countries of Eastern Europe. The Minister of Education and Culture of Czechoslovakia, Dr. Kahuda, who met with the leaders and contestants, stated that the Olympiad would seriously influence the improvement of the quality of mathematics teaching, and would promote the development of the students' mathematical talents.

Below are the solutions [2] to problems 3, 5, 6, and 7, which caused the contestants the most difficulty.

PROBLEM 3. (Solution by I. Bernstein.)

Let us designate the center of face $ABB'A'$ by O_1, the center of face $BB'C'C$ by O_2, and the center of face $ABCD$ by O_3. We will prove that the locus of points Z, the midpoint of the segment XY, is the broken line $O_1O_2CO_3O_1$. Let A be the

origin of a coordinate system, and let AB, AD, and AA' be the x, y, and z axes, with $AB = AD = AA' = 1$. We shall divide the time during which point X traverses the path $ABCDA$ into four equal parts, and take the resulting time as our unit of measurement.

It is known that if a point S moves rectilinearly and uniformly, then its coordinates are linear functions of time and, conversely, if the coordinates of a point S are linear functions of time, then S moves rectilinearly and uniformly.

Recall also that if M is the midpoint of segment P_1P_2, where $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$, then

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Using the statements we have formulated, we can compile the following table, showing the dependence of the coordinates of points X, Y, and Z on time t.

		$0 \leqq t \leqq 1$	$1 \leqq t \leqq 2$	$2 \leqq t \leqq 3$	$3 \leqq t \leqq 4$
	x	t	1	$3-t$	0
X	y	0	$t-1$	1	$4-t$
	z	0	0	0	0
	x	1	1	1	1
Y	y	t	1	$3-t$	0
	z	1	$2-t$	0	$t-3$
	x	$\frac{1+t}{2}$	1	$\frac{4-t}{2}$	$\frac{1}{2}$
Z	y	$\frac{t}{2}$	$\frac{t}{2}$	$\frac{4-t}{2}$	$\frac{4-t}{2}$
	z	$\frac{1}{2}$	$\frac{2-t}{2}$	0	$\frac{t-3}{2}$

For $t = 0, 1, 2, 3, 4$, as is easily seen, Z occupies the positions O_1, O_2, C, O_3, O_1, and on the segments between these points the coordinates of Z change linearly, that is, Z traces in space the segments O_1O_2, O_2C, CO_3, and O_3O_1, so that Z moves along the rhombus $O_1O_2CO_3O_1$.

In other attempts at solving the problem, the contestants tried to prove the

converse assertion, namely: if a point M lies on the broken line $O_1O_2CO_3O_1$, then point M is the midpoint of segment XY. The failure to prove this assertion was a characteristic error in the solution of this problem.

PROBLEM 5. If it is possible to inscribe a circle in a quadrilateral (Figure 1) whose consecutive vertices are the points A, B, C, and D, then

$$AD + BC = AB + CD.$$

Let $AB \geqq BC$, and consequently $AB - BC = AD - CD \geqq 0$.

Then the problem is reduced to the construction of the triangle ACD on the base AC, the angle ADC, equal to $\pi - \angle ABC$, and the difference $AD - CD$. Suppose that the problem has been solved and triangle ACD has been constructed. We mark off from D the segment $DE = CD$. Triangle CDE is isosceles.

$$\angle CED = \frac{\pi - \angle ADC}{2} = \frac{\angle ABC}{2},$$

$$\angle AEC = \pi - \frac{\angle ABC}{2}.$$

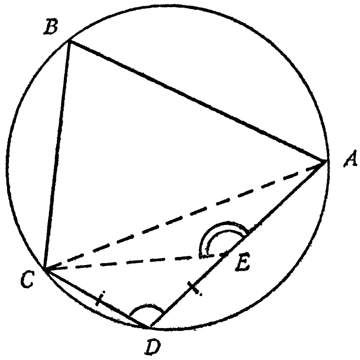

FIG. 1.

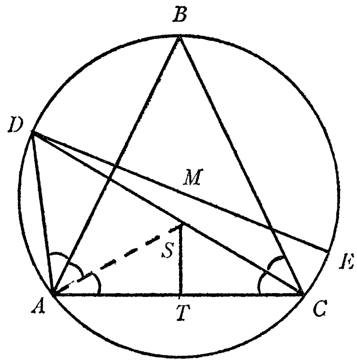

FIG. 2.

From this the following construction results. (By "the segment containing $\angle AEC$" is meant the segment containing $\angle AEC$ of the circle circumscribing $\triangle AEC$. This circle is not shown in Fig. 1.) On the side of chord AC opposite point B, we construct the segment containing angle AEC, which is equal to $\pi - \angle ABC/2$, and from point A we draw a circle with radius $AB - BC$. The point at which the arc containing $\angle AEC$ and the circle intersect (this point always exists, since $AE < AC$, because $AB - BC < AC$) is the desired point E. If we extend AE until it intersects the given circle, we will obtain the desired point D. The problem always has a unique solution on the arc AC (the arc not containing B). If $AB = BC$, then we get point D as the intersection of the bisector of angle ABC with the given circle.

In solving problem 5, several contestants did not demonstrate how the needed segment is chosen. And in using an algebraic method, they did not prove that a solution exists.

PROBLEM 6. Let $AB=BC$. Designate the center of the circumscribed circle by M and the center of the inscribed circle by S. Let D be the second point of intersection of the straight line CS with the circumscribed circle. We will show that triangle ADS is isosceles (Figure 2). Now $\angle BCS=\angle SCA=\angle SAC=\angle SAB$, but $\angle DAB=\angle SCB$ and consequently these five angles are all equal to one another. Therefore, $\angle DAS=\angle DSA$ and $AD=DS$. Let E be the second point of intersection of the straight line DM with the circumscribed circle, and let T be the base of the perpendicular dropped from S to AC.

Examine triangles CST and ADE: $\angle STC=90°$, $\angle DAE=90°$; $\angle SCA=\angle DEA$. Therefore, $\triangle CST\sim\triangle EDA$, and $CS:ST=DE:AD$; $ST=\rho$, $ED=2r$, $AD=SD$, as was shown before. Consequently, $CS\cdot DS=2r\rho$.

We now construct the diameter XY of the circumscribed circle through points M and S. We shall assume that point X is the end of the radius lying nearest to point S. Then $XS=r-d$ and $YS=r+d$, where $d=SM$, the distance between centers.

By a well-known theorem, $SC\cdot SD=SX\cdot SY$, and consequently $2r\rho=(r-d)(r+d)$ and $d^2=r(r-2\rho)$, that is,

$$d = \sqrt{\{r(r - 2\rho)\}}.$$

It can be shown that a slight alteration in the suggested solution yields a derivation of the formula $d=\sqrt{\{r(r-2\rho)\}}$ (known as Euler's formula) for an arbitrary triangle. All the contestants in the Olympiad, in solving this problem, calculated the length of segment SM directly by making use of the fact that it lies on the axis of symmetry BT of isosceles triangle ABC. The students' characteristic error was consideration of the specific sketch with the sequence of points $BMST$, although for $\angle B>60°$ the order is changed to $BSMT$.

PROBLEM 7. Let the sphere Ω be tangent to all the straight lines on which the edges of the tetrahedron $SABC$ lie. Then the sphere intersects each face of the tetrahedron in a circle inscribed in or tangent to one side and the extensions of the other two sides of the corresponding triangles. Moreover, the circles which lie on the faces adjacent to a given edge have a common point at which the straight line containing the given edge is tangent to the sphere. In other words, on the two faces adjacent to a given edge, the point of contact of the circles with this edge or its extension is a common point.

There are two possible cases:

1. The points of contact with the sphere lie inside each edge, i.e., on the tetrahedron itself. In this case all the circles are tangent to all three sides of each of the face triangles; that is, they are inscribed. The tangent sphere Ω must pass through P, Q, and R, the points of contact of the circle inscribed in triangle ABC with corresponding sides BC, CA, and AB, and also through K, the point

of contact of the circle inscribed in triangle SAB with the side SA. Points P, Q, and R do not lie on a straight line, and therefore point K does not lie in the plane PQR. Through the four points K, P, Q, and R, not lying on the same plane, it is possible to construct one and only one sphere. Therefore, if there exists a sphere of the first type, it is unique.

2. Suppose one point of contact with the sphere lies exterior to an edge. For the sake of clarity, we shall assume that it is edge SA and that the point of contact K lies beyond point A (that is, A lies between S and K). Then circle O_1, tangent to lines SA, SB, and AB, is drawn exterior to triangle SAB in plane SAB and lies on the side of AB opposite vertex S. Therefore O_1 is tangent to the edge AB itself at some point R and is tangent to the extension of edge SB at point L beyond vertex B (starting from S). In the same manner, circle O_3, in plane SCA, tangent to line SA at point K, is tangent to edge CA at some point Q and the extension of edge SC at point M beyond vertex C (starting from S). Finally, circle O_2 in plane SBC is tangent to the straight lines SB and SC at points L and M and the edge BC at some point P. In this manner, our sphere of the second type is tangent to the three edges of the face ABC and the extensions of edges SA, SB, and SC, at points lying beyond A, B, and C, respectively (starting from S).

Repeating the reasoning from part 1, we can prove that there exists no more than one sphere of the second type that touches the edges of given side ABC and the extended edges that originate from opposite vertex S. Therefore, the total number of spheres of the second type cannot exceed four.

Assume that all five spheres exist. We designate the length of the edges of tetrahedron $SABC$ thus: $SA=a$, $SB=b$, $SC=c$, $BC=a'$, $AC=b'$, and $AB=c'$.

Let K, L, M, P, Q, R be the points of contact of the first type of sphere. Then $SK=SL=SM$, $AK=AQ=AR$, $BL=BP=BR$, $CM=CP=CQ$, and since $AK+KS=SA$, $BP+PC=BC$, and so forth, we get the equation

$$a + a' = b + b' = c + c'. \tag{1}$$

Let us consider a sphere of the second type, corresponding to face ABC. Then, using the same designations as in case 2, we have $SK-AK=SA$, $BP+PC=BC$, $SL-BL=SB$, $AQ+QC=AC$, $SM-CM=SC$, $AR+RB=AB$. From this we get

$$a - a' = b - b' = c - c', \tag{2}$$

and comparing (1) and (2) we get

$$a = b = c, \qquad a' = b' = c'.$$

If we examine a sphere of the second type, corresponding to side SAB, we get $c'=a=b$.

In this manner we have shown that the tetrahedron is regular. It remains to show that for a regular tetrahedron there exist five such spheres.

Designate O as the center of a regular tetrahedron $SABC$. If sphere Ω with center at point O passes through the center of one edge of the tetrahedron, then it will pass through the centers of the remaining five edges and be tangent to all the edges of the tetrahedron. By a homothetic transformation with center S and coefficient 3, this sphere is transformed into sphere Ω_1, which, as is easily seen, is tangent to all the edges of the tetrahedron and is a sphere of the second type.

In this manner it is possible to construct a sphere for each vertex of a tetrahedron.

In solving this problem many contestants proved only the second assertion. In their proof of the first assertion they began with the assumption that the sphere was already placed in the necessary manner.

This paper is part of a Survey of Recent East European Literature in Elementary, Secondary, and College Mathematics, a project conducted by A. L. Putnam and I. Wirszup, Department of Mathematics, The University of Chicago, under a grant from the National Science Foundation.

References

1. E. A. Morozova, I. S. Petrakov, IV Mezhdunarodnaya Matematicheskaya Olimpiada (The Fourth International Mathematical Olympiad), Matematika v shkole, 6 (1962) 54–55.

2. E. A. Morozova, I. S. Petrakov, Resheniya zadach IV Mezhdunarodnoi Matematicheskoi Olimpiady (Solutions to the Problems of the Fourth International Mathematical Olympiad), Matematika v shkole, 1 (1963) 90–93.

3. Herbert Titze, Bericht über die IV. Internationale Mathematik-Olympiade 1962 in der ČSSR, Mathematik und Physik in der Schule, 12 (1962) 916–922.

4. Izaak Wirszup, The First Two International Mathematical Olympiads for Students of Communist Countries, this MONTHLY, 69 (1962) 150–155.

5. Izaak Wirszup, The Third International Mathematical Olympiad for Students of European Communist Countries, this MONTHLY, 70 (1963) 564–567.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

COLLABORATING EDITOR: C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1671. *Proposed by D. C. Duncan, East Los Angeles College*

If a sphere is inscribed in a trirectangular tetrahedron, show that the ratio of the volumes of the sphere and the tetrahedron is equal to the ratio of their areas. (Ref. inscription on the tomb of Archimedes.)

E 1672. *Proposed by Guy Torchinelli, State University College at Buffalo*

Prove that each prime divisor of 2^p-1, where p is prime, is greater than p. (It is a corollary that the number of primes is infinite.)

E 1673. *Proposed by Alexander Oppenheim, University of Malaya*

The three positive numbers x, y, z lie between the least and greatest of the three positive numbers a, b, c. If $x+y+z=a+b+c$ and $xyz=abc$, show that, in some order, x, y, z are equal to a, b, c.

E 1674. *Proposed by C. A. Nicol, University of South Carolina*

Prove that a necessary and sufficient condition that n be a prime is that $\sigma(n)+\phi(n)=nd(n)$, where $\sigma(n)$ is the sum of the divisors of n, $\phi(n)$ is the Euler totient of n, and $d(n)$ is the number of divisors of n.

E 1675. *Proposed by Andrzej Makowski, Warsaw, Poland*

Prove that $h_1h_2+h_2h_3+h_3h_1 \leqq r_1r_2+r_2r_3+r_3r_1$, where the h_i are the altitudes and the r_i are the exradii of a triangle.

E 1676. *Proposed by J. C. Van Rhijn, Vollenhove, Netherlands*

Given an ellipse E with foci F_1 and F_2, a point P outside E, and a positive number f. PR_1 and PR_2 are tangents to E. Find the locus of P if $(PR_1)(PR_2)=f(PF_1)(PF_2)$.

E 1677. *Proposed by J. E. Potter, Massachusetts Institute of Technology*

If A and B are positive semidefinite symmetric matrices, prove that $C=I+AB$ is nonsingular, if I represents the identity matrix.

E 1678. *Proposed by Reuben Hersh, Stanford University*

What are the eigenvalues of the nth order matrix having $2a$'s along the main diagonal, b's along the two diagonals bordering the main diagonal, and 0's everywhere else?

E 1679. *Proposed by Harry Lass, California Institute of Technology*

Given an $n \times n$ matrix with randomly selected integer elements, what is the probability that the absolute value of the determinant of the matrix is an odd integer? (It is assumed that the even and odd integer elements of the matrix occur with equal probability.)

E 1680. *Proposed by R. A. Struble, University of North Carolina*

Show that the limit of a uniformly convergent subsequence of Picard successive approximations need not satisfy the associated differential equation.

SOLUTIONS

"Able Was I Ere I Saw Elba"

E 1591 [1963, 567.] *Proposed by Frank Hawthorne, Education Department, Albany, N. Y.*

By how many different paths following the king's move in chess can the sentence "Able was I Ere I saw Elba" be traced in the accompanying diagram?

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
A	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	A
A	B	L	L	L	L	L	L	L	L	L	L	L	L	L	L	L	B	A
A	B	L	E	E	E	E	E	E	E	E	E	E	E	E	E	L	B	A
A	B	L	E	W	W	W	W	W	W	W	W	W	W	W	E	L	B	A
A	B	L	E	W	A	A	A	A	A	A	A	A	A	W	E	L	B	A
A	B	L	E	W	A	S	S	S	S	S	S	S	A	W	E	L	B	A
A	B	L	E	W	A	S	I	I	I	I	I	S	A	W	E	L	B	A
A	B	L	E	W	A	S	I	E	E	E	I	S	A	W	E	L	B	A
A	B	L	E	W	A	S	I	E	R	E	I	S	A	W	E	L	B	A
A	B	L	E	W	A	S	I	E	E	E	I	S	A	W	E	L	B	A
A	B	L	E	W	A	S	I	I	I	I	I	S	A	W	E	L	B	A
A	B	L	E	W	A	S	S	S	S	S	S	S	A	W	E	L	B	A
A	B	L	E	W	A	A	A	A	A	A	A	A	A	W	E	L	B	A
A	B	L	E	W	W	W	W	W	W	W	W	W	W	W	E	L	B	A
A	B	L	E	E	E	E	E	E	E	E	E	E	E	E	E	L	B	A
A	B	L	L	L	L	L	L	L	L	L	L	L	L	L	L	L	B	A
A	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	A
A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

Solution by D. I. A. Cohen, Princeton University. First we find the number of paths from R to A. Drawing the two main diagonals we note that a path may not cross them. In each quarter there are 3 choices from R to E, 3 from each E to an I, and so on. The total number of paths in one quarter is 3^9. Multiplying this by 4 we see that we have counted each diagonal twice, and the total number of paths is $16(3^9-1)^2$.

Also solved by Randy Barron, Robert Bart, Charles Conlin, D. W. Cox, Wayne Cutrer, E. R. Deal, J. F. Dillon, T. M. Feder, E. T. Frankel, Robert Gold, A. J. Goldman, S. H. Greene, Sidney Heller, R. A. Jacobson, Roman Kaluzniacki, A. J. Keeping, P. G. Kirmser, Coline M. Makepeace, D. C. B. Marsh, R. M. Meyer and S. G. Mohanty (jointly), Amos Nannini, Rodger Poore and Bruce Walker (jointly), Barry Simon, Arnold Singer, D. E. Stahl, Eric Sturley, Gary Venter, W. C. Waterhouse, and Oswald Wyler.

The proposer pointed out that if knight moves are used instead of king moves, the answer is 0, since there then is no way to get to the central R.

Several solvers called attention to the similar problems in H. E. Dudeney, *Amusements in Mathematics* (Dover Publications, Inc., 1958) pp. 74–5.

The Union of Two Subgroups

E 1592 [1963, 568]. *Proposed by Mira Bhargava, McGill University*

Prove that the union of two subgroups of a group is itself a subgroup if and only if one contains the other.

Solution by E. R. Barnes, Morgan State College. Let A and B be subgroups of a group G. If, say, $A \subset B$, then $A \cup B = B$ is a subgroup. On the other hand, if $A \not\subset B$ and $B \not\subset A$, there exist elements $a \in A - B$ and $b \in B - A$. Now $ab \in A$ implies that $a^{-1}(ab) = b \in A$, which is a contradiction. Similarly $ab \notin B$, and $A \cup B$ is not a group.

Also solved by H. L. Abbott, D. J. Allen, W. A. Al-Salam, N. Abdul-Halim, G. B. Aman, Eileen Appelbaum, K. F. Bailie, Donald Barr, Randy Barron, E. D. Bender, Paul Bennett, Ralph Bennett, Moshe Berlin, D. R. Beuerman, D. K. Bissonnette, Andreas Blass, W. R. Boland, F. L. Bookstein, Robert Bowen, J. J. Bowers, Alan Braver, Brother T. C. Wesselkamper, R. E. Brown, R. J. Bumcrot, R. L. Carmichael, Frederick Carty, P. R. Chernoff, D. I. A. Cohen, Martin Cohen, David Cotton, Wayne Cutrer, R. E. Dalton, Frank Dapkus, K. M. Das, Eleanor G. Dawley, E. R. Deal, L. E. De Noya, J. F. Dillon, J. S. Dombek, P. F. Duvall, Jr., Bruce Erickson, L. S. Evans, G. P. Farrell, C. L. Fefferman, Stephen Fisk, Michael Fried, Michael Friedman, Richard Fritsche, Michael Gemignani, G. J. Giaccai, J. A. Glasenapp, Anton Glaser, J. B. Goebel, Robert Gold, H. H. Green, Ralph Greenberg, S. H. Greene, Arthur Greenspoon, L. J. Grimm, R. L. Guderjohn, J. D. Haggard and D. W. Hight (jointly), P. G. Hahn, D. L. Hansen, Alvin Hausner, Dunstan Hayden, H. E. Heatherly, Sidney Heller, J. C. Hennessey, R. A. Jacobson, J. A. Johnson, Leroy Junker, Erwin Just and Norman Schaumberger (jointly), A. J. Keeping, P. A. Kingston, Alex Koler, J. A. Lambert, E. S. Langford, J. F. Leetch, A. E. Livingston, G. F. Lowerre, Jiang Luh, C. R. MacCluer, S. I. Mack, Thomas Maddock, Coline M. Makepeace, C. F. Marion, D. C. B. Marsh, R. A. Melter, R. M. Meyer, Stephen Montague, J. S. Muldowney, Amos Nannini, R. J. Oberg, C. A. Oster, F. J. Papp, Jr., F. D. Parker, Lewis Parker and J. D. Watson (jointly), C. B. A. Peck, B. E. Petersen, Stanton Philipp, G. W. Polites, J. H. Ramsey, Henry Ricardo, G. S. Rogers, Azriel Rosenfeld, J. P. Ryan, Perry Scheinok, E. M. Scheuer, A. L. Schreiber, L. S. Schulman, R. Sibson, Jr., D. L. Silverman, Barry Simon, R. T. Smythe, Rory Thompson, Andreas Thuswaldner, Roseanna Toretto, E. W. Wallace, W. C. Waterhouse, Albert Wilansky, J. M. Wild, Ran Wilder, J. E. Wilkins, Jr., Kenneth Yanosko, K. L. Yocom, and the proposer.

Dombek pointed out that this problem is proved as Theorem 5, p. 38, of *Introduction to Modern Algebra and Analysis*, by R. Crouch and E. Walker (Holt, Rinehart and Winston, 1962). Rosenfeld called attention to his paper, "Groups as unions of proper subgroups," this MONTHLY, 66 (1959) 491–4. Here it is shown that a group G is the union of three proper subgroups if and only if the Klein 4-group is a homomorphic image of G, and a finite group of order n is not the union of p or fewer of its proper subgroups, where p is the smallest prime dividing n. Wilansky stated the related results: (1) A linear space over a field with n elements cannot be the union of less than n subspaces. (2) A linear space over the reals cannot be the union of finitely many subspaces. He called attention to Bialnicki-Birula, Browkin, and Schinzel, *Colloq. Math.*, 7 (1959) 31–2, and to the discussion and references accompanying Problem 5025 [1963, 579].

Differential Operators

E 1593 [1963, 568]. *Proposed by Reuben Hersh, Stanford University*

If $P(x)$ and $Q(x)$ are arbitrary polynomials, k an arbitrary constant, and D the operator d/dx, prove that

$$P(D)[e^{kx}Q(x)]\,\big|_{x=0} = Q(D)[P(x)]\,\big|_{x=k}.$$

I. *Solution by W. C. Waterhouse, Harvard University.* Since both sides are linear in P and Q, we need prove it only for monomials; and it is easy to check by induction on n that

$$D^n(e^{kx}x^m)\,\big|_{x=0} = m!\binom{n}{m}k^{n-m} = D^m(x^n)\,\big|_{x=k}.$$

II. *Solution by S. I. Mack, Syracuse University.* We first note that for any pair of polynomials $P(x)$, $Q(x)$ we have $P(D)[Q(x)]\,|_{x=0}=Q(D)[P(x)]\,|_{x=0}$. Thus

$$P(D)[e^{kx}Q(x)]\,|_{x=0} = P(D+k)[Q(x)]\,|_{x=0} = Q(D)[P(x+k)]\,|_{x=0} = Q(D)[P(x)]\,|_{x=k}.$$

Also solved by D. S. Ahluwalia, W. A. Al-Salam, Raymond Balbes, Randy Barron, Robert Bart, Brother R. F. Schnepp, P. R. Chernoff, D. I. A. Cohen, Martin Cohen, David Cotton, K. M. Das, Joseph Erbacher, Stephen Fisk, Michael Fried, H. W. Gould, Ralph Greenberg, S. H. Greene, Arthur Greenspoon, M. G. Gruendl, Alvin Hausner, Sidney Heller, J. C. Hickman, A. J. Keeping, Alex Koler, E. S. Langford, A. E. Livingston, Thomas Maddock, E. L. Magnuson, D. C. B. Marsh, Stephen Montague, J. S. Muldowney, R. J. Oberg, B. E. Petersen, Stanton Philipp, B. E. Rhoades, G. S. Rogers, Jean-Pierre Sampson, Perry Scheinok, Barry Simon, Arnold Singer, J. E. Wilkins, Jr., J. S. W. Wong, G. O. Wunderly, K. L. Yocom, Leonard Zacks, David Zeitlin, and the proposer.

A Scalene Oxygon with Minimum Perimeter

E 1594 [1963, 568]. *Proposed by Walter Penney, Navy Department, Washington, D. C.*

Triangle ABC is a scalene triangle with three acute angles, integral sides, and $\sin A=5/13$. What is the minimum possible perimeter?

Solution by J. W. Baldwin, Wayne State University. A perpendicular from C to AB at D gives two right triangles with $DC=5x$ and $AC=13x$, where x is an integer. Let $BC=a$, $DB=m$, and get

$$a^2 = 25x^2 + m^2. \tag{1}$$

Now, for $C<\pi/2$ we must have $\tan(A+B)<0$, which requires that $\tan A \tan B >1$. Hence $\tan B=5x/m>12/5$, or $m<25x/12$. For $x=1, 2, \cdots$, with allowable values of m, we have a, from (1), irrational until we come to $x=7$, $m=12$, when $a=37$, and the triangle fulfilling all conditions has a perimeter of 224.

Also solved by A. N. Aheart, Merrill Barnebey, Wayne Cutrer, Sidney Heller, J. E. Jean, Jr., A. J. Keeping, Esther A. Linfield, D. C. B. Marsh, P. R. Nolan, W. M. Stone, Gary Venter, Oswald Wyler, K. L. Yocom, and the proposer.

Sixteen incorrect solutions were received. The proposer called attention to Problem E 1147 [1955, 40].

An Affine Problem

E 1595 [1963, 568]. *Proposed by R. V. Moody, Toronto, Ontario*

Let ABC be any (nondegenerate) triangle and let the lines parallel to its sides and one-fourth the way from each side to the opposite vertex determine the tri-

angle DEF. Let $A'B'C'$ be any triangle circumscribed about triangle DEF. Prove that at least one vertex of triangle $A'B'C'$ must lie on or within triangle ABC.

Solution by D. C. B. Marsh, Colorado School of Mines. Since this problem is of an affine nature, assume ABC equilateral. Then a vertex of $A'B'C'$ will lie outside of ABC only if the vertex angle is less than 60°. As this cannot be true for all three angles, we conclude that at least one vertex of $A'B'C'$ must lie on or within ABC.

Also solved by Michael Goldberg, Oswald Wyler, and the proposer.

The proposer applied the theorem that states that of the four triangles $A'FE$, $B'DF$, $C'ED$, DEF, the last one does not have the least area.

Covering Triangles of a Convex Polygon

E 1596 [1963, 568]. *Proposed by Carl Evans, Cornell Aeronautical Laboratory, Inc., Buffalo, New York*

Let P_n be a convex polygon of n sides with no four vertices concyclic. Call a triangle ABC whose vertices are among those of P_n a *covering triangle* of P_n if the circumcircle of ABC contains the remaining $n-3$ vertices of P_n in its interior. Show that there are exactly $n-2$ covering triangles of P_n and that they constitute a dissection of P_n by $n-3$ diagonals.

Solution by Randy Barron, McGill University. Select two adjacent vertices A, B in counterclockwise order. Choose point C from the remaining vertices so that $\measuredangle ACB$ is a minimum. This point is unique, because if another such point C' existed, then A, B, C, C' would be concyclic. The circumcircle of ABC encloses all of the remaining $n-3$ vertices, and triangle ABC is a covering triangle. From among the vertices lying clockwise from A to C, select point D such that $\measuredangle ADC$ is the minimum. Then the circumcircle of ADC encloses all the other vertices lying clockwise from A to C. But D is interior to the circumcircle of ABC, whence $\measuredangle ADC>\pi-\measuredangle ABC$. It follows that $\measuredangle ABC>\pi-\measuredangle ADC$. But $\measuredangle ABC$ is the minimum angle subtended by chord AC at any vertex lying counterclockwise from A to C, since B is on the circumcircle of ABC which encloses these points. Therefore all these angles exceed $\pi-\measuredangle ADC$, and all the vertices lying counterclockwise from A to C lie within the circumcircle of ADC. That is, triangle ADC is a covering triangle. We may proceed in this manner, each time selecting one unused vertex from among its neighbors (if any) to construct a covering triangle. We thus get $n-3+1=n-2$ covering triangles. Of these triangles, triangle ABC has three lines. Each successive covering triangle thereafter constructed adds two lines. Of all these lines, n are sides of the polygon, whence $3+2(n-3)-n=n-3$ of them are diagonals.

Also solved by D. I. A. Cohen, Stanton Philipp, E. G. Straus, Oswald Wyler, and the proposer.

The problem is a generalization of Problem 3, Part II, of the 1961 Putnam Competition (see this MONTHLY, (1962) 764).

Maximizing an Escalator

E 1597 (corrected) [1963, 757]. *Proposed by Ralph Greenberg, University of Pennsylvania*

Let $E_1(x_1)=x_1$, $E_2(x_1, x_2)=x_1^{x_2}, \cdots, E_n(x_1, x_2, \cdots, x_n)=x_1^{E_{n-1}(x_2,x_3,\cdots,x_n)}$, and let $a_n>a_{n-1}> \cdots >a_1>e$. Which permutation of the a_i's maximizes $E_n(x_1, x_2, \cdots, x_n)$?

Solution by the proposer. We will show that the permutation $a_1, a_2, \cdots, a_n$ always gives the maximum value to E_n. If $n=2$, this is clear, since

$$a_2/\log a_2 > a_1/\log a_1.$$

If $n>2$, assume, contrary to our assertion, that $E_n(b_1, b_2, \cdots, b_n)$ is maximum, where $b_1, b_2, \cdots, b_n$ is some permutation of $a_1, a_2, \cdots, a_n$ with $b_k>b_{k+1}$. Setting $M=E_{n-k-2}(b_{k+2}, \cdots, b_n)$, and using the fact that

$$b_{k+1}^{M}/\log b_{k+1} < b_k^{M}/\log b_k,$$

we have

$$b_k^{b_{k+1}^{M}} < b_{k+1}^{b_k^{M}}.$$

Thus we easily deduce that

$$E_n(b_1, \cdots, b_{k+1}, b_k, \cdots, b_n) > E_n(b_1, \cdots, b_k, b_{k+1}, \cdots, b_n),$$

which contradicts the assumption that the last expression is maximum. This proves our assertion.

Also solved by D. I. A. Cohen, Wayne Cutrer, Michael Goldberg, D. C. B. Marsh, Jon Peterson, and I. D. Ruggles.

A Polynomial Having Only Real Roots

E 1598 [1963, 568]. *Proposed by John Rainwater, University of British Columbia*

Let m be a positive integer and define the real polynomials $f(x)$ and $g(x)$ by $(1+ix)^m=f(x)+ig(x)$. Prove that for arbitrary real numbers a and b, the polynomial $af(x)+bg(x)$ has only real roots.

I. *Solution by Alex Koler, University of Dayton.* Assume $z=c+id$, $d\neq 0$, and $\bar{z}$ are solutions of $f(x)-kg(x)=0$, where $k=-b/a$, $a\neq 0$. Then

$$g(x) = (k+i)^{-1}(1+ix)^m. \tag{1}$$

Now substituting into $|g(z)|=|\overline{g(\bar{z})}|$ using (1), and since k is real, it follows that $|1+iz|=|1-iz|$. But this implies that $d=0$, which contradicts our assumption.

II. *Solution by J. E. Wilkins, Jr., General Dynamics Corporation, San Diego, California.* The equation $af+bg=0$ may be written in the form

$$(a + ib)(1 + ix)^m + (a - ib)(1 - ix)^m = 0.$$

If a and b are not both zero, let $a+ib=re^{i\theta}$ for a real nonzero r and a real θ. Then $(1 + ix)^m/(1 - ix)^m = e^{2i\theta}$, and consequently $(1 + ix)/(1 - ix) = e^{i\phi}$ if $\phi=(2\theta+2\pi k)/m$ for some integer k. It follows at once that $x=\tan(\phi/2)$ is real. If a and b are both zero, the proposed assertion is obviously false.

Also solved by W. A. Al-Salam, Randy Barron, Leonard Carlitz, P. R. Chernoff, Stephen Fisk, Michael Fried, Charles Goldberg, Michael Goldberg, M. G. Gruendl, Alvin Hausner, Sidney Heller, A. J. Keeping, D. C. B. Marsh, U. Ocnav, Stanton Philipp, G. O. Wunderly, David Zeitlin, and the proposer.

Zeitlin pointed out that the problem is a special case of Problem 33, pp. 149–50, J. V. Uspensky, *Theory of Equations*, McGraw-Hill, 1948.

Functions with the Binomial Property

E 1599 [1963, 568]. *Proposed by David Friedman, University of California at Berkeley*

Let

$$f_0(x) = c^x, \qquad f_n(x) = c^x \prod_{i=0}^{n-1} (ax + bi) \qquad \text{for } n = 1, 2, \cdots,$$

where $c>0$ and a, b are arbitrary real numbers. Show that

$$f_n(x + y) = \sum_{i=0}^{n} \binom{n}{i} f_i(x)f_{n-i}(y).$$

I. *Solution by W. C. Waterhouse, Harvard University.* We proceed by induction on n, the result being obvious for $n=0$. Assuming it true for n, then, we have

$$\begin{aligned}\sum_{j=0}^{n+1}\binom{n+1}{j}f_j(x)f_{n+1-j}(y) &= \sum_{j=0}^{n+1}\left[\binom{n}{j}+\binom{n}{j-1}\right]f_j(x)f_{n+1-j}(y)\\ &= \sum_{j=0}^{n}\binom{n}{j}f_j(x)f_{n+1-j}(y) + \sum_{i=0}^{n}\binom{n}{i}f_{i+1}(x)f_{n-i}(y)\\ &= \sum_{j=0}^{n}\binom{n}{j}f_j(x)f_{n-j}(y)\left[ay + b(n-j)\right] + \sum_{i=0}^{n}\binom{n}{i}f_i(x)f_{n-i}(y)\left[ax + bi\right]\\ &= f_n(x + y)\left[a(x + y) + bn\right] = f_{n+1}(x + y).\end{aligned}$$

II. *Solution by Stanton Philipp, Long Beach, Calif.* It is well known that the factorial polynomials $F_n(u)=u(u-1)\cdots(u-n+1)$ have the binomial property $F_n(x+y)=\sum_{i=0}^{n}\binom{n}{i}F_i(x)F_{n-i}(y)$. For $b\neq 0$, $f_n(x)=c^x(-b)^nF_n(-ax/b)$, whence the desired result follows immediately. For $b=0$, the result is equivalent to the binomial theorem.

III. *Solution by W. A. Al-Salam, Texas Technological College.* It is well known that any set of polynomials $\{f_n(x)\}$ satisfies the addition theorem

$$f_n(x+y) = \sum_{i=0}^{n} \binom{n}{i} f_i(x) f_{n-i}(y)$$

if and only if its generating function

$$F(x, t) = \sum_{n=0}^{\infty} (t^n/n!) f_n(x)$$

is multiplicative, i.e., $F(x+y, t) = F(x, t)F(y, t)$. Now it is easy to show that the given polynomials may be written as

$$f_n(x) = (-b)^n c^x n! \binom{ax/b}{n},$$

which implies that $F(x, t) = c^x(1-bt)^{ax/b}$. This is evidently multiplicative.

Also solved by D. S. Ahluwalia, Randy Barron, Robert Bart, P. R. Chernoff, D. I. A. Cohen, Robert Cohen, K. M. Das, Stephen Fisk, Michael Fried, H. W. Gould, Ralph Greenberg, S. H. Greene, A. J. Gross, Eldon Hansen, Sidney Heller, A. J. Keeping, Alex Koler, J. A. Lambert, T. J. Lee, D. C. B. Marsh, J. W. Moon, J. S. Muldowney, M. G. Murdeshwar, R. J. Oberg, B. E. Petersen, E. D. Rainville, B. E. Rhoades, Perry Scheinok, E. M. Scheuer, Barry Simon, Arnold Singer, J. E. Wilkins, Jr., David Zeitlin, and the proposer.

It is apparent that the restrictions on a, b, c are unnecessary and that we may take a, b, c, x, y as any complex numbers. Gould became interested in other solutions, than the one proposed in the problem, of the equation

$$f_n(x+y) = \sum_{i=0}^{n} \binom{n}{i} f_i(x) f_{n-i}(y).$$

As pertinent references he gave: (1) H. L. Krall, Newtonian polynomials (abstract), this MONTHLY, 55 (1948) 388, (2) H. L. Krall, Polynomials with the binomial property, this MONTHLY, 64 (1957) 342–3, (3) Eri Jabotinsky, On Newtonian sequences of polynomials (abstract), *Bull. Amer. Math. Soc.*, 54 (1948) 470.

A Determinant Whose Elements Are Combinatory Numbers

E 1600 [1963, 569]. *Proposed by J. E. Schneider, Franklin and Marshall College*

Evaluate the determinant

$$\begin{vmatrix} \binom{0}{0} & \binom{1}{1} & \cdots & \binom{n}{n} \\ \binom{1}{0} & \binom{2}{1} & \cdots & \binom{n+1}{n} \\ \cdot & \cdot & \cdots & \cdot \\ \binom{n}{0} & \binom{n+1}{1} & \cdots & \binom{2n}{n} \end{vmatrix}.$$

Solution by D. I. A. Cohen, Princeton University. Subtracting the mth column from the $(m+1)$th, $(m=1, 2, \cdots, n-1)$, and then subtracting the mth row from the $(m+1)$th, $(m=1, 2, \cdots, n-1)$, we get 0's in the first row and first column except for a 1 in a_{11} and the rest of D_n is identical to D_{n-1}. Therefore $D_n=D_1=1$.

Also solved by J. C. Abad, A. N. Aheart, Randy Barron, Robert Bart, Moshe Berlin, E. P. Berger and C. V. Bitterli (jointly), W. R. Boland, J. L. Brenner, Brother T. C. Wesselkamper, J. L. Brown, Jr., G. J. Ciaccai, Leonard Carlitz, Robert Cohen, C. G. Cullen, Wayne Cutrer, K. M. Das, J. A. Erbacher, T. M. Feder, Stephen Fisk, Michael Fried, Ralph Greenberg, S. H. Greene, R. E. Greenwood, M. G. Gruendl, Sidney Heller, W. H. Holter, R. A. Jacobson, Erwin Just, Roman Kaluzniacki, A. J. Keeping, P. G. Kirmser, Alex Koler, J. A. Lambert, T. J. Lee, B. S. Levine, Douglas Lind, R. F. McCoart, E. L. Magnuson, D. C. B. Marsh, S. G. Mohanty, J. W. Moon, M. G. Murdeshwar and V. K. Rohatgi (jointly), J. P. Muskat, R. J. Oberg, F. D. Parker, Lewis Parker and J. D. Watson (jointly), B. E. Petersen, Stanton Philipp, B. E. Rhoades, Perry Scheinok, E. M. Scheuer, Barry Simon, Arnold Singer, F. C. Smith, R. L. Syverson, Rory Thompson, L. T. Van Tassel, J. E. Wilkins, Jr., Oswald Wyler, K. L. Yocom, David Zeitlin, and the proposer.

The problem was located as: (1) a special case of Example 729, p. 679, of Muir, *A Treatise on the Theory of Determinants*, Dover Publications, Inc., 1960, (2) the case $m=1$, $r=n$ of Netto, *Lehrbuch der Kombinatorik*, Chelsea Publishing Co., p. 256, bottom, (3) Example 18, p. 428, Hall and Knight, *Higher Algebra*, Macmillan, 1957.

ADVANCED PROBLEMS AND SOLUTIONS

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editors. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5157. [1963, 1107]. CORRECTION. *Proposed by R. L. Graham, Bell Telephone Laboratories*

Replace "real" by "rational."

5181. *Proposed by E. J. Burr, University of New England, Australia*

Let $f(x)=0$ for x irrational, $f(0)=\epsilon_1>0$, and $f(m/n)=\epsilon_n>0$ for m, n coprime integers with $n>0$. Find, or disprove the existence of, sequences $\{\epsilon_n\}$ with $\lim \epsilon_n=0$ such that (a) $f'(x)$ exists nowhere, (b) $f'(x)$ exists for some x, (c) $f'(x)$ exists for all irrational x.

5182. *Proposed by D. Ž. Djoković, University of Belgrade, Jugoslavia*

Prove or disprove the following inequality

$$x^3\{y^2(z+1)^2 + t(y+z+yz)^2\} \leqq 1,$$

where x, y, z, t are nonnegative real numbers such that $3x+2y+2z+t=5$.

Note. This and the following inequality are connected with Shapiro's inequality, Problem 4603, [1954, 571]. Shapiro's inequality is known to be true for $1 \leqq n \leqq 8$, and false for $n=14$, 16, 18, 20, 22, 24 and also for all $n>25$. The two present inequalities are needed to prove Shapiro's inequality for $n=10$.

5183. *Proposed by D. Ž. Djoković, University of Belgrade, Jugoslavia*

Prove or disprove the following inequality

$$x^2\{v[z(y+1)(u+1)+yu]^2 + u^2(y+z+yz)^2\} \leqq 1,$$

where x, y, z, u, v are nonnegative real numbers such that

$$2(x+y+z+u)+v = 5.$$

5184. *Proposed by A. Oppenheim, University of Malaya, Kuala Lumpur, Malaysia*

Find the complete solution in rational integers of

$$x^3 + y^3 + z^3 + t^3 = 2$$
$$x + y + z + t = 2.$$

5185. *Proposed by Carl Evans, Cornell Aeronautical Laboratory, Buffalo, N. Y.*

Prove: Through any three noncollinear points of the plane there are uncountably many polygons similar to a given simple closed polygon, with each of the three points on a different side.

5186. *Proposed by J. D. Nulton, University of California, Berkeley*

A horticulturist wishes to grow a flower garden on a plot of ground which is large enough for only n flowers. Each of his flower bulbs, however, has a probability $p\neq 1$ of not growing into a flower. How many times (expected value) must he replant his garden before he reaches his goal of n flowers? Naturally he replants only those which failed to grow from the last planting.

5187. *Proposed by D. H. Denniston, University of Canterbury, New Zealand*

A, B, C, D, E, F are general points of a plane. AB', AC', AD' are the tangents at A to the conics $ACDEF$, $ABDEF$, $ABCEF$. Prove that AB, AB'; AC, AC'; AD, AD' are in involution.

5188. *Proposed by J. R. Baugh, S. P. Franklin and T. G. McLaughlin, University of California*

Place the following axioms on a topological space X in the hierarchy of separation axioms and determine what implications exist between them:

A. Every point of X is the intersection of an open set and a closed set.
B. The Boolean algebra generated by the topology of X is 2^X.
C. Every point of X is either open or closed.
D. X is a door space, i.e. every set is either open or closed.

5189. *Proposed by L. Carlitz, Duke University*

Let $a \equiv b \equiv c \pmod{p^{r+1}}$, where p is a prime and $0 \leqq c \leqq p^{r+1}$. Then, if $m \leqq \min(p^r-1, c)$ show that the following quotients are integral (mod p):

$$\frac{a(a-1)\cdots(a-m+1)}{b(b-1)\cdots(b-m+1)}, \quad \frac{b(b-1)\cdots(b-m+1)}{a(a-1)\cdots(a-m+1)}.$$

5190. *Proposed by J. L. Brenner, Stanford Research Institute, Menlo Park, California.*

Let ϕ be a mapping of the full linear group of $n\times n$ matrices (over the field of complex numbers, say) into the underlying field. It is known that if $\phi(AB) = \phi(A)\phi(B)$, then $\phi(A)$ is a multiplicative function of the determinant of A. In this MONTHLY [1963, 163], S. Cater proved that the same conclusion follows if $\phi(ABC) = \phi(CBA)$ for all triples of matrices, but that the hypothesis $\phi(AB) = \phi(BA)$, which is satisfied by the trace function, is not enough. (1). Show that, even if $n=2$, there are infinitely many maps ϕ satisfying this last hypothesis. (2). If both hypotheses $\phi(AB) = \phi(BA)$, $\phi(A^{-1}) = [\phi(A)]^{-1}$ are satisfied, how is the map restricted?

SOLUTIONS

Surface Area and Total Mean Curvature

5050 [1962, 813; 1963, 903]. *Proposed by G. D. Chakerian, California Institute of Technology*

Let K be a convex subset of R^3 with constant width. Let F be the area of the surface of K and let M be the total mean curvature. Then

$$1 \leqq \frac{M^2}{4\pi F} \leqq \frac{\pi}{2(\pi - \sqrt{3})} \approx 1.114.$$

Note by H. Guggenheimer. The proposer observes that, contrary to a statement appearing in the former solution [1963, 903], equality cannot hold for the solid of revolution of a Reuleaux triangle, since the latter has a circular projection onto the plane normal to the axis of revolution. In fact, it seems that there does not exist a solid of constant width all of whose projections are Reuleaux triangles, and that therefore strict inequality holds in

$$2(\pi - \sqrt{3})M^2 < 4\pi^2 F.$$

Multiplicative Seminorm

5052 [1962, 925; 1963, 904]. *Proposed by Seth Warner, Duke University*

A multiplicative seminorm on the algebra E (over the complex numbers) of all entire functions is, by definition, a function p from E into the nonnegative real numbers with the following properties: $p(0)=0$, $p(\lambda f)=|\lambda|p(f)$, $p(f+g)\leqq p(f)+p(g)$, and $p(fg)\leqq p(f)p(g)$ for all entire functions f, g and all complex numbers λ. If E is furnished with the topology of uniform convergence on compact sets (the compact-open topology) is every multiplicative seminorm on E necessarily continuous?

Editorial Note. I. N. Baker points out that his solution to this problem [1963, 904] is in error. If one assigns $p(f)=|a_k|k!$ to $f=\sum_{n=k}^{\infty}a_nz^n$, then it is not always true that $p(fg)\leqq p(f)p(g)$. Indeed, if $g=\sum_{n=l}^{\infty}b_nz^n$, then

$$p(fg)=(k+l)!|a_k||b_l|\geqq k!l!|a_k||b_l|=p(f)p(g)$$

whenever $kl\neq 0$.

No correct solution has been submitted.

Normal Subgroup

5055 [1962, 926; 1963, 1016]. *Proposed by Peter Yff, American University, Beirut, Lebanon*

If p is the smallest prime factor of the order of a finite group G, prove that any subgroup of index p is normal.

Editorial Note. W. T. Reid calls attention to an error which occurs in solution II [1963, 1016]. The phrase "while in the latter, $x\in G=H\times H$, hence $Hx=xH$" should be replaced by "while if $m=p$, then $1\in G=H\times H$ which easily yields the contradiction $H=G$."

Lebesgue Measure

5083 [1963, 335]. *Proposed by Joshua Barlaz, Rutgers, the State University*

Prove that there is no set E in the interval $(0, 1)$ having Lebesgue measure $m(E)$ and satisfying the conditions: (1) $0<m(E)<1$, (2) $m(E\cap(0, c))=r\cdot c$ for all c, $0\leqq c\leqq 1$ and a fixed r.

Solution by W. J. Bruecks, University of Pennsylvania. Suppose that such a set E does exist; then for each interval $I=(a, b)\subset(0, 1)$ we have $m(E\cap I)=rm(I)$. It follows from the definition of $m(E)$ that for any $\delta>0$ we can choose a denumerable sequence of intervals $\{I_\nu\}$, each contained in $(0, 1)$, such that $\bigcup_{\nu=1}^{\infty} I_\nu\supset E$ and $(1+\delta)m(E)>\sum_{\nu=1}^{\infty} m(I_\nu)$. Now the sequence $\{I_\nu\cap E\}$ covers E and hence

$$m(E)\leqq\sum_{\nu=1}^{\infty}m(I_\nu\cap E)=\sum_{\nu=1}^{\infty}rm(I_\nu)=r\sum_{\nu=1}^{\infty}m(I_\nu)<rm(E)(1+\delta).$$

Therefore $r>1/(1+\delta)$ for all $\delta>0$; which implies $r\geqq 1$. Thus $m(E)\geqq m(E\cap(0, 1))=r\cdot 1\geqq 1$, contradicting the hypothesis.

Also solved by D. W. Bailey, J. C. Barron, W. H. Bonney, R. C. Busby, Harley Flanders, D. M. Friedlen, Allan Gibbard, H. A. Guess, R. A. Jacobson, E. S. Langford, A. E. Livingston, D. G. Malm, Solomon Marcus, Katuyosi Matoba, J. C. Mauldon, M. D. Mavinkurve, Veselin Perić, Vivian Pessin, Stanton Philipp, J. E. Potter, J. V. Ryff, Marion Scattergood and George Tiller, H. G. Tucker, W. C. Waterhouse, Alan Weinstein and Alexander Zabrodsky.

Editorial Note. While this interesting result does not seem to be stated anywhere, it is an easy corollary of many known results. See, e.g., Natanson, *Theory of Functions of a Real Variable*, v. I, 260–261; Halmos, *Measure Theory*, p. 68; M. E. Munroe, *Introduction to Measure and Integration*, p. 290.

Optimum Strategy in a Guessing Game

5086 [1963, 336]. *Proposed by B. H. Bissinger, Lebanon Valley College, and Conrad Siegel, F. S. A., Harrisburg, Pa.*

A different integer is written on the face of each of 1000 slips of paper and the slips are placed face down. A player (who has no knowledge of the particular integers used) turns over and reads as many of the slips as he wishes. Success occurs when the integer on the last slip turned over is the largest of the 1000 integers.

If only one slip is turned over, the chance of success is obviously .001, as it is when all the slips are turned over. Devise a system for playing the game which gives maximum probability of success. Determine this probability.

Solution by A. J. Bosch, Technological University, Eindhoven, Netherlands. Suppose that, with $n=1000$, $a_0>a_1> \cdots >a_{n-1}$. To get initial information, in each system we must first turn over x slips. A system which stops when the last number is not larger than all the previous ones is a bad system, because continuing gives a greater probability of success. Hence we continue until we meet (say after k slips more) a number larger than all before. Then we stop. If we do not stop, it is the same as though we replaced x by $x'=x+k$, otherwise following the same system.

Let A be the set of numbers on the x slips turned over, and let B be the set of numbers on the $n-x$ remaining slips. Let $p_i=P(a_0, a_1, \cdots, a_{i-1}\in B$ and $a_i\in A)$, $i=1, \cdots, n-1$. If $a_0, \cdots, a_{i-1}\in B$, then the probability that we meet a_0 the first of these is clearly $1/i$. Hence

$$
\begin{aligned}
P(\text{success}) &= \sum_{i=1}^{n-1} p_i/i \\
&= \frac{n-x}{n}\cdot\frac{x}{n-1} + \frac{1}{2}\cdot\frac{(n-x)(n-x-1)}{n}\cdot\frac{x}{(n-1)}\cdot\frac{x}{n-2} + \cdots \\
&= \frac{x}{n}\left[\left(1-\frac{x}{n}\right)\frac{n}{n-1} + \frac{1}{2}\left(1-\frac{x}{n}\right)\left(1-\frac{x}{n-1}\right)\frac{n}{n-2} + \cdots\right].
\end{aligned}
$$

Putting $x/n=y$ and passing over to the limit as $n\to\infty$, we get $P(\text{success}) = -y \ln y$. This function has its maximum for $dP/dy=-1-\ln y=0$ or $y=1/e$.

Hence: turn over 368 slips and let the largest integer shown be m. Now continue, stopping with the first slip whose number exceeds m. The probability of success is .368.

Also solved by Milton Ash and Wayne Jones, E. L. Ellis, Michael Goldberg, Jamie J. Goode, J. G. Mauldon, Conrad Siegel, Anita Skelton and Michael Pascual, J. H. van Lint, and Richard Zeckhauser and Emmett Keeler.

Editorial Note. Since $1000/e = 367.879$, it is conceivable that 367 slips is the correct number. Skelton and Pascual carry out the numerical computation: $P(367) = .36819507$, $P(368) = .36819561$.

Ellis reports that essentially the same problem was posed in *Scientific American*, February 1960 by Fox and Marnie, and the outline of a solution by Moser and Pounder appeared in the March 1960 issue.

Isomorphic Rings

5087 [1963, 336]. *Proposed by Seth Warner, Duke University*

Given two rings each having m elements including a unity element. If m is a square-free integer, prove that the rings are isomorphic.

Solution by Veselin Perić, Sarajevo, Yugoslavia. Let R be any ring having m elements including the unity element e. If m is a square-free integer, we prove that e (as element of the abelian group R) is of order m, and thus R and the ring Z_m of the integers mod m, must be isomorphic. If, otherwise, n is the order of e, with $1 < n < m$, then $m = m'n$, $m' > 1$. For any prime factor p of m' there is in the abelian group R at least one element a of order p, $pa = 0$. Now p must divide n, since $na = n(ea) = (ne)a = 0a = 0$. Hence p^2 must divide $m = m'n$ so that m would not be square-free.

Also solved by George Bergman, W. H. Bonney, L. Carlitz, I. G. Connell, K. E. Eldridge, Bruce Erickson, B. P. Gill and Alvin Hausner, G. A. Heuer, W. S. Martindale, 3rd, M. D. Mavinkurve, R. A. Melter, Stephen Montague, M. G. Murdeshwar, J. H. Oppenheim and Frank Levin, E. L. Spitznagel, Jr., Art. Steger, H. R. Stevens, B. R. Toskey, Dennis Travis, C. L. Van der Eynden and T. Onishi and C. R. Mac Cluer, W. C. Waterhouse, and the proposer.

Convex Function

5088 [1963, 336]. *Proposed by Joe Lipman, Queen's University, Canada*

In Meschkowski, *Unsolved and Unsolvable Problems of Geometry* (Vieweg & Son, 1960) a function defined on a convex set C of reals is called convex if $2f((x_1+x_2)/2) \leqq f(x_1)+f(x_2)$ everywhere in C.

a) Show that if f is bounded above on some subinterval of C, then this definition agrees with the usual one, viz: for any t in $(0, 1)$, $f(tx_1+(1-t)x_2) \leqq tf(x_1)+(1-t)f(x_2)$ everywhere in C.

b) Find a function on $(-\infty, \infty)$ which is not convex (in the usual sense) but satisfies $2f((x_1+x_2)/2) < f(x_1)+f(x_2)$ for all $x_1 \neq x_2$.

Solution by G. M. Bergman, Harvard University. (a) Suppose we have x_1, x_2, t such that $f(tx_1+(1-t)x_2)-tf(x_1)-(1-t)f(x_2) = \alpha > 0$. (Clearly $t \neq \frac{1}{2}$.) Subtracting a linear function from f will not affect either convexity or boundedness. Hence let us assume $f(x_1) = f(x_2) = 0$. Then $f(tx_1+(1-t)x_2) = \alpha$.

Either $t<\frac{1}{2}$ or $t>\frac{1}{2}$. If $t<\frac{1}{2}$, let $t'=2t$. If $t>\frac{1}{2}$, let $t'=2t-1$. Applying Meschkowski's condition at the points $t'x_1+(1-t')x_2$ and x_2 in the first case, and at x_1 and $t'x_1+(1-t')x_2$ in the second, we find $f(t'x_1+(1-t')x_2)>2\alpha$. Inductively, we can find t'', t''', $\cdots$, such that at corresponding points $[x_1, x_2]$, $f>4\alpha$, 8α, $\cdots$. Thus f is unbounded above.

(b) The construction of nonlinear functions satisfying $f(x+y)=f(x)+f(y)$ is by now a very familiar problem. (See G. Hamel, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung* $f(x+y)=f(x)+f(y)$, Math. Ann., 60 (1905) 459–462.) For such a function, it is easy to see that $2f((x_1+x_2)/2)=f(x_1)+f(x_2)$. Such functions are known to be unbounded above and below in every interval. The sum of such a function and a strictly convex function such as x^2 will satisfy Meschkowski's inequality, but, being unbounded above, cannot satisfy the usual condition.

Also solved by Solomon Marcus, by the proposer and (first part) by Martin Cohen.

Editorial Note. Marcus points out that Jensen had considered convexity in the same sense as Meschkowski. See J. L. Jensen, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Mathematica, v. 30, 1906.

Limit of a sequence of Real Functions

5089 [1963, 336]. *Proposed by R. D. Sinkhorn, R. R. Gordon and Reid June, Boeing Airplane Co., Wichita, Kan.*

Let a sequence of real functions f_n, $n=1, 2, \cdots$, be generated by

$$f_1(x) = \begin{cases} 1/2a & \text{if } |x| < a \\ 1/4a & \text{if } |x| = a, \\ 0 & \text{if } |x| > a \end{cases} \qquad f_{n+1}(x) = \frac{1}{2a}\int_{x-a}^{x+a} f_n(t)dt,$$

where a is a positive constant. Prove that, independent of x,

$$\lim_{n\to\infty} n^{1/2}f_n(x) = \frac{1}{a}\sqrt{\frac{3}{2\pi}}\cdot$$

Solution by J. Boersma, Rijksuniversiteit, Groningen, Netherlands. The function $f_1(x)$ can be represented by the following integral:

$$f_1(x) = \frac{1}{\pi}\int_0^\infty \frac{\sin at}{at}\cos xtdt = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\sin at}{at}\cos xtdt.$$

(cf. Erdélyi, *Integral Transforms* I, form. 1.6(1)). Moreover it is readily noted that $f_{n+1}(x)=\int_{-\infty}^{\infty}f_1(x-t)f_n(t)dt$, i.e., that f_n is the n-fold convolution of f_1 with itself. Substituting in the recurrence relation for the sequence f_n, one can easily derive the following integral representation for the function $f_n(x)$:

$$f_n(x) = \frac{1}{\pi}\int_0^\infty\left(\frac{\sin at}{at}\right)^n\cos xtdt = \frac{1}{\pi a}\int_0^\infty\left(\frac{\sin t}{t}\right)^n\cos\frac{x}{a}tdt.$$

The function $f_n(x)$ can be estimated in the following way:

$$f_n(x) = \frac{1}{\pi a}\int_0^\infty \left(\frac{\sin t}{t}\right)^n \cos\frac{x}{a} t dt + O\left(\frac{1}{n}\right)$$
$$= \frac{1}{\pi a}\int_0^\infty \left(\frac{\sin t}{t}\right)^n \{1 + O(t^2)\} dt + O\left(\frac{1}{n}\right).$$

In the last integral we transform from t to the new variable u, with

$$u = 1 - \frac{\sin t}{t} = \frac{t^2}{6} - \frac{t^4}{120} + \cdots,$$

or inversely

$$t = \sqrt{(6u)} + O(u^{3/2}), \qquad \frac{dt}{du} = \frac{1}{2}\sqrt{(6/u)} + O(u^{1/2}).$$

We obtain the following result:

$$f_n(x) = \frac{1}{\pi a}\int_0^1 (1-u)^n\{1 + O(u)\}\{\tfrac{1}{2}\sqrt{(6/u)} + O(u^{1/2})\} du + O\left(\frac{1}{n}\right)$$
$$= \frac{1}{\pi a}\,\frac{1}{2}\,\sqrt{6}\cdot\frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}\left\{1 + O\left(\frac{1}{n}\right)\right\} + O\left(\frac{1}{n}\right)$$
$$= \frac{1}{a}\sqrt{\left(\frac{3}{2\pi n}\right)} + O\left(\frac{1}{n}\right).$$

Hence we have the desired result.

Also solved by J. Koekoek, Harry Lass, J. E. Potter, and the proposers.

Asymptotic Distribution of Sequences

5090 [1963, 336]. *Proposed by Fred Suvorov, Princeton University*

Let $\{x_n\}$ be a sequence of positive real numbers. Consider the set A of all real numbers a such that $\{x_n\}$ converges to 0 (mod a), and show that this set has measure 0. The sequence $\{x_n\}$ is said to converge to 0 (mod a) if the residue classes of the x_n on the circle R/Ra converge to 0, where R is the real numbers considered as an additive group, and Ra is the subgroup generated by a.

Solution by I. J. Schoenberg, University of Pennsylvania. 1. We add the assumption inadvertently omitted that $x_n \not\to 0$ as $n \to \infty$, for if $x_n \to 0$ then clearly $\lim x_n = 0$ (mod a) for every $a \neq 0$. We first dispose of the case when $\{x_n\}$ has a limit point c which is finite and $\neq 0$, $\lim_{i\to\infty} x_{n_i} = c$, say. If so, then $\lim x_n = 0$ (mod a) implies $\lim x_{n_i} = 0$ (mod a), whence $c = 0$ (mod a). But then $A \subset \{c, \frac{1}{2}c, \frac{1}{3}c, \cdots\}$, so that indeed $m(A) = 0$.

We may therefore assume that

$$\lim_{n\to\infty} |x_n| = \infty. \tag{1}$$

Now $a \in A$ implies that $\lim \exp(2\pi i x_n/a) = 1$ as $n \to \infty$, or

$$\lim_{n\to\infty} e^{2\pi i y x_n} = 1 \quad \text{if } y \in Y = \{y \mid y = a^{-1},\ a \in A\}. \tag{2}$$

Let $Y_z = Y \cap (0, z)$, $z > 0$. From (2) by Lebesgue's bounded convergence theorem we conclude that

$$\lim_{n\to\infty} \int_{Y_z} e^{2\pi i y x_n} dy = m(Y_z).$$

On the other hand in view of (1) we conclude from the Riemann-Lebesgue lemma that the integrals converge to zero. Thus $m(Y_z) = 0$ hence $m(Y) = 0$ and finally $m(A) = 0$.

2. This interesting result and its proof may be generalized by the following

THEOREM 1. *Let $\{x_n\}$ be a sequence of reals satisfying the condition* (1). *Let A be the set of positive numbers a such that $\{x_n\}$ has an asymptotic distribution* mod a. *If $m(A) > 0$, then for almost all $a \in A$ the sequence $\{x_n\}$ is equidistributed* mod a.

Proof. Let $a \in A$ and let $\mu_a(x)$ $(0 \leqq x \leqq a)$ be the asymptotic distribution function of $\{x_n\}$ mod a. Its trigonometric moments

$$\omega_h(a) = \int_0^a \exp\left\{\frac{2\pi i}{a}\, hx\right\} d\mu_a(x), \qquad (h \text{ integer}),$$

are also representable by the limits

$$\omega_h(a) = \lim_{n\to\infty} \frac{1}{n}\left(e^{2\pi i h x_1/a} + e^{2\pi i h x_2/a} + \cdots + e^{2\pi i h x_n/a}\right). \tag{3}$$

Again we consider the set $Y = \{y \mid y = a^{-1},\ a \in A\}$ and its sections $Y_z = Y \cap (0, z)$, $z > 0$. Again the Riemann-Lebesgue lemma and the bounded convergence theorem applied to the limit relation (3) give

$$\begin{aligned} 0 = \lim \int_{Y_z} \exp\{2\pi i y h x_n\}\, dy &= \lim \frac{1}{n} \sum_{\nu=1}^{\infty} \int_{Y_z} \exp\{2\pi i h y x_\nu\}\, dy \\ = \int_{Y_z} \omega_h\left(\frac{1}{y}\right) dy, &\qquad (h = 1, 2, \cdots). \end{aligned}$$

By differentiation with respect to z we see that $\omega_h(y^{-1}) = 0$ for almost all y, and hence $\omega_h(a) = 0$ for almost all $a \in A$, with the exception of a set σ_h, $m(\sigma_h) = 0$. Thus

$$\omega_h(a) = 0 \quad \text{for all} \quad h = 1, 2, \cdots, \text{ and all } a \tag{4}$$

with the exception of the set $\bigcup_h \sigma_h$ of measure zero. Since (4) characterizes uniform distribution, the theorem is established.

Remarks. 1. As an illustration of Theorem 1, let θ be irrational and let $x_n = n\theta$. It is well known that $\{n\theta\}$ is equidistributed, mod a, for every a with the exception of a set $E = \{a \mid a = \theta r,\ r \text{ rational}\}$. If $a \in E$ then $\{n\theta\}$ has an asymptotic distribution, mod a, which, however, is not equidistribution.

2. The condition (1) is essential for the validity of Theorem 1. Thus if $0 \leqq x_n \leqq 1$ and $\{x_n\}$ is equidistributed, mod 1, then it is readily seen that $\{x_n\}$ has an asymptotic distribution, mod a, for every a, which is not uniform if $a \neq 1$.

3. The question whether the exceptional set of Theorem 1 is perhaps always enumerable is left open.

4. Relevant to Theorem 1 is the following proposition: If $\{x_n\}$ is an increasing sequence of positive integers, then $\{x_n\}$ is equidistributed, mod a, for almost all a. (This is due to H. Weyl; see his fundamental paper, Math. Annalen, 77 (1916) 313–352, or Selecta, Basel and Stuttgart, 1956, pp. 111–147, in particular, pp. 140–142.)

Also solved by J. E. Potter.

Evaluation of a Limit

5091 [1963, 337]. *Proposed by W. E. Briggs, University of Colorado*

For integral $k > 1$, evaluate:

$$\lim_{x\to\infty}\left[\frac{k}{\phi(k)}\sum_{\substack{n\leqq x\\(n,k)=1}}\frac{1}{n}-\sum_{n\leqq x}\frac{1}{n}\right].$$

Solution by G. M. Bergman, Harvard University. We shall write $a \sim b$ for $\lim_{x\to\infty}(a-b)=0$. It is known that $\sum_{n\leqq x} 1/n \sim \log x + \gamma$, where γ is Euler's constant. Now

$$\sum_{\substack{n\leqq x\\(n,k)=1}}\frac{1}{n}=\sum_{d\mid k}\sum_{jd\leqq x}\mu(d)\frac{1}{jd}=\sum_{d\mid k}\frac{\mu(d)}{d}\sum_{j\leqq x/d}\frac{1}{j}$$

$$\sim\sum_{d\mid k}\frac{\mu(d)}{d}\left(\log\frac{x}{d}+\gamma\right)=(\log x+\gamma)\sum_{d\mid k}\frac{\mu(d)}{d}-\sum_{d\mid k}\frac{\mu(d)}{d}\log d.$$

Now $\sum_{d\mid k}\mu(d)/d = \phi(k)/k$, so

$$\frac{k}{\phi(k)}\sum_{\substack{n\leqq x\\(n,k)=1}}\frac{1}{n}-\sum_{n\leqq x}\frac{1}{n}\sim-\frac{k}{\phi(k)}\sum_{d\mid k}\frac{\mu(d)}{d}\log d.$$

To simplify this, we recall that $\log d$ is the sum of the logarithms of the prime divisors of d. Given a prime $p \mid k$, we determine the total coefficient of $\log p$ in the above expression. Let $k = p^\alpha k'$, with $(p, k') = 1$. The terms pd', $d' \mid k'$, in-

clude all terms $d|k$ for which $\mu(d)\neq 0$ and log d has a p-part. Thus the coefficient of log p is:

$$-\frac{k}{\phi(k)}\sum_{d'|k'}\frac{\mu(pd')}{pd'}=-\frac{p^a k'}{(p-1)p^{a-1}\phi(k')}\left(-\frac{1}{p}\right)\sum_{d'|k'}\frac{\mu(d')}{d'}.$$

The sum at the end of this formula, again, equals $\phi(k')/k'$, and so the whole expression reduces to $1/(p-1)$. Our limit then, is $\sum(\log p)/(p-1)$, where the summation is taken over all prime divisors p of k.

Also solved by T. M. Apostol, P. T. Bateman, Robert Breusch, L. Carlitz, Stephen Fisk, Ralph Greenberg, Emil Grosswald, D. R. Hayes, A. E. Livingston, J. G. Mauldon, Stanton Philipp, S. L. Segal and S. Chowla, D. Suryanarayana, and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, Carleton College and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. Programmed Materials: K. O. May, Carleton College, Northfield, Minn. Films: E. P. Vance, Oberlin College, Oberlin, Ohio.

Calculus of Vector Functions. By R. H. Crowell and R. E. Williamson. Prentice-Hall, Englewood Cliffs, N. J., 1962. x+484 pp. $8.25.

This text "was written to be used in teaching some topics in the theory of functions of several variables from the point of view of linear algebra." It contains a chapter on linear algebra borrowed from *Finite Mathematical Structures*, by Kemeny, Mirkil, Snell, and Thompson; a chapter on differential calculus that includes study of differentials and Jacobians, tangents to n-dimensional surfaces, and the inverse function theorem; a chapter on real-valued functions centered around maximum value problems with discussions of diagonalization of matrices and the general Taylor's theorem; a chapter on multiple integrals with emphasis on evaluation by iteration, change of variables, and surface area; and an appendix that contains additional material on determinants and proofs of the inverse transformation theorem, the existence of multiple integrals, and the formula for change of variables. The book is written carefully and should be given serious consideration for courses whose aim is that of the authors. The statement, however, that the prerequisite "is the traditional content of a year's course in one-variable calculus" that "need not be covered in any particular way" is misleading (e.g., the student is expected to have a good introduction to properties of limits and continuity, Taylor's theorem, and definite integration). Except for the first chapter, students with only this prerequisite

may find the text difficult, although the sets of relatively simple exercises should help.

Considering the rigorous logical viewpoint of the text, there are some places where more care should have been used. For example, since the curves used are not simple, it should have been made clear that the arc length of a curve is really the arc length of an ordered pair—a curve C and a set of equivalent parametric representations of C. Moreover, the discussion of approximation of arc length by a sum of lengths of chords occurs before the definition of arc length, assumes a concept of arc length, and also (p. 402) ignores the problems of *uniform* approximation of lengths of chords by lengths of tangent vectors at ends of the chords. The definition of continuity implies that a function is not continuous at isolated points of its domain. It is stated (p. 155) that it will be shown that "det A is the constant factor by which A changes volume," although (p. 435) volume is nonnegative. The following criticisms may be a matter of opinion: (1) nth order determinants are defined on page 151 and later (p. 447) it is proved that "determinant functions" exist; (2) the definitions of tangent are such that, although curves apparently are sets of points, whether a curve has a tangent at a particular point depends on the parametric representation being used; (3) the concept of content (or volume) is defined in terms of multiple integrals and is not shown to be independent of the choice of coordinate axes. The discussion is logically consistent, and the change-of-variable formula proved in the appendix could have been used to remove the dependence on choice of axes. It would have helped the less observant students if these facts had been explained carefully.

R. C. JAMES, Harvey Mudd College

Differential Geometry and Symmetric Spaces. By Sigurdur Helgason. Academic Press, New York, 1962. xiv+486 pp. $12.50.

Symmetric spaces are Riemannian manifolds such that the covariant derivative of the curvature tensor is identically zero. They were defined by E. Cartan in 1926, and their theory has been actively developed in the intervening years. The study of such spaces is valuable because of their connection with semi-simple Lie groups, because they can be used as testing grounds for conjectures about general Riemannian manifolds, and because they serve as a framework within which global function theory can be developed.

This book provides a self-contained treatment of Cartan's theory and of the recent developments in the theory of functions on symmetric spaces. It thus provides access to a literature never before collected in a systematic fashion. Since an amazing amount of mathematics is crowded into its pages, the exposition is necessarily compact. This is an excellent book for the mathematician with an adequate background in differential geometry and Lie groups. Novices should be wary.

CARL B. ALLENDOERFER, University of Washington

Sets, Relations, Functions. By Samuel M. Selby and Leonard Sweet. McGraw-Hill, New York, 1963. 226 pp. $5.50.

Designed for use in courses for teachers with little background, this book has a first chapter introducing set language, a last chapter glimpsing a variety of algebraic structures, and three central chapters dealing with real numbers, relations and functions. There is a liberal supply of examples, exercises, and good graphical illustrations. In particular there are many examples and assignable exercises dealing with absolute values; inequalities; sums, products, and compositions of functions with real domains and ranges; plane regions defined by linear and quadratic conditions. No functions essentially more "advanced" than square roots are considered. For some instructors the profusion of useful elementary exercises might well outweigh objections to the book.

There are some unfortunate statements, and some compromises whose appropriateness is a matter for individual judgment. A principal compromise is that while many axioms are stated and used, there is neither a finite induction axiom nor a completeness axiom for the reals. Also, the basic geometry of the plane is tacitly assumed. Some of the extended exercises involve compromises that strike the reviewer as less justifiable.

From page 5, "Enumeration of an infinite set consists in listing a few elements followed by three dots." On page 56 it is asserted that if the coefficients are required to be integers then $x^2+x+\frac{1}{4}$ is not reducible. Each list of axioms begins with a clause which at best is redundant and at worst makes the closure laws perhaps vacuous. An abbreviated version is if a and b are reals combinable under $+$, then $a+b$ is real.

BURROWES HUNT, Reed College

The Dynamics of Automatic Control Systems. By E. P. Popov. Pergamon Press, London, and Addison-Wesley, Reading, Mass., 1962. 761 pp. $10.75.

This book, which is a translation of the original Russian edition published in 1956, is an introduction to the theory of automatic control systems, primarily intended for engineering students and practicing engineers. As with most Russian texts of this kind, it is written in an easily readable leisurely style and abounds in detailed illustrations and completely worked-out examples; there are 335 figures in the text. The treatment of linear control systems, which takes up the first three parts of the book, relies heavily upon the operational calculus for differential equations and uses little more than the most elementary facts of complex function theory. The discussion of nonlinear control systems is on a more advanced level, though methods and techniques are stressed, here too, above the theory upon which they are based. Liapunov's direct method and the construction of Liapunov functions in the manner of Lur'e are treated in detail. Considerable emphasis is also placed on the approximate methods of Krylov and Bogoliubov and their variants due to Bulgakov; no mention is made, however, of the powerful method of averaging developed by Bogoliubov and Mitropolsky.

The headings of the five parts into which the book is divided are as follows: (i) General information about automatic control systems. (ii) Ordinary linear automatic regulation systems. (iii) Special linear automatic regulation systems. (iv) Nonlinear automatic regulation systems. (v) Methods of plotting the regulation-process curve.

H. A. ANTOSIEWICZ, University of Southern California

Methods of Mathematical Physics, Vol. II: Partial Differental Equations. By R. Courant and D. Hilbert. Interscience (Wiley), New York, 1962. xxii +830 pp. $17.50.

This work is *not* a translation of the German book by the same name that appeared first in 1937 and which was reprinted in this country in 1943. Rather, it represents a new edition, prepared by Professor Courant during the past two decades, and available in English as the original language. Like all of Courant's writings, this book is distinguished by its masterful presentation, which combines the intuitive approach so greatly desired by the "consumers" of mathematics (scientists and engineers) with standards of mathematical rigor expected of a distinguished mathematician.

Starting from basic comments concerning general properties of partial differential equations, and including in particular the Cauchy-Kowalewsky existence theorem, Courant goes on to elliptic problems, and hence to hyperbolic equations. There is an appendix devoted to the presentation of the ideal functions, or distributions, which play such an important part in today's quantum mechanics and quantum field theory.

The "Courant-Hilbert" is not primarily a textbook to be studied cover-to-cover, nor is it a handbook in which to look up a forgotten numerical constant; a very detailed table of contents, a good index, and an extensive bibliography, which includes both research papers and conference proceedings, as well as a few monographs, help the reader to discover any particular item within the context in which he might study it and understand the underlying grand design. All in all, it is most fortunate that this work is now available and that it incorporates the results of many papers of very recent date.

PETER G. BERGMANN, Syracuse University

Contemporary Geometry. By André Delachet. Translated from the French by Howard G. Bergmann. Dover, New York, 1962. xix+94 pp. $1.00.

The aim of this little volume (originally La Géométrie Contemporaine, one of the "Que Sais-Je?" Series, Presses Universitaires de France) is to give a rapid survey of developments in the field from the work of Monge to the contributions of Elie Cartan. Some relief from the current proclivity to employ the word "modern" in titles of books on mathematics is afforded here in the use of "contemporary," which use may be justified by the brief mention of Bouligand's so-called "direct infinitesimal geometry." The author guides the reader

from elementary notions through applications of the group concept to cartesian, desarguesian, and projective geometries in Part I. In Part II geometries and abstract spaces appear, and Part III covers some intuitive aspects of both general and combinatorial topology.

In such a brief presentation it is impossible to explain fully all of the concepts introduced. For instance, such a statement as "the principle of duality is not applicable to properties of affine, euclidean, or metric geometry which involve the plane at infinity" with no elucidation must leave the reader curious but without much edification. In spite of a few curiosities such as "if two points are *collinear* on a line D," "a set of ordered *groups* of four numbers," and the statement that projective space is obtained from desarguesian space by considering the plane at infinity as an "ordinary plane," the book has an interesting flavor and serves a very useful purpose. It is recommended as definitely worth the price for all students of mathematics.

C. E. SPRINGER, The University of Oklahoma

Nonlinear Transformations of Random Processes. By Ralph Deutsch. Prentice-Hall, Englewood Cliffs, N. J., 1962. xi+157 pp. $5.95.

The book presents a readable introductory account of some analytical methods that have been employed with reasonable success on a class of problems of interest in communication engineering, involving nonlinear functionals of Gaussian (for the most part) random processes—largely on systems containing a single zero-memory nonlinear device. There are some errors of commission and omission which mar the value of an otherwise useful work.

A. V. BALAKRISHNAN, University of California at Los Angeles

Russian Reader in Pure and Applied Mathematics. By P. H. Nidditch. Interscience, Wiley, New York, 1962. 166 pp. $2.25.

This reader consists of 100 short excerpts from 21 Russian works. Ninety-four of these sections have an interlined, word-by-word English translation, with brief notes on the grammar of more difficult words. The final six sections are given without a translation, but with a vocabulary of new words. Anyone with some knowledge of written Russian should be able to get through this book. When finished he then should be capable of reading any Russian mathematical work.

STEPHEN HOFFMAN, Trinity College, Hartford, Connecticut

A Modern View of Geometry. By Leonard M. Blumenthal. W. H. Freeman and Company, San Francisco, 1961. xii+191 pp. $2.25.

About a half century ago Felix Klein wrote on geometry from a "higher standpoint." The book under review treats the axiomatics of various geometries of the plane, and has little in common with the treatment of Klein. The present author demonstrates very lucidly the correspondence between algebra and geometry by which the geometric structure imposed by postulates on an ab-

stract point set implies an algebraic structure of an abstract coordinate set, and conversely. Geometric properties of the plane are logical consequences of algebraic properties of a ternary ring.

Following a brief but engaging account of historical background in Chapter I, the author prepares the reader further for later developments in devoting the next two chapters to sets and postulational systems. There follows an illuminating chapter on the coordinatizing of the affine plane where a ternary operator replaces the equation of a line to liberate analytical geometry from all intuition. In Chapter V further (desarguesian) properties of the geometric plane are seen to impose additional requirements in the algebraic system to obtain, in turn, a Veblen-Wedderburn system, a division ring, and finally, a field. The coordinatizing of the projective plane is accomplished in Chapter VI, where also the important simplifying role of the duality principle is revealed.

It is interesting to compare the six metric postulates for the euclidean plane in Chapter VII with the six metric postulates for the hyperbolic plane in Chapter VIII. The first five are identical in the two geometries, but the sixth involves a characterizing determinant. The differences between euclidean and hyperbolic geometry are therefore a consequence of the differences in the properties of these two determinants. The six postulates for the euclidean plane are shown to be categorical by the proof that they imply the axioms of H. G. Forder.

This book is surely on the required reading list for all thoughtful students who desire to see mathematics from the "higher standpoint."

C. E. SPRINGER, The University of Oklahoma

Operational Methods for Linear Systems. By Wilfred Kaplan. Addison-Wesley, Reading, Mass., 1962. 577 pp. $12.50.

The treatment of linear systems by transform methods is indicated in book after book. Professor Kaplan's work may very well become the definitive text in the area. At least, prospective authors of books on transform methods will have to ask themselves in what way they believe their work is superior to his.

The first six chapters are devoted to Fourier series, z-transforms, Fourier transforms, Laplace transforms, and Hilbert transforms. The treatment employs engineering terms such as "weighting function" and "transfer function," but such mathematical matters as existence of solutions and uniform convergence of series are not neglected. There are many interesting exercises along with some hints for their solution.

Chapter seven, entitled "Stability," covers the Hurwitz-Routh criterion, the Nyquist criterion and the root-locus method. Time-varying linear systems are treated in the last chapter. In an appendix the increasingly important operational calculus of Mikusiński is sketched. Regrettably, there is no discussion of linear systems with stochastic elements nor of numerical inversion of Laplace transforms.

Along the way the student is introduced to complex variable theory, Wronskians, Laguerre polynomials, perturbation theory, Hermitian matrices, etc., so that the book really provides a rich background in mathematical topics and concepts that are of prime importance in analysis itself and in the applications.

ROBERT E. KALABA, The RAND Corporation

Mathematical Methods in Small Group Processes. By J. Criswell, H. Solomon, and P. Suppes. Stanford University Press, California, 1962. 361 pp. $9.75.

This is a collection of papers that resulted from the Symposium on Mathematical Methods in Small Group Processes held at Stanford University in the summer of 1961. The papers all deal in some way with the problem of applying mathematical techniques to the study of human behavior, with emphasis on small group situations. There are discussions of specific mathematical models such as the Lorge-Solomon model for the efficiency of small groups jointly solving problems and the Cohen model for social conformity exemplified by the Asch experiments. There are several papers which follow up the earlier work of Suppes and Atkinson, and Estes, on applying learning models to analyze learning experiments involving two persons. There are papers that attempt to formalize concepts that occur in small group study. Examples of these are the papers by A. R. Anderson, who uses elementary formal logic to clarify such concepts as rights, duties and powers, and the paper by G. Karlsson, who studies the concept of the power of an individual and develops formulas to express the degree to which one person can punish another.

It is interesting to compare this book with a similar book, *Studies in Mathematical Learning Theory*, Stanford Press, 1959. The latter book resulted from a similar conference on problems in mathematical learning theory, and one finds here well-defined problems being attacked with obvious enthusiasm, using mathematics in a way that contributes both to psychology and to mathematics. The reviewer found little of the same spirit in the study being reviewed. Perhaps the explanation lies in the remarks made in this book in the contribution of R. Bush. He asks, "What direction should we take in our research on two-person interactions?" and answers, "My advice is to find an interesting phenommenon and a clean experimental paradigm first, then to collect data, and finally to develop models for describing data." He closes with the remark "It took individual psychology a century to provide the applied mathematician material to work with. It may take social psychology even longer."

It is clear from these papers that the authors are dealing with difficult and intriguing problems, and that the field of small group research will undoubtedly suggest important new mathematical problems and techniques.

J. LAURIE SNELL, Dartmouth College

Introduction to Set Theory and Topology. By K. Kuratowski. Addison-Wesley, Reading, Mass., 1962. 283 pp. $7.50.

The author, in his foreword, announces his intent to provide an introduction to set theory and topology "for the beginner." His claim is perhaps a bit modest. What he has written is a broad summary of elementary logic, axiomatic set theory, transfinite numbers, point-set topology, dimension theory, and algebraic topology. Each of the 22 chapters ends with a set of exercises which provide examples or extend the theory.

Although much material is presented in fewer than 300 pages, an abundance of detail is supplied. The style is clear, theorems and proofs being interspersed with examples and remarks. The overall impression of this reviewer is that a very large mass of information has been condensed into surprisingly little space.

This book could be used as a text for a "topology plus" course, provided that such a course proceed at breakneck pace for a year's time. Some material on transfinite cardinals in part I (set theory) and on algebraic topology in part II (topology) might be sacrificed to achieve a more normal pace, but such omission would be regrettable. It is more likely that the best use would be as a source book for independent study on the part of the student. N years ago the material here would have been considered what every graduate student should know before finishing. N years from now it will probably be what every undergraduate should know before beginning graduate work. At present, the level at which this excellent work is most useful varies from one institution to another.

STEPHEN HOFFMAN, Trinity College

Proceedings of the International Symposium on Linear Spaces. Jerusalem Academic Press, Jerusalem; Pergamon Press, Oxford, London, New York and Paris, 1961. xii+452 pp. $14.00.

This volume consists of the papers contributed at the International Symposium on Linear Spaces which was held in July 1960 in Jerusalem. These papers may be classified under the following five groups.

General topological vector spaces or Banach spaces: I. Amemiya, On ordered topological linear spaces. N. Aronszajn, Quadratic forms on vector spaces. A. Dvoretzky, Some results on convex bodies and Banach spaces. E. Gagliardo, A unified structure in various families of function spaces, compactness and closure theorems. V. Klee, Relative extreme points. G. Köthe, Probleme der linearen Algebra in topologischen Vektorräumen. W. A. J. Luxemburg, On closed linear subspaces and dense linear subspaces of locally convex topological linear spaces. L. Nachbin, Some problems in extending and lifting continuous linear transformations. A. E. Taylor, Spectral theory and Mittag-Leffler type expansions of the resolvent.

Hilbert spaces and other special classes of spaces: J. A. Dieudonné, Quasi-hermitian operators. R. E. Fullerton, Geometrical characterizations of certain

function spaces. I. Halperin, Function spaces. H. Helson and D. Lowdenslager, Invariant subspaces. P. D. Lax, Translation invariant spaces. J. Mikusiński, Operations on distributions. W. Orlicz, On spaces of ϕ-integrable functions. R. S. Phillips, The extension of dual subspaces invariant under an algebra. A. C. Zaanen, Banach function spaces.

Differential equations: S. Agmon, Remarks on self-adjoint and semi-bounded elliptic boundary value problems. L. Ehrenpreis, A fundamental principle for systems of linear differential equations with constant coefficients and some of its applications. E. Hille, Linear differential equations in Banach algebras. J. L. Massera, Function spaces with translations and their application to linear differential equations. L. Nirenberg, Inequalities in boundary value problems for elliptic differential equations. G. Stampacchia, Régularisation des solutions des problèmes aux limites elliptiques à données discontinues.

Banach algebras: R. F. Arens, The analytic-functional calculus in commutative Banach algebras. F. F. Bonsall, Semi-algebras of continuous functions. J. Wermer, Subalgebras of $C(X)$.

Others: L. Bers, Completeness theorems for Poincaré series in one variable. G. Fichera, Spazi lineari di k-misure e di forme differenziali. J. P. Kahane, Fonctions pseudo-périodiques dans R^p. G. W. Mackey, Induced representations and normal subgroups. M. H. Stone, Hilbert space methods in conformal mapping.

A great number of recent results are to be found among these papers. Some of them (e.g., those of Köthe and Nachbin) present and discuss a group of unsolved significant problems in certain areas. Since a detailed review of the volume would be altogether too lengthy, we mention just one sample. The paper of Dvoretzky is a detailed development of his earlier note in *Proc. Nat. Acad. Sci. USA*, 45 (1959), 223–226. The main result is the following remarkable theorem. For any positive integer k and any ϵ, $0<\epsilon<1$, there exists a positive integer $N=N(k, \epsilon)$ with this property: for every symmetric convex body C in the Euclidean n-space E^n with $n \geqq N$, there exist a k-dimensional subspace E^k in E^n and two concentric Euclidean balls B_1, B_2 in E^k of radii $r(1-\epsilon)$ and r such that $B_1 \subset C \cap E^k \subset B_2$. The proof of this beautiful theorem, which is so easily stated, occupies nearly 30 pages. As applications of this geometric theorem, the author obtains a characterization (in terms of the "metric type") of infinite-dimensional Hilbert spaces, and a stronger version of the well-known Dvoretzky-Rogers theorem.

As should be clear from the above list of titles, the collection of papers represents well the various lines of recent research in the theory of linear spaces. The publication of this volume will certainly serve to disseminate these new developments.

KY FAN, Northwestern University

Matrix Methods for Engineering. By Louis A. Pipes. Prentice-Hall, Englewood Cliffs, N. J., 1963. 427 pp. $9.75.

As the author indicates, very few books in English emphasize the utility of matrix methods for many physical applications. Of course there are books which specialize in applications to some particular field, such as Frame Analysis or Electrical Network Theory. Here the first 100 pages are devoted to the development of the fundamentals of matrix theory, while the remaining 300 pages are devoted to illustrating matrix methods in Elasticity, Structures, Classical Mechanics, Vibration Problems, Electrical Networks and Multiconductor Lines. Three appendices on the Theory of Laplace Transforms are included.

In an introductory text, it would appear desirable to have the mathematical method developed, along with an introduction to a particular physical problem in nonmatrix form, before any extensive listing of a variety of similar problems for a special field. Further, it would be illuminating to have a few nontrivial examples. The various chapters vary with respect to these remarks. The application to Elasticity and Classical Mechanics seems to be developed in a more thorough and satisfactory manner than, for example, the chapter on the Analysis of Structures. Of course, it is certainly impossible to give a complete background course for every physical discipline. It would seem preferable, however, to have somewhat less content in order to work in more fruitful examples. In this respect it should be noted that in the introductory mathematical material there are numerous examples.

By way of comparison, the recent book of Zurmühl (Matrizen und ihre technischen Anwendungen, Springer-Verlag, Berlin, 1961) approximately reverses the extent of coverages. It goes quite a bit deeper into the theory of matrices and associated numerical procedures with a somewhat less extensive list of applications. It must be noted, however, that Zurmühl does manage to work in some fairly explicit and nontrivial examples, for example in Structures, even though in a very brief form.

Misprints do not appear too numerous and are mostly of a trivial sort. The typography is excellent. The reviewer would have preferred to take the risk of possible confusion in writing a matrix equation as $Ax=b$ rather than as $[A](x)=(b)$, although the latter is certainly safer in physical equations where many scalars have to be used along with vector and matrix items. The treatment of multiple roots of the characteristic equation for $(\ddot{x})+[K](x)=(0)$, pp. 249–251, is rather brief and leaves the impression that certain types of solutions are impossible.

All things considered, it is felt that this is a good text, fulfills a definite need, and should be useful to a variety of people; for those who read German fairly well, Zurmühl appears to be somewhat superior. Mention should be made of the useful reference list at the end of each chapter.

A. B. FARNELL, General Dynamics/Astronautics

Multivariate Procedures for the Behavioral Sciences. By W. W. Cooley and P. R. Lohnes. Wiley, New York, 1962. 211 pp. $6.75.

The purpose of this book is to introduce to social scientists some of the better-known procedures for analyzing multivariate data. The procedures introduced include canonical correlation, multivariate analysis of covariance, classification procedures and factor analysis. An introduction to the calculus of matrices is also included, the above procedures being simplified greatly by its use.

The format of the book is basically as follows: a problem is discussed, a solution is proposed and then a FORTRAN programme to carry out the proposed solution is presented. Consequently a reader with some data and access to a computer with a FORTRAN translator could soon find himself with a foot of print-out.

I find this latter fact both encouraging and discouraging within the framework of the book being reviewed. Encouraging because it should get behavioral scientists interested in, and actually analyzing the reams of data that it seems to be so easy for them to obtain. Discouraging, however, because it makes it possible for the reader to come by the results of a lot of complicated calculations, without really contemplating his own particular problem. The book may cause this discouraging state of affairs because it barely mentions the assumptions involved in the various models, gives no indication of a class of problems within which the assumptions are reasonable, and no lead on how at least to mull over the assumptions before carrying out the calculations.

I dislike the mechanical view of data analysis that seems to underlie the book, viz. given a set of data one grinds it through one of the procedures indicated and out comes the correct answer. This mechanical view is evidenced by the fact that anything significant at the 5% level causes an unceremonious quick conclusion.

The criticism above results from things omitted from the book. There is much in the book. In fact the book should provide a good first step for many people into the domain of multivariate analysis.

DAVID R. BRILLINGER, Bell Telephone Laboratories
and Princeton University

BRIEF MENTION

Progress in Control Engineering, vol. I. Edited by R. H. Macmillan, T. J. Higgins and P. Naslin. Academic Press, New York, 1962. vii+260 pp. $10.00.

Multicomponent Distillation. By Charles D. Holland. Prentice Hall, Englewood Cliffs, N. J., 1963. xiii+506 pp. $11.25.

This book is concerned with the use of high speed computers for making multicomponent distillation calculations.

Tables of Lamé Polynomials. (Volume 17 of Mathematical Tables Series.) By F. M. Arscott and I. M. Khabaza. Pergamon Press, New York, 1962. 526 pp. $20.00.

Operations Research in Production and Inventory Control. By Fred Hanssmann. Wiley, New York, 1962. xii+254 pp. $8.50.

Proceedings of the Symposium on Time Series Analysis. Brown University, June 11–24, 1962. Edited by Murray Rosenblatt. Wiley, New York, 1963. xiv +497 pp. $16.50.

The following range of topics is covered: structural questions—representation of processes; interpolation and extrapolation; distribution of zeros and first passage time problems; statistical questions—estimation of spectra; regression analysis with stationary residuals; tests of hypothesis; applications—geophysics, meteorology and oceanography; engineering; economics; biology.

Elementary Mathematics—a logical approach. By Paul Sanders. International Textbook Company, Scranton, Pa., 1963. xiv+264 pp. $6.00.

Introduction to Electromagnetic Fields and Waves. By Charles A. Holt. Wiley, New York, 1963. xvi+583 pp. $12.50.

Analytic Geometry and Calculus, 2nd ed. By William L. Hart. D. C. Heath, Boston, 1963. xvii+750 pp. $9.40.

Topology (Journal of Mathematics), vol. 1. Pergamon Press, New York, $30.00 per year.

An International Journal of Mathematics founded by J. H. C. Whitehead, Oxford.

John von Neumann, Collected Works, I. Editor: A. H. Taub. Pergamon Press, New York, 1961. 654 pp. $14.00.

Volume I: Logic, Theory of sets, and Quantum mechanics. This is the first of six volumes which will contain a reprinting of all of von Neumann's published articles and some of his reports to government agencies and other organizations, and reviews of unpublished manuscripts found in his files. The published papers, with a few exceptions, are given in essentially chronological order.

Handbook of Calculus, Difference and Differential Equations, 2nd ed. By E. J. Cogan and R. Z. Norman. Prentice-Hall, Englewood Cliffs, N. J., 1963. xii+175 pp. $1.95.

An extensively revised version of the companion volume to *Mathematical Methods and Models*, by the Dartmouth College Writing Group.

The Fourier Integral and its Applications, (Electronic Science Series). By Athanasios Papoulis. McGraw-Hill, New York, 1962. ix+318 pp. $10.75.

This graduate-level textbook for electrical engineers and other applied scientists presents a unified treatment of many topics related to the Fourier integral.

Basic Mathematics for Administration. By F. P. Fowler, Jr. and E. W. Sandberg. Wiley, New York, 1962. xvii+339 pp. $7.95.

Theory of Arithmetic. By J. Peterson and J. Hashisaki. Wiley, New York, 1963. xiv+303 pp. $6.95.

Analytic Geometry and Calculus. By Frank Juszli. Prentice-Hall, Englewood Cliffs, N. J., 1961. xii+348 pp. $6.95.

Analytic Geometry, 3rd ed. By R. S. Underwood and F. W. Sparks. Houghton Mifflin, Boston, 1963. viii+303 pp. $4.75.

Modern Business Mathematics. By L. J. Adams. Holt, Rinehart and Winston, New York, 1963. x+348 pp. $5.75.

Mathematical Tables and Formulas. By R. D. Carmichael and E. R. Smith. Dover Publications, New York, 1962. viii+269 pp. $1.00.

Unaltered republication of work first published in 1931.

Composition Methods in Homotopy Groups of Spheres. By Hirosi Toda. Annals of Mathematics Studies, No. 49. Princeton University Press, Princeton, 1962. 193 pp. $4.50.

The Complete Book of Slide Rule Use. By Ira Ritow. Doubleday, New York, 1963. 200 pp. $1.95.

Modern Operational Calculus with applications in technical mathematics. Revised ed. By N. W. McLachlan. Dover Publications, New York, 1962. 218 pp. $1.75.

A revised edition of a well-known work first published in 1948.

General Topology and its Relation to Modern Analysis and Algebra. Proceedings of the Symposium held in Prague in September, 1961. Edited by J. Novek. Academic Press, New York, 1962. 363 pp. $14.00.

Differential Equations-Geometric Theory. 2nd ed. Pure and Applied Mathematics, vol. VI. By Solomon Lefschetz. Interscience, Wiley, New York, 1963. 390 pp. $10.00.

According to the preface, this edition differs from the first chiefly in its extension of the material on Liapunov's direct method and its converse, and in the revision of the treatment of *Stability in Product Spaces* (Chapter VI).

Technical Writing. A guide to manuals, reports, proposals, articles, etc., in industry and the government. By Richard W. Smith. Barnes and Noble, New York, 1963. x+181 pp. $1.25 (paperback).

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Harpur College: Dr. Barbara A. Clinger, University of Texas, has been appointed Assistant Professor; Assistant Professor K. W. Anderson has been promoted to Associate Professor.

Louisiana State University, Baton Rouge: Professor Gordon Pall, University of Arizona, has been appointed Professor; Associate Professor Haskell Cohen has been promoted to Professor; Assistant Professors R. C. Bzoch and J. E. Keisler have been promoted to Associate Professors.

University of Maryland: Dr. Thomas Willke, National Bureau of Standards, Washington, D. C., has been appointed Assistant Professor; Associate Professors J. W. Brace, J. M. Horvath and J. A. Hummel have been promoted to Professors; Assistant Professors Ellen Correl, Carol R. Karp and Guydo Lehner have been promoted to Associate Professors.

Rosary Hill College: Sister Marion Beiter has been promoted to Professor; Mr. Robert McGee has been promoted to Assistant Professor.

Western Reserve University: Mr. A. C. Lazer, Carnegie Institute of Technology, has been appointed Assistant Professor; Visiting Assistant Professor N. P. Bhatia, RIAS Baltimore, Maryland, has been appointed Visiting Assistant Professor.

Xavier University: Rev. L. E. Isenecker, S. J., Catholic University of America, has been appointed Assistant Professor; Dr. W. J. Larkin III has been appointed Chairman of the Department of Mathematics.

Youngstown University: Associate Professor B. J. Yozwiak has been promoted to Professor; Assistant Professor T. M. Dillon has been promoted to Associate Professor; Mr. R. W. Hurd has been promoted to Assistant Professor.

Mr. T. G. Burgess, Idaho State College, has been appointed Assistant Professor at California State Polytechnic College.

Dr. G. C. Caldwell, Georgia Institute of Technology, has been promoted to Associate Director of the School of Mathematics.

Assistant Professor E. O. Hynard, Manhattan College, has been appointed Associate Professor at Nassau Community College.

Assistant Professor E. M. Maletsky, Montclair State College, has been promoted to Associate Professor.

Professor H. J. Weiss, Iowa State University, will become Head of the Department of Engineering Mechanics on June 1, 1964.

Professor C. T. Hazard, Purdue University, died on August 2, 1963. He was a charter member of the Association.

Professor H. A. Meyer, University of Florida, died on September 1, 1963. He was a member for 33 years.

Assistant Professor Aaron Siegel, State University of New York at Buffalo, died on December 13, 1963. He was a member for 7 years.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

The Society for Industrial and Applied Mathematics is planning a Symposium on Matrix Computations to be held in Gatlinburg at the Mountain View Hotel throughout the week of April 12, 1964. It is contingent upon the granting of funds for its support requested from the National Science Foundation. Emphasis will be on informal discussion rather than formal, prepared lectures, and hence attendance will be limited to about 60. This is the second such symposium, the first having been held in April of 1961, supported jointly by the National Science Foundation and the Atomic Energy Commission. Oak Ridge National Laboratory was the host, and will be for this one.

The funds requested are mainly for the purpose of paying expenses for some of the participants. Selections will be made by a committee consisting of F. L. Bauer, G. E. Forsythe, J. W. Givens, Peter Henrici, and J. H. Wilkinson, with A. S. Householder as chairman. Requests for admission as a participant, with or without financial support, should be addressed to the chairman at Oak Ridge National Laboratory, and accompanied by a list of publications relating to the subject of the symposium.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE MINNESOTA SECTION

The annual fall meeting of the Minnesota Section of the Mathematical Association of America was held on November 2, 1963, at North Dakota State University, in Fargo, North Dakota. Professor A. Glenn Hill, North Dakota State University, presided at the morning session, and the Section Chairman, Professor Seymour Schuster, University of Minnesota, presided at the afternoon session. There were 120 persons registered for the meeting, of whom 80 were members of the Association.

By invitation of the section, Professor H. S. M. Coxeter, of the University of Toronto, and first vice-president of the MAA, gave the main address, entitled: *Close-packing and froth.*

The following papers were presented:

1. *Functions continuous at irrationals and discontinuous at rationals*, by Professor G. A. Heuer, Concordia College.

Earlier results of Porter, Fort, and others suggest additional questions about the functions in the title. Differentiability and Lipschitz conditions are considered. Special attention is paid to the ruler function (f) and its powers. Sample results: THEOREM: *If $\alpha<2, f^\alpha$ is nowhere Lipschitzian; f^2 is nowhere differentiable, but is Lipschitzian on a dense subset of the reals.* THEOREM: *If $\alpha>0, f^\alpha$ is continuous but not Lipschitzian at every Liouville number; if $\alpha>2, f^\alpha$ is differentiable at every algebraic irrational.* THEOREM: *If g is continuous at the irrationals and not at the rationals, then there exists a dense uncountable subset of the reals at each point of which g fails to satisfy a Lipschitz condition.*

2. *Root-powering of polynomial equations*, by Professor F. C. Hatfield, Mankato State College.

Four formally-possible methods of transforming a given polynomial equation to one whose roots are any arbitrary positive integral power of the roots of the given equation are presented, with the principal objective of obtaining equations whose roots are consecutive powers of the roots of the given equation. The four methods employ, respectively, convolution sequences, circulant determinants, Newton symmetric functions and the companion matrix of the given equation. With consecutive powers of the roots of the given equation estimated as in the classical Graeffe Root-Squaring method, the roots themselves are obtained with relative ease.

3. *Some aspects of mathematics education in Colombia*, by Professor Arthur Gropen, Carleton College.

The author served as UNESCO Field Expert at the Universidad Industrial de Santander, Bucaramanga, Colombia, from June, 1962 to July, 1963. He advised the Mathematics Department concerning curriculum, teaching methods, library, texts, etc. He also taught students and conducted seminars for professors. Problems encountered included those due to: poor primary and secondary schooling; inadequately trained mathematics professors; scarcity of good Spanish texts; shortage of university degree programs in mathematics. Solutions proposed included: improving elementary and secondary education using summer institutes; institution of degree programs in mathematics and physics; foreign study grants for students and professors; translations of good texts.

4. *The Apollonius circle contact problem*, by Professor C. N. Mills, Sioux Falls College.

A complete analytic solution was given for the determination of the eight possible contact circles of the Apollonius Problems. The author discussed the procedure in the development of the formula yielding the radius of each circle.

5. *Orders for noncommutative rings*, by D. Bruce Erickson, Concordia College.

Any ring of square-free order is commutative. There exists a noncommutative ring of order m if and only if $m=np^2$, where n and p are integers, p a prime. A construction is given which produces a noncommutative ring for each possible (finite) order.

6. *Close-packing and froth*, by Professor H. S. M. Coxeter, University of Toronto. (By invitation)

C. S. Smith, in his contribution to Metal Interfaces (Amer. Soc. for Metals, Cleveland, Ohio, 1952, pp. 96, 106), remarked that the shapes of metal grains and biological cells and bubbles in a soap froth are all virtually indistinguishable. Each is an assemblage of many kinds of polyhedra whose average number of faces approaches 14 (which Fedorow had proved to be the maximum for convex space-fillers). In a later paper (Acta Metallurgica 1, 1953, p. 299) he reduced this average number from 14 to about 13.4, thus anticipating the "statistical honeycomb" of Coxeter, Introduction to Geometry (Wiley, New York, 1961), p. 411. This number provides a theoretical upper bound for the number of solid spheres that can touch another of the same size. Other considerations show that the number of such spheres actually cannot exceed 12. These ideas have been extended to euclidean spaces of more than three dimensions (Proceedings of Symposia in Pure Mathematics, 7, Amer. Math. Soc. 1963, pp. 53–71). For instance, in 8 dimensions, where a sphere can touch 240 others, the general formula gives the upper bound 244.6.

7. *A new approach to the numerical solution of differential equations*, by Lonny B. Winrich, Minneapolis-Honeywell Regulator Co.

The method of z-transforms has been used previously to analyze sampled data control systems. This paper gives a justification of the method based on the complex convolution integral and applies the procedure to the general problem of approximating a differential equation by a difference equation. The results of some numerical experiments are given.

8. *A generalization of the Beta function*, by Professor F. J. Arena, North Dakota State University.

9. *An evaluation of a limit*, by Richard G. Lee, Concordia College.

10. *Some remarks on the preparation of students for College Algebra*, by Professor W. J. Thomsen, Mankato State College.

The following remarks are the results of a two year study made at Mankato State College. A Higher Algebra course taught in college to prepare students for College Algebra is not as effective as second year high school Algebra for the same purpose. A preparatory course for College Algebra which uses a contemporary approach, is as ineffective as a course using the traditional approach of Higher Algebra.

11. *Introduction of vectors in affine geometry*, by Professor W. J. Jonsson, University of Manitoba.

12. *Abstract definitions for the groups $SL(2, p)$, $PGL(2, p)$, and $GL(2, p)$*, by Professor W. O. J. Moser, University of Manitoba.

MURRAY BRADEN, *Secretary*

NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The ninth annual meeting of the Northeastern Section of the Mathematical Association of America was held on November 30, 1963 at the University of Rhode Island, Kingston, Rhode Island. Professor Evans Munroe, Chairman of the section, presided at the morning session and at the business meeting. Professor Harold Dorwart, Vice-chairman of the section, presided at the afternoon session. A total of 115 persons registered for the meetings, including 93 members of the Association.

At the business meeting, the following officers were elected for the coming year: Professor Harold Dorwart, Trinity College, Chairman; Professor Grace Bates, Mt. Holyoke College, Vice-chairman; Mr. Richard S. Pieters, Phillips Academy, Secretary and Treasurer.

The following papers were presented:

1. *Subadditive functions*, by Professor R. P. Gosselin, University of Connecticut.
2. *The Cambridge Conference on School Mathematics*, by Professor E. E. Moise, Harvard University.
3. *Generalization and Specialization in Algebra*, by Professor R. E. Johnson, University of Rochester. (By Invitation)
4. *Recommendations of the CUPM Panel on Pregraduate Training*, by Professor I. M. Singer, Massachusetts Institute of Technology.

R. S. PIETERS, *Secretary*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Haverford College, Haverford, Pennsylvania, on November 23, 1963. The Chairman, Professor C. W. Saalfrank, Lafayette College, presided at the meeting. The meeting was attended by 173 persons including 133 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor C. W. Saalfrank, Lafayette College; Third member of Executive Committee, Professor C. E. Kerr, Dickinson College.

The following papers were presented at the meeting:

1. *Homogeneity*, by Professor R. H. Bing, University of Wisconsin.

2. *On spline interpolation*, by Professor I. J. Schoenberg, University of Pennsylvania and Institute for Advanced Study.

The paper presented a summary of the very recent work on spline interpolation (S.I.) and its relation to the work of A. Sard as presented in his recent book "Linear Approximation," Math. Surveys No. 9 (1963). It described S.I. and its minimal properties due in various stages to Holladay, Walsh-Ahlberg-Nilson, de Boor and Schoenberg. It was also shown that the best approximation, in the sense of Sard, to a linear operator $L(f)$ is obtained by operating with L on the spline function interpolating the function $f(x)$. It follows in particular that the S.I. formula is identical with Sard's best special interpolation formula (l.c., p. 91). Two special S.I. formulae were singled out: interpolation at all nonnegative integers, and interpolation at all integers. Their construction depends on the inversion of certain Toeplitz matrices and their theory allows development of the limit relations conjectured by Sard and found by him from the empirical evidence provided by his numerical tables (l.c., p. 60).

3. *Curricula from K to* 14, by Professor C. O. Oakley, Haverford College.

A short history was given of the "revolution in mathematics" which has taken place since the establishment in 1955 of the Commission on Mathematics by the College Entrance Examination Board and up to the time of the Cambridge Conference on School Mathematics of June–July, 1963. Sample curricula, in the form of available texts, K to 14 (K to 12 plus 2 years of College), were on display and implications for the college curriculum were discussed. Theses: "College algebra" and analytic geometry as college courses, are on the way out. The first two years of college mathematics for most liberal arts students is, or will be very shortly, calculus and linear algebra. And sooner than you think, students will enter college with a solid year of high school calculus.

4. *The search for delightful results*, by Mr. C. K. Brown, Westtown School.

5. *Arzela's theorem*, by Professor F. Cunningham, Jr., Bryn Mawr College.

6. *Integrability of continuous functions*, by Professor N. J. Fine, University of Pennsylvania and Pennsylvania State University.

7. *A little mathematics of the multiplication table variety*, by Professor Marguerite Lehr, Bryn Mawr College.

8. *Some mathematical crumbs*, by Professor P. Schub, University of Pennsylvania.

9. *How using nets simplifies proofs*, by Professor A. Wilansky, Lehigh University.

V. V. LATSHAW, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fifth Summer Meeting, University of Massachusetts, Amherst, August 24–26, 1964.

Forty-eighth Annual Meeting, Denver-Hilton Hotel, Denver, Colorado, January 28–30, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pa., May 2, 1964.

ILLINOIS, Bradley University, Peoria, May 8–9, 1964.

INDIANA, Butler University, Indianapolis, May 2, 1964.

IOWA, Luther College, Decorah, April 17–18, 1964.

KANSAS, Kansas State University, Manhattan, April 18, 1964.

KENTUCKY, University of Kentucky, Lexington, May 1–2, 1964.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Pace College, New York, April 11, 1964.

MICHIGAN

MINNESOTA, College of St. Thomas, St. Paul, May 9, 1964.

MISSOURI, University of Missouri, Columbia, April 18, 1964.

NEBRASKA, University of Nebraska, Lincoln, May 2, 1964.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 7, 1964.

NORTHEASTERN, Worcester Polytechnic Institute, Worcester, Massachusetts, November 28, 1964.

NORTHERN CALIFORNIA

OHIO, University of Akron, May 9, 1964

OKLAHOMA, East Central State College, Ada, April 10–11, 1964.

PACIFIC NORTHWEST, Washington State University, Pullman, Washington, June 19, 1964.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 21, 1964.

ROCKY MOUNTAIN, Colorado College, Colorado Springs, May 1–2, 1964.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, New Mexico State University, University Park, April 10–11, 1964.

TEXAS, Texas Technological College, Lubbock, April 10–11, 1964.

UPPER NEW YORK STATE, New York State Education Department, Albany, May 16, 1964.

WISCONSIN, Wisconsin State College, Whitewater, May 2, 1964.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN MATHEMATICAL SOCIETY, Reno, Nevada, April 18, 1964.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Maine, Orono, June 22–26, 1964.

ASSOCIATION FOR COMPUTING MACHINERY, Philadelphia, August 25–28, 1964.

ASSOCIATION FOR SYMBOLIC LOGIC, Hotel New Yorker, New York, April 21, 1964.

CALIFORNIA MATHEMATICS COUNCIL, Northern Section, Sacramento State College, April 4, 1964.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Detroit, November 26–28, 1964.

INSTITUTE OF MATHEMATICAL STATISTICS, Berne, Switzerland, September 14–16, 1964.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Miami Beach, Florida, April 22–25, 1964.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Queen Elizabeth Hotel, Montreal, May 27–29, 1964.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, The Hotel Shoreham, Washington, D. C., May 11–14, 1964.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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THE AMERICAN MATHEMATICAL MONTHLY

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FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10053.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

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NOTICE TO AUTHORS

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SPHERES IN E^3

R. H. BING, University of Wisconsin and Institute for Advanced Study

1. Introduction. In this paper we shall be concerned with two dimensional spheres embedded in Euclidean three space E^3.

A 2-sphere is a set homeomorphic to the boundary of a ball. It might have an equation like $x^2+y^2+z^2=1$, but in general it is only homeomorphic with something having such an equation. Figure 1 shows some 2-spheres. The first picture shown in Figure 1 is that of a piece of lava. This particular piece of lava was not examined with enough care to determine whether or not its boundary is actually a 2-sphere. Such a 2-sphere (if indeed it is one) seems much more representative of a general 2-sphere, however, than the rather special ones shown in the other parts of Figure 1.

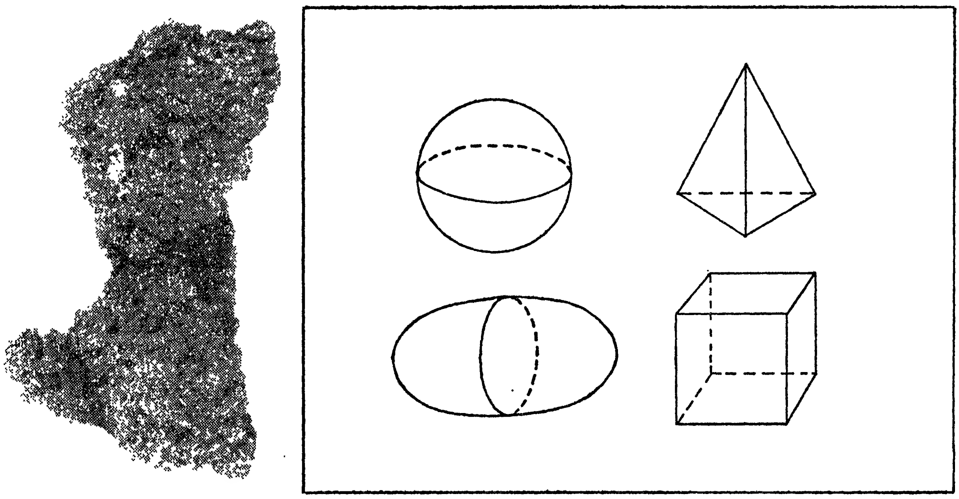

FIG. 1

There are certain properties possessed by all 2-spheres in E^3 irrespective of their shapes. For example, topologists have shown (see for example Theorem 8-38 of [17]) that the Jordan Curve Theorem for the plane E^2 extends to Euclidean spaces of higher dimensions. Hence, if S is a 2-sphere in E^3, S separates E^3 into precisely two pieces. These two pieces are called the interior and exterior of S as follows:

$$\operatorname{Int} S = \text{bounded component of } E^3 - S,$$
$$\operatorname{Ext} S = \text{unbounded component of } E^3 - S.$$

There are certain properties possessed by certain special 2-spheres that are not possessed by 2-spheres in general. The boundaries of the ball and the ellip-

soid shown in Figure 1 are smooth and differentiable whereas the other 2-spheres shown are not. The boundaries of the tetrahedron and the cube are polyhedral while the others are not. The boundary of the rock stands a chance of having infinite area but the others do not. Some of the 2-spheres can be pierced by a straight line interval at each point but one cannot.

It may come as a surprise to some that the solid bounded by one 2-sphere may be topologically different from the solid bounded by another. Those who think the Schoenflies Theorem needs no proof in E^2 because it is intuitively obvious, might also think that the extension to E^3 is true—where indeed it is false. [The Schoenflies Theorem says that if J_1, J_2 are two 1-spheres in E^2 (topological circles, frequently called simple closed curves), then there is a homeomorphism of E^2 onto itself taking J_1 onto J_2. See Corollary 9.25 of [**15**]. The generalization to E^3 (which is false) would say that if S_1, S_2 are two 2-spheres in E^3, then there is a homeomorphism of E^3 onto itself taking S_1 onto S_2.] Once one realizes that 2-spheres can be so exotic, one may be surprised that they behave as well as they do. We mentioned in a preceding paragraph that the Jordan Curve Theorem extends. In Section 5 we discuss some results about retractions that apply to all 2-spheres in E^3. These and other results of this paper extend to Euclidean spaces of higher dimensions and even to abstract spaces of certain sorts but since this is an expository paper, we restrict ourselves to the simplest cases of interest.

A 2-sphere S in E^3 is called *tame* if there is a homeomorphism of E^3 onto itself that takes S onto a polyhedral 2-sphere. Note that for each 2-sphere S' there is a homeomorphism of S' onto a polyhedral 2-sphere but in order that S' be tame we insist that the homeomorphism can be extended to E^3. It is known that, if a 2-sphere S is tame, then there are homeomorphisms of E^3 onto E^3 that take S onto the boundary of a round ball. Examples are known of 2-spheres in E^3 that are not tame. Such 2-spheres are called *wild*. We describe two wild 2-spheres in Sections 3 and 6 of this paper. Sketches of wild 2-spheres are found in [**2**], [**6**], [**7**], [**10**], [**13**], and [**17**] while descriptions of these and others are also found in [**3**] and [**12**].

Topologists are interested in the problem of determining conditions under which a 2-sphere in E^3 is tame. At Stockholm in 1962 one of the addresses [**9**] was devoted to reviewing known sets of conditions that imply that a 2-sphere in E^3 is tame. Some questions were raised, one of which is as follows: *Is a 2-sphere S in E^3 tame if S can be pierced by a straight line segment at each point?*

The question is not wholly topological since the notion of straightness is geometric rather than topological. However, the question seemed of interest even for topologists. A segment axb is said to pierce S at x if $S\cdot axb=\{x\}$ and a, b belong to different components of E^3-S. M. K. Fort, Jr. found [**12**] a negative answer to the question. An example giving this answer is given in Section 3. The example relates to the game of pick-up-sticks which is discussed in the next section.

2. Pick-up-sticks. In pick-up-sticks one is confronted with a tangled pile of sticks and challenged to remove them one at a time in such a way that none of the remaining ones is moved. One finds that there are piles in which each stick has another resting on it so that it is impossible to proceed.

Suppose P_1, P_2 are two horizontal planes with P_2 above P_1. Let X be a collection of mutually exclusive straight line segments such that each has one end in P_1 and the other in P_2. See Figure 2. Is it possible to adjust the segments gradually so that each assumes a vertical position, each keeps its bottom end fixed, and during the movement the segments remain mutually exclusive?

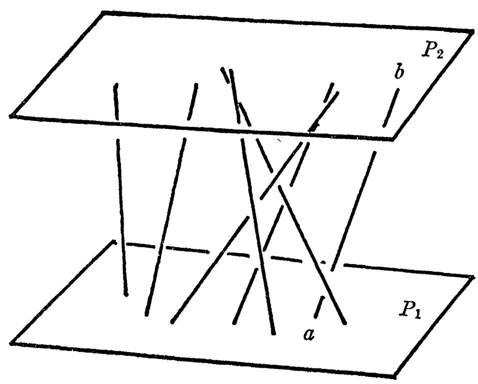

FIG. 2

An affirmative answer to the above question is provided by the following method suggested by R. H. Fox. Consider a segment ab of X with $\{a\}$ in P_1. The lower end of the segment ab is left fixed but at time t the modified segment intersects P_2 at the point which is the vertical projection of the point of ab which divides ab in the ratio $1-t$ to t. Note that at time $t=1$ the modified segment is vertical. In showing that the modified segments do not intersect each other we suppose that a and b have coordinates $(x_a, y_a, 0)$ and $(x_b, y_b, 1)$ respectively. The modified segment at time $t(0 \leqq t < 1)$ is the lower portion of the elongated segment from $(x_a, y_a, 0)$ to $(x_b, y_b, 1/(1-t))$. The elongated segments do not intersect each other since they are the same as the originals under a change of z coordinates. Hence the lower portions of these segments do not intersect each other either.

Let us consider a collection X of segments in a certain instructive position. Let V_1 be the sum of two segments each with ends in P_1, P_2 so that the segments intersect in their upper end points. The segments in V_1 are not elements of X but are merely used as a first approximation to the elements of X. Let N_1 be a tubular neighborhood on V_1. See Figure 3.

Let V_2 be the sum of two inverted V's in N_1 such that the V's hook as shown in Figure 3. Let N_2 be a thin tubular neighborhood of V_2. Let V_3 be the sum of four inverted V's in N_2 so that the part of V_3 in a component of N_2 consists of

two hooked V's linked like V_2 in N_1. We continue defining N_3, V_4, N_4, V_5, $\cdots$. Elements of the set X are the components of the intersection of $\overline{N}_1$, $\overline{N}_2$, $\overline{N}_3$, $\cdots$. Note that X is a Cantor set of segments.

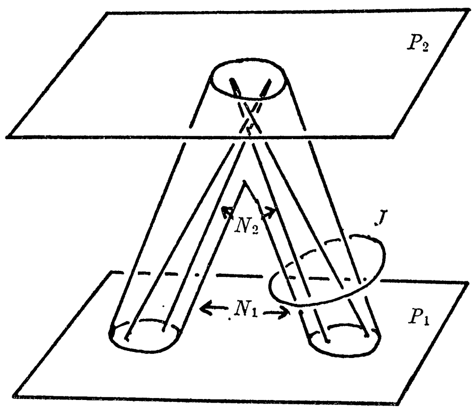

FIG. 3

To see that X is an unusual collection of segments, consider a simple closed curve J as shown in Figure 3 which circles one leg of N_1. In the halfplane above P_1, J cannot be lifted above P_2 without intersecting $\overline{N}_1$. Neither can it be so lifted without intersecting $\overline{N}_2$, nor $\overline{N}_3$ nor . . . . A rigorous proof that it cannot be lifted can be based on Theorem 9 of [7]. Hence it cannot be lifted without intersecting some element of X. If the elements of X were vertical, however, it could be so lifted. Is there a paradox here? Was the argument sound that the elements of X could be lifted into a vertical position?

It was not the verticalizing argument that was in error. The family of homeomorphisms making the segments vertical was defined on the segments only and not on other points of E^3. The family of homeomorphisms could not be extended in a continuous fashion to the rest of space.

3. A wild porcupine. We describe a wild 2-sphere S_0 which can be pierced at each point with a straight line segment. The example is a modification of that given by M. K. Fort, Jr. in [12].

Suppose that S' is the boundary of a cube in E^3 such that the upper face of the cube lies in P_1 of Section 2 and contains the lower portion of N_1 as shown in Figure 4.

Suppose two holes are cut in S' where N_1 intersects P_1 and the holes are replaced by cylinders which run halfway along the vertical sides on N_1 and capped with horizontal disks halfway between P_1 and P_2. Suppose two holes are cut in each cap and cylinders along the sides on N_2 are run up half the remaining distance to P_2. Caps are put on these cylinders but two holes are cut in each cap and cylinders along the sides of N_3 are run up half the remaining distance to P_2. The procedure is continued so as to get a 2-sphere S_0 as shown

in Figure 4 so that all intervals of X lie, except for their upper ends, in Int S_0.

Each tame 2-sphere S has the property that each simple closed curve in E^3-S can be shrunk to a point in E^3-S. The simple closed curve J shown in Figure 3 cannot be shrunk to a point in E^3-S_0; hence S_0 is not tame. It can be shown that the embedding of S_0 is the same as the embedding of Alexander's Horned Sphere [2].

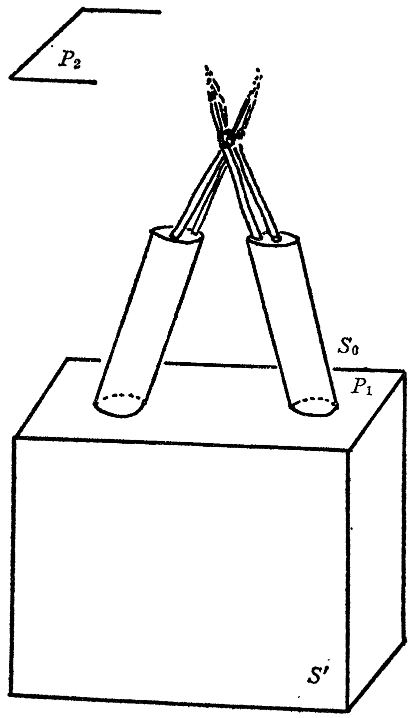

FIG. 4

At each point of S_0, S_0 can be pierced with a straight line segment. The only questionable points are the points of $P_2 \cdot S_0$ and the segments in X show that S_0 can be pierced there.

We can define the segments piercing S_0 so that the directions in which the segments run is continuous. If we do this, however, we will have to permit some of the segments that pierce near $P_2 \cdot S_0$ to be very short. Also, we could have defined the segments so that all of them reach the same distance into Int S_0 and Ext S_0, but under this definition we would not make the directions continuous. That we cannot do both at the same time follows from a result of John Hempel [16].

THEOREM 1. *Any 2-sphere in E^3 for which there is a continuous family of piercing segments is tame.*

The quills of S_0 are tangentially sharp since the cylinders get progressively smaller very fast. If one sharpened a pencil, it would not be possible to touch S_0 from the inside with the tip of the pencil.

Questions. The above considerations suggest the question as to whether or not a 2-sphere S is tame if at each point it can be wedged between two tangent round balls which lie, except for the point of contact, in different components of E^3-S. The left part of Figure 5 suggests the question. The answer is unknown. Similarly we do not know the answer to the corresponding question suggested by the right part of Figure 5 where we show an arbitrary point of S wedged between two cones. These questions go beyond the notions of Topology and involve the idea of straightness from Geometry.

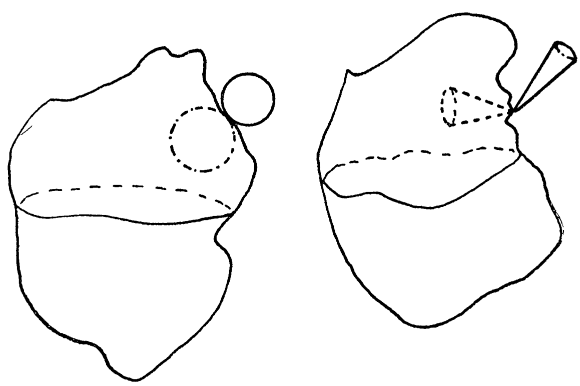

FIG. 5

4. Tietze Extension Theorem. Suppose that one is given an integrable function $y=f(x)$ defined on the line E^1. Even though $f(x)$ is not continuous, for each $\epsilon>0$ the function

$$g_\epsilon(x) = \int_{x-\epsilon}^{x+\epsilon} f(t)dt$$

is continuous. We get a corresponding result if we replace the constant ϵ by a continuous positive function $\epsilon(x)$.

We use the above notion in proving a version of the Tietze Extension Theorem which in turn will be used in the next section. We use calculus (something novel in point set topology) as an averaging tool to give the proof.

Although the Tietze Extension Theorem is normally proved for more abstract spaces (see for example pages 59–61 of [**17**]), we prove it here only for metric spaces since the proof is more elementary in this case and this is the context in which we use it. Furthermore, we get to exhibit an interesting application of calculus. Another elementary proof is found in [**18**].

THEOREM 2. (Tietze Extension Theorem for metric spaces.) *Each map f of a closed subset X of a metric space Y into the segment $[0, 1]$ can be extended to take Y onto $[0, 1]$.*

Proof. Our job is to consider a point p of $Y-X$ and decide what value of $[0, 1]$ to assign to it. If points of X near p are assigned a particular value of $[0, 1]$, we wish to assign p a value nearby. If D is the distance function for Y and

$$V(p, r) = \{q \mid q \in Y, D(p, q) \leqq r\},$$

it does no good to look on the interior of the ball $V(p, D(p, X))$ since there are no points of X there. However, there will be points of X in the hollow ball $V(p, 2D(p, X)) - \text{Int } V(p, D(p, X))$, and we seek a certain average over the f values of X in this annulus or hollow ball. See Figure 6.

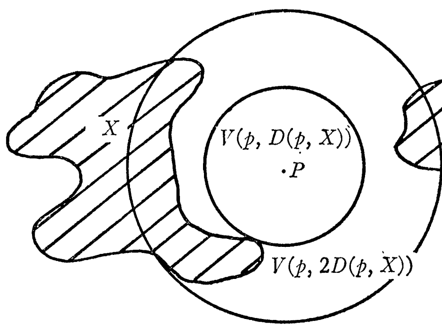

FIG. 6

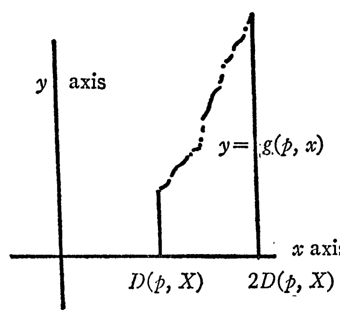

FIG. 7

Consider the function

$$g(p, r) = \text{least upper bound} f(x), \qquad x \in X \cdot V(p, r).$$

There is no reason to suppose that g is continuous in either p or r. It is not defined for $r < D(p, X)$, but for p fixed it is a bounded monotone nondecreasing function for $r > D(p, X)$ and hence integrable. The graph of $y = g(p, x)$ with p fixed and $D(p, X) < x \leqq 2D(p, X)$ might look somewhat like that shown in Figure 7. If we take the area under the curve and divide by the length of the base line, we find an average value of $g(p, x)$. It is immaterial as to whether or not g is defined on the left end of the base line.

For values of p not in X we define

$$f(p) = \frac{1}{D(p, X)} \int_{D(p,X)}^{2D(p,X)} g(p, t)dt.$$

It is an exercise in calculus to show that this function is continuous at each point of $Y-X$. It is a continuous extension of the given function f since for each point q of X and each $\epsilon > 0$, there is a positive number $\delta(q)$ such that if $q' \in X$ $V(q, \delta)$, $|f(q) - f(q')| < \epsilon$. Then for each $p \in V(q, \delta/3)$, $|f(q) - f(p)| \leqq \epsilon$.

It is sometimes convenient to generalize the Tietze Extension Theorem as follows.

THEOREM 3. *Each map of a closed subset of a metric space Y into a disk can be extended to all of Y.*

Proof. We regard the disk as a square in E^2 with opposite vertices at (0, 0) and (1, 1). For each point p on which f is defined let $(f_x(p), f_y(p))$ be the coordinates of $f(p)$. It follows from the preceding theorem that f_x and f_y can be extended to all of Y. If f_x, f_y are used to denote these extensions, the extended f may be defined so that $f(p)$ has coordinates $(f_x(p), f_y(p))$.

5. Retractions onto spheres. A *retraction* of a set Y onto a set X is a map f of Y onto X such that f is fixed on each point of X.

A round 2-sphere has the property that if p is a point of Int S, then there is a retraction of $E^3 - \{p\}$ onto S. As pointed out in Theorem 4, each wild 2-sphere has this property. For the round 2-sphere S, one can take the half lines emanating from p onto the points where these half lines pierce S. For arbitrary 2-spheres more ingenuity needs to be used in getting the retraction.

THEOREM 4. *If S is an arbitrary 2-sphere in E^3 and p is a point of* Int S, *then there is a retraction of $E^3 - \{p\}$ onto S.*

This theorem was proved in [8]. An effort was made to avoid the existence type of proof, but rather to exhibit a method for getting a retraction of $E^3 - \{p\}$ onto S. The proof given in [8] was somewhat constructive but failed to indicate what the retraction of a neighborhood of S into S would look like. The purpose of giving a proof of the Tietze Extension Theorem in the last section by defining the extension rather than by existence techniques was to enable us to show in Theorem 5 what the retraction of the neighborhood of S is.

A set X is an ANR (*absolute neighborhood retract*) if for each embedding of X into a metric space Y there is a neighborhood N of X in Y and a retraction of N onto X. It is known that each 2-sphere is an ANR. See for example Theorem 2-36 of [17]. Hence, no matter how wildly a 2-sphere S is embedded in E^3, there is a neighborhood N of S in E^3 and a retraction of N onto S.

THEOREM 5. *If S is a 2-sphere in E^3, then there is a neighborhood N of S and a retraction of N onto S.*

Proof. Let D_1, D_2, D_3 be three disks in S with $D_1 \subset \text{Int } D_2 \subset D_2 \subset \text{Int } D_3$. Let f be the identity map of D_3 onto itself and use Theorem 3 to extend f so as to obtain a retraction of E^3 onto D^3. Let N be a neighborhood of Bd D_2 such that $\overline{N} \cdot S + f(\overline{N}) \subset D_3 - D_1$. Let N_1, N_2 be neighborhoods of D_2 and $S - \text{Int } D_2$ respectively such that $\overline{N}_1 \cdot \overline{N}_2 \subset N$. Let g be a map of $(S - \text{Int } D_1) + \overline{N}$ onto $S - \text{Int } D_1$ that is the identity on $S - \text{Int } D_1$ and f on $\overline{N}$. It follows from Theorem 3 that the map g can be extended to a retraction of $(S - \text{Int } D_1) + \overline{N} + \overline{N}_2$ onto $S - \text{Int } D_1$. Then the retraction of $N_1 + N + N_2$ given by f on $N_1 + N$ and by the extended g on $N_2 + N$ satisfies the requirements of Theorem 5.

Fixed point properties of cubes imply that if p is not removed in Theorem 4, there is no retraction. Again, this theorem holds as well for arbitrary 2-spheres in E^3 as for nice ones.

THEOREM 6. *There is no 2-sphere in E^3 which is a retract of E^3.*

Proof. Assume there is a retraction r of E^3 onto a particular 2-sphere S. Consider a homeomorphism h of S onto itself that moves every point of S. That here is such a homeomorphism follows from the fact that the antipodal homeomorphism of a round 2-sphere onto itself throws each point into its diametrically opposite point and hence moves every point. The map hr would be a fixed point free map of E^3 onto a compact subset of E^3. Theorems like Theorem 6-39 of [17] say that this is impossible.

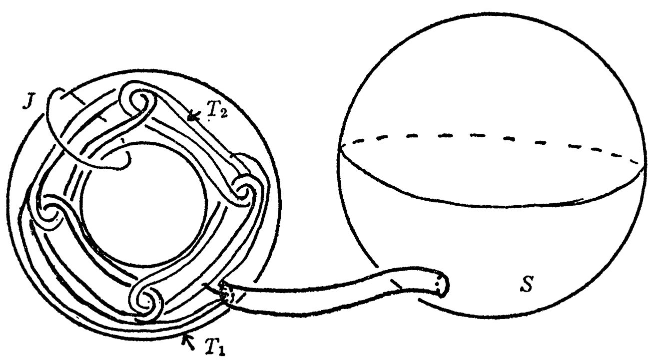

FIG. 8

6. Antoine's necklace. An Antoine's necklace is illustrated in the left portion of Figure 8. It is described in [**3**, **4**, **5**, **6**, **10**, **11**, **17**] and is the intersection of a solid torus T_1, a set T_2 which is the sum of several small solid tori which form a chain circling T_1, a set T_3 which is the sum of some very small solid tori which intersect each component of T_2 in the same fashion that T_2 lies in $T_1, \cdots$. There are countably many layers of T's, and the set of points belonging to their intersection is Antoine's necklace. We denote it by A. It may be shown that A is topologically equivalent to an ordinary Cantor set. Its complement, however, is topologically different from the ordinary Cantor set, because the simple closed curve J shown in Figure 8 cannot be shrunk to a point without hitting A. See [**11**] for proof of this.

Consider a dendron running down to a Cantor set as shown in Figure 9. It looks something like a form used to record results of an elimination tournament in which only winners remain at each stage to engage other winners. There is no starting round, however, since at each stage there was a preceding one. We use a combination of the notion of a dendron of this type and of Antoine's necklace to describe what was perhaps the first known wild 2-sphere [**3**].

Let S be a 2-sphere missing T_1 as shown in Figure 8. Alter S by removing a small disk and replacing the hole with a cylinder leading over to T_1 with a cap on Bd T_1. Holes are cut in this cap and tubes are run in T_1 over to the components of T_2 where the tubes are capped. Holes are cut in these caps and tubes are run in T_2 over to the components of T_3. The process is continued. Note that the tubes get shorter since they lie in the T's which at the later stages get arbitrarily small.

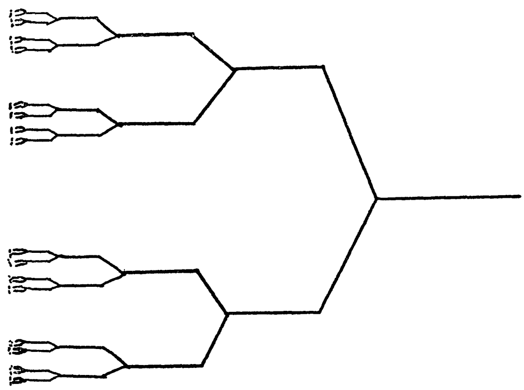

FIG. 9

The resulting 2-sphere is wild because the simple closed curve J shown cannot be shrunk to a point in E^3-A and hence not in the complement of the resulting 2-sphere.

7. The grooved ball. There are many criteria known for determining whether or not a 2-sphere is tame. One of the earliest of these [1] shows that a 2-sphere is tame if it is polyhedral. More understandable proofs of this are found in [**14**] and [**19**].

A condition that is somewhat in doubt is involved in the following question: *Is a 2-sphere S in E^3 tame if each horizontal cross section of S is either a point or a simple closed curve?*

An affirmative answer is suggested in the literature but the proof is very sketchy, saying merely that the result follows by "well-known theorems on logarithmic potential in 2-dimensions."

We give an example to show that even if the above condition does imply that a 2-sphere is tame, it is not enough to ensure that there is a homeomorphism of E^3 onto itself which is invariant on horizontal planes and which takes S onto a round 2-sphere.

Suppose B is a croquet ball to be left out in the rain. One wishes to groove the ball so that as water runs down the ball, one section of the equator does not have water run over it. See Figure 10. The grooves are very shallow so that each horizontal cross section of the ball is a disk. The cross section through the

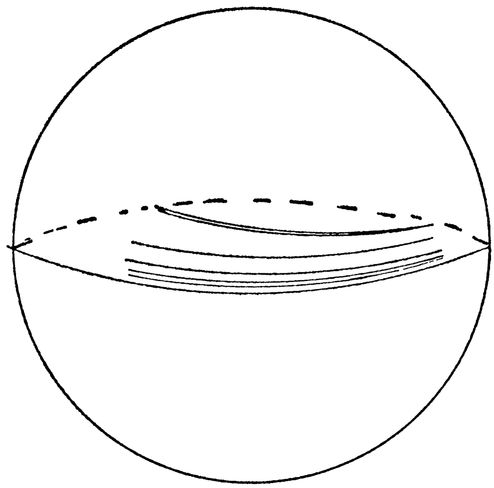

FIG. 10

equator is round but some close to it are disks with long feelers. These disks converge, but not homeomorphically, to the disk containing the equator. If we chose to make the groves spiral we could get even longer feelers.

We have learned much about E^3 but there is much we do not know. For the person with ingenuity and ability this is a fruitful area in which to work.

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References

1. J. W. Alexander, On the subdivision of 3-space by a polyhedron, Proc. Nat. Acad. Sci. U.S.A., 10 (1924) 6–8.

2. ———, An example of a simply connected surface bounding a region which is not simply connected, Proc. Nat. Acad. Sci. U.S.A., 10 (1924) 8–10.

3. ———, Remarks on a point set constructed by Antoine, Proc. Nat. Acad. Sci. U.S.A., 10 (1924) 11–12.

4. M. L. Antoine, Sur l'homéomorphie de deux figures et de leurs voisinages, J. Math. Pures Appl., 86 (1921) 221–325.

5. ———, Sur la possibilité d'étendre l'homéomorphie de deux figures à leurs voisinages, C. R. Acad. Sci. Paris, 171 (1920) 661–663.

6. R. H. Bing, Examples and counter examples, Pi Mu Epsilon J., 1 (1953) 311–317.

7. ———, A wild surface each of whose arcs is tame, Duke Math. J., 28 (1961) 1–16.

8, ———, Retractions onto spheres, this MONTHLY, to appear in May issue.

9. ———, Embedding surfaces in 3-manifolds, Internat. Congress of Math., Outlines of one-hour and half-hour addresses, Stockholm, 1962, 2.11–2.12.

10. W. A. Blankenship and R. H. Fox, Remarks on certain pathological open subsets of 3-space and their fundamental groups, Proc. Amer. Math. Soc., 1 (1950) 618–624.

11. R. P. Coelbo, On the groups of certain linkages, Portugal. Math., 6 (1947) 57–65.

12. M. K. Fort, Jr., A wild sphere which can be pierced at each point by a straight line segment. Proc. Amer. Math. Soc., 14 (1963) 994–995.

13. R. H. Fox and E. Artin, Some wild cells and spheres in three-dimensional space, Ann. of Math., 49 (1948) 979–990.

14. W. Graeub, Die semilinearen Abbildungen, S.-B. Heidelberger Akad. Wiss., part 4 (1950) 205–272.

15. D. W. Hall and G. L. Spencer II, Elementary Topology, Wiley, New York, 1955.

16. J. P. Hempel, A surface in E^3 is tame if it can be deformed into each complementary domain, to appear.

17. J. G. Hocking and G. S. Young, Topology, Addison-Wesley, Reading, Mass., 1961.

18. K. Kuratowski, Topologie, vol. 1, Warsaw, 1948.

19. E. E. Moise, Affine structures in 3-manifolds II, Positional properties of 2-spheres, Ann. of Math., 55 (1952) 172–176.

THE MATHEMATICS USED IN MATHEMATICAL PSYCHOLOGY

R. DUNCAN LUCE, University of Pennsylvania

Introduction. The main issue in applying mathematics to psychological problems today, and most likely for some time to come, is the formulation of these problems in mathematical terms. The solution of difficult but well-formulated mathematical problems and the analysis of complex applied problems in terms of precise and well-confirmed theories are more secondary efforts. We do not yet have the basic concepts and variables staked out in a way that makes the introduction of mathematics the relatively straightforward business that it has become in much of physical science. We are in a situation somewhat analogous to sixteenth, or hopefully seventeenth, century physics, but the analogy is far from complete. We resemble the early physicists in our effort, often fumbling and always slow, to isolate and purify the fundamental variables from the myriad, vague, commonsense psychological ideas and concepts. We differ in the range of techniques available to us. The modern electronics technology, including high speed computers, provides us with a control over experimental conditions and a computational capacity for data analysis incomparably more extensive and subtle than those with which the early physicists had to contend. In addition, most of the mathematics and statistics we now use was quite unknown three centuries ago.

Applications to what psychology? The current applications of mathematics to parts of psychology are precisely that—applications to portions of the total field. To the great satisfaction of many who do not necessarily view with favor the increasing mathematical complication of the psychological literature, huge portions of both academic and applied psychology are essentially free from mathematical inroads. The main areas that have been affected are those usually grouped together as "experimental" psychology, which is a misnomer because

experiments are also performed in the "nonexperimental" areas. By convention, however, experimental psychology equals basic research into such fundamental psychological processes as learning, sensation, perception, and motivation. What is popularly viewed as psychology—abnormal, child, personality, and much of social psychology—has not been seriously influenced by any mathematical developments other than classical statistics, especially testing of hypotheses which is used extensively, if not always well, throughout psychology. (I shall not comment here about the role of either statistics or computers in psychological research. Both are extremely influential, and the mathematical community, whether it likes it or not, is bound to be involved to some extent in the problems that they create, but it is enough for this paper to deal just with mathematical applications.)

With our attention confined to mathematical attempts to understand the data that arise from laboratory experiments that are designed to elucidate fundamental psychological processes, certain features of these experiments should be made explicit because they exert a considerable impact upon the mathematics that we use. Laboratory experiments elicit behavior which differs in various ways from most ordinary, on-going behavior. Although a number of exceptions can be cited, by and large time is rendered discrete in the laboratory, i.e., the temporal pattern of events is structured in some fashion into trials. It is obvious that ordinary experience is not so neatly packaged. The stimulation to the subject—not his total environment, but those aspects of it to which we want him to attend—is usually delivered as discrete, repeatable bursts of some sort or other. The subject's possible responses, either those he is instructed to make or those we choose to observe, are frequently restricted to a discrete set (more often than not, a small finite one). Again, this is sometimes the case in nature, but certainly not the rule. Many ordinary situations possess a certain open-endedness or freedom of response which makes them difficult to study; life's richness is sufficiently impoverished in the laboratory that we can bring to bear our conceptual and calculating tools. Finally, the information feedback and the rewarding or punishing outcomes of an experiment are ordinarily discrete and delivered to the subject immediately after he responds. This is certainly not typical of modern life, in which people continually face long delayed and, more often than not, diffuse feedback and rewards. For a more exact characterization of this class of experiments, (see [4]).

Thus, there is little doubt that significant differences exist between the laboratory experiments we are trying to model in mathematical terms and everyday behavior. Most psychologists hope that one day we may gain some insight into socially significant behavior through the phenomena and laws discovered by means of our restricted and often artificial experimental designs, but the considerable difference between the two types of behavior is a raw fact that cannot be denied, and the bridges linking them do not seem to be easy to construct.

The reasons for these experimental abstractions are not primarily mathematical, although to a minor extent the general dispersion among scientists of

elementary mathematics may have had its impact upon experimental designs. By and large, however, these restrictions to discrete time, stimuli, responses, and outcomes have been imposed by experimentalists so that 1) experiments can be completed in reasonable periods of time, and 2) the resulting data records will be reliable and amenable to certain types of analysis. As disturbing as it may seem, much of what we do in a psychological laboratory is forced upon us by data recording and analysis problems.

From a mathematician's point of view, the effect of these procedures is to limit somewhat the kinds of mathematics he can effectively employ: the continuous mathematics of classical analysis is not particularly well suited to the discreteness of most psychological experiments. For example, psychological problems are rarely formulated as differential equations, although such equations do arise sometimes in an incidental way in the solution of a problem cast in other terms. But as far as actual formulation is concerned, the standard methods of classical physics are little used and when they are the results are not usually very interesting.

Enter numbers. In a way, I have gotten ahead of my story, for to talk about formulating a problem in terms of classical analysis presupposes that the variables can be represented by numbers or vectors. This we take for granted in much of physics, but for psychology and the other social and behavioral sciences one of the most perplexing problems is how to introduce numbers in a meaningful way. Perhaps we are too wedded to the familiar and should not try to bring in numbers at all, and ultimately we may be forced by our subject matter to other mathematical representations, but at present we seem to be relatively helpless until we have a scalar or vector representation. There are reasons to believe that the attempt to represent some psychological notions by numbers is not a totally foolish goal. To be sure, the vector or scalar nature of our variables is far less certain than it is for variables such as mass, velocity, etc. Nevertheless, concepts such as the utility of a commodity, the loudness of a tone, or the brightness of a light—subjective notions that to some extent parallel objective attributes, in these cases, monetary worth, acoustic energy, and light energy—all seem to have an intensive nature reminiscent of numerical scales in physics. Mathematical psychologists and others in related areas are devoting some effort to discovering whether or not this is a poor analogy or, to put it positively, to finding out when we can justify the assignment of numbers to stimuli and outcomes so as to create useful scales of utility, sensation, and the like. I use the phrase "justify the assignment of numbers" intentionally. It is worse than useless for psychologists to parallel superficially textbook problems of physics by saying: let H_i denote the amount of hostility possessed by person i, let A_{ij} denote the amount of aggression expressed by i towards j, etc. There are no reasons that I know of to suppose that either of these notions, hostility or aggression, and many others like them, are scalar quantities. To begin a problem in this way is tantamount to throwing most of it away. At our present state of knowledge,

to "suppose" a measurement problem out of existence is little more than a feeble joke.

So one of our main tasks is to see whether we can sensibly represent some psychological concepts numerically. There are two major techniques: one involves probability concepts and leads to the main body of mathematical psychology; the other, which has been much more incidental, attempts a more fundamental approach. This I discuss first.

Fundamental measurement. If we examine theories of physical measurement, as represented for example by the work of N. R. Campbell [6], we find an important distinction between fundamental and derived measurement. The essential feature of fundamental measurement is that only assumptions about qualitative observations are made: no numbers enter into the axioms of the theory. From these assumptions a numerical representation of the observations is shown to exist. The axioms characterize observables such as the deflection of a beam balance when objects are placed upon the two pans. The best known system, that called extensive measurement, was devised to account for the measurement of mass, length, etc. It involves a set Ω of objects that are to be measured, a binary operation $\circ$ of "combination" or "concatenation" of any two objects in Ω to form a third, and a qualitative ordering $\geqq$ of "not less than" over Ω such as that determined by the deflection of a balance. Typical axioms are: if a and b are in Ω, then $a \circ b$ is also in Ω; the relation $\geqq$ is a weak ordering of Ω; if $a \geqq b$, then $a \circ c \geqq b \circ c$ for all c in Ω; and so on. Ultimately, one states a set of axioms that is sufficient to prove a representation and uniqueness theorem of the following form. There exists a function $\phi: \Omega \rightarrow$ real numbers such that

i. *$a \geqq b$ (qualitatively) if and only if $\phi(a) \geqq \phi(b)$ (numerically);*

ii. *for all a, b in Ω, $\phi(a \circ b) = \phi(a) + \phi(b)$;*

iii. *if ϕ and ϕ^* are two functions that both satisfy* (i) *and* (ii), *then there exists a positive constant α such that $\phi = \alpha\phi^*$. That is, the representation is unique up to a similarity transformation or, in the language of modern measurement theory, the variable in question can be represented as a ratio scale.*

It would be most convenient for psychology to have an analogous theory in which the axioms turned out to be empirically verifiable statements about the behavior of human or animal subjects. Unfortunately, one apparently cannot reinterpret simply the axioms of extensive measurement in psychological terms. It is easy to give sensible psychological interpretations of Ω and $\geqq$, but it is less easy to assign natural meanings to the concatenation operation. Our failure to devise empirically acceptable interpretations of extensive measurement when, nonetheless, we feel strongly that fundamental measurement ought to be possible has forced us to reconsider some of Campbell's ideas [**28**]. He seemed to feel that fundamental measurement is synonymous with extensive measurement, that is, with a system leading to the three assertions above. In this, it is generally agreed today, he was incorrect. As I have suggested, fundamental measurement involves a system of axioms that is stated without any reference to

the real numbers, that has an empirical interpretation which permits the axioms to be checked directly, and from which it is possible to establish a numerical representation of the undefined objects and relations of this system so that condition (i) above is satisfied, (ii) is replaced by some appropriate condition, and (iii) may be weakened somewhat to another fairly restrictive group of transformations such as the positive linear transformations. Extensive measurement is one example of fundamental measurement, but there are others.

Most of these examples of fundamental, nonextensive measurement have arisen from the study of preference, a topic of concern in economics and statistics as well as psychology. One development was triggered by von Neumann and Morgenstern's [**33**] work on the expected utility hypothesis. Although their theory is not an example of fundamental measurement (because probabilities occur in the statement of the axioms), Savage's generalization [**26**] is—in fact, it provides for the simultaneous fundamental measurement of both utility and subjective probability. Suppes and his collaborators [**9**], following up an idea of the philosopher Ramsey [**24**], have devised a system for the fundamental measurement of utility that differs from Savage's in having only one chance event. Pfanzagl [**23**] has presented a rather different axiomatization that involves, essentially, the notion of bisection of two stimuli; it is closely similar to an axiomatization of means given by Aczél [**1**]. Recently, Tukey and I [**20**] have developed a system of fundamental measurement in which the basic ingredients are a weak ordering of objects having at least two independent components. A physical realization that would satisfy our axioms, according to classical physics, is the ordering of objects by momentum as measured by, say, a ballistic pendulum; the two components of the objects are, in ordinary terms, mass and velocity. Potential psychological examples—none of which has yet been explored empirically—involve subject-determined orderings of stimuli that have at least two independent coordinates. Preference of rats between pairs of outcomes consisting of different amounts of food paired with different levels of shock might be an example. We have shown that if the ordering satisfies certain "reasonable" axioms, then a numerical representation exists that is additive over the components and it is unique up to positive linear transformations; the theory of extensive measurement can be extracted as a special case when the two components are the same and one further axiom (involving the existence of a null object) is added.

Work of this type, which I believe promises ultimately to help isolate fundamental psychological variables, constitutes only a tiny portion of current research in mathematical psychology. For the most part, numbers enter in a different way, somewhat analogous to what Campbell called derived measurement. Typical physical examples of derived measurement are the usual definitions of density and momentum in which the measure in question is expressed in terms of two or more quantities that have been fundamentally measured already. His concept of derived measurement has to be stretched somewhat to encompass most of our work, and so I shall not press this point in what follows.

Probabilistic models. The main sources of numbers in most of mathematical psychology are the relative frequencies of responses and the times that it takes for responses to occur. Relative frequencies, which are much the more common measure, are interpreted as estimates of conditional response probabilities. This immediately leads one to consider probabilistic theories of behavior. Such theories are conceptually distinct, and in general quite different, from those postulated in statistics as models for hypothesis testing, analysis of variance, and the like.

Some debate, much of it unpublished, concerns the appropriateness of probability models in psychology. The current trend is viewed with alarm by some, who point out how ready such theories are to incorporate our confusion, ignorance, and experimental errors into what purports to be a description of the subject's behavior. But, as it is very difficult to characterize clearly the conditions under which it is appropriate to view an organism as a probability mechanism, many of us are uncertain just what to do about these criticisms. In one way or another we must cope with the fact that even under carefully controlled laboratory conditions individual subjects do not respond consistently to repeated presentations of exactly the same stimuli and outcomes. In general, the technique in classical physics of adding a dash of normal probability theory to an underlying deterministic theory has not proven very effective. Human and animal behavior appears to exhibit a more complex probabilistic structure than can be encompassed by deterministic theories flavored with a moderately uniform overlay of randomness, and so many of us have been driven to study probability models. I use the word "driven" advisedly because I, at least, was originally unsympathetic to such models. A few examples of these models will serve to illustrate the kinds of mathematics employed.

Learning models. Consider, first, "simple learning." This term should be taken with a goodly dose of salt, for most learning experiments have little immediate bearing upon what we ordinarily call learning—acquisition of concepts, insights, and understanding. Typically, we place the subject in a repetitive choice situation in which his responses are differentially rewarded, and we study the slowly evolving change of performance as the reward schedule unfolds itself to him. If the change evolves too rapidly, then we revamp the experiment to slow it down to the point where we can observe the transient behavior develop. Our problem, then, is to explain the laws whereby the response probabilities change over trials as a function of the subject's past experience *in the experiment*.

A variety of theories have been proposed and are under active study. I shall mention three of them. The first, known as stimulus sampling theory, is due largely to W. K. Estes ([**10**]; for a survey see [**3**]). These models postulate a mechanism whereby the subject "samples" environmental cues which are each "conditioned" or "attached" to particular responses. A decision rule describes how a response is selected on the basis of the cues sampled. Then, depending upon which outcome occurs, the conditioning of cues to responses is changed

in a systematic fashion. Such mechanisms lead to Markov chains to describe the transitions of the response probabilities, and so all of that well-developed theory can be brought to bear upon it. Much attention has been paid to the detailed sequential properties of the responses, which are very complex indeed; it is amazing how well some of the models account for these subtleties in the data.

A second class of learning models begins directly with assumptions about the trial-to-trial changes in the response probabilities. For example, Bush and Mosteller [5] studied stochastic processes generated by path-independent linear operators, and numerous other authors in later publications have extended our knowledge of these processes considerably (see [27]). Let $p_n(r)$ denote the probability of response r on trial n; then they supposed that

$$p_{n+1}(r) = (1 - \theta)p_n(r) + \theta\lambda,$$

where θ and λ are constants that depend upon the particular response and outcome that occurred on trial n. The linearity of the model is obvious; by path-independence is meant the assumption that the relevant past history of the process is completely summarized by the response probability $p_n(r)$ and the events that occurred on trial n. Such nonstationary stochastic processes are relatively complicated, and only a few simple cases are well understood. For example, although the asymptotic mean is known to exist [14], we do not have an explicit expression for it in the most general two response situation.

Another class of path-independent response models are the commutative ones that I have looked into [16], [19]. These learning operators are most simply stated in the form

$$f(p_{n+1}) = \beta f(p_n),$$

where f is a real-valued, strictly monotonic transformation of the unit interval and β is a positive constant that depends upon the response and outcome on trial n. By the form of the operator, this class of models is clearly both path-independent and commutative. Some fairly general results about these operators have been deduced using functional equation techniques, some of which are treated in Aczél's recent book on functional equations [2].

Once a stochastic learning model is formulated, the main task is to determine explicit (computable) expressions for various of its properties, which can then be used to estimate parameters from data and to evaluate the adequacy of the model. Typical properties are asymptotic means and variances, expected number of choices of a particular response, expected number of runs of responses of a given length, etc. In the linear model, it is common to set up the obvious expression for the desired random variable and then to calculate its expectation over the branching process described by the model. When this works, it does so mainly because the operators are linear; it rarely, if ever, works with the nonlinear models. For the commutative models it is possible to formulate and solve functional equations for some properties of interest (see

[13] and [29]). Certain asymptotic results have been obtained by using a technique based upon chains of infinite order [15].

Although parameter estimation and the testing of goodness-of-fit are problems whenever a theory is evaluated by data, these issues have grown especially significant and troublesome for the stochastic processes used in learning. Complicated mathematical problems often develop when we try to work out optimal estimators, and yet simpler procedures have been shown to lead to confusion and misinterpretation of data [27]. This last point blends into the question of measuring the goodness-of-fit of a model to a set of data, which question is not at all well formulated at present. Mostly ad hoc and intuitive evaluations are made and, while everyone agrees that they are not really satisfactory, little in the statistical literature seems helpful. Much interesting work could be done by someone with a strong statistical bent and a taste for unformulated, complex problems.

Preference models. The experimental study of preference is curious for this reason: the outcomes and stimuli of the experiment are either the same or very closely related. On each trial the subject is presented with a set of potential outcomes among which he is to choose. He receives either the one chosen or, in the case of what are known as "risky alternatives," a chance mechanism intervenes to determine which outcome is finally delivered.

Statisticians and economists have studied a variety of algebraic models for preference behavior, among them von Neumann and Morgenstern's expected utility model and Savage's generalization of it which were mentioned earlier. Psychologists who have become interested in preference problems have turned mostly to probabilistic models, primarily because the experimental data exhibit patterns of inconsistencies which are difficult to cope with in a deterministic framework. Broadly speaking, two approaches have been taken. One is to state, in terms of observable choice probabilities, properties that might reasonably be expected to be observed. These do not constitute a complete theory of behavior; rather, they are testable propositions that might become theorems in a more complete theory of behavior. A simple example is strong stochastic transitivity, which is one of several possible generalizations of ordinary transitivity. It asserts that if the probability of choosing a over b, $p(a, b)$, $\geqq 1/2$ and if $p(b, c) \geqq 1/2$, then $p(a, c) \geqq \max [p(a, b), p(b, c)]$. The other approach is to formulate more complete theories of behavior. One familiar example is the class of random utility models in which it is supposed that a person's preferences are determined at any instant of time by a numerical utility function which varies from instant to instant in some random fashion. The outcome having the highest momentary utility is the one that is momentarily preferred. Some of us are trying to deduce testable properties from the different models, to establish the network of relations among the models and properties and to uncover equivalent restatements of models. At the same time, experimentalists are attempting to discover which of the conditions seem to be satisfied by human beings; the

results to date, however, have tended to be ambiguous. I suspect that our experimental studies of preference are not yet adequate to answer the questions we want to ask.

Psychophysical models. The oldest branch of psychology and the one that, over the long haul, has used mathematics more consistently than any other is psychophysics, the study of the way in which responses depend upon continuous physical attributes of the stimulation. These experiments always involve the presentation of one of several possible stimuli on each trial, to which the subject responds from a set of responses that is systematically coordinated with the physical properties of the stimuli. Perhaps the simplest example is a detection experiment in which a faint stimulus or no stimulus is presented on a trial and the subject reports, in effect, "yes" or "no" depending upon whether or not he believes the stimulus to have been presented.

As an example of the problems facing the theorist, consider the plot of the conditional probability of responding "yes" when the stimulus is presented versus the conditional probability of responding "yes" when it is absent. If we vary either the outcomes to the subject or the presentation probability of the stimulus, we find that the resulting data points appear to arise from a continuous underlying curve that goes from (0, 0) to (1, 1) in the unit square and lies above the chance line $y=x$. The various theories proposed for the detection process do indeed predict the existence of such a curve, and different theories predict different ones. For example, signal detectability theory (for summaries see [**11**] and [**18**]) says that the curve becomes a straight line when we plot the probabilities on normal-normal paper, whereas a "low" threshold theory [**17**], [**18**], which postulates a discrete underlying process, says that it consists of two straight line segments in the ordinary plot. Although these predictions are conceptually quite different, considerable data are needed to select between them.

Quite a variety of models now exist for different psychophysical procedures and attempts are being made to systematize and unify them. For a period, information theory seemed to provide a unifying approach, but recent studies suggest that the unity was more apparent than real.

One nice feature of much of mathematical psychophysics, in contrast to other areas, is that it uses comparatively elementary mathematics and yet provides significant psychological insights that probably cannot be achieved otherwise. Therefore, it seems particularly well suited for inclusion in undergraduate courses on mathematical psychology and as a source of simple to intermediate level problems for undergraduate mathematics courses aimed at behavioral scientists.

Latency models. As I pointed out earlier, the elapsed time for a response to occur—the response latency—is a second obvious supply of numbers for the psychologist. In the simplest case, one presents a stimulus and requires the subject to respond to it as rapidly as possible. If the experiment is very carefully

done and if the subject is in good physical and emotional condition, then highly regular and reproducible distributions of response latencies are found. The provisos are important: latency is a much more skitterish measure than response frequency, and correspondingly greater care is needed in the collection of latency data.

The theoretical problem is to devise hypothetical mechanisms to account for the forms of the observed distributions, thereby permitting us to infer something about the nature of the underlying processing of sensory information. The main idea so far is that the resulting latency is built up from a chain of elementary random events—often exponential ones (see [21]). Theories of this type tend to parallel and sometimes use results from the theory of queues. All in all, the main mathematical device is the Laplace transform in one guise or another. I suspect, however, that new ideas will soon enter, because it has recently been observed that when subjects are paid differentially depending upon the latency of the response, the distributions have highly peaked modes and high tails—of the form $t^{-\delta}$ rather than $e^{-\lambda t}$. It is not clear that such distributions can arise from additive chains of independent random variables. Any ideas that arise to account for these data can be expected to have an impact upon our conceptualizations of the choice processes involved in learning, preference, and psychophysics.

Psychometrics. During the 1930's and 40's, the main domain of the mathematically inclined psychologist was psychometrics, which is somewhat like both psychophysics and statistics. Many of the psychometric models are formally the same as those used in psychophysics [32]; others, especially those used in the theory of tests, are basically multivariate statistical models [30], [31]. Throughout psychometrics, one collects relatively little data from each of a relatively large number of subjects. Advantage is taken of the internal consistencies and correlations exhibited within the population of subjects, and the analysis proceeds upon the assumption that a common structure underlies all of the subjects, who differ from one another only in the values of certain parameters. This approach is to be contrasted, for example, with that of much of psychophysics in which relatively large amounts of data are collected from relatively few subjects. There, each subject is studied individually in detail. Most of the psychometric techniques postulate that the subjects can be represented as points in a Euclidean vector space and, of course, one of the main mathematical techniques is matrix algebra. Factor analysis and other multi-dimensional scaling procedures are most readily exposited in the language of matrices, and some moderately deep results of matrix theory have been used.

Nonnumerical models. In the final class of models that I shall mention, numbers play no role. This is not an especially coherent or extensive class of models, and I shall mention only two examples.

In psycholinguistics—which studies the interplay between the speaker, the hearer, and their language—Chomsky and Miller ([8]; see also [7] and [22])

among others, have employed techniques of abstract algebra to describe aspects of the basic grammatical structure of languages. Specifically, they have treated grammars as concatenation algebras and have used results from the theory of recursive functions. The mathematics gets quite involved and a number of interesting new results have been proven. A strong interplay exists between theoretical psycholinguists and those working on the theory of automata. So far, few predictions that are amenable to experimentation have been derived from these pretty mathematical structures, and so relatively little experimentation has yet resulted. But work in this area is active and rapid developments should take place in the next few years.

A different sort of structural model has received attention in the area known as sociometrics, which is, loosely, the study of the structure of qualitative relationships among people. The main tool that Harary [**12**] and others have used in constructing this theory is the topological theory of graphs (for a summary see [**25**]). The results tend to be mathematically sophisticated, but the theory is relatively ineffectual in establishing connections between the mathematics and the empirical world. The main problem, I believe, is the failure to introduce any behavioral assumptions into the model. The structural assumptions do not characterize in any way the entities, the human beings, represented by the points of the graph, and so little in the way of prediction is possible.

Concluding remarks. Up to this point, I have treated the applications of mathematics to psychology from the psychologist's vantage point. Now let me turn about and consider briefly the extent to which different categories of mathematics have been used.

1. FUNDAMENTALS, including set theory, functions, relations, orderings, axiomatics, Boolean algebra, and the like. Fundamental mathematical notions and results are used throughout mathematical psychology; without a moderate grounding in them one cannot read much of the literature. Many problems are formulated initially in terms of sets, relations, and nonnumerical functions. Axiomatic systems are not uncommon and precise, if sometimes relatively elementary, reasoning is usual. Such knowledge is mandatory, and it should be absorbed as early as possible.

2. ANALYSIS. Analysis presents a curious problem because psychological theories are rarely formulated in its terms. A differential or integral equation is hardly ever a starting point, although difference equations sometimes are. Nevertheless, a student must know at least the calculus or suffer an impossible disadvantage, because no one gives a second thought to using derivatives or integrals in the solution of problems cast in other terms. Moreover, one cannot penetrate deeply into probability theory, stochastic processes, or advanced statistics without a firm grip on classical analysis, especially the theory of real variables. As I mentioned earlier, transform theory has proved important in the study of latencies and, along with a little complex variable theory, it is im-

portant in the analysis of tracking behavior. But this topic is sufficiently special that few students need study the theory of functions of a complex variable at present.

It is difficult to know what analysis a student should be advised to take. Although it is handy to have quite a bit, there are as yet few penetrating uses of it except indirectly via probability theory. Nonetheless, a student really should know enough to be able to read some quite sophisticated papers on functional equations, which are beginning to be used in many places, and on stochastic processes. Moreover, to the extent that we begin to formulate and study continuous time processes, classical analysis will prove essential. I mention this because some research of this sort has recently begun to appear.

3. PROBABILITY THEORY. Elementary probability concepts occur extensively in psychology; they are employed with neither comment nor apology. Some of the major theorems—e.g., the central limit theorem—are used frequently. Any well trained mathematical psychologist should have under his belt a solid course on probability and mathematical statistics. For many students, if not most, some additional work on stochastic processes is mandatory. In particular, they should be well grounded in the theory of Markov chains, especially for a finite number of states. These chains arise naturally in some theories of the learning process. In addition, they should be exposed to a number of the nonstationary stochastic processes that are also used. In general, these processes are not yet part of the standard textbook fare and, in many cases, they are not yet completely understood. Psychology offers numerous difficult, well-formulated, but unsolved problems in the theory of stochastic processes. Some interesting functional equations have arisen in the study of stochastic processes for learning; perhaps they are destined to serve in psychology the role of differential equations in physics.

4. ALGEBRA. Elementary algebra is used everywhere and complete facility is presupposed. Matrices are applied in psychometrics and multivariate analysis. Concatenation algebras are used in psycholinguistics. But the main body of abstract algebra—including groups, rings, fields, lattice theory, and more exotic notions, but excluding Boolean algebra—has been little used. Here and there a group or a lattice crops up, but by and large abstract algebra has not proved particularly important except for the sophistication engendered by its study.

5. TOPOLOGY AND GEOMETRY. Neither is much used. The only really serious attempt that I know of to employ topological notions is the application of graph theory to sociometrics. A bit of point set topology occurs now and then and occasionally fixed point theorems have been employed to establish the existence of something or other. Aside from trivial uses of high school geometry and a little work with non-Euclidean geometry in vision, geometry has not played much of a role in mathematical psychology.

In closing, it should be pointed out that no fundamentally new mathematical ideas have yet arisen from work on psychological problems. To date, we are parasites on mathematics. I doubt that this will always be so, but the time span is likely to be such that our contributions to mathematics will have little or no impact on anyone doing research in mathematics today. The reason that I believe an influence is bound to occur ultimately is that psychological problems exist that, on the one hand, seem to be meaningful, if not yet precisely stated, and about which we can experiment, if not yet incisively, and on the other hand, that do not seem to fit comfortably into the existing mathematical language. In some cases this may simply reflect a lack of ingenuity, but in others I suspect that the appropriate mathematics does not yet exist. The history of physics favors such a belief, which of course has the dubious virtue of being capable of neither proof nor disproof. Perhaps a few words about where I see trouble will make it a little more meaningful.

The language of sets does not always seem adequate to formulate psychological problems. Put so baldly, the statement is almost heretical since, in practice, set theory is the accepted way to formulate mathematical problems . . . and, hence, applied mathematical problems. Still, we should not forget that set theory is really quite new—less than a century old. It could be an interim theory. Certainly when I think about certain psychological problems, I wish it weren't the way it is. The boundaries of many of my "sets," and of ones that my subjects ordinarily deal with, are a good deal fuzzier than those of mathematics. Consider an experiment in which the subject is presented with a set of possible responses. This is a nice, unambiguous set of the sort that we are all conditioned to expect. But in a theoretical analysis of the subject's behavior, it often seems far more reasonable to consider not this set, but the one he considered before making his choice. It is quite difficult to pin down just what elements are and are not members of that set, and I am not sure that it is possible in principle. Do we merely lack techniques adequate to answer that question today, or is it basically impossible to answer it? Even supposing that it is impossible, I do not believe that this means that attempts to understand behavior are ridiculous, although ultimately it may be deemed inappropriate to try to cast theories of behavior in current mathematical language.

To take another example, we all deal effectively with the uncertainties of everyday life in terms of extremely imprecise concepts such as "likely," "fairly likely," and so on. As theorists, we often try to cope with this sort of behavior by phrasing it in the language of probability, but I suspect that most of us do not really feel that the mathematics meshes especially well with the problem. The categories of uncertainty are not really well-defined sets and their fuzziness is not particularly well summarized by probability notions. Perhaps we can make the existing concepts work, but I doubt that we should count on it.

Even assuming that there are profound troubles, mathematical psychologists are not about to call a moratorium until the troubles are resolved. Some, indeed most of us, will skirt around those aspects of behavior for which the difficulties

are especially pronounced. Others will try to tackle them more directly and to the extent of their success, they will enrich mathematics as well as psychology.

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References

1. J. Aczél, On mean values, Bull. Amer. Math. Soc., **54** (1948) 392–400.

2. ———, Vorlesungen über Funktionalgleichungen und ihre Anwendungen, Birkhauser, Basel and Stuttgart, 1961.

3. R. C. Atkinson and W. K. Estes, Stimulus sampling theory. In R. D. Luce, R. R. Bush, and E. Galanter (Eds.) *Handbook of mathematical psychology*, vol. II, Wiley, New York, 1963, 121–268.

4. R. R. Bush, E. Galanter, and R. D. Luce, Characterization and classification of choice experiments, *Handbook of mathematical psychology*, vol. I, 77–102.

5. R. R. Bush and F. Mosteller, Stochastic models for learning, Wiley, New York, 1955.

6. N. R. Campbell, Physics; the elements, Cambridge University Press, New York, 1920, reprinted as *Foundations of science: the philosophy of theory and experiment*, Dover, New York, 1957.

7. N. Chomsky, Formal properties of grammars, *Handbook of mathematical psychology*, vol. II, 323–418.

8. N. Chomsky and G. A. Miller, Introduction to the formal analysis of natural languages, *Handbook of mathematical psychology*, vol. II, 269–321.

9. D. Davidson, P. Suppes, and S. Siegel, Decision making, Stanford University Press, 1957.

10. W. K. Estes, Toward a statistical theory of learning, Psychol. Rev., **57** (1950) 94–107. Reprinted in R. D. Luce, R. R. Bush, and E. Galanter (Eds.) *Readings in mathematical psychology*, vol. I, Wiley, New York, 1963, 308–321.

11. D. M. Green, Psychoacoustics and detection theory, J. Acoust. Soc. Amer., **32** (1960) 1189–1203. Reprinted in *Readings in mathematical psychology*, 41–66.

12. F. Harary, Graph theoretic methods in the management sciences, Management Science, **5** (1959) 387–403.

13. L. Kanal, A functional equation analysis of two learning models, Psychometrika, **27** (1962) 89–104. Reprinted in *Readings in mathematical psychology*, 360–375.

14. S. Karlin, Some random walks arising in learning models I, Pac. J. Math., **3** (1953) 725–756. Reprinted in *Readings in mathematical psychology*, 381–403.

15. J. Lamperti and P. Suppes, Chains of infinite order and their applications to learning theory, Pacific J. Math., **9** (1959) 739–754.

16. R. D. Luce, Individual choice behavior, Wiley, New York, 1959.

17. ———, A threshold theory for simple detection experiments, Psychol. Rev., **70** (1963) 61–79.

18. ———, Detection and recognition, *Handbook of mathematical psychology*, vol. I, 103–189.

19. ———, Some one-parameter commutative learning operators. In R. C. Atkinson (Ed.) *Studies in mathematical psychology*, Stanford University Press, 1963.

20. R. D. Luce and J. Tukey, Simultaneous conjoint measurement: a new type of fundamental measurement, J. Math. Psych., **1** (1964) in press.

21. W. J. McGill, Stochastic latency mechanisms, *Handbook of mathematical psychology*, vol. I, 309–360.

22. G. A. Miller and N. Chomsky, Finitary models of language users, *Handbook of mathematical psychology*, vol. II, 419–492.

23. J. Pfanzagl, A general theory of measurement: applications to utility, Naval Res. Logistics Quarterly, **6** (1959) 283–294.

24. F. P. Ramsey, The foundations of mathematics and other logical essays, Harcourt, Brace and Company, New York, 1931.

25. A. Rapoport, Mathematical models of social interaction, *Handbook of mathematical psychology*, vol. II, 493–579.

26. L. J. Savage, The foundations of statistics, Wiley, New York, 1954.

27. S. Sternberg, Stochastic learning theory, *Handbook of mathematical psychology*, vol. II, 1–120.

28. P. Suppes and J. L. Zinnes, Basic measurement theory, *Handbook of mathematical psychology*, vol. I, 1–76.

29. M. Tatsuoka and F. Mosteller, A commuting-operator model. In R. R. Bush and W. K. Estes (Eds.) *Studies in mathematical learning theory*, Stanford University Press, 1959, 227–247.

30. L. L. Thurstone, Multiple factor analysis, University of Chicago Press, 1947.

31. ———, The measurement of values, University of Chicago Press, 1959.

32. W. S. Torgerson, Theory and methods of scaling, Wiley, New York, 1958.

33. J. von Neumann and O. Morgenstern, Theory of games and economic behavior, Princeton University Press, 1944 (1st Ed.) 1947 (2nd Ed.).

THE SOLUTION OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATION NEAR A REGULAR SINGULAR POINT

JOHN W. DETTMAN, Case Institute of Technology

The standard undergraduate course in ordinary differential equations suffers from many ills. In many cases it seems as an anticlimax after an upgraded calculus course. There is usually too much emphasis on specific integration techniques and not enough on existence and uniqueness theorems. It has not kept up with the computer age and therefore does not sufficiently emphasize numerical integration techniques. There is not enough attention paid to qualitative analysis, i.e., the determination of properties of the solution such as bounds, asymptotic expansions, stability, etc., without explicitly solving the equation. It is not necessary to postpone such a course in differential equations until after an advanced calculus course to cure these ills. One possible key is the early introduction of the Green's function and the subsequent reduction of the differential equation to an integral equation. Once this is done the existence and uniqueness theorems usually follow by a simple iteration procedure which uses nothing more sophisticated than the term-by-term integration of a uniformly convergent series. The iteration procedure is very basic, conceptually simple, and sets the tone for a large class of numerical integration techniques. Also, when the problem is cast in this way, many of the properties of the solution can be investigated without explicitly solving the equation.

The purpose of this paper is to illustrate these remarks in connection with the solution of a second order ordinary differential equation near a regular singular point, a case which is very important in applications but one which is commonly mishandled. The present approach is not new, but it does not appear

in undergraduate textbooks. The usual custom is to use series techniques, but to do justice to the problem in this form one should understand the theory of analytic functions, a subject which is not usually taken until the junior or senior year. Our procedure will be to introduce a Green's function which will make it possible to reduce the problem to the solution of a Volterra integral equation, which in all but one case is nonsingular. The iteration technique for solving the integral equation, it will turn out, works for the singular case as well as the nonsingular cases.

One of the most useful tools in the study of initial as well as boundary value problems in both ordinary and partial differential equations is the Green's function. This author believes that it should be introduced at the beginning of the ordinary differential equations course and developed throughout. Consider, for example, the equation $y''(t) = -f(t)$. A first integration yields $y'(t) = y'(0) - \int_0^t f(\tau)d\tau = y'(0) - F(t)$, where $F(t) = \int_0^t f(\tau)d\tau$. A second integration yields

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \int_0^t F(x)dx \\ &= y(0) + ty'(0) - [(x-t)F(x)]_0^t + \int_0^t (x-t)F'(x)dx \\ &= y(0) + ty'(0) + \int_0^t G(x,t)f(x)dx, \end{aligned}$$

where the Green's function $G(x, t) = x - t$. Hence, we have an explicit representation of the solution. The same result could have been found, of course, by "variation of parameters." On the other hand, if we have the homogeneous equation $y''(t) = -y(t)$, then the above procedure yields the integral equation

$$y(t) = y(0) + ty'(0) + \int_0^t G(x,t)y(x)dx.$$

If we multiply $y''(x) + y(x) = 0$ by an unknown function $H(x, t)$ and integrate, we have

$$\begin{aligned} 0 &= \int_0^t [y''(x) + y(x)]H(x,t)dx \\ &= [y'(x)H(x,t)]_0^t - \int_0^t y'(x)H'(x,t)dx + \int_0^t H(x,t)y(x)dx. \end{aligned}$$

To obtain a similar integral equation, $H(x, t)$ should have the following properties: $H'(x, t) \equiv 1$, $H(t, t) = 0$, and $H(0, t) = -t$. It is easy to show that the only function with these properties is $H(x, t) = x - t$.

To be more specific, let $y(0) = 1$ and $y'(0) = 0$. Then we have the integral equation $y(t) = 1 + \int_0^t (x-t)y(x)dx$. As a first approximation to the solution we take $y_0 = 1$. Then a reasonable second approximation is

$$y_1(t) = 1 + \int_0^t (x - t)dx = 1 - t^2/2.$$

A third approximation is

$$y_2(t) = 1 + \int_0^t (x - t)(1 - x^2/2)dx = 1 - t^2/2 + t^4/4!$$

We see that we are generating the terms of the well-known series for cos t, the unique solution of $y''+y=0$, subject to $y(0)=1$, $y'(0)=0$. This situation is relatively simple, but it should serve as a brief introduction to the main part of the paper.

Our main purpose is to consider the equation

$$x^2w'' + xP(x)w' + Q(x)w = 0, \tag{1}$$

where $P(x)$ and $Q(x)$ are real-valued functions of the real variable x, differentiable in the interval $0 \leqq x \leqq b$. For small values of x, the equation resembles

$$x^2u'' + P(0)xu' + Q(0)u = 0. \tag{2}$$

This is Euler's differential equation which has solutions x^{r_1} and x^{r_2}, where r_1 and r_2 are roots of the indicial equation

$$m(m - 1) + P(0)m + Q(0) = 0. \tag{3}$$

We assume that r_1 and r_2 are real and $r_1 \geqq r_2$. It is reasonable to assume that equation (1) has a solution close to x^{r_1} for small values of x. Therefore, we make the transformation

$$w(x) = x^{r_1}y(x). \tag{4}$$

Substituting into (1), we have

$$xy'' + [2r_1 + P(x)]y' + \frac{r_1(r_1 - 1) + r_1P(x) + Q(x)}{x}y = 0.$$

We let $p(x) = 2r_1 + P(x)$ and

$$q(x) = \frac{r_1(r_1 - 1) + r_1P(x) + Q(x)}{x}.$$

Now $p(x)$ is differentiable in $[0, b]$ and $q(x)$ is continuous in $[0, b]$ since

$$\begin{aligned}\lim_{x\to 0+} q(x) &= \lim_{x\to 0+} \frac{r_1(r_1 - 1) + r_1P(x) + Q(x) - [r_1(r_1 - 1) + r_1P(0) + Q(0)]}{x} \\ &= \lim_{x\to 0+} \frac{r_1[P(x) - P(0)]}{x} + \lim_{x\to 0+} \frac{Q(x) - Q(0)}{x} \\ &= r_1P'(0+) + Q'(0+).\end{aligned}$$

Thus we are led to the study of the equation

$$xy'' + p(x)y' + q(x)y = 0, \tag{5}$$

where p and q are continuous in $[0, b]$ and p is differentiable in $[0, b]$. Also since $1-P(0)=r_1+r_2$, $p_0=p(0)=2r_1+P(0)=1+(r_1-r_2)\geqq 1$. We can show that if (5) has a solution $y(x)$ in $(0, b)$ then (1) has the solution $w_1(x)=x^{r_1}y(x)$. To find the second solution of (1), we can then make the usual substitution

$$w = v(x)w_1(x)$$

to obtain

$$w_2 = w_1\int\left[w_1^{-2}\exp\left(-\int(P(x)/x)dx\right)\right]dx.$$

To obtain the integral equation associated with (5) we must develop a Green's function $G(x, t)$. Assume that (5) has a solution $y(x)$. Then

$$\int_0^t (xy'' + py' + qy)G(x, t)dx = 0,$$

$$xy'G(x, t)\Big|_0^t - \int_0^t [(xG)' - pG]y'dx + \int_0^t qyGdx = 0.$$

If $\lim_{x\to 0+} xG(x, t)=0$, $G(t, t)=0$, and $(xG)'-pG=1$, then

$$y(t) = y(0) + \int_0^t q(x)G(x, t)y(x)dx. \tag{6}$$

To show the existence of the Green's function, we solve explicitly. We write

$$(xG)' - \frac{p}{x}(xG) = 1. \tag{7}$$

The integrating factor is

$$\mu(x) = \exp\left(-\int p(x)dx\right) = \exp\left(-\int_1^x (p_0/\xi)d\xi\right)\exp\left(-\int_0^x (f(\xi)/\xi)d\xi\right)$$
$$= \frac{F(x)}{x^{p_0}},$$

where $f(x)/x=\{p(x)-p(0)\}/x$, which is continuous in $[0, b]$ since

$$\lim_{x\to 0+}\frac{f(x)}{x} = \lim_{x\to 0+}\frac{p(x)-p(0)}{x} = p'(0+).$$

Therefore, $F(x)=\exp(-\int_0^x(f(\xi)/\xi)d\xi)$ is positive and continuous in $[0, b]$. Using

the integrating factor we can solve (7) to get

$$G(x,\,t) = \frac{x^{p_0-1}}{F(x)}\left[\int_a^x \frac{F(\xi)}{\xi^{p_0}}\,d\xi + C\right].$$

To satisfy $G(t,\,t)=0$, we put $C=\int_t^a\{F(\xi)/\xi^{p_0}\}d\xi$, and then

$$G(x,\,t) = \frac{x^{p_0-1}}{F(x)}\int_t^x \frac{F(\xi)}{\xi^{p_0}}\,d\xi. \tag{8}$$

Assume $0<A\leqq F(x)\leqq B$ in $[0,\,b]$. Then for $0<x\leqq t\leqq b$

$$|G(x,\,t)| \leqq \frac{B}{A}x^{p_0-1}\int_x^t \frac{d\xi}{\xi^{p_0}} = \frac{B}{A(p_0-1)}\left[1-\left(\frac{x}{t}\right)^{p_0-1}\right]$$

for $p_0>1$. If $p_0=1$, then

$$|G(x,\,t)| \leqq \frac{B}{A}\int_x^t \frac{d\xi}{\xi} = \frac{B}{A}(\ln t - \ln x).$$

In any case, for every t such that $0<t\leqq b$ $\lim_{x\to 0+} xG(x,\,t)=0$.
We have also shown that for $p_0>1$, $G(x,\,t)$ is continuous in $R=\{(x,\,t)\,|\,0\leqq x\leqq t,\ 0\leqq t\leqq b\}$. Therefore, except when $p_0=1$, equation (6) is a nonsingular linear Volterra integral equation. Even when $p_0=1$, however, $G(x,\,t)$ is integrable in R and for this reason our existence and uniqueness theorem will apply in all cases.

Before proving the main theorem, we must show that a solution of the integral equation (6) satisfies the differential equation (5). To this end we need

$$G_t(x,\,t) = -\frac{x^{p_0-1}}{F(x)}\,\frac{F(t)}{t^{p_0}}, \qquad G_t(t,\,t) = -\frac{1}{t}, \qquad G_{tt}(x,\,t) = \frac{x^{p_0-1}}{F(x)}\,\frac{F(t)}{t^{p_0}}\,p(t).$$

Assume that $y(t)$ is a solution of (6) for $0<t<b$. Then

$$y'(t) = \int_0^t q(x)y(x)G_t(x,\,t)dx = -\frac{F(t)}{t^{p_0}}\int_0^t \frac{q(x)y(x)x^{p_0-1}}{F(x)}\,dx,$$

$$y''(t) = -\frac{q(t)y(t)}{t} + \int_0^t q(x)y(x)G_{tt}(x,\,t)dx = -\frac{q(t)y(t)}{t} - p(t)y'(t).$$

Hence, $ty''+p(t)y'(t)+q(t)y(t)=0$.

THEOREM. *Let $p(x)$ be differentiable and $q(x)$ be continuous in $[0,\,b]$. Let $p_0=p(0)\geqq 1$. Then there exists a unique solution of $xy''+p(x)y'+q(x)y=0$ in $(0,\,b)$ such that* $\lim_{x\to 0+} y(x)=1$. *Furthermore, there exists an M such that for all x in $[0,\,b]$ $|y(x)|\leqq I_0(2\sqrt{Mx})$ where $I_0(x)=J_0(ix)$ is a modified Bessel function of the first kind.*

Proof. By the above discussion, the differential equation will have a solution with the given initial value, if and only if the integral equation

$$y(t) = 1 + \int_0^t q(x)G(x, t)y(x)dx \tag{9}$$

has a solution. Let K be the linear operator defined by

$$Kf = \int_0^t q(x)G(x, t)f(x)dx. \tag{10}$$

Then the integral equation can be written $y=1+Ky$. We propose to solve this equation by iteration, i.e., let $\phi(x)$ be any function continuous in $[0, b]$ and let

$$y_0 = \phi(x), \quad y_1 = 1 + K\phi, \quad y_2 = 1 + Ky_1 = 1 + K(1 + K\phi) = 1 + K1 + K^2\phi,$$

$$y_n = 1 + K1 + K^2 1 + \cdots + K^{n-1}1 + K^n\phi.$$

Let $K^0 1=1$, and consider the series $\sum_{j=0}^{\infty} K^j 1$. We show that this series converges uniformly in $[0, b]$ to a solution of the integral equation (9). To this end we show that

$$|K^n 1| \leqq \frac{M^n t^n}{(n!)^2} \leqq \frac{M^n b^n}{(n!)^2} \quad \text{where} \quad M = \frac{B}{A} \max_{0 \leqq x \leqq b} |q(x)|. \tag{11}$$

This inequality is obviously true for $n=0$. Now assume it is true for $n-1$. Then

$$|K^n 1| = |KK^{n-1}1| \leqq \int_0^t |q(x)|\,|G(x, t)|\,|K^{n-1}1|\,dx$$

$$|K^n 1| \leqq \begin{cases} \dfrac{M^n}{[(n-1)!]^2}\displaystyle\int_0^t (\ln t - \ln x)x^{n-1}dx, & p_0 = 1 \\ \dfrac{M^n}{[(n-1)!]^2(p_0-1)}\displaystyle\int_0^t \left[1 - \left(\frac{x}{t}\right)^{p_0-1}\right]x^{n-1}dx, & p_0 > 1. \end{cases}$$

A simple calculation shows that

$$\int_0^t (\ln t - \ln x)x^{n-1}dx = \frac{t^n}{n^2}, \qquad \int_0^t \left[1 - \left(\frac{x}{t}\right)^{p_0-1}\right]x^{n-1}dx = \frac{(p_0-1)t^n}{n(n+p_0-1)}.$$

Therefore, (11) holds for all n. Since $\sum_{n=0}^{\infty} (M^n b^n)/(n!)^2$ converges, $\sum_{j=0}^{\infty} K^j 1$ converges uniformly to a continuous function $y(t)$, which we show is a solution of (9) as follows:

$$1 + \int_0^t q(x)G(x, t)y(x)dx = 1 + \int_0^t q(x)G(x, t)\sum_{j=0}^{\infty} K^j 1 dx = 1 + \sum_{j=0}^{\infty} K^{j+1}1$$

$$= \sum_{j=0}^{\infty} K^j 1 = y(t).$$

The term-by-term integration of the series is justified by the uniform convergence of the series. There is a small point to be settled in the case $p_0=1$ where $G(x, t)$ is not continuous in R. However, the inequality $|G(x, t)| \leqq B/A(\ln t-\ln x)$ allows us to prove the necessary theorem without difficulty.

To show that $y_n(t)$ converges to $y(t)$, we merely observe that, no matter what continuous $\phi(x)$ we start with,

$$|K^n\phi| \leqq C|K^n 1| \leqq C\frac{M^n b^n}{(n!)^2} \to 0 \text{ as } n \to \infty,$$

where $C=\max_{0\leqq x\leqq b}|\phi(x)|$.

Proof of uniqueness is routine.

Finally, inequality (11) shows that

$$|y(t)| \leqq \sum_{j=0}^{\infty}|K^j 1| \leqq \sum_{j=0}^{\infty}\frac{M^j t^j}{(j!)^2} = I_0(2\sqrt{Mt}).$$

It is interesting to note that $I_0(2\sqrt{Mt})$ satisfies the differential equation $tI_0''+I_0'-MI_0=0$ and the initial condition $I_0(0)=1$. This problem corresponds to the integral equation

$$u(t) = 1 + M\int_0^t(\ln t - \ln x)u(x)dx.$$

On the other hand, the integral equation

$$v(t) = 1 + \frac{M}{p_0-1}\int_0^t\left[1-\left(\frac{x}{t}\right)^{p_0-1}\right]v(x)dx \tag{12}$$

corresponds to the associated problem, $tv''+p_0v'-Mv=0$, $v(0)=1$. If $v(t)$ is the unique solution of (12), then

$$v(t) = \sum_{n=0}^{\infty}\frac{M^n t^n\Gamma(p_0)}{n!\Gamma(n+p_0)}.$$

A more refined estimate would show that $|K^n 1| \leqq M^n t^n\Gamma(p_0)/n!\Gamma(n+p_0)$, so that we have the more precise estimate

$$|y(t)| \leqq \sum_{n=0}^{\infty}\frac{M^n t^n\Gamma(p_0)}{n!\Gamma(n+p_0)}.$$

This completes the proof of the theorem. Several related remarks seem appropriate.

Remark 1. If $p(x)$ and $q(x)$ are analytic at $x=0$, then it is a simple matter to show from the integral equation that our solution has a power series expansion valid in some neighborhood of the origin. The coefficients can be computed by repeated differentiation or by operating formally with power series. It is not to be expected that the power series representation will converge in $[0, b]$. For

example, in the case of the equation $xy''+y'+1/(1+x^2)y=0$, $b=\infty$, although the power series representation only converges for $|x|<1$.

Remark 2. The existence theorem for the nonhomogeneous equation $x^2w''+xP(x)w'+Q(x)=F(x)$ can be handled just as easily as the above, provided $F(x)$ is continuous in $[0, b]$ and $O(x^{r_1+1})$ as $x\to 0$. In this case, the equation reduces to $xy''+p(x)y'+q(x)y=f(x)$ where $f(x)$ is continuous in $[0, b]$ and the corresponding integral equation is

$$y(t) = y(0) - \int_0^t f(x)G(x, t)dx + \int_0^t q(x)G(x, t)y(x)dx$$

If $g(t)=y(0)-\int_0^t f(x)G(x, t)dt$, the solution is $y(t)=\sum_{j=0}^{\infty} K^j g$.

Remark 3. We introduced this integral equation point of view in order to avoid the introduction of complex variable methods. There is some advantage, however, to using the same approach in the complex plane. Suppose that the indicial equation has complex roots r_1 and r_2 where $\mathrm{Re}(r_1)\geqq \mathrm{Re}(r_2)$. We transform the equation $z^2w''+zP(z)w'+Q(z)w=0$ by the transformation $w(z)=z^{r_1}y(z)$. The equation becomes $zy''+p(z)y'+q(z)y=0$, and the corresponding integral equation is $y(\zeta)=y(0)+\int_0^\zeta q(z)G(z, \zeta)y(z)dz$, where the path of integration avoids singularities of $p(z)$ and $q(z)$. One advantage of this approach is that we can handle cases where $p(z)$ and $q(z)$ have a branch point at the origin. Also, even if $p(z)$ and $q(z)$ are analytic only for $|z|<R$, the existence theorem is not necessarily limited to $|z|<R$ as in the case of the power series method.

Remark 4. The solution we have obtained is an asymptotic expansion since

$$\left| y(t) - \sum_{j=0}^{n-1} K^j 1 \right| \leqq \sum_{j=n}^{\infty} | K^j 1 | \leqq \sum_{j=n}^{\infty} \frac{M^j t^j}{(j!)^2} \leqq \frac{Ct^n}{(n!)^2}.$$

It is observations of this sort which lead to many interesting recent results in the study of asymptotic solutions. See A. Erdélyi, "Asymptotic Solutions of Ordinary Linear Differential Equations," California Institute of Technology, 1961.

EXTENSION OF GROUPOIDS WITH OPERATORS

T. TAMURA AND D. G. BURNELL, University of California, Davis

It is familiar that a commutative integral domain can be embedded into a field (for example, see [2]). This may also be regarded as an extension of an additive group with operators. As a generalization, to some extent, we present in this note an extension of groupoids with operator semigroup. A groupoid is a system in which a binary operation is defined, and neither associativity nor commutativity is assumed. Our principal results were outlined in [3].

1. Main theorems. Let G be a groupoid and suppose that Γ is a commutative semigroup. Small letters $x, y, \cdots$ denote elements of G and $+$ denotes the operation in G; small Greek letters $\alpha, \beta, \cdots$ denote elements of Γ, and $\alpha\beta$ the product in Γ. A single-valued mapping of the product set $\Gamma \times G$ into G is defined, $(\alpha, x) \rightarrow \alpha x$, such that the following conditions are satisfied:

$$\alpha(x + y) = \alpha x + \alpha y \tag{1.1}$$

$$(\alpha\beta)x = \alpha(\beta x) = (\beta\alpha)x \tag{1.2}$$

$$\alpha x = \alpha y \text{ implies } x = y, \tag{1.3}$$

for every $\alpha, \beta \in \Gamma$; $x, y \in G$. If α is fixed and x runs throughout G, α is a mono-endomorphism, i.e., one-to-one endomorphism of G. We call such a pair G, Γ a groupoid G with Γ and denote it by (G, Γ).

THEOREM 1. *For (G, Γ), there exists $(\overline{G}, \overline{\Gamma})$ such that*

(2.1) *G is embedded into $\overline{G}$,*

(2.2) *Γ and $\overline{\Gamma}$ are isomorphic,*

(2.3) *each $\bar{\alpha} \in \overline{\Gamma}$ is an extension of $\alpha \in \Gamma$ to $\overline{G}$, and $\bar{\alpha}$ is an automorphism of $\overline{G}$, and*

(2.4) *$(\overline{G}, \overline{\Gamma})$ is the smallest extension of (G, Γ) in the following sense: If $(\overline{\overline{G}}, \overline{\overline{\Gamma}})$ is any extension of (G, Γ) satisfying (2.1), (2.2) and (2.3), then $\overline{G}$ can be embedded into $\overline{\overline{G}}$. Furthermore, if G is associative, so is $\overline{G}$; if G is commutative, $\overline{G}$ is also; if G is cancellative, $\overline{G}$ is also so.*

Proof. Consider the product set $G \times \Gamma = \{(x, \alpha); x \in G, \alpha \in \Gamma\}$. Define a relation $\sim$ by

$$(x, \alpha) \sim (y, \beta) \text{ iff } \beta x = \alpha y. \tag{3}$$

Clearly this is reflexive and symmetric. To show transitivity, suppose $(x, \alpha) \sim (y, \beta)$ and $(y, \beta) \sim (z, \gamma)$. Then, by (1.2),

$$\beta\gamma x = \gamma\beta x = \gamma\alpha y = \alpha\gamma y = \alpha\beta z = \beta\alpha z$$

and by (1.3) we have $\gamma x = \alpha z$, i.e., $(x, \alpha) \sim (z, \gamma)$.

Letting $\overline{(x, \alpha)}$ denote the equivalence class containing (x, α), we define $\overline{G}$ to be the set of all such equivalence classes. Now, define an operation $+$ in $\overline{G}$ as follows:

$$\overline{(x, \alpha)} + \overline{(y, \beta)} = \overline{(\beta x + \alpha y, \alpha\beta)}. \tag{4}$$

The operation is single-valued, since if $\overline{(x, \alpha)} = \overline{(x', \alpha')}$ and $\overline{(y, \beta)} = \overline{(y', \beta')}$, we have $\alpha' x = \alpha x'$, $\beta' y = \beta y'$ and $\alpha'\beta'(\beta x + \alpha y) = \alpha'\beta'\beta x + \alpha'\beta'\alpha y = \beta'\beta\alpha' x + \alpha'\alpha\beta' y = \beta'\beta\alpha x' + \alpha'\alpha\beta y' = \alpha\beta\beta' x' + \alpha\beta\alpha' y' = \alpha\beta(\beta' x' + \alpha' y')$, that is, $\overline{(x, \alpha)} + \overline{(y, \beta)} = \overline{(x', \alpha')} + \overline{(y', \beta')}$.

To prove (2.1) we define a mapping $\sum$ of G into $\overline{G}$ by

$$\sum x = \overline{(\alpha x, \alpha)}.$$

Since $\overline{(\alpha x, \alpha)} = \overline{(\beta x, \beta)}$, $\sum x$ is independent of α. $\sum$ is one-to-one, since, if we assume $\overline{(\alpha x, \alpha)} = \overline{(\beta y, \beta)}$, then we have $\alpha\beta x = \alpha\beta y$ and hence $x = y$ by (1.3).

Now

$$\sum(x + y) = \overline{(\alpha(x + y), \alpha)} = \overline{(\alpha(\alpha x) + \alpha(\alpha y), \alpha\alpha)} = \overline{(\alpha x, \alpha)} + \overline{(\alpha y, \alpha)}$$
$$= \sum x + \sum y.$$

Thus we see that $\sum$ is an isomorphism of G onto $G' = \{\overline{(\alpha x, \alpha)}; x \in G\} \subset \overline{G}$.

For each $\alpha \in \Gamma$ we define a mapping $\bar{\alpha}$ of $\overline{G}$ into itself as follows:

$$\bar{\alpha}\overline{(z, \gamma)} = \overline{(\alpha z, \gamma)}. \tag{5}$$

The set of all mappings $\bar{\alpha}$ is denoted by $\bar{\Gamma}$. The mapping $\bar{\alpha}$ is single-valued because if $\overline{(z, \gamma)} = \overline{(z', \gamma')}$, then $\gamma' z = \gamma z'$ and $\gamma' \alpha z = \gamma \alpha z'$ which implies that $\overline{(\alpha z, \gamma)} = \overline{(\alpha z', \gamma')}$. Also $\bar{\alpha}$ is one-to-one since if $\bar{\alpha}\overline{(y, \beta)} = \bar{\alpha}\overline{(z, \gamma)}$, then $\overline{(\alpha y, \beta)} = \overline{(\alpha z, \gamma)}$ and we have $\gamma\alpha y = \beta\alpha z$ which gives us $\gamma y = \beta z$ by (1.2) and (1.3). Now $\bar{\alpha}(\overline{(z, \gamma)} + \overline{(y, \beta)}) = \bar{\alpha}\overline{(\beta z + \gamma y, \gamma\beta)} = \overline{(\alpha(\beta z + \gamma y), \gamma\beta)} = \overline{(\alpha\beta z + \alpha\gamma y, \gamma\beta)} = \overline{(\alpha z, \gamma)} + \overline{(\alpha y, \beta)} = \bar{\alpha}\overline{(z, \gamma)} + \bar{\alpha}\overline{(y, \beta)}$. For any $\overline{(y, \beta)} \in \overline{G}$, we have $\bar{\alpha}\overline{(y, \alpha\beta)} = \overline{(\alpha y, \alpha\beta)} = \overline{(y, \beta)}$, and $\bar{\alpha}\overline{(\beta y, \beta)} = \overline{(\beta\alpha y, \beta)}$. Thus we see that each $\bar{\alpha}$ is an extension of $\alpha \in \Gamma$ and is an automorphism of $\overline{G}$. Now Γ is isomorphic onto $\bar{\Gamma}$ by the mapping $\alpha \to \bar{\alpha}$, since

$$\bar{\alpha} = \bar{\beta} \Rightarrow \bar{\alpha}\overline{(z, \gamma)} = \bar{\beta}\overline{(z, \gamma)} \Rightarrow \overline{(\alpha z, \gamma)} = \overline{(\beta z, \gamma)},$$

which means that $\gamma\alpha z = \gamma\beta z$ and, by (1.3), $\alpha z = \beta z$ for all $z \in G$. The property of preserving multiplication is a trivial consequence of (5) and (1.2).

Finally, to prove (2.4), let $(\overline{\overline{G}}, \overline{\overline{\Gamma}})$ satisfy (2.1), (2.2) and (2.3). Define a mapping $\tau: \overline{G} \to \overline{\overline{G}}$ by $\tau\overline{(x, \alpha)} = y$, where $\bar{\bar{\alpha}} y = \sum x$ and $\sum$ is the embedding map from G into $\overline{\overline{G}}$, and $\bar{\bar{\alpha}} \in \overline{\overline{\Gamma}}$ is the automorphism extension of $\alpha \in \Gamma$. To show τ is single-valued, we assume $\overline{(x', \alpha')} = \overline{(x, \alpha)}$ and $\tau\overline{(x', \alpha')} = y$ and $\tau\overline{(x, \alpha)} = y'$. Then $\bar{\bar{\alpha}}' y = \sum x'$, $\bar{\bar{\alpha}} y' = \sum x$ and we get

$$\bar{\bar{\alpha}}\bar{\bar{\alpha}}' y = \bar{\bar{\alpha}}(\sum x') = \sum(\alpha x') = \sum(\alpha' x) = \bar{\bar{\alpha}}' \sum x = \bar{\bar{\alpha}}'\bar{\bar{\alpha}} y' = \bar{\bar{\alpha}}\bar{\bar{\alpha}}' y'.$$

Therefore $y = y'$. Next, to show that τ is one-to-one, suppose $\tau\overline{(x, \alpha)} = y = \tau\overline{(z, \gamma)}$, $y \in \overline{\overline{G}}$. Then we have $\bar{\bar{\gamma}} \sum x = \bar{\bar{\alpha}} \sum z$. Since $\bar{\bar{\alpha}}$ and $\bar{\bar{\gamma}}$ are extensions of α and γ respectively, $\sum(\gamma x) = \sum(\alpha z)$ and, of course, $\gamma x = \alpha z$, i.e., $\overline{(x, \alpha)} = \overline{(z, \gamma)}$. Now, to show that τ is a homomorphism, suppose that $\tau\overline{(x, \alpha)} = y$, $\tau\overline{(z, \gamma)} = y'$ and $\tau(\overline{(x, \alpha)} + \overline{(z, \gamma)}) = y''$. Then $\bar{\bar{\alpha}}\bar{\bar{\gamma}} y'' = \sum(\gamma x + \alpha z)$ and

$$\bar{\bar{\alpha}}\bar{\bar{\gamma}} y + \bar{\bar{\alpha}}\bar{\bar{\gamma}} y' = \bar{\bar{\gamma}} \sum x + \bar{\bar{\alpha}} \sum z = \sum(\gamma x) + \sum(\alpha z) = \sum(\gamma x + \alpha z),$$

and hence $\bar{\bar{\alpha}}\bar{\bar{\gamma}} y'' = \bar{\bar{\alpha}}\bar{\bar{\gamma}}(y + y')$; we have $y'' = y + y'$.

We shall prove that if G is cancellative, then so is $\overline{G}$. Suppose that

$$\overline{(x, \alpha)} + \overline{(y, \beta)} = \overline{(x, \alpha)} + \overline{(z, \gamma)}.$$

Then $\overline{(\beta x+\alpha y, \alpha\beta)}=\overline{(\gamma x+\alpha z, \alpha\gamma)}$ which implies that $\alpha\gamma\beta x+\alpha\alpha\gamma y=\alpha\gamma\beta x+\alpha\alpha\beta z$ and we have $\alpha^2\gamma y=\alpha^2\beta z$ since G is cancellative; consequently, $\gamma y=\beta z$, or $\overline{(y, \beta)}=\overline{(z, \gamma)}$. Right cancellation is proved in a similar way. Associativity holds in $\overline{G}$ if it holds in G:

$$\begin{aligned}\overline{(x, \alpha)} + (\overline{(y, \beta)} + \overline{(z, \gamma)}) &= \overline{(x, \alpha)} + \overline{(\gamma y + \beta z, \beta\gamma)} = \overline{(\beta\gamma x + (\alpha\gamma y + \alpha\beta z), \alpha\beta\gamma)}\\ &= \overline{((\gamma\beta x + \gamma\alpha y) + \alpha\beta z, \alpha\beta\gamma)} = \overline{((\beta x + \alpha y), \alpha\beta)} + \overline{(z, \gamma)}\\ &= (\overline{(x, \alpha)} + \overline{(y, \beta)}) + \overline{(z, \gamma)}.\end{aligned}$$

One can easily see that commutativity of G implies commutativity of $\overline{G}$. Thus the theorem has been completely proved.

Since $\bar{\Gamma}$ in Theorem 1 is commutative and cancellative, it is possible to embed $\bar{\Gamma}$ into a group.

THEOREM 2. *For each* (G, Γ) *there exists* $(\overline{G}, \Gamma^*)$ *such that*

(3.1) *G is embedded into $\overline{G}$.*

(3.2) *Γ^* is the smallest commutative group into which Γ can be embedded.*

(3.3) *Each $\beta^*\in\Gamma^*$ is an automorphism of $\overline{G}$ and if $\alpha\in\Gamma$ is mapped to $\alpha^*\in\Gamma^*$ under the embedding of Γ into Γ^*, then each α^* is an extension of $\alpha\in\Gamma$ to $\overline{G}$.*

(3.4) *If $(\overline{\overline{G}}, \Gamma^{**})$ is any extension satisfying* (3.1), (3.2) *and* (3.3), *then $\overline{G}$ and Γ^* are embedded into $\overline{\overline{G}}$ and Γ^{**} respectively.*

Proof. By Theorem 1, we can obtain $(\overline{G}, \bar{\Gamma})$ as an extension of (G, Γ). For each pair $(\bar{\alpha}, \bar{\beta})$ of elements of $\bar{\Gamma}$, we define a mapping $((\bar{\alpha}, \bar{\beta}))$ by

$$((\bar{\alpha}, \bar{\beta}))\overline{(z, \gamma)} = \overline{(\alpha z, \beta\gamma)}.$$

The set of all $((\bar{\alpha}, \bar{\beta}))$ is denoted by Γ^*. Clearly $((\bar{\alpha}, \bar{\beta}))=((\bar{\alpha}', \bar{\beta}'))$ iff $\bar{\beta}'\bar{\alpha}=\bar{\beta}\bar{\alpha}'$. The mapping is single-valued. For if $\overline{(z, \gamma)}=\overline{(z', \gamma')}$, then $((\bar{\alpha}, \bar{\beta}))\overline{(z, \gamma)}=\overline{(\alpha z, \beta\gamma)}=\overline{(\alpha z', \beta\gamma')}=((\bar{\alpha}, \bar{\beta}))\overline{(z', \gamma')}$, since $\gamma' z=\gamma z'$. To show that $((\bar{\alpha}, \bar{\beta}))$ is an automorphism,

$$\begin{aligned}((\bar{\alpha}, \bar{\beta}))(\overline{(z, \gamma)} + \overline{(w, \delta)}) &= ((\bar{\alpha}, \bar{\beta}))\overline{(\delta z + \gamma w, \gamma\delta)} = \overline{(\alpha\delta z + \alpha\gamma w, \beta\gamma\delta)}\\ &= \overline{(\beta\delta\alpha z + \beta\gamma\alpha w, \beta\gamma\beta\delta)} = \overline{(\alpha z, \beta\gamma)} + \overline{(\alpha w, \beta\delta)}\\ &= ((\bar{\alpha}, \bar{\beta}))\overline{(z, \gamma)} + ((\bar{\alpha}, \bar{\beta}))\overline{(w, \delta)}.\end{aligned}$$

Suppose that $((\bar{\alpha}, \bar{\beta}))\overline{(z, \gamma)}=((\bar{\alpha}, \bar{\beta}))\overline{(w, \delta)}$. Then $\overline{(\alpha z, \beta\gamma)}=\overline{(\alpha w, \beta\delta)}$, and so $\beta\alpha\delta z=\beta\delta\alpha z=\beta\gamma\alpha w=\beta\alpha\gamma w$; therefore $\delta z=\gamma w$ or $\overline{(z, \gamma)}=\overline{(w, \delta)}$. Now as an inverse image for $\overline{(z, \gamma)}$, look at $\overline{(\beta z, \alpha\gamma)}$. Then $((\bar{\alpha}, \bar{\beta}))\overline{(\beta z, \alpha\gamma)}=\overline{(\alpha\beta z, \beta\gamma\alpha)}=\overline{(z, \gamma)}$. Thus it has been proved that the map $((\bar{\alpha}, \bar{\beta}))$ is an automorphism of $\overline{G}$. To embed $\bar{\Gamma}$ into Γ^* use the correspondence $\bar{\alpha}\rightarrow((\bar{\gamma}\bar{\alpha}, \bar{\gamma}))=\alpha^*$ where α^* is easily seen to be an extension of α to $\overline{G}$. As is well known, Γ^* is the smallest group containing Γ.

Instead of Γ, we may consider $\Gamma_0=\Gamma\cup\{0\}$, where 0 is a zero-mapping, i.e., a mapping of G onto some idempotent element of G; then 0 is a two-sided zero element of Γ_0, and Γ_0 has no proper zero-divisor. Under Γ_0, we get theorems similar to Theorems 1, 2. For this case the subsemigroup Γ of Γ_0, consisting of nonzero one-to-one endomorphisms, plays the same rôle as in previous theorems, and 0 can be extended to a zero-mapping of $\overline{G}$.

2. **Applications and examples.** (1) We give an example of (G, Γ) in which G is not a semigroup. Let G be the set of all positive integers and define two binary operations, i.e., addition $a+b$ and multiplication $a\cdot b$ as follows: Let $n>1$ be a fixed element of G and define

$$a + b = na, \qquad a\cdot b = ab,$$

where na and ab denote the usual multiplication. Clearly an operator ϕ_a defined by $\phi_a(x)=x\cdot a$ is a mono-endomorphism of the additive groupoid G, and $\Gamma=\{\phi_a: a\in G\}$. Addition, however, is not associative. The extension $(\overline{G}, \overline{\Gamma})$ is given as follows: $\overline{G}$ is the set of all rational numbers in which the addition and operators are defined in the same way.

(2) Suppose that S is a commutative semigroup. For every positive integer n, we consider an endomorphism $\mathfrak{n}$ of S:

$$\mathfrak{n}\cdot x = \underbrace{x + \cdots + x}_{n}.$$

The operator semigroup Γ of endomorphisms of this kind is isomorphic to a subsemigroup of the multiplicative semigroup of positive integers. We shall say that a commutative semigroup T is uniquely Γ-divisible if for any $x\in T$ and any $\mathfrak{n}\in\Gamma$, there is exactly one $y\in T$ such that $\mathfrak{n}\cdot y=x$; and T is said to be Γ-cancellative if Γ is one-to-one, that is, $nx=ny$ implies $x=y$ for every $\mathfrak{n}\in\Gamma$.

As an application of Theorem 1, we immediately have the following result. (Hancock obtained the same result in [1].)

Corollary 1. *If a commutative semigroup S is Γ-cancellative, then S is embedded into the smallest uniquely Γ-divisible semigroup where $\Gamma=\{\mathfrak{n}; n\in J\}$, J being a multiplicative semigroup of positive integers.*

(3) We define a semiring R to be an algebraic system with two binary operations—addition and multiplication—such that for every x, y, $z\in R$,

$$(x + y) + z = x + (y + z) \tag{4.1}$$

$$(xy)z = x(yz) \tag{4.2}$$

$$x(y + z) = xy + xz \quad \text{and} \quad (y + z)x = yx + zx. \tag{4.3}$$

Using the previous theorem, we obtain

Corollary 2. *If the multiplicative semigroup of a semiring R is commutative and if, for nonzero $a\in R$, $ab=ac$, implies $b=c$, then R can be embedded into the*

smallest semiring R^* such that the multiplicative semigroup of R^* is a commutative group or a commutative group with zero.

3. Extensions of commutative semigroup with operators. From now on, let S be a commutative semigroup and Γ be the same as in Corollary 1. Also suppose that S is cancellative and Γ-cancellative, that is,

$$x + y = x + z \quad \text{implies} \quad y = z$$
$$\mathfrak{n}x = \mathfrak{n}y \quad \text{implies} \quad x = y \qquad \text{for } \mathfrak{n} \in \Gamma.$$

Let $S^{\mathfrak{g}}$ denote the quotient group of S, i.e., the smallest group containing S, and $S^{\mathfrak{d}}$ denote the smallest uniquely Γ-divisible semigroup containing S. As is easily seen, $S^{\mathfrak{g}}$ is Γ-cancellative and $S^{\mathfrak{d}}$ is cancellative, and hence we can consider the uniquely Γ-divisible extension of $S^{\mathfrak{g}}$ and the group extension of $S^{\mathfrak{d}}$.

THEOREM 3. $(S^{\mathfrak{g}})^{\mathfrak{d}} \cong (S^{\mathfrak{d}})^{\mathfrak{g}}$.

Proof. We shall let x/y denote an element of $S^{\mathfrak{g}}$ where $x, y \in S$. Define a mapping $\sum$ of $(S^{\mathfrak{g}})^{\mathfrak{d}}$ onto $(S^{\mathfrak{d}})^{\mathfrak{g}}$ as follows:

$$\sum \overline{(x/y,\ n)} = \overline{(x,\ n)}/\overline{(y,\ n)},$$

where $\mathfrak{n} \in \Gamma$. Now,

$$\begin{aligned}\overline{(x/y,\ n)} &= \overline{(x'/y',\ n')} \rightleftarrows n'(x/y) = n(x'/y') \rightleftarrows n'x/n'y = nx'/ny' \\ &\rightleftarrows n'x + ny' = nx' + n'y \rightleftarrows n'n(n'x + ny') = nn'(nx' + n'y) \\ &\rightleftarrows \overline{(n'x + ny',\ nn')} = \overline{(nx' + n'y,\ n'n)} \rightleftarrows \overline{(x,\ n)} + \overline{(y',\ n')} \\ &= \overline{(x',\ n')} + \overline{(y,\ n)} \rightleftarrows \overline{(x,\ n)}/\overline{(y,\ n)} = \overline{(x',\ n')}/\overline{(y',\ n')}.\end{aligned}$$

Thus $\sum$ is single-valued and injective. Suppose $\overline{(x,\ m)}/\overline{(y,\ n)} \in (S^{\mathfrak{d}})^{\mathfrak{g}}$. Then $\sum \overline{(nx/my,\ nm)} = \overline{(x,\ m)}/\overline{(y,\ n)}$ and so $\sum$ is surjective. Also

$$\begin{aligned}\sum\{\overline{(x/y, m)} + \overline{(x'/y',\ m')}\} &= \sum \overline{((m'x + mx')/(m'y + my'),\ mm')} \\ &= \overline{(m'x + mx',\ mm')}/\overline{(m'y + my',\ mm')} = \overline{(x, m)}/\overline{(y, m)} + \overline{(x', m')}/\overline{(y', m')} \\ &= \sum \overline{(x/y, m)} + \sum \overline{(x'/y',\ m')}.\end{aligned}$$

Thus it has been proved that $\sum$ is an isomorphism.

Finally we consider the relationship between $\mathfrak{d}$ and direct product. Suppose that S_1 and S_2 satisfy the same condition as Theorem 3.

THEOREM 4. $(S_1 \times S_2)^{\mathfrak{d}} \cong S_1^{\mathfrak{d}} \times S_2^{\mathfrak{d}}$.

Proof. Any element of $S_1 \times S_2$ is denoted by (x_1, x_2), $x_1 \in S_1$, $x_2 \in S_2$. If a mapping $\sum$ of $(S_1 \times S_2)^{\mathfrak{d}}$ onto $S_1^{\mathfrak{d}} \times S_2^{\mathfrak{d}}$ is defined by

$$\sum \overline{((x_1, x_2), m)} = (\overline{(x_1,\ m)},\ \overline{(x_2,\ m)}),$$

we obtain the result in the same way as Theorem 3.

Incidentally, since $(S^{\mathfrak{d}})^{\mathfrak{d}} \cong S^{\mathfrak{d}} \cdot (S^{\mathfrak{g}})^{\mathfrak{g}} \cong S^{\mathfrak{g}}$, combining them with Theorem 3, we can express the product of the operations as follows:

$$\mathfrak{g}\mathfrak{d} = \mathfrak{d}\mathfrak{g}, \qquad \mathfrak{g}^2 = \mathfrak{g}, \qquad \mathfrak{d}^2 = \mathfrak{d}$$

which means that $\mathfrak{g}$ and $\mathfrak{d}$ generate a semilattice of order 3.

References

1. V. R. Hancock, Commutative Schreier extensions of semigroups, Dissertation, 1960.
2. N. Jacobson, Lectures in abstract algebra, vol. 1, Van Nostrand, Princeton, 1951.
3. T. Tamura and D. G. Burnell, A note on the extension of semigroups with operators, Proc. Japan Acad., 38 (1962) 495–498.

A MATRIX APPROACH TO NUMERICAL INTEGRATION

L. R. BRAGG, Case Institute of Technology

1. Introduction. Let $w(x)$ be a given real-valued continuous function on some basic interval, say $[-1, 1]$ and let $I(f) = \int_{-1}^{1} w(x)f(x)dx$, where f is an arbitrary real continuous function on this interval. For fixed n, let $\{w_j\}_{j=1}^{n}$ and $\{x_j\}_{j=1}^{n}$ denote a pair of number sequences with x_j distinct and, in general, restricted to $[-1, 1]$. We shall be concerned with numerical integration formulas of the form

$$J_n(f) = \sum_{j=1}^{n} w_j f(x_j) \tag{1}$$

that are, in some sense, related to $I(f)$. The w_j are referred to as weight factors. The selection of these sequences is, of course, dependent upon how $I(f)$ and $J_n(f)$ are to be related. Generally speaking, it is desirable to have these sequences independent of f. Even so, much freedom remains in selecting them. It may be that certain of the w_j and x_j are to be preselected according to some fixed criterion while those remaining are so chosen that $J_n(f)$ integrates exactly all polynomials up to a certain maximum degree.

We shall be concerned with precisely the problem of selecting such sequences. For this purpose, the algebraic method ([1], [3]) will be adopted. This will make use of (a) triangularizing a matrix and (b) the theorem that a system of linear algebraic equations formulated in matrix form has a solution if and only if the ranks of the coefficient and augmented matrices are the same. In section 2, this is used to obtain consistency conditions that must be satisfied by members of the sequence $\{x_i\}_{i=1}^{n}$ when certain of these x_i are to be preselected while no members of $\{w_j\}_{j=1}^{n}$ are to be prescribed. These relations are modified in section

3 to obtain similar conditions when certain terms of both sequences are to be preselected. Finally, these consistency conditions are used in section 4 for constructing examples of numerical integration formulas. As we shall see, a set of consistency conditions may be too stringent and form a nonsolvable system or else force certain members of $\{x_i\}_{i=1}^n$ to be complex. In these cases it will be necessary to lower the degree of precision relating $J_n(f)$ to $I(f)$ to permit the construction of a meaningful formula $J_n(f)$. Also, the consistency conditions may lead to a real sequence $\{x_i\}_{i=1}^n$ in which certain of the x_i fall outside of the interval $[-1, 1]$. This is permissible in those cases where the points x_i fall within the domains of definition of f and w. Formulas involving such points may be useful in the numerical solution of differential equations. If, however, f and w are only defined in $[-1, 1]$, then the degree of precision must be selected so that all x_i fall within $[-1, 1]$. Remainder terms in such formulas, if desired, may then be determined by mean value estimates ([2], [3]) or by use of the influence (kernel) function [3]. These will require, of course, additional differentiability assumptions concerning f.

2. Consistency relations. We now assume that s $(0 \leqq s \leqq n)$ of the points x_i are to be specified (with restrictions) and that the w_i have not been fixed. The condition that $I(f)$ be related to $J_n(f)$ is furnished by requiring that $I(x^k) = J_n(x^k)$, $k = 0, 1, \cdots, 2n-s-1$. In terms of the formula (1) and our definition of $I(f)$, this condition gives the set of equations

$$\sum_{j=1}^{n} w_j x_j^k = M_k, \qquad k = 0, 1, \cdots, 2n - s - 1, \tag{2.1}$$

where M_k is the kth moment defined by

$$M_k = \int_{-1}^{1} w(x) x^k dx. \tag{2.2}$$

The system (2.1) can be expressed more conveniently in the matrix form:

$$\begin{array}{c} \begin{array}{cccccc} w_1 & w_2 & w_3 & \cdots & w_n & 1 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & \cdots & 1 & M_0 \\ x_1 & x_2 & x_3 & \cdots & x_n & M_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_1^j & x_2^j & x_3^j & \cdots & x_n^j & M_j \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_1^{2n-s-1} & x_2^{2n-s-1} & x_3^{2n-s-1} & \cdots & x_n^{2n-s-1} & M_{2n-s-1} \end{array}\right] \end{array} \tag{2.3}$$

With this formulation, the weight factors w_j may be regarded as the unknowns

with the coefficient matrix lying to the left of the broken line. Upon triangularizing this coefficient (Vandermonde) matrix, we obtain the form

$$
(2.4)\qquad
\begin{array}{cccccc|c}
w_1 & w_2 & w_3 & \cdots & w_n & & 1 \\
\end{array}
$$
$$
\left[\begin{array}{ccccc|l}
1 & 1 & 1 & \cdots & 1 & M_0 \\
0 & \phi(2,1) & \phi(3,1) & \cdots & \phi(n,1) & \overset{1}{\psi}_2(x_1) \\
0 & 0 & \phi(3,2) & \cdots & \phi(n,2) & \overset{2}{\psi}_3(x_1, x_2) \\
\vdots & & & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \phi(n,n-1) & \overset{n-1}{\psi}_n(x_1, \cdots, x_{n-1}) \\
0 & 0 & 0 & \cdots & 0 & \overset{n}{\psi}_{n+1}(x_1, \cdots, x_n) \\
\vdots & & & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \overset{n}{\psi}_{2n-s}(x_1, \cdots, x_n)
\end{array}\right]
$$

with

$$
(2.5)\qquad \phi(i, j) = \prod_{\substack{k=1 \\ j<i}}^{j} (x_i - x_k), \qquad i = 1, 2, \cdots, n
$$

and

$$
(2.6)\qquad \begin{cases} \overset{1}{\psi}_k(x_1) = M_k - M_{k-1}x_1, & k \geqq 1 \\ \overset{i}{\psi}_k(x_1, \cdots, x_i) = \overset{i-1}{\psi}_k(x_1, \cdots, x_{i-1}) - x_i\overset{i-1}{\psi}_k(x_1, \cdots, x_{i-1}) \\ \text{with } k \geqq 2 \text{ and } i \leqq k-1. \end{cases}
$$

From our assumptions on the x_i, $\phi(i, j) \neq 0$ and the equations (2.4) have a unique solution for the w_i if and only if the conditions

$$
(2.7)\qquad \overset{n}{\psi}_{n+i}(x_1, \cdots, x_n) = 0, \qquad i = 1, 2, \cdots, n-s,
$$

are satisfied. Now it is easy to show from (2.6) that

$$
(2.8)\qquad \overset{n}{\psi}_{n+i}(x_1, \cdots, x_n) = \sum_{j=0}^{n} (-1)^j M_{n+i-j-1}\sigma_j(x_1, \cdots, x_n),
$$

$i = 1, 2, \cdots, n-s$, where $\sigma_j(x_1, \cdots, x_n)$ is the jth elementary symmetric function of $(x_1, \cdots, x_n)$.

From the conditions (2.7), we determine the $n-s$ unspecified x_i. If $s=n$, all points have been selected and no difficulties occur. If, however, $0 \leqq s \leqq n-1$, the conditions (2.7) may give rise to a set of equations that (a) are inconsistent, (b) have complex roots, or (c) have real roots some of which fall outside $[-1, 1]$.

A notable exception to this is when $s=0$ and $w(x)\equiv 1$. In this case, we are led to the Gaussian quadrature ([1]). On the other hand, if $1\leqq s\leqq n-1$, one can frequently prevent the situations (a), (b), and, if desired, (c) from occurring by determining permissible ranges in which the points x_i to be specified may be given. This may necessitate lowering the degree of precision to get a solvable system. In general, there is no unique way of selecting these ranges. If the sequence $\{x_j\}_{j=1}^{s}$ is to be specified, a consistent set of conditions (2.7) permits solving for the $\sigma_k(x_{s+1}, \cdots, x_n)$, $k=1, 2, \cdots, n-s$, as rational functions of the $\sigma_k(x_1, \cdots, x_s)$, $k=1, 2, \cdots, s$. The remaining $n-s$ points x_j, $j=s+1, \cdots, n$, must then satisfy

$$
\begin{aligned}
x^{n-s} - \sigma_1(x_{s+1}, \cdots, x_n)x^{n-s-1} + \sigma_2(x_{s+1}, \cdots, x_n)x^{n-s-2} + \cdots \\
+ (-1)^{n-s}\sigma_{n-s}(x_{s+1}, \cdots, x_n) = 0
\end{aligned}
\tag{2.9}
$$

and be restricted to the interval $[-1, 1]$ or to real values lying within the domains of definition of f and w. Permissible ranges for the specifiable x_i may then be determined from the relations connecting $\sigma_k(x_1, \cdots, x_s)$, $k=1, \cdots, s$, with $\sigma_k(x_{s+1}, \cdots, x_n)$, $k=1, \cdots, n-s$, and (or) the equation (2.9).

Remark. If $w(x)$ is (i) an even function or (ii) an odd function, then the criterion used above can be modified to give symmetric forms for the $J_n(f)$. That is, each point x_i is accompanied by an image point $-x_i$ and each of these has equal accompanying weight factors in (1). In case (i), we require that (1) integrate exactly all odd continuous functions while in (ii) we require that (1) integrate exactly all even continuous functions. Then (1) takes the form

$$
J_n(f) = \sum_{j=1}^{m} w_j[f(x_j) \pm f(-x_j)], \qquad m = \left[\frac{n+1}{2}\right],
\tag{2.10}
$$

with the sequence $\{x_i\}_{i=1}^{n}$ restricted to nonnegative real values and possibly to $[0, 1]$. The plus sign is used for case (i) while the minus sign is used for case (ii). Furthermore, if s of the $x_i\geqq 0$ are to be specified (and hence their images), we require that (2.10) precisely integrate the polynomials x^{2k}, $k=0, 1, \cdots, 2m-s-1$, for case (i) and the polynomials x^{2k+1}, $k=0, 1, \cdots, 2m-s-1$, for case (ii). With some modification, to be noted in one of the examples, the set of equations thereby determined can again be fitted into the matrix scheme (2.3) and conditions similar to those in (2.7) can be derived.

3. Further relations. We now assume that s of the x_i and that r of the weight factors are to be specified with $r\leqq n-2$. Since no ordering of the x_i according to indices has been presupposed, it may be assumed that $\{w_i\}_{i=n-r+1}^{n}$ is the sequence of weight factors and $\{x_{j_i}\}_{i=1}^{s}$ is the sequence of points to be assigned. Here, $\{j_1, \cdots, j_s\}$ is some subset of s distinct elements of $(1, 2, \cdots, n)$. Then the possible determination of the remaining w_j and x_j requires that, at most, $2n-s-r$ conditions be satisfied. Taking these to be $J_n(x^k)=I(x^k)$, $k=0, 1, \cdots, 2n-r-s-1$, the matrix (2.3) can be rewritten

$$
\begin{array}{c}
\begin{array}{ccccc|c}
w_1 & w_2 & w_3 & \cdots & w_{n-r} & 1
\end{array}\\
\left[\begin{array}{ccccc|c}
1 & 1 & 1 & \cdots & 1 & \theta_0 \\
x_1 & x_2 & x_3 & \cdots & x_{n-r} & \theta_1 \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
x_1^{2n-r-s-1} & x_2^{2n-r-s-1} & x_3^{2n-r-s-1} & \cdots & x_{n-r}^{2n-r-s-1} & \theta_{2n-r-s-1}
\end{array}\right]
\end{array}
\tag{3.1}
$$

Here, $\theta_i = M_i - \sum_{k=1}^{r} w_{n-r+k} x_{n-r+k}^{i}$, $i=0, 1, \cdots, 2n-r-s-1$. By triangularizing the coefficient matrix as in (2.4), we obtain

$$
\begin{array}{c}
\begin{array}{ccccc|c}
w_1 & w_2 & w_3 & \cdots & w_{n-r} & 1
\end{array}\\
\left[\begin{array}{ccccc|c}
1 & 1 & 1 & \cdots & 1 & \theta_0 \\
0 & \phi(2,1) & \phi(3,1) & \cdots & \phi(n-r,1) & \Theta_2^{1} \\
0 & 0 & \phi(3,2) & \cdots & \phi(n-r,2) & \Theta_3^{2} \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & \cdots & \phi(n-r, n-r-1) & \Theta_{n-r}^{n-r-1} \\
0 & 0 & \cdot & & 0 & \Theta_{n-r+1}^{n-r} \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
0 & 0 & \cdot & \cdots & 0 & \Theta_{2n-r-s}^{n-r}
\end{array}\right]
\end{array}
\tag{3.2}
$$

with the $\phi(i, j)$ as in (2.5) and the Θ_j^k determined from the θ_j in precisely the same way as the ψ_j^k were determined from the M_j in (2.6). Then the consistency requirements for the solvability of (3.2) is given by

$$
\Theta_{n-r+i}^{n-r} = 0, \qquad i = 1, 2, \cdots, n - s. \tag{3.3}
$$

We must require, as before, that each x_i satisfy $-1 \leqq x_i \leqq 1$ or else be restricted to real values lying within the domains of definition of f and w.

4. Some examples. In the following, use is made of the consistency conditions (2.7) and (3.3) to construct numerical integration formulas. We omit the usual Newton-Cotes and Gaussian quadrature formulas that are easily handled by this method. The examples will illustrate how the relaxing of a condition may permit the construction of a formula not otherwise obtainable and how ranges may be selected in which certain of the x_i can be specified. In our examples, we restrict the x_i to lie within $[-1, 1]$. This condition may be relaxed, however, if one desires somewhat more general formulas. The problem of determining such ranges for high order formulas appears to be well suited for electronic computers. The remainder terms in such formulas can be estimated by means of the influence function [3]. Unless the formula is symmetric, such a remainder usually cannot be given a simple structure such as those in the New-

ton-Cotes formulas. In our last example, we give a remainder formula that requires an estimate of the mean value type.

(a) Let $w(x)=\frac{3}{4}+\frac{17}{4}x-\frac{3}{4}x^2-\frac{35}{6}x^3$. Then, from (2.2), $M_0=1$ $M_1=\frac{1}{2}$, $M_2=\frac{1}{5}$, and $M_3=\frac{1}{30}$. If none of the w_j or x_j are to be prescribed in the formula $J_2(f) = w_1f(x_1)+w_2f(x_2)$, then the consistency conditions (2.7) and (2.8) require that

$$(4.1)\qquad \begin{aligned} \frac{1}{5}-\frac{1}{2}(x_1+x_2)+x_1x_2&=0\\ \frac{1}{30}-\frac{1}{5}(x_1+x_2)+\frac{1}{2}x_1x_2&=0. \end{aligned}$$

Then the equation satisfied by x_1 and x_2 is $x^2-\frac{4}{3}x+\frac{7}{15}=0$ and the roots of this are complex. However, if we require one point to be prescribed, say x_1, then only the first of conditions (4.1) need be satisfied. Thus $x_2=(5x_1-2)/(10x_1-5)$. If we take $x_1\in[-1, 0]$, it follows that $x_2\in[\frac{2}{5}, \frac{7}{15}]$ and

$$J_2(f)=\left\{\frac{x_2-\frac{1}{2}}{x_2-x_1}\right\}f(x_1)+\left\{\frac{\frac{1}{2}-x_1}{x_2-x_1}\right\}f(x_2).$$

We have sacrificed a degree of precision to obtain a usable formula.

(b) Let $w(x)=x^2$ and $x_1<x_2<x_3$ with $x_1=-\frac{2}{3}$ and w_2 to be specified in the formula $J_3(f)=\sum_{j=1}^{3}w_jf(x_j)$. In this case, $M_0=\frac{2}{3}$, $M_1=0$, $M_2=\frac{2}{5}$, and $M_3=0$. By (3.3) and the construction procedure (2.6), it follows that

$$(4.2)\qquad \begin{cases} \Theta_3^2=\left(\dfrac{2}{5}-\dfrac{4}{9}x_3\right)+w_2\left(x_2+\dfrac{2}{3}\right)(x_3-x_2)=0\\[2ex] \Theta_4^2=\left(\dfrac{4}{15}-\dfrac{2}{5}x_3\right)+w_2x_2\left(x_2+\dfrac{2}{3}\right)(x_3-x_2)=0. \end{cases}$$

Eliminating w_2 from these, we obtain $x_2=(9x_3-6)/(10x_3-9)$ so that if $(9-\sqrt{21})/10<x_3<\frac{36}{47}$, then $-\frac{2}{3}<x_2<(9-\sqrt{21})/10$. The w_2 may then be selected in (4.2) to be consistent with these conditions on x_2 and x_3.

(c) Let $w(x)=1$ and consider a symmetric integration formula of the form

$$J_6(f)=\sum_{j=1}^{3}w_j\{f(x_j)+f(-x_j)\}$$

with $0\leqq x_1<x_2<x_3\leqq 1$. Let $X_i=x_i^2$, $i=1, 2, 3$. Then, as noted in the remark of section 2, the matrix patterns (2.3) and (2.4) can be readily modified for this example by replacing each appearing x_i by X_i and each M_j by $\frac{1}{2}M_{2j}$. The same replacements are needed in the conditions (2.7) and (2.8). In our case, $M_{2i} =2/(2i+1)$, $i=0, 1, \cdots, 4$. We shall note two situations.

Case (i). *Two points prescribed.* Let x_1 and x_2, and hence X_1 and X_2 be prescribed. From (2.7) and (2.8), we obtain the consistency condition

$$\sum_{j=0}^{3} \frac{(-1)^j}{7-2j}\, \sigma_j(X_1, X_2, X_3) = 0$$

or

$$X_3 = \left[\frac{1}{3}\left(X_1 - \frac{3}{5}\right)\left(X_2 - \frac{3}{5}\right) + \frac{4}{175}\right] \Big/ \left[\left(X_1 - \frac{1}{3}\right)\left(X_2 - \frac{1}{3}\right) + \frac{4}{45}\right].$$

In this form, it is easily checked that if $0 \leqq X_1 \leqq \frac{1}{3}$ and $\frac{1}{3} \leqq X_2 \leqq \frac{3}{7}$, then $\frac{3}{7} \leqq X_3 \leqq 1$. Similarly, if $\frac{3}{7} \leqq X_2 \leqq \frac{3}{5}$ and $\frac{3}{5} \leqq X_3 \leqq 1$, then $0 \leqq X_1 \leqq \frac{1}{5}$.

Case (ii). *One point prescribed.* Let the point X_1 be prescribed. Then the consistency relations are

$$\begin{cases} X_2X_3\left\{\dfrac{1}{3} - X_1\right\} + (X_2 + X_3)\left\{\dfrac{1}{3}X_1 - \dfrac{1}{5}\right\} = -\dfrac{1}{7} \\[2ex] X_2X_3\left\{\dfrac{1}{5} - \dfrac{1}{3}X_1\right\} + (X_2 + X_3)\left\{\dfrac{1}{5}X_1 - \dfrac{1}{7}\right\} = -\dfrac{1}{9} \end{cases}$$

and

$$\begin{cases} \sigma_1 = X_2 + X_3 = \dfrac{5}{7D}\left(\dfrac{2}{15} - X_1\right) \\[2ex] \sigma_2 = X_2X_3 = \dfrac{1}{21D}\left(\dfrac{3}{7} - 2X_1\right) \end{cases}$$

with $D = X_1^2 - \frac{6}{7}X_1 + \frac{3}{35}$. By the use of (2.9), we find, for example, that if $0 \leqq X_1 \leqq \frac{1}{30}$, then $0 \leqq X_1 < X_2 < X_3 \leqq 1$. If $X_1 = 0$, the solution to the above consistency conditions yields the required points in the 5-point Gaussian.

(d) *Example of a Remainder Term.* If, in case (i) of (c), we select $x_1 = 0$, $x_2 = \sqrt{3/7}$, and $x_3 = 1$, we obtain, after determining the w_i,

$$J_6(f) = \frac{1}{90}\left[9f(-1) + 49f(-\sqrt{3/7}) + 64f(0) + 49f(\sqrt{3/7}) + 9f(1)\right],$$

The five point Lobatto formula [3].

This is a modification to the interval $[-1, 1]$ of the example (d) given in [2]. We now make use of the notation of [2] with only slight changes for obtaining a remainder for the above formula.

Let $Q_0(t) = \int_{-1}^{t} P(x)dx$ and let $Q_i(t) = \int_{-1}^{t} Q_{i-1}(x)dx$, $i = 1, 2$. Let $x_0, x_1, \cdots, x_4$ denote the sequence of points -1, $\sqrt{3/7}$, 0, $\sqrt{3/7}$, 1. For the above example, we have $P(t) = t(t^2 - 1)(t^2 - 3/7)$, $Q_0(t) = \frac{1}{42}(7t^2 - 1)(t^2 - 1)^2$, $Q_1(t) = \frac{1}{42}t(t^2 - 1)^3$, and $Q_2(t) = \frac{1}{336}(t^2 - 1)^4$. It will be observed that $Q_i(\pm 1) = 0$, $i = 0, 1, 2$, but that $Q_2(t) \neq 0$ for $t \in (-1, 1)$. If we apply formula (3.6) of [2] and integrate by parts, we obtain

$$R(f) = \sum_{j=0}^{2} (-1)^j Q_j(t)(j+1)!\,g(x_0, \cdots, x_4, \underbrace{t, \cdots, t}_{j+1})\Big|_{-1}^{1}$$

$$- 6\int_{-1}^{1} Q_2(t) g(x_0, \cdots, x_4, t, t, t, t)dt$$

$$= -6g(x_0, \cdots, x_4, \eta, \eta, \eta, \eta)\int_{-1}^{1} Q_2(t)dt, \qquad \eta \in (-1, 1)$$

$$= -\frac{6}{8!} g^{(8)}(\xi)\int_{-1}^{1} Q_2(t)dt, \qquad \xi \in (-1, 1)$$

$$= -\frac{.0145}{8!} f^{(8)}(\xi).$$

References

1. R. E. Beard, Some Notes on Approximate Product Integration, J. Inst. Actuar., 73 (1947) 356–416.

2. L. R. Bragg and E. B. Leach, The Remainder Terms in Numerical Integration Formulas, this Monthly, 70 (1963) 70–76.

3. F. B. Hildebrand, Introduction to Numerical Analysis, McGraw-Hill, New York, 1956.

PRODUCTS OF SEPARABLE SPACES

K. A. ROSS AND A. H. STONE, University of Rochester

The purpose of this paper is to give a unified treatment of some theorems concerning products of separable spaces. Some of the proofs appear to be new. The theorems are all essentially known.

In this paper A will always denote a nonvoid index set and, for each a in A, X_a will denote a Hausdorff topological space having at least two points. We shall denote the product of the spaces X_a by $\prod_{a\in A} X_a$ or X; X consists of all functions x on A whose values $x(a) = x_a$ belong to X_a. A base for the product topology on X consists of all sets

$$U(V_a;\, a \in F) = \{x \in X \colon x_a \in V_a \text{ for } a \in F\},$$

where F is a finite subset of A and each V_a is an open subset of X_a. If F contains just one element b, we shall write $U(V_b)$ in place of $U(V_a;\, a \in \{b\})$. By a separable space we mean one that contains a countable dense subset. For other topological notions, we refer the reader to Kelley [5].

THEOREM 1. *The product space X is separable iff* $\text{card}(A) \leqq c$ *and each factor* X_a *is separable.*

Proof. Suppose that X is separable. Since each X_a is a continuous image of X under projection, each X_a is separable. Now let D be a countable dense subset of X. For each a in A, there exist disjoint nonvoid open subsets V_a and W_a of X_a. We define the function f_a on D by the rule:

$$\begin{aligned} f_a(x) &= 0 \qquad \text{for } x \in D \text{ and } x \in U(V_a), \\ &= 1 \qquad \text{for } x \in D \text{ and } x \notin U(V_a). \end{aligned}$$

To see that the mapping $a \to f_a$ is one-to-one, consider a, b in A, $a \neq b$. Some member y of D belongs to the open set $U(V_a) \cap U(W_b)$; hence $f_a(y) = 0$ and $f_b(y) = 1$. Thus the mapping $a \to f_a$ is one-to-one and

$$\text{card}\,(A) \leqq \text{card}\,\{f : f \text{ maps } D \text{ into } \{0, 1\}\} = 2^{\aleph_0} = c.$$

Suppose now that each X_a is separable and that $\text{card}(A) \leqq c$. The case in which A is finite is trivial. With no loss of generality, therefore, we may assume that A is a dense subset of the set of all real numbers. For a in A, let $D_a = \{x_k^a\}_{k=1}^{\infty}$ be a countable dense subset of X_a. Let T be the set of all tuples $\tau = (r_1, r_2, \cdots, r_{n-1}; k_1, k_2, \cdots, k_n)$, where each r_m is a rational number, each k_m is a positive integer, $r_1 < r_2 < \cdots < r_{n-1}$, and $n \geqq 2$. Clearly T is countable. For τ in T, define x^τ by the rule:

$$\begin{aligned} x_a^\tau &= x_{k_1}^a \qquad \text{for } a \leqq r_1, \\ &= x_{k_m}^a \qquad \text{for } r_{m-1} < a \leqq r_m, \\ &= x_{k_n}^a \qquad \text{for } r_{n-1} < a. \end{aligned}$$

Then $\{x^\tau : \tau \in T\}$ is a countable dense subset of X. In fact, consider a nonvoid subset $U(V_a;\, a \in F)$ of X. Then $F = \{a_1, \cdots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$, and we can choose rational numbers $r_1, \cdots, r_{n-1}$ for which $a_1 < r_1 < a_2 < r_2 < \cdots < a_{n-1} < r_{n-1} < a_n$. Since D_{a_m} is dense in X_{a_m}, there is an integer k_m such that $x_{k_m}^{a_m}$ belongs to $V_{a_m} (1 \leqq m \leqq n)$. It is easy to see that x^τ belongs to $U(V_a;\, a \in F)$, where $\tau = (r_1, r_2, \cdots, r_{n-1}; k_1, k_2, \cdots, k_n)$. This completes the proof.

Theorem 1 is essentially due to Pondiczery [8] in 1944. A special case of his theorem asserts that if all of the spaces X_a are homeomorphic to the same separable space, then $\prod_{a \in A} X_a$ is separable iff $\text{card}(A) \leqq c$; Theorem 1 follows easily from this result. The proof given here is due to Marczewski [6], page 138.

We now introduce three definitions. As in Kelley [5], page 60, we shall say that a space satisfies the *countable chain condition* (briefly, the CCC) if every family of pairwise disjoint open sets is countable. A (K)-*space* is one for which every uncountable family of open sets contains an uncountable subfamily in which every two sets have nonvoid intersection. Finally, a space has *caliber* $\mathfrak{m}$ in every family $\mathfrak{U}$ of open sets for which $\text{card}(\mathfrak{U}) = \mathfrak{m}$ admits a subfamily $\mathfrak{V}$ such that $\text{card}(\mathfrak{V}) = \mathfrak{m}$ and $\bigcap_{V \in \mathfrak{V}} V \neq \varnothing$. Then all separable spaces have caliber $\aleph_1$; all spaces having caliber $\aleph_1$ are (K)-spaces; and all (K)-spaces satisfy the CCC.

THEOREM 2. *If each factor X_a is separable, then X has caliber $\aleph_1$. In particular, X is a (K)-space and satisfies the CCC.*

Proof. Let $\mathfrak{U}=\{U_\gamma:\gamma\in\Gamma\}$ be a family of nonvoid open sets in X such that $\operatorname{card}(\mathfrak{U})=\aleph_1$. For each U_γ in $\mathfrak{U}$, let $U(V_a^\gamma;\, a\in F_\gamma)$ be a nonvoid basic open subset of U_γ. Let

$$\mathfrak{W} = \{U(V_a^\gamma;\; a \in F_\gamma): \gamma \in \Gamma\}.$$

If $\mathfrak{W}$ is countable, then one of the sets W in $\mathfrak{W}$ is contained in $\aleph_1$ members of $\mathfrak{U}$ and $\mathfrak{V}=\{U\in\mathfrak{U}:\, W\subset U\}$ clearly satisfies the relation $\bigcap_{V\in\mathfrak{V}} V\neq\emptyset$. Suppose next that $\operatorname{card}(\mathfrak{W})=\aleph_1$. Let $B=\bigcup_{\gamma\in\Gamma}F_\gamma$; define $Y=\prod_{a\in B}X_a$ and $Z=\prod_{a\in A-B}X_a$, so that $X=Y\times Z$. Then $\operatorname{card}(B)\leqq\aleph_1\leqq c$ and Y is separable by Theorem 1. Since $F_\gamma\subset B$ for each $\gamma\in\Gamma$, we have

$$U(V_a^\gamma;\, a \in F_\gamma) = U'(V_a^\gamma;\, a \in F_\gamma) \times Z,$$

where $U'(V_a^\gamma;\, a\in F_\gamma)$ is a basic open set in Y. The family $\{U'(V_a^\gamma;\, a\in F_\gamma):\gamma\in\Gamma\}$ of subsets of Y also has cardinality $\aleph_1$. Since Y has caliber $\aleph_1$, this family has a subfamily $\mathfrak{W}'$ such that $\operatorname{card}(\mathfrak{W}')=\aleph_1$ and $\bigcap_{W\in\mathfrak{W}'} W\neq\emptyset$. Clearly these relations still hold for the family $\{W\times Z:\, W\in\mathfrak{W}'\}$ of subsets of X. Finally, defining $\mathfrak{V}=\{U\in\mathfrak{U}:\, W\times Z\subset U$ for some W in $\mathfrak{W}'\}$, one readily verifies that $\operatorname{card}(\mathfrak{V})=\aleph_1$ and $\bigcap_{V\in\mathfrak{V}} V\neq\emptyset$.

NOTE. Theorem 2 was first stated by Šanin [9] in 1946; a proof appears in [10], page 65. In fact, Šanin proved that *if each X_a has caliber $\mathfrak{m}$ and $\mathfrak{m}$ is a regular uncountable cardinal, then X also has caliber $\mathfrak{m}$.* In 1947, Marczewski [6], page 139, also proved a theorem more general than Theorem 2: *if each X_a is a (K)-space, then X is a (K)-space.* It is not known whether the product of *two* spaces satisfying the CCC must also satisfy the CCC.

THEOREM 3. *If each factor X_a is separable, then the closure U^- of an open subset U of X is determined by countably many indices. That is, there exists a countable subset C (depending upon U) of A such that*

$$x \in U^-,\ y \in X,\ \textit{and}\ x_a = y_a \textit{ for } a \in C \Rightarrow y \in U^-.$$

Proof. By Zorn's lemma, U contains a maximal family $\mathfrak{V}$ of pairwise disjoint basic open sets. By Theorem 2, $\mathfrak{V}$ is countable:

$$\mathfrak{V}=\{U(V_a^n;\, a\in F_n):\, n=1, 2, \cdots\}.$$

Let

$$V = \bigcup_{n=1}^{\infty} U(V_a^n;\, a \in F_n) \quad \text{and} \quad C = \bigcup_{n=1}^{\infty} F_n.$$

Then C is countable and the set V is determined by the indices in C. It follows that V^- is also determined by the indices in C. Because of the maximality of $\mathfrak{V}$, we have $U^-=V^-$ so that U^- is determined by the indices in C.

THEOREM 4. *Suppose that each factor X_a is separable and that f is a continuous function from X into a regular second countable space Y. Then f is determined by countably many indices: there is a countable subset C (depending upon f) of A such that*

$$x, y \in X \text{ and } x_a = y_a \text{ for } a \in C \Rightarrow f(x) = f(y).$$

Proof. Let $\mathcal{B}$ be a countable base for the open subsets of Y. For B in $\mathcal{B}$, $f^{-1}(B)$ is open in X and, by Theorem 3, $f^{-1}(B)^-$ is determined by countably many indices C_B. Let $C=\bigcup_{B\in\mathcal{B}} C_B$; obviously C is countable.

Now suppose that x, y are in X, $x_a=y_a$ for $a\in C$, and $f(x)=\alpha\in Y$. For a suitable sequence $\{B_k\}_{k=1}^{\infty}$ in $\mathcal{B}$, we have

$$\{\alpha\} = \bigcap_{k=1}^{\infty} B_k = \bigcap_{k=1}^{\infty} B_k^-,$$

where $B_{k+1}^-\subset B_k$ for all k. Using continuity of f, we have

$$f^{-1}(\alpha) = f^{-1}\left(\bigcap_{k=1}^{\infty} B_k^-\right) = \bigcap_{k=1}^{\infty} f^{-1}(B_k^-) \supset \bigcap_{k=1}^{\infty} f^{-1}(B_k)^- \supset f^{-1}(\alpha),$$

and hence $f^{-1}(\alpha)=\bigcap_{k=1}^{\infty} f^{-1}(B_k)^-$. Since each set $f^{-1}(B_k)^-$ is determined by the indices in C_{B_k}, the set $f^{-1}(\alpha)$ is determined by the indices in C. Obviously x belongs to $f^{-1}(\alpha)$. Since $x_a=y_a$ for $a\in C$, y also belongs to $f^{-1}(\alpha)$; i.e., $f(y)=\alpha=f(x)$.

REMARKS. Theorem 3 may be regarded as a generalization of the following theorem of Bockstein [2], proved in 1948: *If each X_a has a countable basis for open sets and if U and V are disjoint open subsets of X, then there are open sets U' and V' which are countable unions of basic open sets $U(V_a;\ a\in F)$ and satisfy $U'\supset U$, $V'\supset V$, and $U'\cap V'=\emptyset$.* To prove this, note that U^- and V^- are determined by countably many indices C, define $U'=\{x\in X\colon$ for some y in U, $y_a=x_a$ for all $a\in C\}$, and define V' analogously. That these are countable unions of basic open sets follows from the fact that $\prod_{a\in C} X_a$ is second countable.

In 1952, Mazur [7] proved a theorem similar to Theorem 4. His Theorem III states that if X and the X_a's satisfy certain mild topological conditions and if card(A) is less than the first inaccessible cardinal, if any, then a *sequentially* continuous function on X is determined by countably many indices. In 1960, A. M. Gleason announced in conversation a result stronger than Theorem 4. In Gleason's theorem the image space Y is required only to be first countable. We wish to thank Professor Gleason for communicating to us his theorem and proof; the proof we offer here is different from his. Theorem 4 for metric spaces is proved by Corson and Isbell [4], page 24.

The hypothesis "each factor X_a is separable" in Theorems 3 and 4 can be replaced by the hypothesis "each factor X_a is a (K)-space." The proofs of the resulting statements are the same as those given here except that we apply Marczewski's theorem, mentioned above, instead of Theorem 2.

Note also that if each factor X_a of X is compact and Y is the real line or

complex plane, then the conclusion of Theorem 4 follows by an elementary Stone-Weierstrass argument: Let $\mathfrak{A}$ consist of all continuous functions on X that are determined by countably many indices and show that $\mathfrak{A}$ is a separating, uniformly closed algebra of functions closed under complex conjugates. See, for example, Bishop [1], page 632.

THEOREM 5. *Suppose that each factor* X_a *consists of the discrete set* Z *of integers and that* $\operatorname{card}(A) \geqq \aleph_1$. *Then* X *is not normal.*

Proof. Assume that X is normal. Let

$$F_0 = \{x \in X\colon \text{for } n \neq 0, \text{ there is at most one index } a \text{ for which } x_a = n\}$$

and

$$F_1 = \{x \in X\colon \text{for } n \neq 1, \text{ there is at most one index } a \text{ for which } x_a = n\}.$$

If $y \notin F_0$, then $y_a = y_b = n$ for some n in Z, $n \neq 0$, and a, b in A, $a \neq b$. Hence $\{x \in X\colon x_a = x_b = n\}$ is an open set containing y and disjoint from F_0. Thus F_0 is closed in X; likewise F_1 is closed. Obviously F_0 and F_1 are disjoint. By Urysohn's lemma, there is a continuous function f mapping X into $[0, 1]$ such that $f(F_0) = 0$ and $f(F_1) = 1$. By Theorem 4, f is determined by a countable set $C = \{c_1, c_2, \cdots\}$ of indices. If z is defined by

$$\begin{aligned} z_a &= k + 1 && \text{for } a = c_k, \\ &= 0 && \text{for } a \notin C, \end{aligned}$$

and z' is defined by

$$\begin{aligned} z_a' &= k + 1 && \text{for } a = c_k, \\ &= 1 && \text{for } a \notin C, \end{aligned}$$

then clearly $f(z) = f(z')$. On the other hand, $f(z) = 0$ since $z \in F_0$ and $f(z') = 1$ since $z' \in F_1$. This contradiction shows that X is not normal.

Theorem 5 was proved by Stone [11] in 1948. He used the same sets F_0 and F_1 used above and showed directly that any two open sets U_0 and U_1 such that $U_0 \supset F_0$ and $U_1 \supset F_1$ must have nonvoid intersection. Corson [3], page 791, gave a proof very like ours using Bockstein's theorem.

Finally, we note that from Theorem 5 it follows easily that *whenever* $X = \prod_{a \in A} X_a$ *is a normal space, then at most countably many of the factors* X_a *contain infinite discrete closed subsets.* See, for example, [11].

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References

1. E. Bishop, A minimal boundary for function algebras, Pacific J. Math., 9 (1959) 629–642.

2. M. Bockstein, Un théorème de séparabilité pour les produits topologiques, Fund. Math., 35 (1948) 242–246.

3. H. H. Corson, Normality in subsets of product spaces, Amer. J. Math., 81 (1959) 785–796.

4. H. H. Corson and J. R. Isbell, Some properties of strong uniformities, Quart. J. Math. Oxford, (2) 11 (1960) 17–33.

5. J. L. Kelley, General topology, Van Nostrand, New York, 1955.

6. E. Marczewski, Séparabilité et multiplication cartésienne des espaces topologiques, Fund. Math., 34 (1947) 127–143.

7. S. Mazur, On continuous mappings on Cartesian products, Fund. Math., 39 (1952) 229–238.

8. E. S. Pondiczery, Power problems in abstract spaces, Duke Math. J., 11 (1944) 835–837.

9. N. A. Šanin, On intersection of open subsets in the product of topological spaces, C. R. (Doklady) Acad. Sci. URSS (N. S.) 53 (1946) 499–501. (Russian)

10. ———, On the product of topological spaces, Trudy Mat. Inst. Steklov, 24 (1948) 112 pages. (Russian)

11. A. H. Stone, Paracompactness and product spaces, Bull. Amer. Math. Soc., 54 (1948) 977–982.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables 46, Florida

A NOTE ON THE ZEROS OF BESSEL FUNCTIONS

ADAM CZARNECKI, Illinois Institute of Technology

An elegant argument in [1] for the existence of zeros of Bessel functions $J_n(x)$ of first kind of (real) order n seems to have a minor error.

The following proof is submitted, however, in order to validate the statement that for real n, $J_n(x)$ has an infinite number of positive roots.

Let $J_n(x)$ be the Bessel function of the first kind of (real) order n, and let

$$Y(x) = x^{1/2}J_n(x), \qquad x \geqq 0. \tag{1}$$

Then Y satisfies the differential equation

$$\frac{d^2Y}{dx^2} + gY = 0, \qquad g = 1 - \frac{4n^2-1}{4x^2}, \qquad x > 0. \tag{2}$$

Next, the equation

$$\frac{d^2Z}{dx^2} + Z = 0 \tag{3}$$

admits $Z = \sin x$, as a solution. Then

$$Z\frac{d^2Y}{dx^2} - Y\frac{d^2Z}{dx^2} = \frac{d}{dx}\left[Z\frac{dY}{dx} - Y\frac{dZ}{dx}\right] = (1-g)YZ. \tag{4}$$

so that, for $0<\alpha<\beta<+\infty$, we have, upon integrating

$$(5)\quad \left[Z\frac{dY}{dx} - Y\frac{dZ}{dx}\right]_\alpha^\beta = \int_\alpha^\beta (1-g)YZdx = -\frac{(1-4n^2)}{4}\int_\alpha^\beta \frac{1}{x^2}YZdx.$$

Let $\alpha=2k\pi$, $\beta=(2k+1)\pi$, $k=1, 2, \cdots$. Then $Z(\alpha)=Z(\beta)=0$, $Z'(\alpha)=1$, $Z'(\beta)=-1$, and hence

$$(6)\quad Y(\alpha)+Y(\beta) = -\frac{(1-4n^2)}{4}\int_{2k\pi}^{(2k+1)\pi}\frac{1}{x^2}YZdx = -\frac{(1-4n^2)}{4}\pi\frac{1}{\xi^2}Y(\xi)Z(\xi),$$

where $\alpha<\xi<\beta$, by the mean value theorem for integrals.

Let $-\frac{1}{2}\leqq n\leqq\frac{1}{2}$, so that $1-4n^2\geqq 0$. Evidently $(1/\xi^2)Z(\xi)\geqq 0$ on $[\alpha, \beta]$.

Suppose that $Y(x)>0$ on $[\alpha, \beta]$. Then

$$(7)\qquad Y(\alpha)+Y(\beta)>0, \qquad -(1-4n^2)\frac{\pi}{4}\frac{1}{\xi^2}Y(\xi)Z(\xi)\leqq 0.$$

This is impossible, in view of equation (6).

Also if $Y(x)<0$ on $[\alpha, \beta]$, then

$$(8)\qquad Y(\alpha)+Y(\beta)<0, \qquad -(1-4n^2)\frac{\pi}{4}\frac{1}{\xi^2}Y(\xi)Z(\xi)\geqq 0.$$

This is a contradiction also.

Consequently $Y(x)$ must change sign on $[\alpha, \beta]$. Since $Y(x)$ is continuous, it follows that $Y(x)$ and hence $J_n(x)$ must have at least one zero on $[\alpha, \beta]$, that is, at least one zero on every interval of length 2π on the positive x-axis.

This establishes the theorem for the case $-\frac{1}{2}\leqq n\leqq\frac{1}{2}$. By the usual argument, using recurrence relations for $J_n(x)$ and Rolle's Theorem (cf. [2]), the result follows for all real values of n.

References

1. J. Pierpont, Functions of a Complex Variable, Dover, New York, 1959.
2. R. Churchill, Fourier Series and Boundary Value Problems, 2nd ed., McGraw-Hill, New York, 1963.

A FIBONACCI PERFECT SQUARED SQUARE

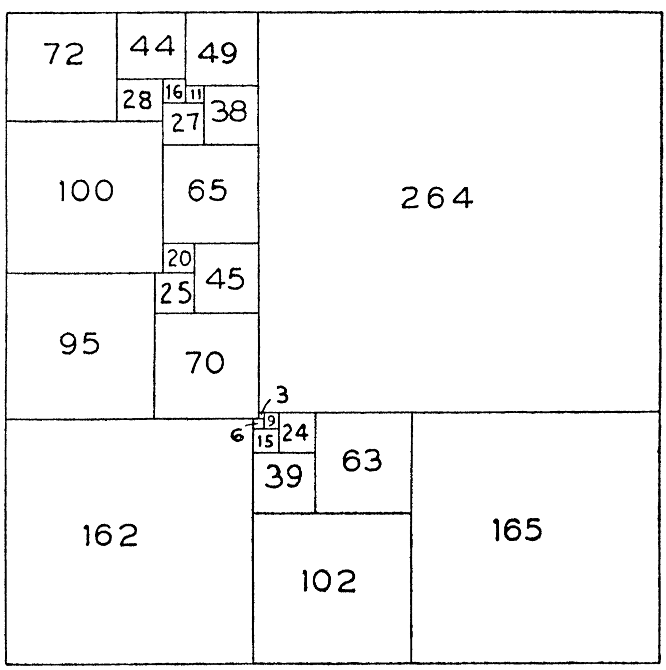
P. J. FEDERICO, Washington, D. C.

S. L. Basin's paper [1] prompts reporting what might be called a "Fibonacci" perfect squared square, which was discovered too late for inclusion in a recent paper on low-order perfect squared squares [2].

As in Basin's paper, assume that each number of the Fibonacci sequence represents a square with that number as its side. Add the second square to the first, obtaining a 1×2 rectangle, add the next square, forming a 2×3 rectangle, then the next square, forming a 3×5 rectangle, etc., always keeping one of the

unit squares in a corner (as in Fig. 1b of Basin's paper). This gives a sequence of squared rectangles, the sides of which are adjacent terms of the Fibonacci sequence; each rectangle is "totally" trivially compound, and also trivially imperfect as two adjacent component squares are equal. While they are not very interesting in themselves, it is possible to derive perfect squared squares from some of these totally trivially compound trivially imperfect rectangles.

A perfect squared square may be formed if a perfect squared rectangle exists which has the same ratio of sides as a squared rectangle having a corner square and an adjacent square equal but otherwise having no other equal components [2], [3]. The two are brought to the same size by multiplying by the appropriate factor if necessary, the duplicate corner square of one rectangle is removed and a corner of the other rectangle fitted in, then two squares are added to complete the squared square which will be perfect if no squares are duplicated. This method was first used most effectively by Willcocks [3] who obtained the lowest order (number of component squares) perfect square yet known, of order 24.



The ninth member of the sequence of squared rectangles formed from the Fibonacci sequence has sides 55 and 89. There is a perfect squared rectangle with sides 165 and 267 ([4], p. 241) which can be fitted with it, on multiplying it throughout by 3, to form a perfect squared square. This square is shown in the figure, from which the method of construction may also be seen. It is of side

429 and is divided into 26 component squares, which is unusual as only one perfect square of lower order, and only two other perfect squares of the same order, appear to have been published before the last paper.

Compound squared rectangles which are not trivially compound may be formed from any given squared rectangle in many ways. The simplest manner is as follows: if x, y are the sides of a squared rectangle $(y>x)$, add four squares of sides $(y-x)/4$, $(3x+y)/4$, $(2x+2y)/4$ and $(x+3y)/4$, forming a larger squared rectangle of sides $(5x+3y)/4$ and $(3x+5y)/4$. If the fifth rectangle in the Fibonacci series of squared rectangles (sides 8, 13) is treated in this manner, there results a squared rectangle (a) of sides 79, 89, after multiplying each element by 4 to clear of fractions. There is a conformal perfect squared rectangle (b) of sides 237, 267 ([4], p. 290) which can be fitted with (a) in the same manner described above, forming the 26-order perfect square of side 492 first described by Willcocks in [3].

More perfect squares can be formed from rectangles (a) and (b). If (a) is compounded by adding a square of side 89 and the process continued five more times (another additive series) forming a rectangle of sides 1107, 1789, there is a congruent perfect rectangle (c) of the same size ([4], p. 270) with which it forms a perfect square of order 32 and side 2892. If (b) is compounded in the same manner another rectangle conformal to (c) is formed and these two can be fitted together to make four perfect squares of order 37; since both rectangles are perfect, they can be reoriented and different corner squares removed to form the four perfect squares.

References

1. S. L. Basin, Generalized Fibonacci sequences and squared rectangles, this MONTHLY, 70 (1963) 372–379. (The explanations of terms on pages 372–3 are not repeated here.)

2. P. J. Federico, Note on some low-order perfect squared squares, Can. J. Math., 15 (1963) 350–362.

3. T. H. Willcocks, A note on some perfect squared squares, Can. J. Math., 3 (1951) 304–308.

4. C. J. Bouwkamp, A. J. W. Duijvestijn, and P. Medema, Tables relating to simple squared rectangles of orders nine through fifteen, Eindhoven, Netherlands, 1960.

A NOTE ON THE RELATION BETWEEN PERIODIC AND ORTHOGONAL FUNDAMENTAL SOLUTIONS OF LINEAR SYSTEMS

F. S. VAN VLECK, University of Kansas

We wish to consider the linear system $dx/dt=Ax$ where x is the n-vector $(x^1, \cdots, x^n)$ and A is a constant real n by n matrix. The unique solution $X(t)$ (X an n by n matrix) of the associated matrix equation $X'=AX(' = d/dt)$ such that $X(0)=I$ (the identity matrix) will be called the fundamental solution of $x'=Ax$. As is well known, $X(t)=e^{At}$ where e^{At} is the matrix defined by the convergent series $I+At+A^2t^2/2!+ \cdots$.

Let B^T be the transpose of a matrix B and C^{-1} be the inverse of a nonsingular matrix C.

Suppose the fundamental solution $X(t)$ is orthogonal ($XX^T=I$) for all t, then under what conditions is $X(t)$ periodic ($X(t+\sigma)=X(t)$ for some $\sigma>0$)? The purpose of this note is to give an answer to this question, as well as to characterize orthogonal fundamental solutions $X(t)$. In fact, the following theorem completely resolves the question.

THEOREM 1. *Let β_j denote the imaginary parts of the characteristic roots λ_j of A. An orthogonal fundamental solution $X(t)$ is periodic if and only if there is a nonzero number β and integers m_j such that $m_j\beta_j=\beta$ for each nonzero β_j.*

In order to prove this we need a characterization of orthogonal fundamental solutions. To that end, we state the following lemmas.

LEMMA 1. $(e^{At})^T=e^{A^Tt}$.

Proof. e^{A^Tt} is the unique matrix solution of $y'=A^Ty$ having the property $Y(0)=I$. Also,

$$[(e^{At})^T]' = [(e^{At})']^T = [Ae^{At}]^T = [e^{At}A]^T = A^T(e^{At})^T,$$

and hence $(e^{At})^T$ is also such a solution. Therefore $(e^{At})^T=e^{A^Tt}$.

LEMMA 2. *e^{At} is orthogonal if and only if A is skew-symmetric ($A^T=-A$).*

Proof. Suppose e^{At} is orthogonal. Then, since $(e^{At})^{-1}=e^{-At}$, $e^{A^Tt}=(e^{At})^T=(e^{At})^{-1}=e^{-At}$. Differentiating the right and left hand sides and letting $t=0$, we have $A^T=-A$.

Conversely, if A is skew-symmetric, $(e^{At})(e^{At})^T=e^{At}e^{A^Tt}=e^{At}e^{-At}=(e^{At})(e^{At})^{-1}=I$, and hence e^{At} is orthogonal.

Using Lemma 2 and the fact that a real skew-symmetric matrix Q is real orthogonally similar to a diagonal block matrix

$$\tilde{Q} = \left\{\begin{bmatrix} 0 & \beta_1 \\ -\beta_1 & 0\end{bmatrix}, \cdots, \begin{bmatrix} 0 & \beta_k \\ -\beta_k & 0\end{bmatrix}, 0, \cdots, 0\right\},$$

where the characteristic roots of Q are $\pm i\beta_j$ and 0 (Gantmacher [1], p. 262), we can characterize orthogonal fundamental solutions by the following theorem.

THEOREM 2. *The fundamental solution $X(t)$ of $x'=Ax$ is orthogonal if and only if there exists a real orthogonal matrix P such that*

$$PAP^{-1} = \tilde{A} = \left\{\begin{bmatrix} 0 & \beta_1 \\ -\beta_1 & 0\end{bmatrix}, \cdots, \begin{bmatrix} 0 & \beta_k \\ -\beta_k & 0\end{bmatrix}, 0, \cdots, 0\right\}.$$

Here the characteristic roots of A are $\pm i\beta_j$ and 0.

We can now prove Theorem 1. By Theorem 2, there is a real orthogonal matrix P such that $PAP^{-1}=\tilde{A}$. Therefore

$$X(t) = e^{At} = P^{-1}e^{PAP^{-1}t}P = P^{-1}e^{\tilde{A}t}P,$$

and we see that $X(t)$ is orthogonally similar to $e^{\tilde{A}t}$. Now

$$e^{\tilde{A}t} = \left\{\begin{bmatrix} \cos\beta_1 t & \sin\beta_1 t \\ -\sin\beta_1 t & \cos\beta_1 t\end{bmatrix}, \cdots, \begin{bmatrix} \cos\beta_k t & \sin\beta_k t \\ -\sin\beta_k t & \cos\beta_k t\end{bmatrix}, 1, \cdots, 1\right\}$$

and hence $e^{\tilde{A}t}$ (and e^{At}) is periodic if and only if there is a $\beta \neq 0$ such that β is an integral multiple of each nonzero β_j.

This also shows that, in general, an orthogonal fundamental matrix $X(t)$ is not periodic but almost periodic.

Reference

1. F. R. Gantmacher, Matrizenrechnung, I, VEB Deutscher Verlag der Wissenschaften, Berlin, 1958.

THE SPECTRAL THEOREM

ISRAEL HALPERIN, Queen's University

1. The Riesz-von Neumann proof of the spectral theorem for a bounded Hermitian operator A (defined everywhere in a Hilbert space) is based on the following theorem:

THEOREM. *Suppose that*
(i) *α, β are real numbers and $\alpha\|x\|^2 \leqq (Ax|x) \leqq \beta\|x\|^2$ for all x,*
(ii) *p is a real polynomial and $p(t) \geqq 0$ for $\alpha \leqq t \leqq \beta$.*
Then $p(A) \geqq 0$.

The proofs given previously for this Theorem seem to use the fundamental theorem of algebra combined with an appeal either to the spectral theorem for operators in a finite dimensional space [3, page 112], or to the theorem of positivity for commuting positive operators [2, page 270], or to the properties of the spectrum [1].

We shall prove (in Lemma 1 below) a simple property of real polynomials which yields an elementary proof for the Theorem above (that is, a proof which does not use the fundamental theorem of algebra).

LEMMA 1. *Suppose that p is a real polynomial, α and β are real numbers, and $p(t) \geqq 0$ for $\alpha \leqq t \leqq \beta$. Then p is the sum of a finite number of polynomials each of one of the following forms:*

$$\text{(i) } (t-\alpha)(q(t))^2, \qquad \text{(ii) } (\beta - t)(q(t))^2, \qquad \text{(iii) } (q(t))^2,$$

with $q(t)$ a real polynomial.

Proof of the Lemma. If p is a constant, necessarily the constant (c, say) is positive and $p = (q(t))^2$ with q the constant, c.

Now we proceed by induction on the degree of p, so that we need only consider the case of p of degree $n \geqq 1$ assuming that the Lemma has been established for polynomials of degree less than n.

Let $a=\min\,(p(t)\,|\,\alpha\leqq t\leqq\beta)$. Clearly $a\geqq 0$ and $p(t)=(p(t)-a)+a$, so we need only consider the addend $p(t)-a$ in place of the original p. This means: we may suppose $p(t_0)=0$ for at least one t_0 satisfying $\alpha\leqq t_0\leqq\beta$. Hence $t-t_0$ is a factor of $p(t)$.

If $\alpha<t_0<\beta$ then $(t-t_0)^2$ must be a factor of $p(t)$ since otherwise $p(t)$ would change sign at t_0. Hence $p(t)=(t-t_0)^2\,p_1(t)$ for some real polynomial $p_1(t)$ of degree $n-2$; by the inductive assumption, the Lemma holds for p_1 and this clearly implies its validity for p.

If $t_0=\alpha$, then $p(t)=(t-\alpha)p_1(t)$ and p_1 is of degree $n-1$. By the inductive assumption, the lemma holds for p_1 and hence to show its validity for p we need only consider the case that p_1 is of the form $(\beta-t)(q(t))^2$. Then

$$p=(t-\alpha)(\beta-t)(q(t))^2$$

and it is sufficient to note that

$$\begin{aligned}(t-\alpha)(\beta-t)&=(t-\alpha)(\beta-t)((\beta-t+t-\alpha))c^2 \qquad (\text{with } c=(\beta-\alpha)^{-1/2})\\ &=(t-\alpha)(c(\beta-t))^2+(\beta-t)(c(t-\alpha))^2.\end{aligned}$$

The case $t_0=\beta$ can be treated in the same way, and this establishes the Lemma.

Proof of the Theorem. To prove $p(A)\geqq 0$ that is, $(p(A)x\,|\,x)\geqq 0$ for all x, it is sufficient to prove this under the additional assumption that p is of the form (i), (ii) or (iii). For the case (i),

$$\begin{aligned}(p(A)x\,|\,x)&=((A-\alpha)q(A)x\,|\,q(A)x)\\ &=((A-\alpha)y\,|\,y)\geqq 0\end{aligned}$$

(here y denotes $q(A)x$). The cases (ii), (iii) are treated similarly.

Note. If it is *assumed* that the real polynomial p has been factored into linear and quadratic factors

$$p(t)=c\prod_i(t-t_i)\prod_j((t-t_j)^2+r_j^2)$$

with c, all t_i, s_j, r_j real constants, the Lemma can be deduced as follows:

We can assume that for each i, $t_i\leqq\alpha$ or $t_i\geqq\beta$ (since the factors $(t-t_i)$ with $\alpha<t_i<\beta$ must occur to even powers and can be absorbed as quadratic factors). Hence p may be rewritten:

$$p(t)=(\text{positive constant})\textstyle\prod_l(t-a_l)\prod_m(b_m-t)\prod_j((t-s_j)^2+r_j^2),$$

with $a_l\leqq\alpha$, $b_m\geqq b$. Now we replace

$$\begin{aligned}t-a_l &\quad\text{by}\quad (t-\alpha)+(\alpha-a_l)\\ b_m-t &\quad\text{by}\quad (b_m-\beta)+(\beta-t).\end{aligned}$$

After this replacement, we multiply out the expression for p and obtain a sum of terms each of one of the forms (i), (ii) or (iii) above.

References

1. S. K. Berberian, The spectral mapping theorem for a Hermitian operator, this MONTHLY, 70 (1963) 1049–1051.
2. F. Riesz and B. Sz-Nagy, Functional analysis, Unger, New York, 1955.
3. J. von Neumann, Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, Math. Ann., 101 (1929) 49–131.
4. For propositions close to Lemma 1, see Polya-Szegö, Aufgaben and Lehrsätze, Springer, Berlin, 1925, vol. II, Chapter 6, Ex. 47, 49. I am indebted to Paul Halmos for this reference.

INVERSES OF VANDERMONDE MATRICES

F. D. PARKER, University of Alaska (now at SUNY at Buffalo)

It is frequently useful to be able to produce the inverse of a Vandermonde matrix for curve fitting, numerical differentiation, and difference equations. Explicit formulas have been given in [1], and this note simplifies those formulas to the point where pencil and paper calculation for the inverse of a matrix of order six takes about twenty minutes.

Let $V(n)$ be the Vandermonde matrix

$$V(n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

We define $F(x)$ to be a polynomial whose roots are x_i, $i=1, 2, \cdots, n$, and $f_k(x)$ to be the reduced polynomial when the factor $x_k - x$ is taken from $F(x)$. Thus

$$F(x) = \prod_{i=1}^{n} (x_i - x), \quad \text{and} \quad f_k(x) = F(x)/(x_k - x).$$

A useful result follows immediately, namely that

$$f_k(x_j) = 0 \text{ for } j \neq k, \quad \text{and} \quad f_k(x_k) = -F'(x_k) \tag{1}$$

where indicates the first derivative. The first part of this result follows from the definition of $f_k(x)$ (only the factor $(x_k - x)$ is taken from $F(x)$). $F'(x)$ will be the sum of n terms, each one of which contains $n-1$ factors, and each term except one contains the factor $x_k - x$. When $x = x_k$, this nonzero term is precisely $-f_k(x_k)$, which establishes the second part.

To form the inverse, write the co-efficients of $-f_k(x)/F'(x_k)$, in row k, $k=1, 2, \cdots, n$ to form the matrix M. We see that the element in row k, column j of MV is precisely $-f_k(x_j)/F'(x_k)$, and result (1) establishes that M is the required inverse.

In actual practice, we can calculate $F(x)$, then divide synthetically by x_1-x to find a vector orthogonal to each of the column vectors of $V(n)$ except the first. This vector can be normalized by dividing each element by $-f_1(x_1)$. The remaining rows can be quickly calculated in a similar manner.

Reference

1. N. Macon and A. Spitzbart, Inverses of Vandermonde Matrices, this MONTHLY, 65 (1958) 95–100.

ON PERFECT MAPPINGS

GEORGE W. DAY and S. ROY SCHUBERT, Philco Corporation, Palo Alto, California

We call a mapping f from a topological space X into a topological space Y *perfect* in case $f[P]$ is dense-in-itself whenever the set P is dense-in-itself in X. Similarly, we say that f is *connected* if the image under f of every connected set in X is connected.

THEOREM. *If X is a locally connected topological space without one point open sets, f: $X\to Y$, and Y has a neighborhood system of closed sets, then the statements*:

(1) *f is perfect*;

(2) *f is continuous*;

(3) *f is connected*;

(4) *if U is an open set in X and $x\in U$, then $f(x)\in\overline{f[U-\{x\}]}$*; *are related in the following way*:

$$(1) \to (2) \to (3) \to (4) \quad and \quad (4) \not\to (3) \not\to (2) \not\to (1).$$

The relation of interest is (1)→(2). The implication (2)→(3) is well-known; the implication (1)→(4), which will be used below, is easily verified, since X has no one point open set. In proving that (1)→(2), we first establish the following lemma.

LEMMA. *Let X and Y be topological spaces, and let f: $X\to Y$. Then* (4) *is equivalent to the condition that the inverse image under f of each nonempty open set of X is dense-in-itself.*

Proof. Suppose that f satisfies (4), that U is an open set in Y, that $x\in f^{-1}[U]$, and that V is an open neighborhood of x in X. Then, by condition (4), $f(x)\in\overline{f[V-\{x\}]}$. Therefore, $U\cap f[V-\{x\}]$ is not empty, and hence $f^{-1}[U]\cap[V-\{x\}]$ is not empty. Thus x is a limit point of $f^{-1}[U]$, and $f^{-1}[U]$ must be dense-in-itself.

Now suppose that f does not satisfy (4). There then exists an open set U in X and a point $x\in U$ such that $f(x)\notin\overline{f[U-\{x\}]}$. Thus the open set $V=Y-\overline{f[U-\{x\}]}$ is such that $f^{-1}[V]$ is not dense-in-itself.

Next, to prove that (1)→(2) under the hypotheses of the theorem, let us suppose that f is perfect, that $x\in X$, and that U is a closed neighborhood of $f(x)$ in

Y. Then, since (1)→(4), $f^{-1}[Y-U]$ is dense-in-itself. Now $x\notin\overline{f^{-1}[Y-U]}$, for otherwise the set $D=\{x\}\cup f^{-1}[Y-U]$ would be dense-in-itself, while $f[D]$ would not. Hence, $V=X-\overline{f^{-1}[Y-U]}$ is an open neighborhood of x, and $f[V]\subset U$. Thus f is continuous.

To show that (3)→(4), let U be an open set in X, let x be a point of U, and let f be connected. Since X is locally connected, there is a connected neighborhood V of x which is contained in U. $f[V]$ is also connected, and we may conclude that $f(x)\in\overline{f[V-\{x\}]}$, for otherwise, since Y has a neighborhood system of closed sets, $\overline{\{f(x)\}}\cap\overline{f[V-\{x\}]}$, would be empty, and $f[V]$ would not be connected. Since $\overline{f[V-\{x\}]}\subset\overline{f[U-\{x\}]}$, we have that $f(x)\in\overline{f[U-\{x\}]}$, which completes our proof.

Now, three simple examples establish the relations (4)$\nrightarrow$(3)$\nrightarrow$(2)$\nrightarrow$(1). First, to show that (4)$\nrightarrow$(3), consider the real function f, defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational;} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

f clearly satisfies condition (4), but is not connected. The real function g, defined by

$$g(x) = \begin{cases} \sin\dfrac{1}{x} & \text{if } x > 0; \\ 0 & \text{if } x \leqq 0, \end{cases}$$

shows that (3)$\nrightarrow$(2). Finally, note that any constant mapping is continuous but not perfect.

This completes the proof of the theorem. As an immediate corollary we see that f is perfect if and only if f is continuous and is not constant on any open set. It is also worth noting that the local connectedness of X was invoked only in proving the implication (3)→(4).

SEPARABILITY AND COMPACTNESS IN POINTWISE PARACOMPACT SPACES

J. N. YOUNGLOVE, University of Missouri

The following two properties of subsets of a topological space are of considerable interest in many settings.

(S) Every subset of a separable set is separable.

(C) The closure of a conditionally compact set is conditionally compact.

Both of the above properties hold for subsets of a metric space. Armentrout and Martin have shown in [1] that (C) holds in a normal T_1 space which satisfies the first axiom of countability. It is the purpose of this paper to demonstrate the relationship between (S), (C), and pointwise paracompactness in developable topological spaces. The terminology and definitions used in this paper are consistent with that of [2].

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland.

EVALUATION OF SURFACE AND VOLUME INTEGRALS OVER A SPHERE

MURRAY S. KLAMKIN, State University of New York at Buffalo

Let us consider the integrals

$$\iint x^a y^b z^c dA, \qquad \iiint x^a y^b z^c dV,$$

for positive integral values of a, b, c, where the region of integration is a sphere of radius R. These integrals arise in determining centroids, moments, moments and products of inertia, etc. of a spherical surface or volume.

We need consider only the case in which a, b, c are all even. Otherwise the integrals will be zero by symmetry. The evaluation of the nonvanishing cases will also be accomplished by using the symmetry of the sphere, i.e.,

$$\iint x^2 dA = \frac{1}{3}\iint (x^2+y^2+z^2)dA = \frac{R^2A}{3} = \frac{4}{3}\pi R^4. \tag{1}$$

$\iint x^4 dA = \iint\{(x+y)/\sqrt{2}\}^4 dA$ (by rotation of axes), or simplifying,

$$\iint x^4 dA = 3\iint x^2y^2 dA. \tag{2}$$

Also,

$$R^4A = \iint R^4 dA = \iint (x^2+y^2+z^2)^2 dA = 3\iint x^4 dA + 6\iint x^2y^2 dA. \tag{3}$$

Hence

$$\iint x^4 dA = \frac{R^4A}{5}, \qquad \iint x^2y^2 dA = \frac{R^4A}{15}. \tag{4}$$

Similarly,

$$\iint x^6 dA = \iint \left\{\frac{x+y}{\sqrt{2}}\right\}^6 dA = \iint \left\{\frac{x+y+z}{\sqrt{3}}\right\}^6 dA,$$

or

$$\iint x^6 dA = 5\iint x^4y^2 dA = 15\iint x^2y^2z^2 dA. \tag{5}$$

(This result corresponds to problem E 1582, this MONTHLY.) Also,

$$R^6A = \iint (x^2+y^2+z^2)^3 dA = 3\iint x^6 dA + 18\iint x^4y^2 dA + 6\iint x^2y^2z^2 dA.$$

$$\iint x^6 dA = \frac{R^6A}{7}, \qquad \iint x^4y^2 dA = \frac{R^6A}{35}, \qquad \iint x^2y^2z^2 dA = \frac{R^6A}{105}. \tag{7}$$

The corresponding volume integrals can be obtained from the following: If $\iint x^ay^bz^c dA = F(R)$ and $\iiint x^ay^bz^c dV = G(R)$, then

$$G(R) = \int_0^R dr \iint x^ay^bz^c dA_r = \int_0^R F(r)dr. \tag{8}$$

Although the higher order integrals of $x^ay^bz^c$ can be evaluated in a similar fashion, it is not worthwhile to pursue it this way. They can be evaluated immediately by Dirichlet's theorem [1] i.e.,

$$\iiint_R x_1^{i_1-1}x_2^{i_2-1}x_3^{i_3-1}dx_1dx_2dx_3 = \frac{a_1^{i_1}a_2^{i_2}a_3^{i_3}}{p_1p_2p_3}\,\frac{\Gamma\left(\frac{i_1}{p_1}\right)\Gamma\left(\frac{i_2}{p_2}\right)\Gamma\left(\frac{i_3}{p_3}\right)}{\Gamma\left(\frac{i_1}{p_1}+\frac{i_2}{p_2}+\frac{i_3}{p_3}+1\right)} \tag{9}$$

where R is the region

$$\left(\frac{x_1}{a_1}\right)^{p_1} + \left(\frac{x_2}{a_2}\right)^{p_2} + \left(\frac{x_3}{a_3}\right)^{p_3} \leqq 1, \qquad x_1,\ x_2,\ x_3 \geqq 0.$$

From (9) and (8) it follows that

$$G(R) = R^{a+b+c+3}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{a+b+c+5}{2}\right)^{-1}, \tag{10}$$

$$F(R) = 2R^{a+b+c+2}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{a+b+c+3}{2}\right)^{-1}. \tag{11}$$

What has been done previously for three space (E_3) can be done for E_n.

At this time it may be pertinent to make the comment that in many calculus courses a lot of time is spent in evaluating 2 and 3 dimensional integrals of the form (9). Consequently, it would be advantageous to introduce Dirichlet's integral theorem even without proof to be used after a few such integrals were

calculated in the usual manner. One would of course also have to include the following properties of the Gamma function:

$$\Gamma(1) = 1, \qquad \Gamma(n+1) = n\Gamma(n), \qquad \Gamma(x)\Gamma(1-x) = \pi \csc \pi x.$$

Reference

1. E. T. Whittaker and G. N. Watson, A course of modern analysis, Cambridge University Press, New York, 1946, p. 258.

LINEAR INEQUALITIES AND ANALYSIS

A. J. Hoffman and M. H. McAndrew, Thomas J. Watson Research Center

The student who first meets the ideas of the theory of linear inequalities in, say, a course in linear programming is sometimes perturbed that his recently acquired proficiency in calculus and analysis generally seems to play almost no role. In fact, apart from the theorem that a continuous function defined on a compact set attains its minimum (used in some proofs of the existence of separating hyperplanes), he will probably see no use of analysis. As a palliative for this cultural shock, we offer two examples (inspired by a proposal of T. S. Motzkin, at least ten years old, of an "exponential" method for solving systems of linear inequalities) of analytic demonstrations of theorems on systems of linear inequalities. As supplements to standard proofs (or as challenges to students to imitate their spirit in proving other results on inequalities), they may be pedagogically useful.

Theorem 1. *Let $a_1, \cdots, a_m$ be vectors in R^n such that $(a_i, x) \leqq 0$ for all i only if $x=0$. Then there exist positive numbers λ_i such that $\sum \lambda_i a_i = 0$.*

Proof. Here (x, y) denotes the scalar product of x and y. Let

$$f(x) = \sum_{i=1}^{m} \exp\,(a_i, x). \tag{1}$$

The function $\max_i(a_i, x)$ is positive everywhere on the unit sphere, hence has a positive minimum δ. From this it readily follows that if x is on a sphere of radius larger than $(\log m)/\delta$, $f(x) > f(0)$. Hence, the minimum of $f(x)$ on that spherical disk is attained in the interior of the disk, at a point x^0, where all partial derivatives are 0.

If we write $a_i = (a_{i1}, \cdots, a_{in})$, then $0 = f_j(x^0) = \sum_i a_{ij} \exp\,(a_i, x^0)$, for every j. Setting $\lambda_i = \exp\,(a_i, x^0)$ yields the theorem.

To prove the next theorem, we invoke a simple lemma.

Lemma. *Let $f(x)$ be a real function defined for all $x \in R^n$, with continuous first partial derivatives everywhere and bounded below. Then, given any $\epsilon > 0$, there exists a point x_0 such that $|f_j(x_0)| < \epsilon$ for all j.*

Proof. Since $f(x)$ is bounded below, we may add a suitable constant and assume that $f(x) > 0$. Choose δ so that

$$0 < \delta < \frac{\epsilon^2}{4f(0)} \cdot \tag{2}$$

Let $g(x) = f(x) + \delta(x, x)$. Clearly, $g(x)$ is large if x is, so there exists a point x^0 at which g attains its minimum, and

$$0 = g_j(x^0) = f_j(x^0) + 2\delta x_j^0. \tag{3}$$

But

$$\delta(x^0, x^0) \leqq f(x^0) + \delta(x^0, x^0) \leqq f(0). \tag{4}$$

The left inequality follows from $f(x) > 0$, and the right inequality from $g(x^0) \leqq g(0) = f(0)$. From (4), we conclude that

$$(x^0, x^0) \leqq \frac{f(0)}{\delta} \cdot$$

From (3), we have

$$|f_j(x^0)| = 2\delta |x_j^0| \leqq 2\delta\sqrt{\{(x^0, x^0)\}} \leqq 2\sqrt{\{\delta f(0)\}}, \tag{5}$$

which, combined with (2), yields the lemma.

THEOREM 2. *Let $\{a_i\}$ be a set of vectors in R^n such that $(a_i, x) < 0$ for every i has no solution. Then there exist $\lambda_i \geqq 0$, $\sum \lambda_i = 1$ such that $\sum \lambda_i a_i = 0$.*

Proof. The function

$$f(x) = \sum_i \exp(a_i, x)$$

is bounded below. For any $\epsilon > 0$, there exists a point x^0 such that

$$\epsilon > |f_j(x^0)| = \left|\sum_i a_{ij}\mu_i\right|,$$

where $\mu_i = \exp(a_i, x^0)$. By hypothesis, at least one $\mu_i \geqq 1$, hence, setting $\lambda_i(\epsilon) = \mu_i / \sum \mu_k$, we obtain

$$\epsilon > \left|\sum_i a_{ij}\lambda_i(\epsilon)\right|,$$

where

$$\lambda_i(\epsilon) > 0, \qquad \sum \lambda_i(\epsilon) = 1.$$

If we choose a sequence of ϵ's approaching 0, a convergent subsequence of $\{\lambda_i(\epsilon)\}$ exists, satisfying the conclusion of the theorem.

Most of the familiar "dual" theorems on systems of linear inequalities can be deduced from Theorems 1 and 2. It might be a worthwhile exercise for a student to approach them directly in imitation of the attack described above. An example is the following theorem:

Let $\{a_i\}$ *and* b *be given vectors in* R^n *such that every* x *satisfying* $(a_i, x) \leqq 0$ *also satisfies* $(b, x) \leqq 0$. *Then there exist numbers* $\lambda_i \geqq 0$ *such that* $b = \sum \lambda_i a_i$.

One way of proving this theorem is to consider the function

$$f(x) = -(b, x) + \sum \exp(a_i, x).$$

Since the hypotheses imply that $f(x)$ is bounded below, the approach used in Theorem 2 is applicable.

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ON THE EQUIVALENCE OF TWO POINT SETS

ALEXANDER R. BEDNAREK, University of Florida

The exhibition of a mapping f effecting the equivalence of the intervals $(0, 1]$ and $(0, 1)$ (see, for example, Kamke [1], p. 15, where $f(x) = 3/2^n - x$ for $1/2^n < x \leqq 1/2^{n-1}$ and $n = 1, 2, \cdots$) suggests, rather immediately, the question as to whether or not it is possible to construct such a mapping having fewer than a denumerable number of discontinuities. This question must be answered negatively.

Suppose, to the contrary, that the mapping $f: (0, 1] \to (0, 1)$ had a finite number of discontinuities in $(0, 1)$ but was one-to-one and onto. Denote these points of discontinuity by $x_1, x_2, \cdots, x_n$. Thus the restriction of f to $E = (0, 1] - \{x_1, \cdots, x_n, 1\}$ is a one-to-one continuous mapping of E onto $F = (0, 1) - \{f(x_1), \cdots, f(x_n), f(1)\}$. The set E is the union of $n+1$ disjoint open intervals while the set F is the union of $n+2$ disjoint open intervals, denoted by E_i, $i = 1, \cdots, n+1$ and F_j, $j = 1, \cdots, n+2$ respectively. Since the restriction of f to E is a continuous real function, each set $f(E_i)$, $i = 1, \cdots, n+1$, must be an interval (the continuous image of a connected space is connected) and, therefore, a subset of F_j for some $1 \leqq j \leqq n+2$. But this implies that $f^{-1}(F_{j^*}) = \varnothing$ for some $1 \leqq j^* \leqq n+2$. In view of this contradiction, f cannot have a finite number of discontinuities in $(0, 1]$.

Reference

1. E. Kamke, Theory of sets, Dover, New York, 1950.

A "PRACTICAL" APPLICATION FOR DIGITAL COMPUTERS

R. A. JACOBSON, South Dakota State College

The purpose of the following paper is to present a "practical" application in computer programming which is easily visualized, simple in nature, and yet can be made quite diverse and extremely challenging to prospective students who are interested in writing their own programs.

The problem of finding the expected score for a solitaire game involving n distinct cards, discussed in a recent paper [1], is readily generalized by con-

sidering a deck in which card $[q]$ occurs m_q times; that is, a deck of m cards, numbered 1 to n, such that $m_1+m_2+\cdots+m_n=m$. Although this problem can be handled in much the same manner as the problem in the previous paper, the recursion relation which is developed is often most readily evaluated by a digital computer. Indeed, a familiarity with the following problem will enable one to construct a multitude of "practical" computer problems utilizing many programming techniques in varying degrees of complexity, simply by changing the distribution of cards in the deck or the method of scoring the game. It is assumed that the reader will become well acquainted with the method of attack employed in the previous note in order to avoid an unnecessary amount of repetitive explanation.

The deck of cards, as described above, is shuffled and the cards are placed face up, one at a time. A number of points are to be scored each time a card is turned up, and the score is to be based on the card just turned up and all preceding cards. The first card turned up will contribute a score of zero. Our problem is to find the expected score, assuming that all arrangements of the deck are equiprobable.

Example. Let a 10 card deck contain 2 one's, 4 two's, 1 three, and 3 four's. Score two points for each preceding card which is odd numbered and less than or equal to the card just turned up. Score one point for each preceding card which is greater than the card just turned up.

In the following, $f(a, b)$ is the score achieved whenever card $[a]$ precedes card $[b]$. The expectation and the sum for all $k!$ equiprobable outcomes for a deck of k cards are denoted by E_k and S_k, respectively. In finding E_m, we consider sub-decks of the original deck and employ the following recursion relation.

THEOREM 1. *Let m_q be the quantity of cards $[q]$ in a sub-deck of k cards. The expectation E_{k+1} for a sub-deck of $k+1$ cards consisting of the previous sub-deck and the card $[a]$ is given by*

$$E_{k+1} = E_k + \frac{1}{2}\sum_{q=1}^{n} m_q\{f(a, q) + f(q, a)\}, \qquad k = 1, \cdots, m-1. \tag{1}$$

Proof. Consider the $k!$ by k array (matrix) of equiprobable outcomes of a deck of k cards where each row indicates one ordering in which cards are turned face up, one at a time. The $(k+1)!$ by $k+1$ array obtained by including card $[a]$ in the deck can be found by inserting card $[a]$ between two successive cards in the original array. Noting that card $[q]$ occurs in $m_q(k-1)!p$ positions in columns $1, \cdots, p$ and in $m_q(k-1)!(k-p)$ positions in columns $p+1, \cdots, k$ of the original matrix, we apply the reasoning developed in the previous paper and find that the sum of the scores obtained by all possible insertions of the card $[a]$ between the cards in columns p and $p+1$ is

$$S_k + \sum_{q=1}^{n} m_q(k-1)!pf(q, a) + \sum_{q=1}^{n} m_q(k-1)!(k-p)f(a, q). \tag{2}$$

Sum (2) also holds if $p=0$, k; in other words, if card $[a]$ is first or last. Summing (2) for $p=0, 1, \cdots, k$, we find S_{k+1}. Equation (1) follows directly from the fact that $S_k = k!E_k$ and $S_{k+1}=(k+1)!E_{k+1}$.

In the example above, $f(x, y) = 1-(-1)^x$, $x \leqq y$; $f(x, y) = 1$, $x > y$. To calculate E_{10} we might use the sub-decks $\{1\}$, $\{1, 1\}$, $\{1, 1, 2\}$, $\{1, 1, 2, 2,\}$, $\cdots$, etc., chosen by placing the cards of the original deck in ascending order; that is, if $[a]$, $[b]$ are the numerically greatest cards in the sub-decks containing j, k cards, respectively, then $j<k$ implies $a \leqq b$.

Calculating successive E_k's, we let $m_1=1$, $m_2=m_3=m_4=0$ and find

$$E_2 = E_1 + \frac{1}{2}\sum_{q=1}^{4} m_q\{f(1, q) + f(q, 1)\} = 2.$$

Next, $m_1=2$, $m_2=m_3=m_4=0$, and it follows that

$$E_3 = E_2 + \frac{1}{2}\sum_{q=1}^{4} m_q\{f(2, q) + f(q, 2)\} = 5.$$

Proceeding, we have $m_1=2$, $m_2=1$, $m_3=m_4=0$, and

$$E_4 = E_3 + \frac{1}{2}\sum_{q=1}^{4} m_q\{f(2, q) + f(q, 2)\} = 8.$$

The iterative process continues, giving $E_5=11$, $E_6=14$, $E_7=19$, $E_8=51/2$, $E_9=32$. Finally, $m_1=2$, $m_2=4$, $m_3=1$, $m_4=2$, and we see that

$$E_{10} = E_9 + \frac{1}{2}\sum_{q=1}^{4} m_q\{f(4, q) + f(q, 4)\} = 77/2.$$

Relation (1) also permits us to calculate the expected score for a different deck without necessarily recalculating the expectations of all sub-decks. For instance if the deck in the example were to contain 2 one's, 3 two's, 2 three's, and 3 four's, we can find the new expectation, E_{10}^*, by utilizing the previous result. Since E_{10} can be obtained by adding a two to the 9-card deck $\{1, 1, 2, 2, 2, 3, 4, 4, 4\}$ it follows that the expected value, E_9^*, for this deck can be found by considering the relation

$$E_{10} = E_9^* + \frac{1}{2}\sum_{q=1}^{4} m_q\{f(2, q) + f(q, 2)\}$$

where $m_1=2$, $m_2=m_4=3$, $m_3=1$. Since $E_{10}=77/2$, it is evident that $E_9^*=67/2$. Then adding a three to this 9-card deck, we have $m_1=2$, $m_2=m_4=3$, $m_3=1$; and thus

$$E_{10}^* = E_9^* = \frac{1}{2}\sum_{q=1}^{4} m_q\{f(3, q) + f(q, 3)\} = 89/2.$$

By basing the score on the card just turned up and its immediate predecessor *only*, we find a more challenging recursion relation given in the following theorem.

THEOREM 2. *Let m_q be the quantity of cards $[q]$ in a sub-deck of k cards. The expectation E_{k+1} for a sub-deck of $k+1$ cards consisting of the previous sub-deck and the card $[a]$ is given by*

$$\begin{aligned} E_{k+1} = E_k &+ \frac{1}{k+1}\sum_{q=1}^{n} m_q\{f(q,a)+f(a,q)\} \\ &- \frac{1}{k(k+1)}\sum_{q=2}^{n}\sum_{r=1}^{q-1} m_q m_r\{f(q,r)+f(r,q)\} \\ &- \frac{1}{k(k+1)}\sum_{q=1}^{n} m_q(m_q-1)f(q,q); \quad k = 1, 2, \cdots, m-1. \end{aligned} \tag{3}$$

The proof entails a reapplication of the reasoning in the above paragraphs and the reference [1].

Reference

1. R. A. Jacobson, Expectation for solitaire, this MONTHLY, 71 (1964) 65–69.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

MINNESOTA MATHEMATICS AND SCIENCE TEACHING PROJECT (MINNEMAST)

P. C. ROSENBLOOM, University of Minnesota and Minnesota State Dept. of Education

This project aims to produce a coordinated mathematics and science curriculum for grades K-9, and undergraduate courses for the preservice education of teachers. The project began in 1961, and has produced mathematics for K-3, nine science units for various grades, mathematics tests for K-1, and outlines and sample chapters of college mathematics and science subject matter and methods courses.

During the summer of 1963, there were 55 scientists, mathematicians, educators, and schoolmen at the MINNEMAST writing conference. The school materials are being tried out in about 250 classes this year at 20 centers managed by cooperating colleges around the country. It is expected that, when the first draft of the MINNEMAST undergraduate program is ready in September, 1964, these colleges will try out the courses with prospective teachers.

A. Douglis, of the University of Maryland, is in charge of the MINNEMAST college mathematics courses. Our first aim here is to produce a one-year course suitable for the liberal education of any college student. This will include an adequate sampling of the content recommended by CUPM for elementary teachers, so as to give the students some background in arithmetic, algebra, geometry, probability, as well as some intuitive calculus. Attention will be given to the connections between mathematics and science, and to the role of mathematics in the history of our culture.

After we are confident that we have a satisfactory course of this type, we shall consider the production of a second year course, aimed primarily at elementary education majors. This should enable the cooperating colleges ultimately to implement the full CUPM recommendations for Level I courses.

Last summer, D. E. Meyers, of the University of Arizona, and L. G. Woodby, of the U. S. Office of Education, collaborated with the author in supervising a team of school teachers who revised and extended our school materials.

In general, the recommendations of the Cambridge *Conference on School Mathematics* (see this MONTHLY 70 (1963) 888) which were made independently of our previous work, agree with our school curriculum quite well, except for comparatively minor details.

Like the Cambridge Conference, we take as our first main goal the early introduction of the entire real number system. We lead the children, by third grade, to work with coordinates, and give them geometrical algorithms for the arithmetical operations which apply to all real numbers.

Their expected experience with measurement in the science curriculum, now under development, will give the children interesting data to represent graphically and algebraically. The children will use their graphs and equations to make predictions which can be tested by further observation.

We have also introduced sampling and random walk experiments to lay the groundwork for an early introduction of probability.

We have also introduced games based on Post's canonical systems, to lay the groundwork for the concept of a deductive science, and the comparison of systems with each other.

Space does not permit any detailed description here of our plans for the mathematics curriculum in grades 4–9. We hope that other mathematicians, who are thinking along the lines of the Cambridge report, will be interested in joining us next summer.

THE POLARIS MATHEMATICS PROGRAM

FRANCIS SCHEID, Boston University

The U. S. Navy has begun an extensive effort to provide educational opportunities for members of its Polaris submarine crews. Through the Harvard University Commission on Extension Courses it has planned to offer a program which is equivalent to two years of college work, and to provide extensive lec-

ture series on film so that a classroom situation can be approximated during the long undersea tours of duty. About one quarter of the program (eight quarter-courses) is mathematics.

Each course includes fifteen films to be viewed and problems to be worked. Before the cruise and after the cruise, shore sessions with instructor offer some chance to get questions answered, but the courses are largely self-study. The opportunity to plan and produce this mathematics series, including 120 thirty-minute films, is an unusual one, and I thought that others might wish to hear what has already been done. I will certainly be interested in hearing comments or criticism, favorable or not, from any reader of this brief report.

The Polaris men are believed to be among the best educated of naval personnel. Even so, I felt it would be wise to begin at precalculus level and, though calculus was especially requested, to include other parts of mathematics as well. I have kept the recommendations of both CUPM and SMSG constantly in mind, but have taken the opportunity to implement some personal convictions as well.

For example, I have found the view of mathematics as a collection of games, each game having its own rules (axioms) to be well received by the average student. It gives a compact but realistic answer to the question of what mathematics *is*, a question which more and more people ask nowadays when school curricula begin to include unfamiliar things under the mathematics label. The game-like nature of mathematics can be used to emphasize:

1. the importance of knowing the axioms (you should know which game you are playing),
2. the importance of knowing the theorems (they are the strategy of the game, separating the skillful player from the amateur).

It can also be used to point up two of the principal goals of mathematics teaching, which are heavily emphasized in the series:

1. Mathematics is honest. (You play strictly by the rules.)
2. Mathematics is useful. (Many of the games have serious applications.)

Thinking of mathematics as game-like leads almost automatically to an effort at total integration of the usual algebra and calculus content. After all, algebra and calculus are parts of the same "game of numbers." The basic rules are no different for the one or the other. This appears to suggest a thorough reordering of our topic presentation, and in a preliminary way I have tried to do some reordering. The real number story is told first, not avoiding the idea of limit, without which the complete rule book of the game of real numbers cannot be written out. Numerous ideas of "algebra" (such as logarithm and exponent, polynomial and graph) are postponed, to be brought in when actually needed in an integrated approach. The general idea is to offer a totally integrated, more or less self-contained sequence through algebra, coordinate geometry, trigonometry and calculus. In addition, work in modern algebra and probability is to be included, but these will be treated as other games, having their own basic rules.

The topics chosen are obviously those which currently find place both in

school and college programs. Whether the ultimate product of this effort falls in the school or the college category remains to be seen, but in these days of violent curriculum permutation it is sometimes difficult to locate the borderline.

Descriptions of the first two (of eight) series follow.

The game of numbers. This series in college algebra presents an introductory but mature treatment of the arithmetic of real numbers, from the point of view of modern mathematics. Many of the ideas presented are, therefore, old familiar ideas; but to catch the spirit of mathematics, the presentation is from what may be an unfamiliar point of view. Numbers are treated as the elements, or pieces, of an abstract "game of numbers," and the fact that mathematics is definitely game-like is a major objective. The honesty and usefulness of mathematics are emphasized. As the series develops, the different kinds of real number enter the game one by one, always because applications have called for them. The positive integers naturally enter first, then zero and the negative integers, followed by the rational numbers. Finally the irrationals appear, the need for them having been carefully detailed by conceptual experiments involving the problems of exact measurement. The connection between the concepts of number and limit appears naturally in these experiments, and is the climax of the series. It receives heavy emphasis, because an understanding of the number-limit relationship is critical for understanding modern applications of the game of numbers. As the different kinds of number enter the game, theorems which permit computations with them are included, but this series is far from a technical manual for algebraic maneuvers. It is aimed at those who wish to *understand* the real number system, either to teach it at elementary or secondary school level, or because it is essential to an understanding of calculus.

Coordinate geometries. This is a beginning course in analytic geometry, which means geometry by a coordinate approach. Principal attention is given to two-dimensional Euclidean geometry. The coverage of this material is entirely independent of school work. A fresh start is made, using coordinates, so that geometry grows out of algebra. The ideas of position, distance, straight line, parallel, perpendicular, intersection, circle, parabola, length, angle, area, over, under, inside and outside are introduced analytically, by means of coordinates. Different observers, using different frames of reference and getting different coordinates for the same point, are introduced; the question of what is relative to the observer and what is invariant, the same for all observers, gets special attention. Numerous applications are suggested. Other, less useful, more exotic geometries are also mentioned for purposes of comparison. These include taxicab geometry and a chessboard geometry. They serve to emphasize that there is no one absolutely true geometry, but many geometries, all of which are man-made games useful under the right circumstances. The overall spirit is that geometry can be made an outgrowth of the game of numbers, and in most applications this is exactly what is wanted. The course is suitable for nonscience students, for teachers of mathematics, and as preparation for calculus.

WHITHER MATHEMATICS CONTEST WINNERS?

NURA D. TURNER, State University of New York at Albany

What happens to the top-ranking high school students who enter the Annual High School Mathematics Contest sponsored by the Mathematical Association of America and the Society of Actuaries? This question arose during my continuing study of the academic progress of the students who ranked in the top one per cent in the Upstate New York Section in the 1958 Contest, a study expanded first to include a number of students who ranked at the top nationally, and later to include students who ranked in the top one per cent in the Upstate Section in both the 1959 and 1960 Contests. Plans are to continue communication with these students through graduate years and beyond.

Although one hundred twenty students are involved, first reference here will be to only the fifty-nine of the 1958 groups. Thirty-six of these students are known to have at least bachelor's degrees. Among the thirty-three who are in graduate school, the frequencies of careers planned are as follows: actuarial science 1; aeronautical engineering 1; college teaching and/or research 7, (engineering 1, mathematics 4, physics 2); computer programming 1; electrical engineering 3; management consulting 1; mathematics 5; medicine 2; music 1 (violin); physics 4; public accounting 1; zoology 1; undecided 5.

Among the three remaining students with bachelor's degrees, the one who is employed in industry is still undecided. He is doing actuarial work because he is not sure what he wants to do for his "life's work." The other two are a housewife married to a high school teacher of mathematics and an electrical engineer with IBM. Of the thirty-six who hold bachelor's degrees, twenty-three are in science, twelve in arts, and one in music.

There remain twenty-three of the fifty-nine in the 1958 groups for whom to account. Fifteen are still undergraduates (twelve of these were high school sophomores or juniors in 1958) and three are involved in five year undergraduate programs (two in architecture and one in electrical engineering). Eight have not replied to the inquiries sent; one of these applied to medical school.

As early as the fall of 1961, the study aimed to follow progress on the graduate level. At that time two students had entered graduate school after three years of undergraduate study, one to study medicine and the other to study toward a doctorate in mathematics. The latter received an M.S. in mathematics in June, 1962. In the fall of 1962 three others entered graduate school after no more than three years of undergraduate study. One began work toward a doctorate in pure mathematics. A second who had received a B.S. in electrical engineering, began a program for a master's degree in physics. The third began work in astro-physics. These accomplishments have been made possible by the advanced placement program and credit given for college caliber work done in high school.

Twenty of the graduate students have teaching assistantships or fellowships

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of from $1350 to $3650 per year, nine of which are NSF fellowships. Two of the fifty-nine members flunked out of college, but one has made a good comeback after having been unsuccessful with mechanical engineering, and having changed from engineering to the ministry to programming or teaching. The average grade over all subjects for the Upstate New York group is 2.99. That for the National group is 3.26. It is to be understood, however, that the students in the Upstate New York group ranked in the top one per cent in that section, while those in the National group ranked in the top .03 per cent nationally.

Already there is evidence of the contribution these students are likely to make in mathematics. An article by Alan Zame, graduate student in mathematics at the University of California at Berkeley, appeared in this MONTHLY 70 (1963) 531–535.

FINAL REPORT ON NASDTEC-AAAS STUDIES

NASDTEC-AAAS Studies started in December 1960 and terminated in the fall of 1963. The American Association for the Advancement of Science and the National Association of State Directors of Teacher Education jointly sponsored the studies leading to the publication of guidelines for the preparation of secondary and elementary school teachers in the areas of science and mathematics. The Committee on the Undergraduate Program in Mathematics cooperated very closely with the Studies and assisted in obtaining the participation of a considerable number of mathematicians in the various conferences and preparation of the final guideline statements. A number of publications which resulted from the studies can be obtained upon request from the American Association for the Advancement of Science. These are:

1. Guidelines for Preparation Programs of Teachers of Secondary School Science and Mathematics.
2. Guidelines for Science and Mathematics in the Preparation Program of Elementary School Teachers.
3. New School Science. A Report of Regional Conferences of School Administrators.
4. Secondary School Science and Mathematics Teachers: Characteristics and Service Loads. (Also available from NSF.)

The preparation of the Guidelines (1 and 2) was supported by the Carnegie Corporation of New York. The studies which resulted in publications 3 and 4 were supported by the National Science Foundation.

NSF COOPERATIVE COLLEGE—SCHOOL SCIENCE PROGRAM GRANTS FOR 1964

The National Science Foundation has recently announced the awarding of $749,000 in grants to 41 educational and research institutions as a part of a program to improve the teaching of science and mathematics in the nation's high schools. The grants will help to provide training for 934 secondary school teachers and 2115 secondary school students of high ability.

Under this program funds are granted to colleges, universities, and non-

profit research organizations which in turn use their facilities and faculties to assist teachers and students from schools planning improvements in science and mathematics offerings. In some cases college and university scientists work with school administrators to plan introducing new course material. In most cases improvement calls for additional training of teachers followed by a year or more of regular consultations. In many cases able young students are chosen to participate in the training programs.

Twenty-three of the programs will take place in the summer of 1964; 13 during the 1964–1965 school year and an additional 6 will take place both during the summer and the coming school year. There will be 22 courses in the biological sciences, 16 in chemistry, 23 in mathematics including some in the use of electronic computers, 6 in engineering, 11 in physics, 3 in oceanography, 1 in psychology, and 3 in the history of science. Many of the programs will be organized around research projects and in some cases teams of high school students and teachers will join university professors and their graduate students in fundamental new research.

Among the colleges or universities receiving a grant for programs in mathematics only are the following:

University of California, Berkeley
University of Miami, Coral Gables, Florida
University of Minnesota
Northeast Missouri State Teachers College, Kirksville
University of New Hampshire
Austin Peay State College, Clarksville, Tennessee
St. Mary's University, San Antonio, Texas
Emory and Henry College, Emory, Virginia
Virginia Polytechnic Institute, Blacksburg
Fairmont State College, Fairmont, West Virginia
University of Puerto Rico

NEW FEDERAL SUPPORT FOR VOCATIONAL EDUCATION

On December 12 and 13 of 1963, the House and the Senate passed and sent to President Lyndon Johnson major legislation for the improvement of education with specific reference to vocational education. The $1,560,000,000 package increases federal funds for vocational education and extends and expands the National Defense Education Act of 1958. The new legislation authorizes a permanent program with support for state vocational education programs amounting to $60 million for fiscal year 1964 and increased amounts during subsequent fiscal years. Funds will be allotted among the states on the basis of population groups and a per capita income factor. The new funds may be expended for state and local vocational education programs without categorical limitation under a broadened definition of vocational education to fit individuals for gainful employment, embracing all occupations, including business and office occupations not now covered under existing law. The new law requires the state administering agency to review vocational education programs peri-

odically in the light of current and projected manpower needs. Ten per cent of each year's appropriation will be reserved for grants by the Commissioner of Education for research and demonstration projects in vocational education.

The bill is of interest to mathematicians because of an indication of increased federal support for education and also because city superintendents and others responsible for technical education programs see in the new programs an important place for science and mathematics quite unlike science and mathematics programs which have been a part of traditional vocational education.

Among provisions of the one-year extension of the National Defense Education Act were modifications in the student loan and graduate fellowship programs. The Language Development Title in the new legislation will include teachers engaged in teaching English as a second language. The so-called "Goldwater amendment," prohibiting psychological testing, was deleted in conference.

PLANS FOR NEW FILMS AND FILMSTRIPS

The Panel on the Preservice Training of Elementary Teachers of the Committee on Educational Media is planning a workshop in the summer of 1964. The goal of this Panel is to prepare a course on motion picture film with ancillary text material, filmstrips, etc., on the "Foundations of the Number System and Arithmetic," generally adhering to the spirit of the CUPM Level I recommendations for the education of elementary school teachers. Thus, the course will be designed for the education of college students who will become elementary school teachers. While methodology will be a consideration, the mathematical subject matter will have paramount importance.

The summer workshop is expected to begin June 22, 1964 and to last eight weeks, until August 15. It is hoped that this can be held in conjunction with workshops of our Panel on Programmed Learning and our Panel on the Calculus course. This will enable a simultaneous education of the participants in certain aspects of motion picture production and techniques. Generally, participants will be paid travel expenses, a subsistence allowance, and a premium salary computed by a formula similar to that used by other curriculum reform groups.

It is expected that this activity will be supported by a grant from the National Science Foundation. The workshops will be held, it is hoped, in the San Francisco Bay Area, or nearby.

STUDY OF ACCREDITATION IN TEACHER EDUCATION

The National Commission on Accrediting and the Carnegie Corporation of New York have announced the establishment of a study of the influence on higher education of accreditation in teacher education. This study is planned for a 15-month period. Headquarters of the study will be 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005. John R. Mayor will be in charge of the study, assisted by Dean Willis Swartz of Southern Illinois University. The study is intended to include teacher accreditation policies of the National Council on Accreditation of Teacher Education (NCATE), the regional accrediting

associations, and accreditation by the state departments of education in the 50 states. At the termination of the study a report will be made to the National Commission on Accrediting and, at the same time, the report will become a public document.

PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (other than proposers') should be sent to Professor Starke.

ELEMENTARY PROBLEMS

All solutions of Elementary Problems should be sent to A. E. Livingston, Dept. of Math. University of Alberta, Edmonton, Alberta, Canada. To facilitate their consideraiion, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before July 31, 1964.

E 1681. *Proposed by Azriel Rosenfeld, The Budd Company, Silver Spring, Maryland*

Find the unique solution to the cryptarithm

$$\begin{array}{rrrrrr} & & T & H & I & S \\ & & & I & S & A \\ & G & R & E & A & T \\ & & T & I & M & E \\ \hline W & A & S & T & E & R \end{array}$$

E 1682. *Proposed by V. R. Rao Uppuluri, Oak Ridge National Laboratory*

Show that

$$\left(\sum_{i=1}^{n} a_i/i\right)^2 \leqq \sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j/(i+j-1).$$

E 1683. *Proposed by Eugène Ehrhart, Strasbourg, France*

Let $\alpha=\pi/7$. Show that

$$\sum_{n=1}^{6} \sin n\alpha/\sin 5n\alpha = 4, \qquad \sum_{n=1}^{6} \sin 5n\alpha/\sin n\alpha = 2.$$

E 1684. *Proposed by E. A. Lee, Albuquerque, New Mexico*

If p and q are given positive prime numbers, show that there is only one power of q which is representable as a sum of nonnegative integral powers of p.

E 1685. *Proposed by F. D. Parker, State University of New York at Buffalo*

A square matrix M of order n has the properties that $a_{ii}=1$ and $a_{ik}a_{kj}=a_{ij}$ for all i, j, k. What are the characteristic values of M?

E 1686. *Proposed by Richard Sinkhorn, University of Houston*

If p is a nonnegative integer, evaluate

$$\lim_{x\to 1-} (1-x)^{p+1} \sum_{n=1}^{\infty} n^p x^n.$$

E 1687. *Proposed by Daniel Pedoe, Purdue University*

UVW is an equilateral triangle; A, B, C are the respective midpoints of the sides VW, WU, UV; A' is any point on line VW, B' any point on line WU, and C' any point on line UV. If P is the intersection of BC and $B'C'$, Q of CA and $C'A'$, R of AB and $A'B'$, prove that

(1) the lines $A'P$, $B'Q$, $C'R$ are concurrent,

(2) the areal coordinates of the point of concurrency with respect to triangle ABC are, with a suitable sign convention, $(AA')^{-1}:(BB')^{-1}:(CC')^{-1}$.

Generalize both (1) and (2) by means of an affine projection, and generalize (1) by a general projection.

E 1688. *Proposed by R. A. Jacobson, South Dakota State College*

The graph of the relation $|x+y|+|x-y|=2$ is a square with sides of length 2. Find a relation of the form $\sum_i |a_ix+b_iy+c_i|=A$ such that its graph is a regular octagon with sides of length 2.

E 1689. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College*

Let f be an integral-valued function defined for all nonnegative integers x such that $0\leqq f(x)\leqq x$. Must the graph of f have an infinite subset whose points are collinear?

E 1690. *Proposed by D. E. Daykin, University of Reading, England*

Let n be a positive integer, p a prime, $q=p^n$, and $E=(q^p-1)/(q-1)$. Prove that $f(x)=x^q-x-1$ divides x^E-1 modulo p if and only if $n=1$.

SOLUTIONS OF ELEMENTARY PROBLEMS

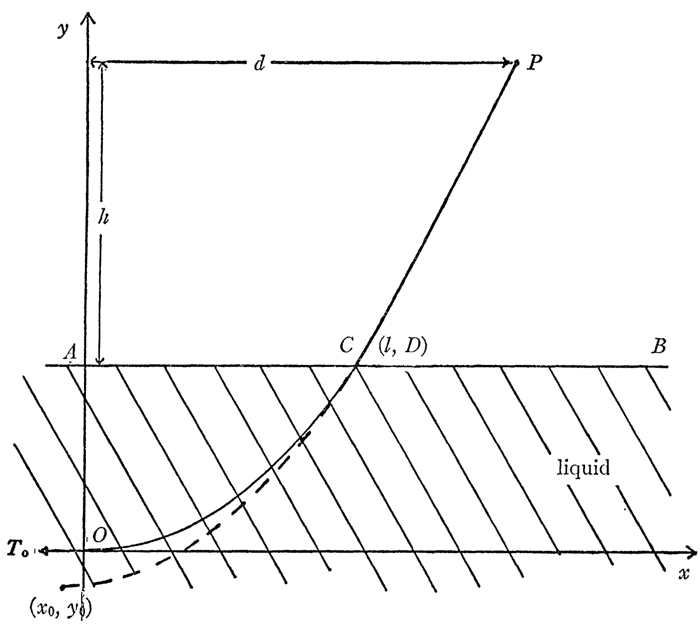
Partially Submerged Hanging Chain

E 1020 [1952, 329]. *Proposed by John Disch, Cleveland, Ohio*

A perfectly flexible inextensible chain of uniform density is hanging from

two supports at equal heights above the surface of a liquid. If the chain sags into the liquid find the dip in the chain in terms of those factors that affect the problem.

Solution by Neil Ashby, University of Colorado. See the accompanying figure. Since the shape of the chain will be symmetric about a vertical drawn through the vertex, we need only consider half of the chain. For simplicity, let the length of the arc of chain which hangs between the fixed support P and the origin O be unity. The coordinates of P are $(d, h+D)$, where h is the distance between the level of P and the surface AB of the liquid, D is the distance below



AB of the lowest point of the cable where the origin is placed. We suppose that the chain has a weight per unit length of unity when outside the liquid, and an effective weight per unit length $\lambda<1$ when inside the liquid. We let T_0 be the tension in the cable at (0, 0). In mechanics texts it is proved that the form of a hanging chain is a catenary. Thus, inside the liquid, the chain has the form

$$y = c(\cosh x/c - 1), \tag{1}$$

and outside the liquid,

$$y - y_0 = c'[\cosh (x - x_0)/c' - 1], \tag{2}$$

where the parameter c of the catenary inside the liquid is defined as

$$c = T_0/\lambda, \tag{3}$$

and where (x_0, y_0) would be the position of the vertex of the portion CP of the

catenary if extrapolated into the liquid, and c' is some constant to be determined. We let the coordinates of the point where the cable breaks the surface be (l, D), and the length of the cable within the liquid be s, where, from the study of the catenary,

$$s = c \sinh l/c. \tag{4}$$

With our choices of parameters, we regard λ, d, h as given and ask to find the parameters x_0, y_0, l, D, s, c, c', and T_0 as functions of λ, d, and h. In addition to equations (2) and (3), we need six more equations. These are obtained as follows: From (1) we obtain, obviously,

$$D = c(\cosh l/c - 1). \tag{5}$$

The two catenaries must pass through the point C with the same slope; otherwise the point C would not be in static equilibrium. We thus obtain from (1) and (2),

$$c(\cosh l/c - 1) = y_0 + c'[\cosh (l - x_0)/c' - 1] \tag{6}$$

and

$$l/c = (l - x_0)/c'. \tag{7}$$

It is known from mechanics that the tension T at any point of a hanging chain of unit weight per unit length, at a distance $y-y_0$ above the origin, is $T=y-y_0+c'$, so the vertical component of this tension at P, where $y=h+D$, must support the effective weight of the entire chain, which is $1-s+\lambda s$. Hence

$$T_P \sin \theta = (h + D - y_0 + c') \tanh (d - x_0)/c' = 1 - s + \lambda s. \tag{8}$$

The horizontal component at any point must equal T_0 since the supporting force of the liquid is entirely vertical. Hence, at P,

$$(h + D - y_0 + c')/\cosh (d - x_0)/c' = T_0, \tag{9}$$

while along CP we find

$$T_0 = [c' \cosh (x - x_0)/c']/\cosh (x - x_0)/c' = c'. \tag{10}$$

Equations (3) through (10) constitute eight simultaneous equations in the eight unknowns x_0, y_0, l, D, s, c, c', and T_0. Eliminating all unknowns except l and c we find the two equations

$$h + \lambda c \cosh l/c = \lambda c \cosh [d + (1 - \lambda)l]/\lambda c, \tag{11}$$

$$\begin{aligned} 1 - (1 - \lambda)c \sinh l/c & \\ &= [h + \lambda c \cosh l/c - (1 - \lambda)c] \tanh [d + (1 - \lambda)l]/\lambda c. \end{aligned} \tag{12}$$

Neither of these transcendental equations may be solved explicitly for l in terms of c or vice versa, and therefore only graphical or numerical solutions are possible. Having obtained l and c graphically, it is a simple matter to obtain the values of all the other unknowns.

Thus the problem has no solution which can be given in closed form. One may verify this fact in another way by introducing the variable $u = e^{l/c}$. Equations (11) and (12) may then be combined in such a way as to give an eighth-order algebraic equation for u, which, as is well known, cannot be solved in closed form.

A Pronounced Failure to Generalize

E 1601 [1963, 668]. *Proposed by A. A. Mullin, University of Illinois*

If d is a nonnegative decimal integer, let d_n be the decimal integer obtained from d by keeping only the last n digits of d. Prove that $2^n \mid d$ if and only if $2^n \mid d_n$.

Solution by J. C. Abad, San Francisco, Calif. Any d can be expressed in the form $d = 10^n a + d_n$ for some integer a. Since $2^n \mid 10^n$, the result follows.

Also solved by W. M. Angel, K. F. Bailie, Raymond Balbes, Randy Barron, Robert Bart, E. P. Berger, Andreas Blass, Walter Bluger, W. R. Boland, Robert Bowen, J. J. Bowers, W. G. Brady, D. A. Breault, Brother T. C. Wesselkamper, P. G. Carr, Yuan Chang, Gene Chase, D. I. A. Cohen, D. M. Cohen, Martin Cohen, Wayne Cutrer, E. R. Deal, H. J. de St. Germain, P. F. Duvall, Jr., E. S. Eby, R. J. Eckert, J. A. Erbacher, C. G. Fain, C. L. Fefferman, Andrew Feldmar, Stephen Fisk, Hyman Gabai, Anton Glaser, G. S. Glazer, J. B. Goebel, J. S. Golan, Ira Goldstein, Jerry Goodman, R. M. Grassl, Ralph Greenberg, Cornelius Groenewoud, Andras Gyarfas, J. D. Haggard, W. J. Hansen, H. W. Hickey, D. W. Hight, R. A. Jacobson, J. E. Jean, Jr., A. W. Johnson, Jr., A. F. Kaupe, Jr., B. G. Klein, Kenneth Kramer, Sidney Kravitz, Joel Kugelmass, S. Lajos, E. S. Langford, S. J. Lawrence, J. F. Leetch, C. J. Mabee, D. C. B. Marsh, Stephen Montague, Sam Newman, P. R. Nolan, Michael Ohnick, Lewis Parker, M. J. Pascual, Stanton Philipp, Anatol Rapoport, Thomas Renfrow, Henry Ricardo, J. T. Robbins, G. S. Rogers, Dorothy S. Rutledge, Perry Scheinok, J. A. Schumaker, Clyde Schwartz, D. L. Silverman, Arnold Singer, E. M. Stone, R. P. Tapscott, Rory Thompson, A. M. Vaidya, Simon Vatriquant, Gary Venter, John Vinson, Julius Vogel, Charles Wexler, W. C. Waterhouse, Ron Wilder, D. G. Wilson, P. O. Wood, Jr., K. L. Yocom, Anthony Zee, David Zeitlin, and the proposer.

Editorial Note. A pronounced tendency in mathematics today is that of generalization. The above very simple problem was proposed, with certain misgivings, at the request of a group of interested mathematicians who were curious to see how many solvers would, in some way or other, generalize the problem. Of the 96 submitted solutions, only 4 offered generalizations. The broadest generalization submitted was: If d is a nonnegative integer to base p, d_n is the nonnegative integer to base p obtained from d by keeping only the last $n>0$ digits of d, and q is any positive divisor of p, then $q^n \mid d$ if and only if $q^n \mid d_n$.

Disconnecting a Graph

E 1602 [1963, 668]. *Proposed by M. C. Gemignani, University of Notre Dame*

Given a set of n points in Euclidean space such that no three of the points are collinear, prove that the set of all line segments determined by these n points can be disconnected by no set of less than $n-1$ points, and there is at least one set of $n-1$ points disconnecting the set of segments, where we require that no more than one point of the disconnecting set can come from any one segment.

Solution by D. I. A. Cohen, Princeton University. Any m points are connected to the other $n-m$ by $m(n-m) \geqq n-1$ lines. So to disconnect a group of m

points requires at least $n-1$ disconnecting points. The minimum occurs when we disconnect 1 point by taking a point from each of the $n-1$ lines issuing from it.

Also solved by Ralph Bennett, D. C. B. Marsh, Stephen Montague, and the proposer. Three of these solutions employed induction on n.

A Generalization of Wilson's Theorem

E 1603 [1963, 668]. *Proposed by R. S. Luthar, University of Illinois*

If p is a prime not less than n, where n is a given positive integer, show that $(n-1)!(p-n)!\equiv(-1)^n$, mod p.

Solution by W. C. Waterhouse, Harvard University. Since $(-1)k\equiv p-k$, we have $(n-1)!\equiv(-1)^{n-1}(p-1)\cdots(p-n+1)$, and $(n-1)!(p-n)!\equiv(-1)^{n-1}(p-1)!$. The result then follows by Wilson's theorem, which states that $(p-1)!\equiv-1$.

Also solved by J. C. Abad, A. N. Aheart, Joseph Arkin, K. F. Bailie, Randy Barron, Robert Bart, H. E. Bell, Walter Bluger, W. J. Blundon, D. A. Breault, Brother T. C. Wesselkamper, P. G. Carr, Yuan Chang, D. I. A. Cohen, Martin Cohen, D. J. Corliss, Wayne Cutrer, John De Vore, F. J. Duarte, E. S. Eby, W. F. Feeny, Andrew Feldmar, Stephen Fisk, J. M. Gandhi and S. Sharma (jointly), M. L. Chachere, G. S. Glazer, Michael Goldberg, L. D. Goldstone, Jerry Goodman, Cornelius Groenewoud, Mark Hayamizu, D. W. Hight, Bernard Jacobson, R. A. Jacobson, A. W. Johnson, Jr., Erwin Just, Irving Katz, E. D. Kinkade, B. G. Klein, Kenneth Kramer, Joel Kugelmass, J. G. Lakin, Douglas Lind, A. E. Livingston, Jiang Luh, M. R. Lund, Paul Machuca, D. C. B. Marsh, E. V. Martin, Stephen Montague, Sidney Penner, Stanton Philipp, Robert W. and Ronald W. Prielipp (jointly), Dorothy S. Rutledge, Perry Scheinok, Sister Marion Beiter, D. L. Silverman, E. M. Stone, M. C. Thornton, A. M. Vaidya, Simon Vatriquant, John Vinson, J. D. Watson, Charles Wexler, F. D. Wilder, K. S. Williams, Anthony Zee, David Zeitlin, and the proposer.

Most proofs employed induction on n; Goldstone gave a proof by finite descent. The problem was located as Prob. 38, p. 27, in Niven and Zuckerman, *An Introduction to the Theory of Numbers* (Wiley, 1960), and as Prob. 3, p. 157, in Uspensky and Heaslet, *Elementary Number Theory* (McGraw-Hill, 1939). It is also the subject of the article, "A generalization of Wilson's theorem," by F. G. Elston, *Mathematics Magazine*, Jan.–Feb. 1957, pp. 159–62.

A Vanishing Integral

E 1604 [1963, 668]. *Proposed by A. P. Boblett, U. S. Naval Ordnance Laboratory, Corona, California*

Evaluate $\displaystyle\int_0^\infty[\sin^2 x/(\pi^2-x^2)]dx.$

I. *Solution by A. E. Livingston, University of Alberta.* Write $1/(\pi^2-x^2)$ as a sum of partial fractions, distribute the integral over this sum, make the obvious changes of variable, and collect terms to get $(2\pi)^{-1}\int_{-\infty}^{\infty}t^{-1}\sin^2 t dt=0$ ($t^{-1}\sin^2 t$ is an odd function) for the value of the given integral.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* Choose r and R

such that $0<\pi-r<\pi+r<R$ and consider in the first quadrant the quadrantal region of radius R from which a semicircle of center $(\pi, 0)$ and radius r has been removed. By standard methods of integration about the contour of this region one finds, by Cauchy's residue theorem, that the given integral has value 0.

III. *Solution by Arnold Singer, Institute of Naval Studies, Cambridge, Mass.* By Bierens de Haan, Table 166, Integral 1,

$$\int_0^\infty [\sin^2 px/(q^2 - x^2)]dx = -(\pi/4q)\sin 2pq.$$

Set $p=1$ and $q=\pi$ to obtain the value 0 for the given integral.

IV. *Solution by W. M. Stone, Oregon State University.* In terms of arbitrary parameters write

$$F(a, t) = \int_0^\infty \frac{\sin(x - a\pi)t \sin(x + a\pi)t}{x^2 - (a\pi)^2}\, dx$$
$$= (1/2)\int_0^\infty \frac{\cos 2\pi at - \cos 2xt}{x^2 - (a\pi)^2}\, dx, \qquad t \geqq 0.$$

The Laplace transformation with respect to t yields

$$f(a, s) = (s/2)\int_0^\infty \left[\frac{1}{s^2 + (2\pi a)^2} - \frac{1}{s^2 + (2x)^2}\right]\frac{dx}{x^2 - (a\pi)^2}$$
$$= \frac{2s}{s^2 + (2\pi a)^2}\int_0^\infty \frac{dx}{s^2 + 4x^2} = \frac{\pi/2}{s^2 + (2\pi a)^2}.$$

Inversion yields $F(a, t) = (\sin 2\pi at)/4a$. If $2at = n$, an integer $\neq 0$, the immediate result is

$$\int_0^\infty \frac{\sin^2 xt \cos^2(n\pi/2) - \cos^2 xt \sin^2(n\pi/2)}{x^2 - (n\pi/2t)^2}\, dx = 0, \qquad t \neq 0,$$

and the problem as stated requires that $n=2$, $t=1$. The case $F(0, t) = \pi t/2$, $t \geqq 0$, is well known.

Also solved by Randy Barron, Robert Bart, C. M. Becker, C. R. Berndtson and C. G. Fain (jointly), J. L. Brown, Jr., M. M. Chawla, D. I. A. Cohen, Martin Cohen, David Colton, J. F. Duarte, J. A. Faucher, Andrew Feldmar, Michael Goldberg, Ralph Greenberg, S. H. Greene, Arthur Greenspoon, R. Hermann and W. Weidekamm (jointly), E. S. Eby and Georgianna T. Klein (jointly), C. B. A. Peck, Stanton Philipp, Perry Scheinok, F. C. Smith, John Stout, J. E. Wilkins, Jr., Oswald Wyler, David Zeitlin, and the proposer.

Marsh pointed out that if we replace $\sin^2 x$ by $(1-\cos 2x)/2$, the problem takes on the form of Ex. 6.2 (f), p. 168, K. S. Miller, *Advanced Complex Calculus* (Harper and Brothers, 1960), where the value 0 is also given.

Trace of an Involutoric Matrix

E 1605 [1963, 668]. *Proposed by R. E. Mikhel, Ball State Teachers College*

If A is a 3×3 involutoric matrix ($A^2=I$) with no zero elements, prove that the trace of A is $+1$ or -1.

Solution by C. G. Cullen, University of Pittsburgh. Since $A=I$ and $A=-I$ are ruled out, the minimum polynomial of A must be x^2-1, so that the characteristic polynomial is either $(x+1)^2(x-1)$ or $(x+1)(x-1)^2$. Since the trace of A is the negative of the sum of the zeros of the characteristic polynomial, the result is apparent.

The result holds for all involutoric 3×3 matrices except I and $-I$. An example of such a matrix with some zero elements is $dg(1, -1, 1)$.

Also solved by Randy Barron, E. D. Bender, Marjorie R. Bicknell, M. M. Chawla, D. I. A. Cohen, Wayne Cutrer, E. R. Deal, J. H. De Vore, P. F. Duvall, Jr., Stephen Fisk, Marshall Freimer and Arnold Singer (jointly), Otomar Hájek, J. C. Hickman, Irving Katz, A. F. Kaupe, Jr., G. J. Kurowski, J. F. Latimer, A. E. Livingston, Jiang Luh, Raul Machuca, D. C. B. Marsh, M. G. Murdeshwar, P. J. Nikolai, Stanton Philipp, Lester Rubenfeld, Perry Scheinok, Hans Schwerdtfeger, Robert Singleton, M. Stojaković, T. Teichmann, J. E. Wilkins, Jr., Oswald Wyler, David Zeitlin, and the proposer.

Number of Nonzero Elements in a Matrix

E 1606 [1963, 668]. *Proposed by Leopold Flatto, Yeshiva University*

Let n denote the number of nonzero elements in a rectangular matrix of numbers each row and each column of which has a zero sum. What possible values can n assume?

I. *Solution by D. I. A. Cohen, Princeton University.* Consider the $(m+1)\times 2$ matrix with the first column all $+1$'s except for $a_{m+1,1}=-m$, and the second column all -1's except for $a_{m+1,2}=+m$. We have a matrix with the given property with $2m+2$, $m>0$, nonzero elements. Consider the $(m+2)\times 3$ matrix with the first column all $+1$'s except $a_{11}=0$ and $a_{m+2,1}=-m$, the last column all -1's except $a_{m+2,3}=m+1$, the second column all 0's except $a_{12}=+1$ and $a_{m+2,2}=-1$. This matrix has the given property and $2m+5$, $m>0$, nonzero elements. It follows that n can be any even number greater than 2 and any odd number greater than 5. Since there cannot be exactly one nonzero element in a row or column, it is clear that n cannot be 1, 2, 3, or 5.

II. *Solution by Michael Goldberg, Washington, D. C.* Clearly n cannot be 1, 2, 3, or 5. All other values of n are possible. These possibilities are illustrated by the use of the following matrices for $n=4$, 6, 7, and 9:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & -2 \\ -1 & -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

All even values of n (except $n=2$) are obtainable by combining the appropriate

number of matrices for $n=4$ and $n=6$, and filling the remainder of the places with zeros. Odd values of n (except $n=1$, 3, or 5) are obtainable by combining these with the matrix for $n=7$ or $n=9$.

Also solved by Randy Barron, Robert Bart, Frank Dapkus, C. L. Fefferman, Marshall Freimer and Arnold Singer (jointly), W. J. Hansen, Erwin Just, A. F. Kaupe, Jr., D. C. B. Marsh, Sidney Penner, Perry Scheinok, Robert Singleton, Anthony Zee, and the proposer.

On Real Zeros of a Polynomial

E 1607 [1963, 668]. *Proposed by Ralph Greenberg, University of Pennsylvania*

Show that, for all integers $t\geqq 0$,

$$x^n/(n+1)^t + x^{n-1}/n^t + \cdots + x/2^t + 1 = 0$$

has no real root if n is even and exactly one real root if n is odd.

Solution by D. C. B. Marsh, Colorado School of Mines. Denote the left side of the given equation by $f(x, n, t)$. We note that $f(x, n, 0)\equiv(x^{n+1}-1)/(x-1)$ has as zeros the $(n+1)$st roots of unity excepting $+1$, and thus has no or one real zero according as n is even or odd. We now use simple induction on t, noting that $D_x\{xf(x, n, t+1)\}=f(x, n, t)$. Since $xf(x, n, t+1)$ has one real zero $(x=0)$, it has precisely one or two real zeros according as its derivative has no or one real zero. Thus $f(x, n, t+1)$ has no or one real zero according as n is even or odd. The proposition therefore holds for each n and all nonnegative integral t.

Also solved by E. R. Barnes, Randy Barron, H. E. Bell, W. G. Brady, P. G. Carr, D. I. A. Cohen, Martin Cohen, Robert Cohen, Evelyn Frank, B. G. Klein, A. E. Livingston, Stanton Philipp, Oswald Wyler, and the proposer.

Evelyn Frank pointed out that the problem is an easy corollary to a theorem of Van Vleck, *Am. Journ. of Math.* (2) 4 (1903) 191–2, namely: If the A_p are real and the terms of the sequence

$$A_0, \quad \begin{vmatrix} A_0 & A_1 \\ A_1 & A_2 \end{vmatrix}, \quad \begin{vmatrix} A_0 & A_1 & A_2 \\ A_1 & A_2 & A_3 \\ A_2 & A_3 & A_4 \end{vmatrix}, \cdots, \quad \begin{vmatrix} A_0 & A_1 & \cdots A_n \\ A_1 & A_2 & \cdots A_{n+1} \\ \cdot & \cdot & \cdots \cdot \\ A_n & A_{n+1} & \cdots A_{2n} \end{vmatrix}$$

are positive, then all the zeros of

$$P_{2n}(z) = A_0 + A_1z + \cdots + A_{2n}z^{2n}$$

are imaginary, and all but one of the zeros of

$$P_{2n+1}(z) = A_0 + A_1z + \cdots + A_{2n+1}z^{2n+1}$$

are imaginary.

Approximating tan x with a Linear Fraction

E 1608 [1963, 669]. *Proposed by G. A. Heuer and Dean Knudson, Concordia College*

Find the linear fraction $f(x)=(x+b)/(cx+d)$ which best approximates $\tan x$ in the interval $[0, \pi/2]$ in the sense that $\int_0^{\pi/2}|f(x)-\tan x|\,dx$ is minimized.

Solution by Craig Comstock, Harvard University. It is clear that we must

have $d=-(\pi/2)c$, $b=-c-\pi/2$, and $f(x)=1/c-1/(x-\pi/2)$. Let α be the one value of x for which $\tan x=f(x)$. Then our integral to be minimized is

$$\int_0^\alpha [\tan x - 1/c + 1/(x - \pi/2)]dx + \int_\alpha^{\pi/2} [1/c - 1/(x - \pi/2) - \tan x]dx = h(c).$$

Setting $\partial h/\partial c=0$ we obtain

$$\int_0^\alpha (1/c^2)dx + \int_\alpha^{\pi/2} (-1/c^2)dx = \alpha/c^2 + \alpha/c^2 - \pi/2c^2 = 0.$$

Therefore $\alpha=\pi/4$, and since α is the solution of $\tan x=1/c-1/(x-\pi/2)$ we obtain

$$c = \pi/(\pi - 4), \qquad d = -\pi^2/2(\pi - 4), \qquad b = \pi(\pi - 2)/2(4 - \pi).$$

Also solved by Randy Barron, Robert Bart, P. G. Carr, M. S. Demos, Michael Goldberg, Otomar Hájek, D. C. B. Marsh, Perry Scheinok, T. Teichmann, J. E. Wilkins, Jr., Oswald Wyler, and the proposers.

Probability that the Product of Two Numbers Exceed Their Sum

E 1609 [1963, 669]. *Proposed by Richard Sinkhorn, University of Houston*

Two numbers are chosen independently and at random from the closed interval $[-a, a]$. Show that the chance that their product will exceed their sum is the least when $a=[x_0/(x_0-1)]^{1/2}$, where x_0 is the real root of the equation $x=1+e^{-x}$.

Solution by D. C. B. Marsh, Colorado School of Mines. If we let the two numbers be x and y, then the probability that $xy>x+y$ is given by the fractional part of the $2a\times 2a$ square which is occupied by the convex side of the hyperbola $xy=x+y$. There are two cases, both readily computed by integration:

$$P = \{a^2 + 2a - 2\ln(a+1)\}/4a^2, \qquad 0 < a \leqq 2, \tag{1.1}$$

$$P = \{2a^2 - 2\ln(a^2-1)\}/4a^2, \qquad a \geqq 2. \tag{1.2}$$

The derivative of (1.1) is

$$\{2\ln(a+1) - a - 1 + 1/(a+1)\}/2a^3;$$

the bracketed factor is 0 at $a=0$ and has a negative derivative in the open interval (0, 2); thus, in the open interval, the derivative of (1.1) is negative so that (1.1) is monotone decreasing and has no minimum in the open interval. The derivative of (1.2) is

$$\{\ln(a^2-1) - a^2/(a^2-1)\}/a^3;$$

the bracketed factor is negative at $a=2$ and has positive derivative, whence there exists precisely one real value of a for which (1.2) has zero derivative. At this point the second derivative of (1.2) is $2/(a^2-1)^2$, indicating the existence of a relative (and the absolute) minimum. Setting $x=a^2/(a^2-1)$, $a=x^{1/2}/(x-1)^{1/2}$

and $\ln(a^2-1)=a^2/(a^2-1)$ becomes $-\ln(x-1)=x$ or $x-1=e^{-x}$, substantiating the proposer's assertion.

Also solved by J. C. Abad, Randy Barron, Robert Bart, J. F. Dillon, Michael Goldberg, W. J. Hansen, P. G. Kirmser, Eric Langford, Stanton Philipp, G. S. Rogers, Perry Scheinok, T. Teichmann, Rory Thompson, Charles Wexler, J. E. Wilkins, Jr., David Zeitlin, and the proposer.

Editorial Note. The above solution supposes (as the proposer probably intended) that $a>0$. If $a=0$, then $x=y=0$ and $xy \not> x+y$, and the minimum probability for $a\geqq 0$ is 0, occurring when $a=0$. We note, from (1.1) and (1.2), by l'Hospital's rule, $\lim_{a\to 0} P = \lim_{a\to\infty} P = 1/2$. The minimum P in the solution above is approximately 0.3607.

Impossibility of Imbedding a Finite Projective Plane

E 1610 [1963, 669]. *Proposed by R. A. Olshen, University of California at Berkeley*

Prove that it is impossible to imbed a finite projective plane in a real affine plane.

Solution by Oswald Wyler, University of New Mexico. Nothing is lost if we replace the real affine plane by the real projective plane P^2. A finite degenerate plane P obviously can be imbedded in P^2; hence we assume that P is a nondegenerate plane. In P, and through the imbedding of P in the real projective plane P^2, we carry out the following construction. Let p_0, p_1, a be three collinear points, let b not be on the line p_0a, let q_0 be a third point of the line p_0b, and let c be the intersection of p_1q_0 and ab. We construct $q_1, q_2, \cdots$, and $p_2, p_3, \cdots$, by letting q_n be the intersection of aq_0 with bp_n, and p_{n+1} the intersection of ap_0 with cq_n. In the finite plane P, the sequence $p_0, p_1, p_2, \cdots$ must have repetitions. In the real projective plane P^2, we take a homogeneous coordinate system in which coordinate triples $(0, 0, 1)$, $(u, 0, 1)$, and $(1, 0, 0)$ are assigned to p_0, p_1, and a. Then $(nu, 0, 1)$ is a coordinate triple for p_n, as is easily verified, and thus the sequence $p_0, p_1, p_2, \cdots$ has no repetitions. This proves the impossibility of the proposed imbedding.

Also solved by the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before October 31, 1964.

5143 [1963, 1013]. *Corrected. Proposed by J. B. Roberts, Reed College*

Let $n_1, n_2, \cdots$ be a sequence of integers each greater than unity. Put $p_0=1$, $p_j=n_1\cdots n_j$ for $j\geqq 1$. Let ϕ be a function belonging to L^2, having period 1, and satisfying

$$\sum_{j=1}^{n_k}\phi\left(x+\frac{j-1}{n_k}\right)=0 \qquad k=1,2,\cdots.$$

Then the sequence $\{\theta_n\}$ is orthogonal on (0, 1) when $\theta_n(x)=\phi(p_n x)$.

(This generalizes a problem, p. 43 of Kaczmarz and Steinhaus, *Theorie der Orthogonal Reihen.*)

5191. *Proposed by Kwangil Koh, University of North Carolina*

Given that R is a ring without divisors of zero but that R is not a right uniform ring. Prove that there is a right quotient ring Q of R which is a primitive ring without a minimal right ideal. (If, for arbitrary nonzero elements a, b of R we have $aR\cap bR\neq\{0\}$, then R is called a right uniform ring. Q is a right quotient ring of R if Q has a subring isomorphic to R such that to every nonzero element q of Q there exist x, $y\in R$ such that $qx=y\neq 0$.) Refer to N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Colloquium Publications, vol. xxxvii.

5192. *Proposed by J. M. Horner, University of Alabama*

Let $\wp(z)$ be a Weierstrass elliptic function whose invariants are of the form $g_2=-3r^2$, $g_3=2s^3$, where r and s are rational, $s\neq 0$, $r^2-s^2\neq 0$. Prove that $\wp(\omega+\omega')$ is real, sgn $(\wp(\omega+\omega'))=\operatorname{sgn}(s)$, and $\wp(\omega+\omega')$ is irrational.

5193. *Proposed by Lewis Batson, University of Southeastern Louisiana*

Find the zeros of the function

$$f(x) = 1 + \sum_{n=1}^{\infty}\left[(2x)^n \Big/ \prod_{k=1}^{n}(2^k - 1)\right].$$

5194. *Proposed by John V. Ryff, Harvard University*

If u is a solution of the Airy equation $u''+tu=0$ which satisfies $u(0)=0$, show that ξ, the first positive zero of u, is such that $\pi^{2/3}<\xi<2^{1/3}\pi^{2/3}$.

5195. *Proposed by A. C. Lazer, Carnegie Institute of Technology*

Let $p(x)$ be any function positive and continuous for x in an interval I. Show that no nontrivial integral of the differential equation $y'''-p(x)y=0$ can have more than one point of inflection in I, nor can it be tangent to the line $y=0$ more than once in I, nor can it have both an inflection point and a point of tangency at the line $y=0$ in I.

5196. *Proposed by L. Carlitz, Duke University*

Prove the identities

$$\begin{aligned}(1)\qquad &\sum_{n=0}^{\infty} q^{n(n+1)/2}\,\frac{(1-a)(1-qa)\cdots(1-q^{n-1}a)}{(1-q)(1-q^2)\cdots(1-q^n)}\\ &= \sum_{n=0}^{\infty}(-1)^n\,\frac{q^{n^2}a^n}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}\cdot\prod_{r=1}^{\infty}(1+q^r),\end{aligned}$$

$$(2)\qquad \begin{aligned}&\sum_{n=0}^{\infty}(-1)^n q^{n^2}\frac{(1-a)(1-q^2a)\cdots(1-q^{2n-2}a)}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}\\ &=\sum_{n=0}^{\infty}\frac{q^{n^2}a^n}{(1-q)(1-q^2)\cdots(1-q^{2n})}\prod_{r=1}^{\infty}(1-q^{2r-1}).\end{aligned}$$

5197. *Proposed by H. Guggenheimer, University of Minnesota*

Let $r=r(\theta)$ be the polar equation of a smooth, convex, simple closed curve, the origin being placed at the area centroid of the curve. Show that for any constant α the equation $r(\theta)=r(\theta+\alpha)$ has at least four distinct solutions.

5198. *Proposed by H. S. Shapiro, New York University and the University of Michigan*

Let $\|a_{ij}\|$ be a square matrix of complex numbers, and $\Delta(\lambda)=\lambda^n+c_1\lambda^{n-1}+\cdots+c_n$ its characteristic polynomial. Show that

$$|c_k| \leqq n^{k/2}\binom{n}{k}M^k,$$

where $M=\text{Max }|a_{ij}|$.

5199. *Proposed by H. S. Shapiro, New York University and the University of Michigan*

Prove or disprove the following conjecture: A closed subspace of L^2 [0, 1] which is a subset of L^p for some $p>2$ is finite dimensional.

SOLUTIONS OF ADVANCED PROBLEMS

Maximal Nonsingular Subspaces

5027 [1962, 438; 1963, 580; 1963, 1016]. *Proposed by A. J. Goldman, National Bureau of Standards*

Let $M_n(F)$ be the set of $n\times n$ matrices over the field F, considered as an n^2-dimensional vector space over F. Call a vector subspace of $M_n(F)$ nonsingular if all its nonzero members are nonsingular matrices. Find maximal nonsingular subspaces of $M_n(F)$.[1]

Editorial Note. Unfortunately, solution II [1963, 1016] is not valid but involves the same basic error as solution I [1963, 580]. E. J. Taft comments that we can "choose C such that $C^{-1}(A^{-1}B)C$ is superdiagonal $\cdots$" only if the field F is algebraically closed. Another way of saying this is that the scalar d_{11} which appears in the argument may not be in F, but rather in the algebraic closure of F. Thus the final "$B=d_{11}A$" is not really a contradiction.

A simple counterexample is provided in the case where F is the real number. The set of all matrices of the form

$$\begin{bmatrix} a & b\\ -b & a\end{bmatrix}$$

is a 2-dimensional nonsingular subspace of $M_2(F)$.

Similar criticisms were submitted by G. E. Bredon, G. J. Janusz, J. L. Pietenpol, Thomas Ströhlein, Seith Warner, and R. Westwick. No satisfactory answer to the original question has been received—it may well depend on the nature of the particular field F.

A Converse of the Divergence Theorem

5071 [1963, 97]. *Proposed by Peter Ungar, New York University*

Let $F(u, u_{x_1}, \cdots, \partial^k u/\partial x_n^k)$ depend continuously on its arguments. Assume that for every region R

$$I(R, u) = \int \underset{R}{\cdots} \int F(u, \cdots)dx_1 \cdots dx_n$$

depends only on the boundary values of u and its derivatives of order $\leqq k$, i.e., $I(R, u) = I(R, v)$ whenever $u - v$ vanishes on the boundary of R together with its derivatives of order $\leqq k$. Then F is a divergence, i.e., there exist expressions $F_i(u, u_{x_1}, \cdots)$ such that

$$F(u, \cdots) = F(0, 0, \cdots, 0) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i(u, u_{x_1}, \cdots).$$

Solution by the proposer. We use induction with respect to the number of dimensions n. In order to make the induction work more easily we will alter the hypothesis and prove that if $I(R, u)$ vanishes for all infinitely often differentiable functions u which vanish, along with all of their derivatives, on the boundary of R, then F is a divergence.

We shall use a function $g(t)$ defined as follows:

$$g(t) = \begin{cases} \exp\left[-\exp\left(\dfrac{1}{1+t} + \dfrac{1}{t}\right)\right] & -1 < t < 0, \\ 0 & t \leqq -1, \\ 1 & t \geqq 0. \end{cases}$$

The function $g(t)$ has continuous derivatives of all orders. In the proof the extension from n to $n+1$ dimensions follows precisely the steps used in passing from $n=1$ to $n=2$, and we shall describe the latter. Finally, there is no loss of generality in assuming $F(0, 0, \cdots, 0) = 0$.

The case $n=1$. For any function $u(x)$ of compact support, and an arbitrary number X, set

$$v(x; X) = v(x) = \begin{cases} g(x - X) \displaystyle\sum_{i=0}^{k} D^i u(X) \frac{(x - X)^i}{i!} & \text{for } x \leqq X \\ u(x) & \text{for } x \geqq X. \end{cases}$$

$v(x)$ has only k continuous derivatives, for the higher order derivatives may

jump at $x=X$. Nevertheless, it follows from the hypothesis that $\int_{-\infty}^{\infty} F(v)dx=0$, for we can obtain a sequence of infinitely differentiable functions $v_i(x)$ which vanish outside a preassigned fixed interval and such that the sequence of kth derivatives of $v_i(x)$ converges uniformly to $D^k v$. The integral, clearly, need be written only over any finite interval containing the support of v.

Now $u(x)=v(x)$ for $x\geqq X$; and our hypothesis therefore yields

$$\int_{-\infty}^{X} F(u)dx = \int_{-\infty}^{X} F(v)dx = G[u(X), u'(X), \cdots, D^k u(X)], \tag{1}$$

where we observe that

$$G(a_0, a_1, \cdots, a_k) = \int_{-1}^{0} F\{V(x, a_0, a_1, \cdots, a_k), \cdots, D_x^k V\}dx,$$

$$V(x; a_0, \cdots, a_k) = g(x)\sum_{i=0}^{k} a_i x^i/i!.$$

Differentiating with respect to the upper limit in (1) we have

$$F(u(x), \cdots, u^{(k)}(x)) = \frac{d}{dx} G[u(x), \cdots, D^k u(x)], \tag{2}$$

and that (2) is an identity now follows from the fact that $u(x)$ and its derivatives may be assigned arbitrary independent values.

The case $n=2$. We let $u(x, y)$ be any function of compact support with infinitely many continuous derivatives. For an arbitrary number Y, set

$$v(x, y; Y) = v(x, y) = \begin{cases} g(y-Y)\sum_{i=0}^{k} D_y^i u(x, Y)\dfrac{(y-Y)^i}{i!}, & y \leqq Y \\ u(x, y) & y \geqq Y \end{cases}$$

and then we get, as in the one dimensional case,

$$\begin{aligned} \int_{-\infty}^{\infty} dx \int_{-\infty}^{Y} F(u)dy &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{Y} F(v)dy \\ &= \int_{-\infty}^{\infty} G[u(x, Y), \cdots, D_x^k D_y^k u]dx. \end{aligned} \tag{3}$$

Applying a similar process once again to the function $\int_{-\infty}^{Y} F - G$ we have

$$\int_{-\infty}^{X}\int_{-\infty}^{Y} F(u(x, y))dxdy = H(u(X, Y), u_x(X, Y), \cdots, D_x^k D_y^k u(X, Y)), \tag{4}$$

and the result now follows by taking $\partial^2/\partial X \partial Y$.

Triangle Inequalities

5092 [1963, 444]. *Proposed by A. Oppenheim, University of Malaya, Kuala Lumpur*

Suppose that A_i, B_i, C_i $(i=1, 2)$ are triangles with sides a_i, b_i, c_i, area Δ_i, and altitudes p_i, q_i, r_i. Define numbers a_3, b_3, c_3 by the equations $a_3=(a_1^2+a_2^2)^{1/2}$, etc. Show that

(i) a_3, b_3, c_3 are the sides of a triangle;

(ii) $p_3^2 \geqq p_1^2+p_2^2$, $q_3^2 \geqq q_1^2+q_2^2$, $r_3^2 \geqq r_1^2+r_2^2$, equality occurring in all three if and only if the original two triangles are similar;

(iii) $\Delta_3 \geqq \Delta_1+\Delta_2$, with equality if and only if the triangles are similar;

(iv) $\Delta_3^2 \geqq 4\Delta_1\Delta_2$, with equality if and only if the triangles are congruent.

Solution by P. R. Nolan, Department of Education, Dublin, Ireland. We will use the familiar relation

$$(1) \qquad (\alpha x + \beta y)^2 \leqq (\alpha^2 + \beta^2)(x^2 + y^2),$$

with equality if and only if $\beta x = \alpha y$.

(i) $a_3^2 = a_1^2+a_2^2 < (b_1+c_1)^2+(b_2+c_2)^2$
$= b_3^2+c_3^2+2(b_1c_1+b_2c_2)$.

But $b_1c_1+b_2c_2 \leqq (b_1^2+b_2^2)^{1/2}(c_1^2+c_2^2)^{1/2}=b_3c_3$ by (1). Therefore $a_3^2<(b_3+c_3)^2$, i.e. $a_3<b_3+c_3$, and the full result follows by symmetry.

(ii) The cosine formula gives $c_3a_3 \cos B_3 = c_1a_1 \cos B_1 + c_2a_2 \cos B_2$. Squaring, applying (1), and dividing by $a_3^2=(a_1^2+a_2^2)$, we obtain

$$c_3^2 \cos^2 B_3 \leqq c_1^2 \cos^2 B_1 + c_2^2 \cos^2 B_2.$$

Subtraction then gives

$$c_3^2 \sin^2 B_3 \geqq c_1^2 \sin^2 B_1 + c_2^2 \sin^2 B_2$$

which is $p_3^2 \geqq p_1^2+p_2^2$ as required. Similarly for q_3^2 and r_3^2. Equality holds for p_3^2 if and only if $a_1/a_2 = c_1 \cos B_1/c_2 \cos B_2$, similarly for r_3^2 if and only if $c_1/c_2 = a_1 \cos B_1/a_2 \cos B_2$, giving

$$\cos B_1 = \cos B_2, \qquad a_1/a_2 = c_1/c_2.$$

Therefore for equality for one altitude, similarity is sufficient but not necessary (e.g., both triangles isosceles is sufficient). Equality for two or three altitudes holds if and only if the original triangles are similar.

(iii) $2(\Delta_1+\Delta_2) = p_1a_1+p_2a_2 \leqq (p_1^2+p_2^2)^{1/2}(a_1^2+a_2^2)^{1/2}$
$\leqq p_3a_3 = 2\Delta_3$,

as required, equality occurring if and only if $a_1/a_2=c_1 \cos B_1/c_2 \cos B_2$ as in(ii), and $a_1/a_2=p_1/p_2=c_1 \sin B_1/c_2 \sin B_2$, giving $\tan B_1=\tan B_2$, etc., i.e., triangles are similar.

(iv) $\Delta_3^2 \geqq (\Delta_1+\Delta_2)^2 = (\Delta_1-\Delta_2)^2+4\Delta_1\Delta_2 \geqq 4\Delta_1\Delta_2$,

with equality if and only if the triangles are similar, as in (iii), and $\Delta_1=\Delta_2$, i.e., the triangles are congruent.

Also solved by Robert Breusch, L. Carlitz, T. R. Curry, K. M. Das, H. Guggenheimer, R. A. Jacobson, G. Laman, and the proposer (whose solution is based on his paper, *Inequalities connected with definite Hermitian forms*, J. London Math. Soc., 5(1930) 114–119).

A Convergent Series

5093 [1963, 444]. *Proposed by S. M. Shah, University of Kansas*

Let $0<\lambda_n\leqq\lambda_{n+1}$, $n=1, 2, \cdots$; let $\phi(x)$ be positive and nondecreasing for $x\geqq\lambda_1$; and suppose that $\int_{\lambda_1}^{\infty}dt/t\phi(t)<\infty$. Prove that

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{\phi(\lambda_n)}\left(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\right)<\infty.$$

Solution by Ralph Greenberg, University of Pennsylvania. First, we observe that

$$S=\sum_{n=1}^{\infty}\frac{\lambda_n}{\phi(\lambda_{n+1})}\left(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\right)=\sum_{n=1}^{\infty}\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}\phi(\lambda_{n+1})}\leqq\sum_{n=1}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}}\frac{dt}{t\phi(t)}<\infty,$$

so that S converges. Secondly, letting T equal the given series, we have

$$T-S=\sum_{n=1}^{\infty}\left(1-\frac{\lambda_n}{\lambda_{n+1}}\right)\left(\frac{1}{\phi(\lambda_n)}-\frac{1}{\phi(\lambda_{n+1})}\right)<\sum_{n=1}^{\infty}\left(\frac{1}{\phi(\lambda_n)}-\frac{1}{\phi(\lambda_{n+1})}\right)\leqq 1/\phi(\lambda_1).$$

Hence the stated result follows.

Also solved by Robert Breusch, S. Carley, K. M. Das, Roy O. Davies, G. Di Antonio, Fred Gross, A. S. S. Sastry, L. Seshu, W. C. Waterhouse, and the proposer.

Editorial Note. E. G. Straus remarks that it is not essentially more difficult to prove the following generalization: Let $0<\lambda_n\leqq\lambda_{n+1}$, $n=1, 2, \cdots$; let $\varphi(x)$ and $\psi(x)$ both be positive and nondecreasing for $x>\lambda_1$; and suppose that $\int_{\lambda_1}^{\infty}dt/\varphi(t)\psi(t)<\infty$; then

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{\phi(\lambda_n)}\left(\frac{1}{\psi(\lambda_{n+1})}-\frac{1}{\psi(\lambda_n)}\right)<\infty.$$

Reversible Matrices

5094 [1963, 444]. *Proposed by Albert Wilansky, Lehigh University*

Find two reversible matrices whose product is not reversible. ("Reversible" is defined by Banach, p. 90.)

Solution by S. T. M. Ackermans, Technological University, Eindhoven, Netherlands. According to Banach, an infinite matrix A with elements $a_{ij}(i, j=0, 1, 2, \cdots)$ is called reversible if and only if for every converging sequence $\{y_n\}$ $(n=0, 1, 2, \cdots)$ there exists exactly one sequence $\{x_n\}$ such that $\sum_{n=0}^{\infty}a_{in}x_n=y_i$ $(i=0, 1, 2, \cdots)$. Let A be the diagonal matrix with elements $a_{ij}=2^{-i}\delta_{ij}$ $(i, j=0, 1, 2, \cdots)$ and B the matrix with elements $b_{0j}=2^{-j}$ $(j=0, 1, 2, \cdots)$; $b_{ij}=\delta_{ij}$ $(i=1, 2, \cdots; j=0, 1, 2, \cdots)$. The reversibil-

ity of A is trivial; for B it follows from the fact that if $\{y_n\}$ is a converging sequence, then $\sum_{n=1}^{\infty} 2^{-n}y_n$ is a convergent series and so $x_0+\sum_{n=1}^{\infty} 2^{-n}y_n=y_0$ can be solved. The product $C=AB$ has elements $c_{0j}=2^{-j}$ $(j=0, 1, 2, \cdots)$; $c_{ij}=2^{-i}\delta_{ij}$ $(i=1, 2, \cdots; j=0, 1, 2, \cdots)$. C is not reversible for there exists no sequence $\{x_n\}$ such that $\sum_{n=1}^{\infty} c_{in}x_n=1$ $(i=0, 1, 2, \cdots)$.

Also solved by the proposer.

Riemann Zeta Function

5095 [1963, 444]. *Proposed by Gregory J. Lodge, Rensselaer Polytechnic Institute*

Prove that

$$\zeta(n) = \frac{(-1)^n}{(n-1)!}\Gamma_n(k) + S_n(k),$$

where n and k are positive integers $\neq 1$, $\zeta(n)$ is the Riemann zeta function, $\Gamma_n(x)$ is the nth poly-gamma function defined by $\Gamma_n(x)=d^n \ln \Gamma(x)/dx^n$, and $S_n(k)=\sum_{i=1}^{k-1} i^{-n}$.

Solution by Robert Breusch, Amherst College. From the canonical product for $1/\Gamma(x)$,

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{m=1}^{\infty}\left[\left(1+\frac{x}{m}\right)e^{-x/m}\right]$$

it follows, for $x>0$, that

$$\log \Gamma(x) = -\log x - \gamma - \sum_{m=1}^{\infty}\left[\log\left(1+\frac{x}{m}\right)-\frac{x}{m}\right],$$

$$\frac{d}{dx}\log\Gamma(x) = -\frac{1}{x} - \sum_{m=1}^{\infty}\left[\frac{1}{m+x}-\frac{1}{m}\right],$$

and, for $n>1$,

$$\Gamma_n(x) \equiv \frac{d^n}{dx^n}\log\Gamma(x) = (-1)^n(n-1)!\sum_{m=0}^{\infty}\frac{1}{(m+x)^n}\cdot$$

The termwise differentiations are clearly legitimate, since $1/\Gamma(z)$ is an entire function, and all the derived series converge uniformly in every finite interval of the positive x-axis. Thus

$$\frac{(-1)^n}{(n-1)!}\Gamma_n(k) = \sum_{m=0}^{\infty}\frac{1}{(m+k)^n} = \sum_{i=k}^{\infty}\frac{1}{i^n},$$

and

$$S_n(k) + \frac{(-1)^n}{(n-1)!}\Gamma_n(k) = \sum_{i=1}^{k-1}\frac{1}{i^n} + \sum_{i=k}^{\infty}\frac{1}{i^n} = \sum_{i=1}^{\infty}\frac{1}{i^n} = \zeta(n).$$

Also solved by W. E. Briggs, R. G. Buschman, L. Carlitz, A. E. Danese, J. A. Faucher, Stephen Fisk, Ralph Greenberg, Emil Grosswald, Eldon Hansen, S. Heller, W. C. Janes, J. Koekoek, A. E. Livingston, Stanton Philipp, and John Vinson.

Editorial Note. The proposed relation, or its equivalent, is found in many places. The following references were supplied by readers: Jahnke and Emde, *Tables of Functions*, pp. 9, 18; Erdelyi, et al., *Higher Transcendental Functions*, v. I, p. 22 (iv); Whittaker and Watson, *A Course in Higher Analysis*, (1954), p. 241; H. T. Davis, *Tables of Higher Mathematical Functions*, v. II, p. 10.

Impossible Congruences

5096 [1963, 445]. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime. Is it possible to find a set of p integers $a_1, \cdots, a_p$ such that $\Pi_{j=1}^{p}(x+a_j)\equiv x^p+1 \pmod{p^2}$?

Is it possible to find a set of p^2 integers $a_1, \cdots, a_{p^2}$ such that $\Pi_{j=1}^{p^2}(x+a_j)\equiv x^{p^2}+1 \pmod{p^2}$?

Solution by Robert Breusch, Amherst College. If $p=2$, either one of the two congruences would imply, for $x=-a_j$, that a_j^2 or $a_j^4\equiv -1 \pmod 4$, which is impossible.

Assume now that p is odd. Since $x^{p^2}\equiv x^p\equiv x \pmod p$, both given congruences imply, with $x=-a_j$, that $a_j\equiv 1 \pmod p$,

$$a_j = 1 + p\cdot b_j. \tag{1}$$

Also, both imply that $\Pi a_j\equiv 1 \pmod{p^2}$, $1+p\sum b_j\equiv 1 \pmod{p^2}$,

$$\sum b_j \equiv 0 \pmod p. \tag{2}$$

If S_r is the rth elementary symmetric function of the a_j, it follows from (1) and (2) that $S_r\equiv\binom{p}{r} \pmod{p^2}$ in the first congruence, and $S_r\equiv\binom{p^2}{r} \pmod{p^2}$ in the second. The given congruences would imply that $S_r\equiv 0 \pmod{p^2}$ for $1\leqq r\leqq p-1$ in the first case, $1\leqq r\leqq p^2-1$ in the second. But $\binom{p}{1}\not\equiv 0 \pmod{p^2}$, and $\binom{p^2}{p}\not\equiv 0 \pmod{p^2}$. Thus both congruences are impossible in every case.

Also solved by George Bergman and by the proposer.

Determinant of a Special Matrix

5097 [1963, 445]. *Proposed by Melvin Hausner, New York University*

Let an $n\times n$ matrix have positive entries along the main diagonal, and negative entries elsewhere. Assume that it is normalized so that the sum of each column is 1. Prove that its determinant is greater than 1.

Solution by Sidney Heller, Brookhaven National Laboratory. Use induction. Let the $n\times n$ matrix be $A=(a_{ij})$. The result is immediately true for $n=2$; assume that it is true for $(n-1)\times(n-1)$ matrices with the required properties. Then, using a_{11} as a pivotal element, eliminate column 1 of A, to give:

$$a_{ij}\to a_{ij} - a_{1j}\cdot\frac{a_{i1}}{a_{11}} = b_{ij} \qquad i, j = 2, \cdots, n.$$

Note that $b_{ij} < 0$, $i \neq j$, and $\sum_{i=2}^{n} b_{ij} = 1 - a_{1j}/a_{11} = c_j > 1$. Thus $b_{ii} > 1$, $i=2, 3, \cdots, n$ and (b_{ij}) is an $(n-1)\times(n-1)$ matrix having the required properties except for normalization. Hence

$$|A| = a_{11}\left(\prod_{j=2}^{n} c_j\right)|B|,$$

where $|B|$ is the determinant of the matrix which is obtained from (b_{ij}) by dividing each jth column by c_j $(j=2, \cdots, n)$, and is thus normalized. Since $a_{11} > 1$, the induction is complete.

Also solved by Alfred Brauer, J. L. Brenner, Robert Breusch, L. Carlitz, Robert Cohen, K. M Das, Roy Feriman, Stephen Fisk, A. S. Housholder, R. A. Jacobson, Fulton Koehler, A. G. Konheim and T. J. Rivlin and Sam Winograd, Marvin Marcus and William Gordon, A. W. Marshall, F. D. Parker, Stanton Philipp, J. E. Potter, D. Ramakotayya, Olga Taussky Todd, Robert Vermes, and the proposer.

Editorial Note. The result follows easily from known, more general theorems. See, e.g., G. B. Price, *Proc. Amer. Math. Soc.*, 2(1951) 497–502; Ostrowski, *Bull. Sci. Math.*, (2), 61(1937) 19, 32; *Proc. Amer. Math. Soc.*, 3(1952) 26–30; A. Brauer, *Duke Math. J.*, 13(1946) 387–395; S. Gersgorin, *Izv. Akad. Nauk, SSSR.*, 7(1931) 749–754.

Summation of Products of Bessel Functions

5099 [1963, 445]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Let k be an even number and $J_n(x)$ be the Bessel's function of the first species. Find the sum $\sum_{n=0}^{\infty} (2n+k)J_n(x)J_{n+k}(x)$.

Solution by Eldon Hansen, Lockheed Aircraft Corp. In what follows we shall use the abbreviation J_n for the Bessel's function $J_n(x)$. We define formally the sums

$$S(k) = \sum_{n=0}^{\infty} nJ_nJ_{n+k}, \qquad T(k) = \sum_{n=0}^{\infty} J_nJ_{n+k}.$$

Using the known recursion relation

$$2mJ_m = x(J_{m+1} + J_{m-1}) \tag{1}$$

with $m=n$, we rewrite $S(k)$ as

$$\begin{aligned} S(k) &= \tfrac{1}{2}x\sum_{n=0}^{\infty} (J_{n+1} + J_{n-1})J_{n+k} \\ &= \tfrac{1}{2}x\left[\sum_{n=1}^{\infty} J_nJ_{n-1+k} + \sum_{n=-1}^{\infty} J_nJ_{n+1+k}\right] \qquad (2) \\ &= \tfrac{1}{2}x[T(k-1) - J_0J_{k-1} + T(k+1) + J_{-1}J_k]. \end{aligned}$$

Also

$$S(k) + kT(k) = \sum_{n=0}^{\infty} (n+k)J_nJ_{n+k}. \tag{3}$$

From (1) with $m=n+k$, (3) yields

$$\begin{aligned} S(k) + kT(k) &= \tfrac{1}{2}x \sum_{n=0}^{\infty} J_n(J_{n+k+1} + J_{n+k-1}) \\ &= \tfrac{1}{2}x[T(k+1) + T(k-1)]. \end{aligned} \tag{4}$$

Subtracting (2) from (4) we have

$$kT(k) = \tfrac{1}{2}x(J_0J_{k-1} - J_{-1}J_k), \tag{5}$$

and substituting (5) into (2) and using (1) to reduce J_{-1}, J_{k-2}, J_{k+1}, J_{k+2} we obtain

$$S(k) = \frac{x}{2(k^2-1)}\{x(J_1J_{k-1} - J_0J_k) - (k-1)J_1J_k\}, \tag{6}$$

$k \neq \pm 1$. So, finally, the desired sum reduces to

$$2S(k) + kT(k) = \frac{x^2}{k^2-1}\{J_1J_{k-1} - J_0J_k\} + \frac{x}{2}\left\{J_0J_{k-1} + \frac{k-1}{k+1}J_1J_k\right\}.$$

Also solved by L. Carlitz, Donald Childs, P. R. Khandekar, J. Koekoek, and J. Ernest Wilkins, Jr.

Editorial Note. Khandekar finds the result in terms of hypergeometric functions, viz.

$$\frac{(\frac{1}{2}x)^k}{(k-1)!}\,{}_2F_3\left[\begin{matrix} 1+\frac{1}{2}k, & \frac{1}{2}k-\frac{1}{2}; & -x^2 \\ 1, & k, & 1+k \end{matrix}\right].$$

The proposer restricted the problem to even values of k since the result for odd k has been published in the form

$$\sum_{n=1}^{k-1} (-1)^n nJ_n(x)J_{k-n}(x),$$

see Stojanović et Djoković, *Publications de la Faculté d'Électrotechnique de l'Université de Belgrade,* séries Mathématiques et physique, No. 51 et 52(1961).

Rings with Cyclic Additive Group

5100 [1963, 445]. *Proposed by Seth Warner, Duke University*

To within isomorphism, find the number of rings there are whose additive group is cyclic of order m.

I. *Solution by W. C. Waterhouse, Harvard University.* Let R be such a ring, with x a generator of the additive group; R is necessarily commutative. Clearly the ring is determined by giving the integer n such that $x^2 = nx$. Let S be another such ring, with generator y satisfying $y^2 = py$. It is easily checked that the mapping which carries x into ky is an isomorphism of R onto S if and only if $(k, m) = 1$ and $n \equiv kp \pmod{m}$. But these conditions can be satisfied simultaneously if and

only if $(n, m) = (p, m)$. Thus the number of nonisomorphic rings is the same as the number of different values of (n, m), i.e., the number of divisors of m. Only one of the rings, of course, has a unit.

II. *Solution by G. A. Heuer and D. B. Erickson, Concordia College.* L. Fuchs in *Abelian Groups*, Pergamon Press, N. Y., 1960, p. 263, shows that the number of such rings is the number of positive divisors of m.

Also solved by K. F. Bailie, R. A. Beaumont, George Bergman, Robert Bowen, K. E. Eldridge, D. P. Giesy, R. W. Gilmer, Jr., Veselin Perić, and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, Carleton College, and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. Programmed Materials: K. O. May, Carleton College, Northfield, Minn. Films: E. P. Vance, Oberlin College, Oberlin, Ohio.

Introduction to Topology and Modern Analysis. By George F. Simmons. McGraw-Hill, New York, 1963. xv+372 pp. $8.95.

Now that topology is fast becoming an integral part of the undergraduate mathematics curriculum, every publisher is eager to have available on his list a suitable textbook for such a course. Of the ones that have already appeared, I would have no hestitation in selecting the present one. It wisely restricts itself to point-set topology which it develops axiomatically; experience has shown that a rigorous treatment of combinatorial topology is best left for graduate study after the student has acquired the necessary algebraic background. A sufficient amount of abstract set theory is presented to enable the author to present in succession metric spaces, topological spaces, compactness, separation, connectedness, and approximation (a good treatment of the Stone-Weierstrass theorem appears here). The stage is now set to present some of the topics in analysis which are currently of central interest; this occupies the second part of the book where, following a preliminary chapter on algebraic systems, the author proceeds to deal with Banach spaces, Hilbert spaces, and Banach algebras.

The textbook is eminently readable, uses present-day commonly accepted notation and terminology, and is well documented with good illustrative examples and exercises. A good undergraduate course in classical analysis should serve as an adequate prerequisite for its study. The first part of the book could then constitute a one-semester course in topology; the second part of the book could serve as an excellent introduction to modern analysis, supplemented with the modern theory of integration, if this becomes desirable at this point.

ARTHUR E. DANESE, State University of New York at Buffalo

Principles of Abstract Algebra. By Richard W. Ball. Holt, Rinehart and Winston, New York, 1963. 290 pp. $6.00.

This book is designed for undergraduates who have completed a course in calculus. The author has chosen what he considers to be the path of least resistance to abstract algebra. As explained in the preface "the author has tried to emulate the exciting clarity and precision found in experimental grade-school and high-school texts today." I believe he has succeeded admirably in this respect.

Instead of confronting the reader with a complete set of postulates for the integers, the author devotes a separate section to each postulate and establishes the fundamental properties of the integers with detailed references to the postulates. The elementary theory of numbers, occupying about one-third of the book, is then developed in a leisurely way. Generalizing, the reader is introduced to the concepts of rings, fields, and integral domains; but only number-fields are explored at length. Two chapters are devoted to group theory, but only cyclic groups are treated in detail. There is no mention of isomorphism, and no example is given of an abstract group defined solely by a multiplication table.

The theory of polynomials in one indeterminate is presented lucidly without mentioning the word "indeterminate." This is followed by two chapters on the theory of equations at the level of the now obsolete college algebra textbooks. The author states that, in practice, the real heart of the theory of equations is the theory of real roots of polynomials with real coefficients; accordingly the Cardan formulas are omitted. Since approximations to the roots can be handled efficiently by a computing machine, the main problem is that of isolating the roots. The principal tools recommended are the rules of signs and Rolle's theorem. A brief chapter is devoted to the algebra of matrices. The final chapter treats systems of linear equations without using determinants, the solution being obtained by reducing a matrix to echelon form.

The author asserts that the completeness of the real number system and the fundamental theorem of algebra, whose proofs are omitted, may be regarded as additional postulates. Since nothing is said about the completeness, consistency or independence of the postulates introduced, an alert reader may well wonder why one may not likewise regard as postulates other theorems which are stated without proof, such as the fundamental theorem on symmetric functions and the invariance of the rank of a matrix. Silence on these topics can hardly aid the beginner to appreciate the nature of a deductive science.

While an author has the right to delimit the scope of his book, the present author has exercised his right to the point of failing to fulfill the promise implied by the title. I am sure that many students will enjoy an algebra course based on this textbook; but I doubt whether they will have acquired more than a superficial acquaintance with the principles of abstract algebra.

LOUIS WEISNER, University of New Brunswick

The Real Numbers in an Algebraic Setting. By J. B. Roberts. Freeman, San Francisco, 1962. x+145 pp. $1.75 (paper), $3.50 (hardcover).

This book consists of a detailed construction of the real numbers, starting from the natural numbers, and using Cauchy sequences rather than Dedekind cuts. It is intended to provide material for a course which is, in the author's words, "of great cultural value to nonscience students." Such students will certainly be convinced that mathematics consists of proving lemmas—a rough count shows 85 lemmas and 15 theorems.

There are some bad points: In the preface, the student is told that, *in general*, he should work every exercise (no answers are provided); on page 2, he is told that, *in general*, two sets are equal if and only if they have exactly the same elements. On page 28 the axioms for the natural numbers are given (with the least integer principle replacing the induction axiom—the latter becoming a lemma), with no mention of the word "trichotomy"; on p. 33 a hint for exercise 3 calls for the use of trichotomy. On page 39, exercise 1 asks about a solution of $a+x=b$; on page 41 the word "solution" is defined. Exercise 9 on page 53 asks for square roots, which are defined on page 62.

There are some good points: On page 11, i is said to be one of the complex square roots of -1. Pages 29–30 have a fine short essay on the role played by proofs in mathematical creation. The appendix to Chapter 3 on cardinality is very well done. The appendix on continued fractions should appeal to the beginner.

Most of the arguments in the proofs are intricate enough so that the average nonscience student would find them very difficult to follow on his own. An instructor using this book as a text would have to work hard to prevent the trees from obscuring the forest. On the other hand, a mathematics student in his second or third year ought to be able to go through the book almost entirely on his own, and very likely should be made to do so.

This book is considerably longer than Landau's classic "Grundlagen der Analysis." It is less ambitious than Olmsted's "The Real Number System," and seems to be more detailed than Thurston's "The Number System." Algebraists may wonder at the title—there is a curious reluctance to use such words as equivalence relation, group, semi-group, ring, and field, although there is no hesitation in using equivalent, identity, closed operation, and isomorphic.

On the whole, the book seems to be an adequate presentation of the material, but this reviewer finds it difficult to consider it as a cultural document.

STEPHEN HOFFMAN, Trinity College

Games, Gods and Gambling. By F. N. David. Hafner, New York, 1962. xvi+275 pp. $6.50.

This book will be valued by all persons interested in the history of probability. In her own words, the author has "tried to supplement Todhunter on the early development of ideas about chance and to fill in a certain amount of the background of ideas and controversies which attended the creation of the mathe-

matical theory of probability." She has done this job well.

The first four chapters (39 pp.) are concerned with the earliest developments. Here we find references to board games at the time of the First Dynasty in Egypt (c. 3500 B.C.), archaeological finds of astragali and knucklebones, Herodotus' reference to the playing of games at the time of a famine in Lydia (c. 1500 B.C.) "on one day so entirely as not to feel any craving for food, and the next day to eat and abstain from games. In this way they passed eighteen years"; divination by lots, astragali, or dice; an enumeration in Latin verse of the number of ways in which three dice can fall (c. 1250 A.D.); and the earliest known publication of the famous problem of points in 1494 (with an incorrect solution).

Chapters 5–15, pp. 40–178, give both the probabilistic subject material associated with their names and illuminating biographical data about Tartaglia and Cardano, Galileo, Fermat, Pascal, Graunt, Huygens, Wallis, Newton, Pepys, James Bernoulli, Montmort, and de Moivre (who died in 1754).

The book ends with five appendices (pp. 179–267) and an index. The appendices are translations of works by or about Buckley, Galileo, Mersenne, de Moivre, and the famous letters between Fermat and Pascal and Carcavi.

The author had originally hoped to bring the account down to the present day. We are glad she has decided to publish this much and look forward to the monographs she has in mind writing on "the great triumvirate, James Bernoulli-Montmort-de Moivre, and another on Laplace."

GEORGE B. THOMAS, JR., MIT

Continued Fractions. By C. D. Olds. New Mathematical Library, Random House, New York, 1963. 162 pp. $1.95.

This paperback, written under the auspices of the Monograph Project of the School Mathematics Study Group, is intended for high school students and laymen. The four chapters containing proofs are concerned with the expansion of rational numbers, the linear diophantine equation, the expansion of irrational numbers, and periodic continued fractions. Here the development is leisurely, well motivated and highly readable; it should certainly be accessible to the intended readers. The fifth chapter contains the statement of Hurwitz's theorem, that for every irrational number α there are infinitely many fractions p/q such that $|\alpha-p/q|<1/(\sqrt{5}q^2)$, and that this is false for some α if $\sqrt{5}$ is replaced by a larger number. There are two appendixes, one on the equation $x^2-3y^2=-1$, and one giving the continued fraction expansions of a variety of numbers and functions, and the book ends with solutions to the problems.

The monograph was carefully written and well proof-read. The one error noted by the reviewer occurs in the first half of Problem 7 on page 26, where the possibility that $a_1=0$ necessitates certain modifications. The author chose to denote a continued fraction by $[a_1, a_2, \cdots]$, and was therefore forced to use braces instead of brackets to designate the greatest integer function. This violates well established notation as badly as if one used $f^*(x)$ to denote the de-

rivative of $f(x)$; logically, of course, both are impeccable. But on the whole, the reviewer feels that the book is an excellent example of its kind, and that it is a significant contribution to the growing list of books designed to enrich and diversify the talented high school student's mathematical education.

W. J. LEVEQUE, University of Michigan

Sets, Sequences and Mappings: The Basic Concepts of Analysis. By Kenneth W. Anderson and Dick Wick Hall. Wiley, New York-London. 1963. 191 pp. $5.00.

Aimed at undergraduates who have completed a calculus course but have not yet met much rigor or abstraction, this book hits the target, perhaps a little above center. Except for the last chapter, on metric spaces, the central topics are real sequences and continuous real-valued functions on real domains. The axioms making the reals an ordered field are tacitly assumed, but there are three explicit axioms: the least upper bound axiom, the well-ordering of the positive integers, and an axiom of choice for sequences of subsets. A minor flaw in a well-written book is the unproved parenthetical assertion (pp. 89–90) of the equivalence of Cauchy completeness to the L.U.B. axiom; without a complete list of other axioms, this could be misleading.

Sequences are made basic; the definition of continuity is in terms of preservation of convergence of sequences. Other standard characterizations of continuity are obtained via a chain of theorems about "suburbs." A suburb of a point p is the complement of a neighborhood of p. Clearly a sequence converges to p iff no suburb of p contains a subsequence. Although some of the language of the first five chapters will be unfamiliar to the intended reader, he should recognize the relevance of most of the content to his calculus course. Perhaps the least obviously relevant theorem is the equivalence of "compact" and "sequentially compact."

This is a worthwhile addition to the library of any undergraduate major. It also merits consideration as a text by any department wanting to give a course which makes no obvious progress toward useful goals such as solving differential equations. There are ample sets of exercise, many of which fill in details in proofs.

BURROWES HUNT, Reed College

The Laplace Transform: an Introduction. By Earl D. Rainville. Macmillan, New York, 1963. vi+106 pp. $2.50.

This book is an introduction to Laplace transforms which can be read by a student who has had calculus, and at least an introduction to differential equations. The exposition is quite elementary, so that integrations by parts, for example, are worked out in full detail. Although the theorems are stated explicitly, the proofs are generally of the "it can be shown" type. In several instances, however, the author gives descriptions of possible mathematical behavior which may well be of more value to the tyro than a detailed proof of one relatively weak

theorem. The book contains a great many examples worked out in detail, and a generous number of exercises with answers.

After giving the basic information in the first two chapters, the book treats in the third chapter the applications to vibrating springs, electrical circuits, and bending beams. The remaining chapter headings are: systems of differential equations, additional properties of the transform, partial differential equations.

The author appears to have accomplished admirably his aim of providing an elementary treatment of the Laplace transform for those who are interested primarily in the applications.

H. S. BEAR, University of California, Santa Barbara

Mathematical Models of Economic Growth. By J. Tinbergen and H. C. Bos. McGraw-Hill, New York, 1962. 131 pp. \$6.90.

This book considers a series of economic models which are thought to be of practical use in designing economic development in underdeveloped countries.

The authors favor the method of planning in stages as a general principle to be followed in constructing mathematical models for economic development. Two main stages are distinguished. The first consists in setting up a macromodel of the economy, including equations referring to production, investment, consumption, and international transactions. The evolution of the endogenous variables, as e.g., the volume of production, is obtained as soon as specific values of the various instruments of economic policy, as e.g., the rate of savings, are given. If additional data referring to specific sectors of the economy are available, then a second stage becomes feasible. Namely, one can then determine how the volumes of aggregate investment and consumption arrived at in the first stage can be carried through by operating the various sectors at appropriate levels. This results in the specification of particular targets of operation of the various sectors for the whole planning period. In this spirit a series of successively more detailed macro- and multi-sector models are presented.

In general the book is well proportioned. This reviewer, however, feels that more space should be devoted to a linear programming formulation of an intertemporal development model. It is admitted that lack of data, of computing facilities, and even deficiencies in the quality of the planning staffs, may render a full-blown linear programming model inapplicable for many under-developed countries at the present. However, the "planners" should at least be taught that in most cases there exist several alternative feasible plans, and should be acquainted with a general method of finding the optimal ones.

EMMANUEL DRANDAKIS, Yale University

Handbook of Statistical Tables. By Donald B. Owen. Addison Wesley, Reading, Massachusetts, 1962. xii+580 pp. \$12.50.

This is an excellent collection of tables for use in applications. Much of the tabular material was reproduced directly from the output of digital computers. The sections of the book are entitled: 1. Normal distribution; 2. Student's

t-distribution; 3. Chi-square distribution; 4. F-distribution and multiple comparison; 5. Noncentral t and tolerance limits; 6. Range, Studentized range, and mean square successive difference; 7. Order statistics from the normal distribution; 8. Multivariate normal and t-distributions; 9. Logistic, Poisson, and binomial distributions; 10. Nonparametric tolerance limits; 11. Wilcoxon (Mann-Whitney) tests; 12. Sign, runs, and quadrant tests; 13. Rank correlation; 14. Nonparametric analysis of variance; 15. Kolmogorov-Smirnov statistics; 16. Cramér-von Mises and random division of an interval distribution; 17. Matching and multinomial distributions; 18. Hypergeometric distribution; 19. Product moment correlation coefficient, 20. Orthogonal polynomials, random numbers, and constants. Each section contains from 2 to 14 tables. Tables which are widely available are deliberately given rather cursory treatment. For example, the cumulative Poisson distribution takes up only two pages. The introductory matter given for each table is brief and to the point and includes references to the literature. The book closes with a comprehensive and, therefore, useful index.

FRANK L. WOLF, Carleton College

Statistical Treatment of Experimental Data. By Hugh D. Young. McGraw-Hill, New York, 1962. xv+172 pp. $2.95.

This small paperback text (which is also available cloth-bound) is intended for the use of "sophomore science and engineering students with little mathematical sophistication and no previous exposure to the subject" in a "sub-course—fitted into any course—in which quantitative laboratory work plays an important part." It begins with a discussion of the propagation of errors, proceeds to define the mean and various measures of dispersion, and then presents probability in terms of relative frequencies. There follow discussions of the binomial, Poisson, and normal distributions; the χ^2 goodness of fit test; the method of least squares; and correlations. Exercises appear at the end of each chapter. A summary of formulas and a few mathematical derivations appear in appendices. Appropriate tables are included.

The manner of presentation is relaxed, informal, and appealing. Unfortunately, however, the content suffers in quite a few places from careless or misleading statements. Some of these arise from (a) the lack of any notational device to distinguish between parameters and statistics, (b) what appears to be an occasional identification of the notions "infinite" and "very large," and (c) the lack of any clear explanations of the meaning of "event," "independent events," "mutually exclusive events," and "expected value" although all these notions are involved in the discussion. It also seems unfortunate to the reviewer that the connection between moments and the mean and variance is never mentioned, especially since the book is intended for students of the physical sciences.

FRANK L. WOLF, Carleton College

Elementary Concepts of Mathematics, Second Edition. By Burton W. Jones. Macmillan, New York, 1963. 344 pages. $6.00.

An engagingly written potpourri of mathematics. In addition to the usual material included in a terminal course at the college level, chapters on mirror geometry, Lorentz geometry and topology provide stimulus for the better students. Each chapter concludes with a list of *Topics for Further Study* where specific page references are given to many of the sixty-nine books listed in the *Bibliography.*

The choice of problems is excellent. Many are thought provoking and far from routine. The lack of answers to all but approximately ten problems in the entire text will prove disconcerting, for the problems vary widely in difficulty and will require active class discussions. A teacher adopting the text would be well advised to work carefully through the book not only to choose the topics most suited to the abilities of the students, but also to absorb the "spirit" in which mathematical ideas are presented. The author's approach, varying from the fairly rigorous to the ultra-casual, is perhaps best summed up in his own words at the end of Chapter IX, page 288: "It would be difficult to pin down completely the meaning of this concept, but this can also be said about a number of other things in this chapter and throughout this book."

The format is appealing and the few misprints are easy to spot. The most serious error is the omission of parentheses in limit statements on pages 172–174. One wonders, too, at the use of ${}_mC_n$ instead of $\binom{m}{n}$. Diagrams are plentiful and clearly marked. Now and then the word "numbers" is used in different senses, e.g. page 120: 9 versus the last paragraph of page 126. Set notation, interestingly presented in the first chapter, is rarely used in the remainder of the book.

These criticisms are minor, for the book is well suited to the purpose for which it is written.

WINIFRED ASPREY, Vassar College

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo, New York 14214. *Items must be submitted at least two months before publication can take place.*

PERSONAL ITEMS

University of Minnesota: Associate Professors Elizabeth Carlson, S. A. Gal, and L. W. Green have been promoted to Professors; Assistant Professors C. A. McCarthy and J. M. Slye have been promoted to Associate Professors.

U. S. Naval Postgraduate School: Professor M. C. Wicht, North Georgia College, has been appointed Professor; Dr. U. R. Kodres, I.B.M., Poughkeepsie, New York, has been appointed Associate Professor.

Dr. Evelyn B. Collins, North American Aviation, has accepted a position as Senior Mathematician with the Space Systems Department of the Federal Systems Division of I. B. M., Los Angeles, California.

Assistant Professor H. C. Kennedy, Providence College, has been promoted to Associate Professor.

Professor J. B. Rosser, Cornell University, has been appointed Director of the U. S. Army Mathematics Research Center, University of Wisconsin.

Professor Alberto Saez, Universidad de Los Andes, Merida, Venezuela, has been appointed Professor at the Universidad Central de Venezuela, Caracas, Venezuela.

Professor Emeritus E. F. Allen, Oklahoma State University, died on November 20 1963. He was a member of the Association for 47 years.

Professor Emeritus Archibald Henderson, University of North Carolina, died on December 7, 1963. He was a member of the Association for 42 years.

Visiting Professor Harry Langman, Clarkson College of Technology, died on December 16, 1963. He was a charter member of the Association.

Professor Emeritus Sophia L. McDonald, University of California at Berkeley, died on December 6, 1963. She was a member of the Association for 39 years.

Mr. Joseph Tajen, General Electric, Pittsfield, Massachusetts, died on October 27, 1963. He was a member of the Association for 11 years.

Professor R. C. Yates, University of South Florida, died on December 18, 1963. He was a member of the Association for 34 years.

GRADUATE SUMMER SESSION OF STATISTICS IN THE HEALTH SCIENCES

The Department of Biostatistics, School of Public Health, University of North Carolina is the host institution for the seventh cooperative training program of statistics in the health sciences, to be held June 29 to August 7, 1964.

The Summer Session, under a grant from the National Institute of Health, is offering courses at nearly every academic and experience level. It is designed to meet some of the educational and training needs of those engaged in health and health-related work, and those preparing themselves for such work.

Stipends are available to qualified persons. Inquiries should be made to: Summer Session, Department of Biostatistics, School of Public Health, University of North Carolina, Chapel Hill, North Carolina 27515.

OPERATIONS RESEARCH SOCIETY OF AMERICA—HAWAII MEETING

The Tenth Annual Meeting of the Western Section of the Operations Research Society of America will be held at Honolulu, Hawaii, on September 14–18, 1964. Dr. John E. Walsh, System Development Corporation, Santa Monica, California, is Meeting Chairman. The Honorable Charles J. Hitch, Assistant Secretary of Defense, Comptroller, is scheduled to deliver the keynote address. The meeting will be directed primarily toward the uses of operations research (OR) in technical fields and professions. Practitioners in technical fields and professions are encouraged to contribute papers which illustrate uses, or potential uses, of the OR approach in their areas. This meeting should be valuable, as an introduction to OR methods, for persons in virtually all technical areas. Abstracts of contributed papers should reach the Meeting Chairman by June 15, 1964.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-SEVENTH ANNUAL MEETING OF THE ASSOCIATION

The Forty-seventh Annual Meeting of the Mathematical Association of America was held at the University of Miami, Coral Gables, Florida, from Saturday to Monday, January 25 to 27, 1964, in conjunction with the Annual Meeting of the American Mathematical Society. There were registered 1,493 persons, including 982 members of the Association.

Sessions of the Association were held on Saturday morning and on Sunday morning in the Ponce de Leon Junior High School Auditorium, across Dixie Highway from the University of Miami, and on Monday morning and afternoon in Room 120 of the University College Building. Presiding officers were Professor Herman Meyer for the session on "Goals for School Mathematics" and President Bing for the Retiring Presidential Address on Saturday morning, Professor A. D. Wallace for the session on "Content of the First Course in Real Variables" and President Bing for the Business Meeting on Sunday morning, Professor J. Aczel for the session on "Functional Equations" on Monday morning and First Vice-President H. S. M. Coxeter for the session on "Convexity" on Monday afternoon. The Program Committee for the meeting consisted of Herman Meyer, Chairman; M. K. Fort, Jr., Leo Moser, C. E. Rickart and A. D. Wallace.

FIRST SESSION OF THE ASSOCIATION

Goals for School Mathematics

(Report of the Cambridge Conference)

The Earliest Grades, by Professor A. M. Gleason, Harvard University.

Children should be exposed from the earliest grades to the entire real number system. They should have direct experience with geometry through construction exercises and the study of symmetry. Geometry and analysis should be interrelated through the use of coordinates in elementary situations and interpretations of numbers as lengths, areas and volumes. These ideas should not be presented rigorously but so as to impart true familiarity. Experiments and games should be used in addition to direct teaching, textbooks and workbooks. Particularly in the earliest grades, excessive drill seems to stand in the way of learning concepts which are ultimately more important than algorithmic skill.

Viewpoints of a User of Mathematics, by Professor G. F. Carrier, Harvard University.

A re-emphasis of some features of the conference deliberations which, in the speaker's view, are of enormous importance in the teaching of mathematics either to future mathematicians or to potential users of mathematics.

Implications for Teacher Education, by Professor R. J. Wisner, New Mexico State University.

The mathematics program under discussion is not curriculum revision—it is curriculum construction. As such, there seems to be little hope of preparing large numbers of teachers for the program who themselves were trained to teach the "standard" curriculum even in (GCMP-, SMSG-, UICSM-, etc.) revised form: their attitudes toward mathematics are at variance with what is obviously required by the suggestions under consideration. This seems especially true for teachers who are responsible for students in the early years. If this argument is in line with reality, we are forced to conclude (once again?) that the proper preservice education of teachers is critical to the success of the proposals and that, without such a foundation, the curriculum under construction will eventually collapse.

Panel Discussion, Critique, Rebuttal, Discussion from the Floor.

Retiring Presidential Address: Pivotal Methods in Linear Algebra, by Professor A. W. Tucker, Princeton University.

Pivotal methods, such as Gaussian elimination and G. B. Dantzig's simplex method in linear programming, are basic algorithmic tools of numerical linear algebra. They yield constructive proofs of fundamental theorems, such as John von Neumann's minimax theorem for matrix games. Pivotal (or exchange) methods can be profitably employed in linear algebra courses and texts, especially to unite theory and practice.

This paper will be published in an early issue of this MONTHLY.

SECOND SESSION OF THE ASSOCIATION

Content of the First Course in Real Variables

The session was opened with presentations by Professor E. J. McShane of the University of Virginia, Professor Edwin Hewitt of the University of Washington and Professor Walter Rudin of the University of Wisconsin.

Professor McShane regarded the course in real analysis not as an end in itself but as a utility for all students of mathematics. A generalization is not truly understood as such unless the student sees its applicability in particular instances; the road from special examples to general theory must permit easy passage in both directions. Choice of material should be dictated by the needs and interests of a wide spectrum of today's students, no topic having a claim for inclusion either because of modernity or because of tradition.

Professor Hewitt first reminded his audience that a student of mathematics in U. S. universities at the present time frequently is exposed to topics in real analysis in a variety of courses which he may take over a three-year period. It seems pointless to try to parcel out these topics course by course. Professor Hewitt would rather list a table of "what every young analyst should know," with more details given for the topics in a canonical real variables course than for some others. Certainly students have to know about continuous functions—equicontinuity, the Stone-Weierstrass theorem and uniform continuity. That is, they need the fundamentals of set-theoretic topology. They also need to know Zorn's lemma, the well-ordering theorem, and ordinal numbers. They need in addition to know cardinal arithmetic. Obviously integration theory is essential. Probably the soundest pedagogic approach is through measure theory, although the neatest way he knows to construct countably additive measures is to begin with Riemann-Stieltjes integrals and extend them à la Daniell. Undoubtedly both constructions are needed. Differentiation is not taught as much as it should be, and, Professor Hewitt added, "at least in my real variables courses." He has tried various devices to avoid Vitali's theorem, but has concluded that every student should know Vitali's theorem—there just is no reasonable substitute for it, although F. Riesz had a good try at getting around it. The Radon-Nikodým theorem is another classic that should be taught. Professor Hewitt personally likes to compute the conjugate space of $L_p(p>1)$ by some elementary means, for example, by McShane's technique, and then get the Radon-Nikodým theorem from knowing the conjugate space of L_2.

Professor Rudin suggested that the course should rapidly develop the facts of life concerning Lebesgue integration and differentiation by whatever route the instructor prefers. It should introduce some fundamental ideas from the theory of Banach spaces and Hilbert space, and it should apply this machinery to other parts of analysis. Fourier series and transforms can effectively be used as illustrative material. The relation between Radon-Nikodým derivatives and Jacobians furnishes a nice topic. The interplay with the theory of analytic functions can be discussed. Purely topological or set-theoretic topics should be soft-pedalled; as far as infinite cardinals are concerned, for instance, the difference between "countable" and "uncountable" is all that is required at this stage.

Panel Discussion, Critique, Rebuttal, Discussion from the Floor.

Annual Business Meeting of the Association; the Association's Third Award for Distinguished Service to Mathematics, and the Award of the 1964 Chauvenet Prize.

THIRD SESSION OF THE ASSOCIATION

Functional Equations

Mean Values and Transfinite Diameters, by Professor Einar Hille, Yale University.

Consequences of the Kolmogoroff-Nagumo postulates for mean values were examined, in particular, the averaging of averages and the related oscillation reducing properties. Applications were given to (i) maxima and minima problems with side conditions involving polytopes, and (ii) the existence of transfinite diameters of bounded sets in metric spaces. The transfinite diameters of unit spheres of function spaces were discussed. It was shown that the maximum value 2 is reached in several cases. For Hilbert space the transfinite diameter is $\sqrt{2}$ under suitable assumptions. In this discussion the maximizing properties of regular simplexes in R_n play an important role.

Analytic Properties of the Solutions of Certain Functional Equations, by Professor J. H. B. Kemperman, University of Rochester.

The existence and uniqueness problems, for a given functional equation, are often greatly simplified if one knows beforehand the analytic properties (such as continuity and differentiability) of the solutions of this equation. As will be shown, such information is available for an important class of functional equations, closely related to the so-called semigroup equation. If necessary, one is usually willing to assume that the solution on hand is measurable. Often it suffices to assume that the solution is bounded (or bounded above) on a sufficiently thick set. On occasion, any nonnegative solution is measurable (and hence continuous, etc., depending on the situation).

The Functional Equation of Associativity, by Professor Berthold Schweizer, University of Arizona.

The solution (1) $F(x, y)=f(f^{-1}(x)+f^{-1}(y))$ of the functional equation of associativity (2) $F(F(x, y), z)=F(x, F(y, z))$ was first found by Abel in 1826. Since then it has been obtained under successively weaker hypotheses by Brouwer (1909), Cartan (1930) and Aczel (1949). In all cases, however, the function F was either assumed or shown to be strictly increasing. By replacing f^{-1}, the inverse of f, in (1) by a suitably chosen right-subinverse, nondecreasing solutions of (2) may be obtained. J. Ling, again working with right-subinverses rather than inverses, has recently solved (2) when F is nondecreasing and Archimedean rather than increasing. These hypotheses are the weakest on F to date.

FOURTH SESSION OF THE ASSOCIATION

Convexity

The Polyhedron Inequality, by Professor Herbert Busemann, University of Southern California.

Let a measure (r-area) $\alpha(M)$ be defined for every Borel set M lying in the intersection of an r-flat with an open nonempty convex set C in the n-dimensional affine space $A^n(1 \leqq r \leqq n-1)$. The area satisfies the polyhedron inequality if $\alpha(F_0) \leqq \sum_{i=1}^{k} \alpha(F_i)$ for the r-faces $F_0, \cdots, F_k$ of a closed r-dimensional polyhedron in C. The lecture deals with theorems and principally with problems concerning the polyhedron inequality with emphasis on the case where $C=A^n$ and $\alpha(M)$ is invariant under translation of M. The close connection of this field with convexity questions is discussed.

Some Problems of Elementary Euclidean Geometry, by Professor H. G. Eggleston, University of London and University of Washington.

Attention was drawn to a number of problems, both solved and unsolved, which can be easily stated in terms of elementary Euclidean geometry. The problems were largely those which involve a rotation or which had some connection with rotations, and it seems that this fact is largely responsible for their peculiar difficulties. It was pointed out that the introduction of more restrictive conditions in a problem may simplify it whilst a closely allied problem is made more difficult. The lecture concluded with some remarks on the differing roles that the concept of convexity can play in elucidating these problems.

Facial Structure of Convex Polytopes, by Professor Victor Klee, University of Washington.

The combinatorial study of convex polytopes and their facial structure was initiated by Euler's

famous theorem and was later carried on by several other investigators. In recent years, interest in the subject has revived due to its close connections with linear programming. The speaker reviewed recent progress in the field and discussed the status of various problems that are still unsolved.

SPECIAL SESSIONS OF THE ASSOCIATION

There was a film showing of color animations by Bruce Cornwell on Saturday at 1:00 P.M. in the Ponce de Leon Junior High School Auditorium. The following films were shown: "Seven Bridges of Königsberg," "Possibly so, Pythagoras," "How Do We Count?," "Big Numbers, Little Numbers," and "Sets, Crows, and Infinity." This was followed by the showing of "Mathematical Peepshows," five 2-minute color animations by Charles and Ray Eames for IBM: "Eratosthenes," "Topology: Jordan's Curve Theorem," "Symmetry," "Something about Functions," and "2^n—A Story of the Power of Numbers." The following PSSC physics films were shown on Saturday, beginning at 7:00 P.M., in the Everglades Room of the Everglades Hotel in Miami: "Straight Line Kinematics" (black and white), "Frames of Reference" (black and white), and "Vector Kinematics" (black and white). This was followed by a showing of the ETS film "Thinking Machines" (color). There was a showing of Part I of the film produced by the Association's Committee on Production of Films, "Theory of Limits," with Professor E. J. McShane as lecturer on Sunday at 7:00 P.M. in the Everglades Room, followed at 7:40 P.M. by "The Kakeya Problem" (color and with animation), with Professor A. S. Besicovitch as lecturer and also produced by the same Committee of the Association. This film was followed at 8:45 P.M. by a Madison Project film (in black and white) showing the teaching, by Professor Robert Davis, of complex numbers via matrices to seventh-grade children.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Friday morning and afternoon in the Board Room of the Ashe Building (Room 226) of the University of Miami with 29 members present.

The Board approved the appointment by President Bing of the following Nominating Committee for 1964: R. D. Anderson, Chairman; Paul Eberhart, and Victor Klee.

The Board elected Professor G. B. Price as a member of the Finance Committee. It also elected Professor Gertrude Ehrlich as an Associate Editor of the MONTHLY, effective January 1, 1964, to fill the unexpired part of the term ending December 31, 1966, of Professor A. L. Shields, who had resigned. The Board also elected Professor Sam Perlis as an Associate Editor of the MATHEMATICS MAGAZINE, effective January 1, 1964, in place of Professor V. H. Haag, who had been unable to serve in that capacity. The Board also elected Professors Ruth B. Rasmusen and Raoul Hailpern as additional Associate Editors of the MATHEMATICS MAGAZINE, effective January 1, 1964, the latter to be in charge of the editorial work in the Buffalo office.

On the recommendation of the Committee on a Yearbook, the Board authorized the preparation of a *Handbook of Undergraduate Mathematics Programs*, designed for high school students and for junior college and college students who expect to transfer to another university. The Board referred the matter to the Committee on Advisement and Personnel for further definition and implementation and requested the President of the MAA to explore with the NCTM cooperation on the preparation and distribution of this material.

The Board voted to invite Professor E. E. Floyd of the University of Virginia to deliver the thirteenth series of Earle Raymond Hedrick Lectures at the 1964 Summer Meeting.

The Board approved amendments to the regulations governing the Association's Award of the Chauvenet Prize (a history of the Chauvenet Prize and the new regulations will appear in the May issue of this MONTHLY).

The Board approved a proposal by the Committee on the Undergraduate Program in Mathematics for continued support of the Committee for the two-year period starting July 1, 1964.

The Executive Director reported to the Board that a member, who prefers to remain anonymous, has made a gift to the Association of more than $4,000. This is to be used to establish a new fund, whose income is unrestricted. The donor's preference is that the income be spent for publication purposes.

The Board instructed the Secretary to prepare an amendment to the By-Laws providing for an increase in dues to $6 annually, effective January 1, 1965. This amendment will be voted on by the membership of the Association at the Business Meeting this summer.

The Board approved the following schedule of future meetings: University of Massachusetts, August 24–26, 1964; Denver, Colorado, January 28–30, 1965; Cornell University, August, 1965; Chicago, January 26–28, 1966; Rutgers—The State University, New Brunswick, New Jersey, August, 1966; Houston, January 26–28, 1967; University of Toronto, August, 1967; University of Wisconsin, Madison, August 26–28, 1968.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Sunday, January 26, 1964, in the Ponce de Leon Junior High School Auditorium, with President Bing presiding. The Association's third Award for Distinguished Service to Mathematics was made to Professor E. J. McShane of the University of Virginia. The citation (which appeared on pages 1–2 of the January issue of this MONTHLY) was read by Professor W. L. Duren, Jr., of the University of Virginia; the award was presented by President Bing. An autographed copy of the January issue of this MONTHLY containing the citation was presented to Mrs. McShane.

The 1964 Chauvenet Prize was awarded to Professor Leon A. Henkin of the University of California, Berkeley, for his paper "Are Logic and Mathematics Identical?," published in SCIENCE, 138 (1962) 788–794. The award was presented by President Bing. (For further details on this award see the January issue of this MONTHLY, page 3). An autographed copy of the January issue was presented to Mrs. Henkin.

The Secretary reported the membership of the Association as 14,256, an increase of 1464 since the corresponding date last year.

The balloting for officers in which 2219 votes were cast resulted in the election of Professor R. L. Wilder of the University of Michigan as President-Elect for 1964, and of Professor E. E. Moise of Harvard University as First Vice-President for the two-year term 1964–1965, and of Professor Roy Dubisch of the University of Washington and Professor K. O. May of Carleton College as Governors for the three-year term 1964–1966.

The Secretary then reported on some of the actions taken by the Board of Governors on Friday. He announced that the Executive and Finance Committees had approved a recommendation of the Committee on Membership concerning appointment of "MAA representatives" at all university and college departments of mathematics in the United States and Canada. The Buffalo office of the Association will shortly request from the chairman of each Section of the Association a list of MAA representatives in each college within the Section (including junior colleges). When the chairman of the department is a member of the MAA, the suggested procedure is to write him asking if he would nominate a representative, preferably someone other than himself. If the department chairman is not a member, direct solicitation of some MAA member is suggested; if no one in the department is an MAA member, every effort, including personal contact, should be made to remedy this situation. The duties of the MAA representative will be described in a statement to be supplied by the Buffalo office.

The Secretary also reported that a sixth edition of the brochure *Professional Opportunities in Mathematics* has been prepared by the Association's Committee on Advisement and Personnel. The material of the fifth edition was updated, and some sections were completely rewritten. Publication is expected in April, 1964.

The Secretary expressed the deepest gratitude to the local members of the Committee on Arrangements for having done so much to insure the success of the meeting and singled out Professor J. H. Curtiss for his untiring efforts to coordinate its many aspects.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Thursday, January 23, to Sunday, January 26. The thirty-seventh Josiah Willard Gibbs Lecture was delivered by Professor Lars Onsager of Yale University on Friday evening at 8:00 P.M. in the Everglades Room of the Everglades Hotel on "Mathematical Problems of Cooperative Phenomena." Professor Deane Montgomery of the Institute for Advanced Study delivered the Retiring Presidential Address entitled "Compact Groups of Transformations" on Friday at 9:00 A.M. ProfessorMorton Brown of the University of Michigan delivered an address entitled "Topological Manifolds" on Thursday at 9:00 A.M. and Professor Heisuke Hironaka of Brandeis University one on "Singularities in Algebraic Varieties" on Friday at 2:00 P.M.

At the Business Meeting of the Society on Thursday at 1:30 P.M., the First Veblen Prize in Geometry was awarded to Professor C. D. Papakyriakopoulos of Princeton University for his papers "On Solid Tori," Proc. London Math. Soc., 7 (1957) 281–299, and "On Dehn's Lemma and the Aspherity of Knots," Ann. of Math., 66 (1957) 1–26, and the Second Veblen Prize to Professor R. H. Bott of Harvard University for his papers "The Space of Loops on a Lie Group," Mich. Math. Jour., 5 (1958) 35–61, and "The Stable Homotopy of the Classical Groups," Ann. of Math., 70 (1959) 313–337. The awards were presented by Professor R. H. Bing. The tenth Bocher Prize was awarded to Professor P. J. Cohen of Stanford University for his paper "On a Conjecture of Littlewood and Idempotent Measures," Amer. Jour. Math., 82 (1960) 191–212.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of J. H. Curtiss, Chairman; H. L. Alder, R. W. Bagley, A. T. Butson, M. L. Curtis, Mrs. Georgia K. Del Franco, Edwin Duda, E. F. Low, Jr., Herman Meyer, Mrs. Agnes Y. Rickey, Andrew Sobczyk, G. L. Walker.

Registration headquarters were located in the lobby of the 730 Building on the University of Miami campus, beginning on Thursday. On Wednesday, a Registration Desk was maintained in the Mezzanine of the Everglades Hotel from 2:00 P.M. to 8:00 P.M. On Monday, the Registration Desk was located in the lobby of the University College Building. Books and other exhibits were located in the Recreation Room of the 730 Building. The Mathematical Sciences Employment Register was located in the Westminster Chapel Fellowship Room on the University of Miami campus. Accommodations for the meeting were handled by the Housing Bureau of the Convention Bureau of the City of Miami which made reservations for members from a listof 14 hotels in downtown Miami.

The University of Miami was host to a President's tea and reception on Friday from 4:30 P.M. to 6:00 P.M. in the Richter Library Lecture Hall. A bus left the 720 Dormitory at 12:30 P.M. on Sunday for Flamingo (Everglades National Park), from where a sightseeing boat trip through the Everglades National Park started at 3:00 P.M. for a two-hour bird watchers' and photographers' trip. A tour to the International Design Center left the Everglades Hotel at 9:00 A.M. on Saturday.

HENRY L. ALDER, *Secretary*

OFFICERS AND COMMITTEES AS OF FEBRUARY 1, 1964

General Offices: SUNY at Buffalo, Buffalo, New York 14214
Executive Director: H. M. GEHMAN

OFFICERS

President, R. H. BING, University of Wisconsin, Madison (1963–1964)
President-Elect, R. L. WILDER, University of Michigan (1964)
First Vice-President, E. E. MOISE, Harvard University (1964–1965)
Second Vice-President, MINA S. REES, City University of New York (1963–1964)
Editor, F. A. FICKEN, New York University (1962–1966)
Secretary, H. L. ALDER, University of California, Davis (1960–1964)
Treasurer, H. M. GEHMAN, SUNY at Buffalo (1963–1967)
Associate Secretary, RAOUL HAILPERN, SUNY at Buffalo (1963–1967)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

G. B. PRICE, University of Kansas (1959–1964)
C. B. ALLENDOERFER, University of Washington (1961–1966)
A. W. TUCKER, Princeton University (1963–1968)

Elected Members of the Finance Committee (not already otherwise members of the Board)

E. A. CAMERON, University of North Carolina (1962–1965)

Governors at Large

T. M APOSTOL, California Institute of Technology (1962–1964)
R. P. BOAS, Northwestern University (1963–1965)
R. P. DILWORTH, California Institute of Technology (1963–1965)
ROY DUBISCH, University of Washington (1964–1966)
K. O. MAY, Carleton College (1964–1966)
HANS RADEMACHER, University of Pennsylvania (1962–1964)

Sectional Governors (July 1, 1961–June 30, 1964)

Kansas, PAUL EBERHART, Washburn Univeristy
Missouri, R. J. MICHEL, Southeast Missouri State College
New Jersey, H. O. POLLAK, Bell Telephone Laboratories, Inc.
Northeastern, D. E. RICHMOND, Williams College
Ohio, R. R. STOLL, Oberlin College
Pacific Northwest, D. C. MURDOCH, University of British Columbia
Southeastern, H. S. THURSTON, University of Alabama
Southwestern, J. B. GIEVER, New Mexico State University
Upper New York State, HARRIET F. MONTAGUE, SUNY at Buffalo

Sectional Governors (July 1, 1962–June 30, 1965)

Illinois, M. M. DAY, University of Illinois
Iowa, J. J. L. HINRICHSEN, Iowa State University
Louisiana-Mississippi, R. D. ANDERSON, Louisiana State University
Maryland-D.C.-Virginia, M. GWENETH HUMPHREYS, Randolph-Macon Woman's College
Michigan, L. E. MEHLENBACHER, University of Detroit
Minnesota, J. E. HAFSTROM, University of Minnesota, Duluth
Philadelphia, D. W. WESTERN, Franklin and Marshall College
Southern California, P. J. KELLY, University of California, Santa Barbara
Texas, H. C. PARRISH, North Texas State University

Sectional Governors (July 1, 1963–June 30, 1966)

Allegheny Mountain, EVAN JOHNSON, JR., Pennsylvania State University
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Kentucky, A. W. GOODMAN, University of Kentucky
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Terms of office of members expire, except where otherwise noted, at the Annual Meeting in January following the last year of service listed below. For temporary committees, no terms of office are listed, since they are automatically discharged at the expiration of the President's term of office which is the Annual Meeting in January, 1965.

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On the National Research Council:
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ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendments adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting in Miami on January 24, 1964, elected to membership in the Association the fifth set of applicants for academic membership (for election of the other four sets, see pages 337–38 of the April 1962, page 953 of the November 1962, pages 479–80 of the April 1963, and page 1044 of the November 1963 issues of this MONTHLY). Approval for election to membership was given to the following 37 applicants for academic membership.

University of Akron
University of Alberta
Andrews University
Brown University
Bucknell University
Central Michigan University
Central Missouri State College
Clarkson College of Technology
Concordia College
University of Dayton
Eastern Illinois University
Eastern New Mexico University
Florida State University
Frederick College
Geneva College
Hofstra University
Loyola College
Missouri School of Mines and Metallurgy
Université de Montréal
University of Nebraska
University of Nevada
University of Omaha
University of Oregon
Pace College
University of Pittsburgh
Rutgers, The State University
San Diego State College
Seton Hall University
Southwestern at Memphis
Stanford University
College of Steubenville
Susquehanna University
Tennessee Agricultural and Industrial State University
Tennessee Polytechnic Institute
Texas A & M University
Winthrop College
University of Wyoming

HENRY L. ALDER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fifth Summer Meeting, University of Massachusetts, Amherst, August 24–26, 1964.

Forty-eighth Annual Meeting, Denver-Hilton Hotel, Denver, Colorado, January 28–30, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pa., May 2, 1964.

ILLINOIS, Bradley University, Peoria, May 8–9, 1964.

INDIANA, Butler University, Indianapolis, May 2, 1964.

IOWA

KANSAS

KENTUCKY, University of Kentucky, Lexington, May 1–2, 1964.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 19–20, 1965.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Westinghouse Electric Corp., Friendship International Airport, Baltimore, Maryland, May 2, 1964.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, College of St. Thomas, St. Paul, May 9, 1964.

MISSOURI

NEBRASKA, University of Nebraska, Lincoln, May 2, 1964.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 7, 1964.

NORTHEASTERN, Worcester Polytechnic Institute, Worcester, Mass., November 28, 1964.

NORTHERN CALIFORNIA

OHIO, University of Akron, May 9, 1964.

OKLAHOMA

PACIFIC NORTHWEST, Washington State University, Pullman, Washington, June 19, 1964.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 21, 1964.

ROCKY MOUNTAIN, Colorado College, Colorado Springs, Colorado, May 1–2, 1964.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS

UPPER NEW YORK STATE, New York State Education Department, Albany, May 16, 1964.

WISCONSIN, Wisconsin State College, Whitewater, May 2, 1964.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN MATHEMATICAL SOCIETY, Pullman, Washington, June 20, 1964.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Maine, Orono, June 22–26, 1964.

ASSOCIATION FOR COMPUTING MACHINERY, Philadelphia, August 25–28, 1964.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Detroit, November 26–28, 1964.

INSTITUTE OF MATHEMATICAL STATISTICS, Berne, Switzerland, September 14–16, 1964.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Queen Elizabeth Hotel, Montreal, Canada, May 27–29, 1964.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, The Hotel Shoreham, Washington, D. C., May 11–14, 1964.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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CONTENTS

MAY 1964

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(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10053.

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For nonmembers the subscription price is $8.00.

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GRAPH THEORY

CLAUDE BERGE, Institut Henri Poincaré

1. Introduction. Graph Theory has its origin in a great number of old problems (in the work of Euler, Kirchhoff, *et al.*) and in recent years its range has become vastly greater. We draw a graph each time we want to represent by points a number of individuals, cities, chemical substances, strategic positions, and to join certain pairs of them by arrows, symbolizing a definite relationship. These diagrams are used in different fields under different names: sociograms (psychology), simplexes (topology), electrical circuits (physics), communication or transportation networks (operational research), family trees, etc. Thanks to the process of abstraction, which is so characteristic of twentieth-century mathematics, the properties of all these diagrams have been systematically analysed and a uniform theory has arisen which is applicable to all these fields.

Our aim here is to review some topics which are of particular interest today and are undergoing intensive development. We shall not deal with the problems of enumerating different kinds of graphs, this being considered as a part of Combinatorial Analysis. We shall just mention here, without giving proofs, certain delicate theorems which are the result of numerous researches, and we shall limit ourselves to finite graphs.

2. General definitions. Let us consider an abstract set X, and a multi-valued function Γ, mapping X into X; that is by definition, a law which associates to each element $x \in X$ a set $\Gamma x \subset X$. We shall define a *graph* as the pair (X, Γ) consisting of the set X and the function Γ. As an example, let X be a set of people, and if $x \in X$, let Γx be the set of children of the individual x. The pair (X, Γ) defines a graph usually called the family tree. It is usually convenient to represent all the individuals by points, and if x is y's father, to join them by an arrow going from point x to point y.

Each time we have a graph, we shall think of such a representation; but, of course, set X may be infinite and, in fact, in no case shall we have recourse to a diagram for our reasonings.

For a graph (X, Γ) an element of X is called a *vertex* while the pair (x, y) with $y \in \Gamma x$, is called an *arc* of the graph. In the following the set of the arcs of a graph will be designated by U, the arcs themselves being labelled u, v or w (with indices if necessary). One can see that the set of the arcs of a graph completely determines the associated function Γ, just as this multi-valued function determines the set U; consequently, it is equally valid to express a graph in the form $G=(X, \Gamma)$ or in the form $G=(X, U)$.

A subgraph of a graph (X, Γ) is by definition a graph of the form (A, Γ_A), where $A \subset X$, and where the function Γ_A is defined by

$$\Gamma_A x = \Gamma x \cap A.$$

A *partial graph* of (X, Γ) is by definition a graph of the form (X, Δ), where

$\Delta x \subset \Gamma x$ for all x. A *partial subgraph* is by definition a graph of the form (A, Δ_A), where $A \subset X$, and where

$$\Delta_A x \subset \Gamma_A x \qquad (x \in A).$$

If we consider as an example the graph representing a road map of the United States: X is the set of the cities in the States, and (x, y) is an arc of U if a road of any sort goes from city x to city y; a map of the highways would be a partial graph, while a road map of the State of New York would be a subgraph. A map of the highways of the State of New York would be a partial subgraph.

An arc whose initial vertex is x and whose terminal vertex is not x is said to be *incident out* from x; an arc *incident into* x may be similarly defined. If A is a subset of the set of vertices, an arc u is said to be *incident out* from A if

$$u = (a, x), \qquad a \in A, \qquad x \notin A.$$

To complete the list of the most fundamental terms we shall use, let us define a *path* as a sequence $(u_1, u_2, \cdots,)$ of arcs in which the terminal vertex of each arc is the initial vertex of the following arc. The *length* of this path is the number of arcs in the sequence. A *circuit* is a finite path in which the initial vertex coincides with the terminal vertex.

A graph is *strongly connected* if for any two different vertices x and y there is a path going from x to y. As an example, let us consider a graph where X is a group of people and where $(x, y) \in U$ when x can send a message directly to y. Such a graph is called a *communication network*, and, if it is well organized, it must be possible for any individual to send a message to any other individual in the group, either directly or by successive retransmissions; in other words, the graph must be *strongly connected.*

When a graph is not strongly connected, one can sub-divide it into several disjointed subgraphs which are strongly connected. To be more precise, given a vertex x, let us denote by B_x the set of vertices y for which there exists a circuit passing through x and y; the subgraph generated by a set B_x is called a *strong component* of the graph and it is obvious that the different strong components constitute a *partition* of X; that is: $B_x \neq \emptyset$, $B_x \neq B_y$ implies $B_x \cap B_y = \emptyset$, $\bigcup_{x \in X} B_x = X$.

For some of the concepts we shall study here, we do not take into account the direction taken by the arrows of the graph, but only consider whether two vertices are joined together or not; in the unoriented theory, we shall not speak of *arcs*, *paths*, *circuits*, or *strong components*, and we must define corresponding concepts which do not imply an orientation.

An *edge* (or a *link*) of the graph is by definition a set of two vertices x and y such that either $y \in \Gamma x$ or $x \in \Gamma y$.

An edge will be denoted here by a Roman letter in heavy type: $\mathbf{u}$, $\mathbf{v}$, etc., and their set is written $\mathbf{U}$. The set of all vertices joined to x (or "adjacent" to x) is written $\boldsymbol{\Gamma} x$, the Greek letter $\boldsymbol{\Gamma}$ being here in heavy type. A *chain* is a sequence of edges, each edge being attached to the preceding one by one extremity and

to the following one by the other. A *cycle* is a finite chain which starts and finishes at the same vertex. A graph is *connected* if there is a chain between every pair of distinct vertices; a strongly connected graph is connected, but the converse is not necessarily true. If the graph is not connected, one can subdivide it into several disjointed subgraphs which are connected; let us denote by C_x the set of all vertices which can be joined to vertex x by a chain. The subgraph generated by a set C_x is called a *component* of the graph and it is obvious that the different components constitute a partition of X. The components are easily recognisable since they are not linked together. Let us consider, for example, the graph of Fig. 1.

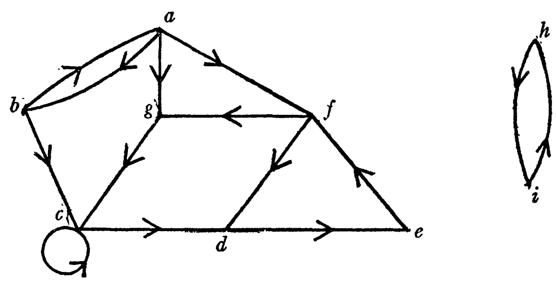

FIG. 1

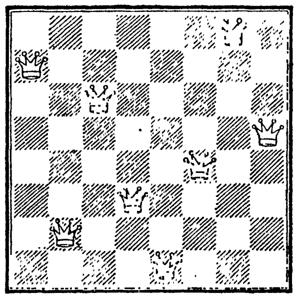

FIG. 2

In this graph there are two components: $\{abcdefg\}$ and $\{hi\}$; there are three strong components: $\{ab\}$, $\{cdefg\}$, $\{hi\}$; there are fourteen arcs, but only twelve edges: for example, the two arcs (a, b) and (b, a) correspond to only one edge $[a, b]$. The edge (c, c) whose two extremities are the same vertex c, is called a *loop*. The chain formed by the sequence $[d, c]$, $[c, c]$, $[c, b]$, $[b, a]$, is also written $[d, c, c, b, a]$.

3. Internal stability number of a graph. Consider a graph $G=(X, \Gamma)$. A set S is said to be *internally stable* if no two vertices of S are adjacent; in other words, if $x \in S$ implies $\Gamma x \cap S = \emptyset$. The *internal stability number* of the graph G is defined to be the maximum number of elements of an internally stable set. We shall denote it by $\alpha(G)$. The determination of this number $\alpha(G)$ is a problem which has often recurred in the history of mathematics.

Gauss set the following well-known problem: *is it possible to place eight queens on a chess board so that no one queen can be taken by any other?*

Consider a symmetric graph with 64 vertices representing the squares of the board and join vertex x to vertex y when the squares x and y are on the same rank or file or on the same diagonal: the Gauss problem reduces to that of finding a maximum internally stable set of this graph. This problem is not as simple as it may appear at first and, in fact, it took a whole century to find out that there are 92 solutions to it. We give one of them in Fig. 2.

Another well-known problem was set by Cayley: *if you have fifteen schoolgirls, can you form* 35 *distinct triads in such a way that no two girls shall be together in a triad more than once?*

To solve this problem, form a graph G whose vertices are the 455 possible triads, two triads being joined if they have two girls in common: then find a maximum internally stable set.

A third problem was set by Shannon and occurs in *Information Theory*: *a transmitter can send several signals and we know that certain given pairs of these can be confused at the receiving end; what is the maximum number of signals which can be used so that there is no possibility of confusion on reception?*

This reduces to finding a maximum internally stable set of a graph G, where two vertices are adjacent if they represent two signals liable to confusion.

As a result of recent works, it is now rather easy to determine the internal stability number by means of the theory of *Boolean operations*. But in certain cases we need also theorems and several problems on internal stability remain unsolved. We shall just quote the following result:

THEOREM. *If S is a subset of vertices, if $p(S)$ denotes the number of uneven components of the subgraph generated by the set $X-S$, and $n(S)$ denotes the number of elements of S, let us define*

$$\xi = \max_{S\subset X} [p(S) - n(S)]$$

then the internal stability number of the graph G verifies $\alpha(G) \leqq \frac{1}{2}[n(X)+\xi]$.

4. External stability number. Given a graph $G=(X, \Gamma)$, a subset T of vertices is said to be *externally stable* if

$$x \notin T \quad \text{implies} \quad \Gamma x \cap T \neq \emptyset.$$

By definition the *external stability number* of the graph G is the minimum number of elements of an externally stable set. The problem of determining this number is easier than the problem considered in the preceding paragraph. It also appears frequently; in a game of chess, for instance, what is the smallest number of queens which can be placed on the board so that every square is dominated by at least one of the queens? This reduces to the problem of finding a minimum externally stable set for a graph with 64 vertices (the squares on the board), and with an arc (x, y) each time squares x and y are on the same rank, or file or diagonal. The external stability number of this graph is 5, as shown on Fig. 3.

The problem of finding minimum externally stable sets of a graph can also be solved by means of the *Boolean operations.*

For a graph G a subset S of vertices is called a *kernel* of the graph if S is both internally and externally stable; we have therefore:

$$x \in S \quad \text{implies} \quad \Gamma x \cap S = \emptyset, \tag{1}$$

$$x \notin S \quad \text{implies} \quad \Gamma x \cap S \neq \emptyset. \tag{2}$$

From condition (1) we deduce that the kernel S is free from loops; and from condition (2) we deduce that S contains every vertex x for which $\Gamma x = \emptyset$; further, the empty set is not a kernel.

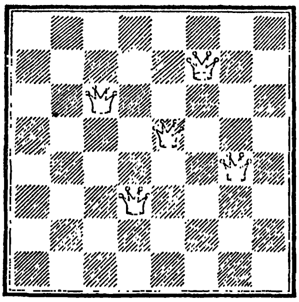

FIG. 3

The notion of a kernel was first introduced by von Neumann and Morgenstern into the theory of games, under the name of "solution." Suppose that n players, whom we shall designate (1), (2), $\cdots$, (n), can by their choice of action select a situation x from a set X; the situation a is said to be *effectively preferred* to the situation b, written $a \succ b$, if there exists a set of players who judge a to be better than b, and who can, if they so wish, make their point of view prevail; the relation $\succ$ is not necessarily transitive. Let us now consider the graph (X, Γ), where Γx represents the set of situations which are effectively preferred to x. Let S be a kernel of the graph, if one exists. Von Neumann and Morgenstern proposed that the game be limited to the elements of S. Since S is internally stable, no one situation of S can be effectively preferred to another, which makes for a certain consistency; since S is externally stable to any situation x which is not in S, one can find a situation in S which is effectively preferred to x, so that x can immediately be discarded.

Another case in which the concept of *kernel* is of great interest is the theory of *nim type games*: given a graph (X, Γ), and a selected vertex x_0, two players (1) and (2) play in turn; player (1) first chooses a vertex x_1 from the set Γx_0, then (2) chooses a vertex x_2 from the set Γx_1, then (1) chooses a vertex x_3 from Γx_2, etc. If a player selects a vertex x_k such that $\Gamma x_k = \emptyset$, the game terminates; the player who chooses the last vertex x_k wins, and his opponent loses. In honor of the familiar pastime of which this is a generalization, we shall call that game a *nim type game*, and the problem now arises of how to characterize the winning positions, that is to say, the vertices which ought to be chosen in order to win the game.

It is obvious that if the graph (X, Γ) possesses a kernel S, and if a player chooses a vertex in S, this choice assures him of a win or a draw. In fact, if player (1) chooses $x_1 \in S$, one has either $\Gamma x_1 = \emptyset$, in which case he has won; or else his opponent will be forced to choose a vertex x_2 in $X - S$, and then at the next move, player (1) again chooses a vertex x_3 in S, and so on. Should the game

be terminated at some stage by one of the players choosing a vertex x_k such that $\Gamma x_k=\varnothing$, we have $x_k \in S$, and hence the winning player is necessarily (1).

Two theorems must be quoted. A graph is *symmetric* if $y \in \Gamma x$ implies $x \in \Gamma y$.

THEOREM 1. *A symmetric graph without loops possesses a kernel.*

THEOREM 2 (M. RICHARDSON). *If a graph has no circuits of uneven length, then it possesses a kernel.*

5. Chromatic number of a graph. Given a positive integer p, a graph G is said to be *p-chromatic* if the vertices can be painted with p distinct colors in such a way that no two adjacent vertices are of the same color. The smallest number p for which the graph is p-chromatic is called the *chromatic number* of the graph G and is written $\gamma(G)$. This concept appears in mathematical statistics (balanced incomplete block design) and in certain old combinatorial problems. Let us consider as an example this variant of the schoolgirls problem: fifteen schoolgirls whom we represent by the letters $a, b, \cdots, m, n, o$, go for a walk each day in groups of three; is it possible to choose the groups so that on seven consecutive days no two girls walk together more than once? In the following table is the Cayley solution:

SUNDAY	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY	SATURDAY
afk	*abe*	*alm*	*ado*	*agn*	*ahj*	*aci*
bgl	*cno*	*bcf*	*bik*	*bdj*	*bmn*	*bho*
chm	*dfl*	*deh*	*cjl*	*cek*	*cdg*	*dkm*
din	*ghk*	*gio*	*egm*	*fmo*	*efi*	*eln*
ejo	*ijm*	*jkn*	*fhn*	*hil*	*klo*	*fgj*

Let us determine first a set of 35 triads such that no two of them have two girls in common (this is a problem we have already examined). To find out whether this set can be divided into 7 walks, let us construct a graph G whose vertices are the 35 chosen triads and whose edges join any two triads having one girl in common. If the chromatic number of this graph is $\gamma(G)=7$, the seven colors used define the 7 walks; if $\gamma(G)>7$, another set of 35 triads must be selected.

The device for determining a chromatic number uses again the Boolean operations: the problem of coloring the vertices with p colors is equivalent to the problem of finding a partition of X into p internally stable sets; thus, construct an auxiliary graph with points $x, x', x'', \cdots$ representing the vertices of G and other points $s, s', s'', \cdots$ representing the maximal internally stable sets of G. Draw the arc (x, s) from a point x representing a vertex of G to a point s representing an internally stable set of G containing it. The minimum coloration of G will be defined by a minimum externally stable set of the auxiliary graph. A first result is:

THEOREM 1 (KÖNIG). *A graph is 2-chromatic* (*or bichromatic*) *if and only if it contains no cycles of uneven length.*

We show first that a graph (X, U) with no uneven cycles is bichromatic. We can suppose that the graph is connected (if not, we treat each component separately). We shall color the vertices according to the following rule: 1) An arbitrary vertex a is colored blue; 2) if a vertex x is colored blue, the vertices adjacent to it and not yet colored will be colored red, and if a vertex y is colored red, the vertices adjacent to it and not yet colored will be colored blue.

Since the graph is connected, sooner or later every vertex will be colored; a vertex x cannot be colored both blue and red, for this would imply that vertex x and vertex a are on a cycle of uneven length. The graph is therefore bi-chromatic.

Conversely, if a graph is bi-chromatic, it clearly cannot contain any cycles of uneven lengths, for it would not be possible to color the vertices of such a cycle with two colors according to the given rule.

Consider a graph G and define $q = \max_{x \in X} n(\Gamma x)$. The number q is called the *maximum degree* of the graph. It is obvious that $q+1$ colors are sufficient to color the graph; in other words, the graph G is $(q+1)$-chromatic. In fact, if $q=2$, and if the graph consists of one cycle of uneven length, the coloration will require exactly $q+1=3$ different colors; similarly, if the graph consists of $q+1$ vertices each of which is joined to the other q vertices, we also need exactly $q+1$ colors. We have:

THEOREM 2 (BROOKS). *A graph G of maximum degree q is q-chromatic except in either of the two following cases:*

1) $q=2$, *and one of the components of G is a cycle of uneven length;*

2) *a component of G consists of $q+1$ vertices, each of which is joined to the other q vertices.*

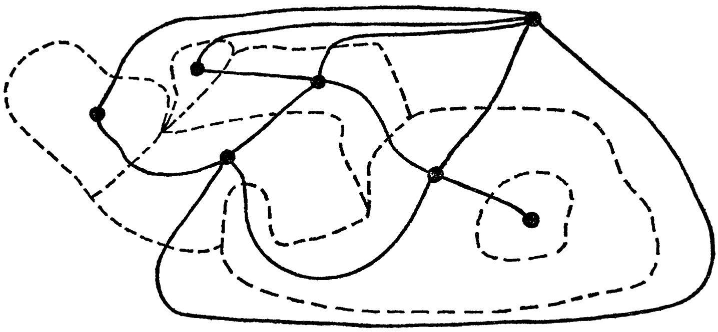

FIG. 4

The chromatic number of a graph is also related to one of the most puzzling of unsolved problems, namely the *four-color problem*: we want to color a map so that two countries having a strip of boundary in common must have different colors; is it true that every map can be colored with only four colors? Despite

many efforts, no one has ever found a map requiring more than four colors and it can be proved that such a map would have at least 480 countries on it. On the other hand, many people have proposed proofs that every map can be colored with four colors, but all these proofs have been erroneous.

A given map defines a graph G in the following way: each vertex of G will represent a country and we shall draw it on the map inside the country it represents; two vertices x and y will be joined if they represent adjacent countries on the map, and we shall draw the edge (x, y) completely inside the two countries involved (see Fig. 4).

Notice that this graph G has a very special property: it can be drawn on a plane so that no edge will cross another. Any graph with this property is said to be *planar*; and any planar graph defines a class of maps. The four-color problem now reduces to: is it true that a planar graph is 4-chromatic? One can easily prove the weaker result:

THEOREM 3 (HEAWOOD). *Every planar graph is 5-chromatic.*

For a graph G of any kind, it is often convenient to consider, in addition to its chromatic number, another coefficient called the *chromatic index* of G, which is by definition the minimum number of colors required to color the edges of G so that no two adjacent edges have the same color.

The chromatic index of a graph $G=(X, \mathbf{U})$ is the chromatic number of an auxiliary graph $(\mathbf{U}, \mathbf{\Gamma})$ defined in the following way: its vertices are the edges of G and $\mathbf{v} \in \mathbf{\Gamma u}$ when the edges $\mathbf{u}$ and $\mathbf{v}$ are adjacent in the graph G. The problem of finding the chromatic index is therefore associated with the study of the chromatic number, but usually it is much easier to solve.

We find the problem of the chromatic index, for example, in the theory of the *latin squares*, invented by Euler. A *latin square of order n* is a square array of n numbers such that each one occurs exactly once in each row and once in each column; for instance, with $n=5$, we have the solutions:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{matrix} \qquad\qquad \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 2 & 3 & 1 \\ 5 & 3 & 1 & 2 & 4 \end{matrix}$$

An important problem is: given a rectangular array of n numbers, with n columns and $r<n$ rows, each number occurring once in each row and at most once in each column; is it possible to add $n-r$ rows so that we obtain a latin square? Let us construct a simple graph with two sets of vertices, $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$; vertex x_i represents the ith column, vertex y_j represents number j, and we draw the edge (x_i, y_j) when the ith column does not contain the number j. This graph is bi-chromatic. If its chromatic index is $n-r$ we shall

color the edges with colors $(r+1)$, $(r+2)$, $\cdots$, (n); if edge $[x_i, y_j]$ is colored with color (k), we shall put the number j at the intersection of the ith column and the kth row: finally we shall obtain a latin square. An important result is:

THEOREM 4. *The chromatic index of a bi-chromatic graph is exactly the maximum degree* $q = \max_{x \in X} n(\Gamma x)$.

6. Matching of a graph. Given a graph $(X, \mathbf{U})$ (without its orientations), a set of edges $\mathbf{V} \subset \mathbf{U}$ is said to be a *matching* if two edges of $\mathbf{V}$ have no vertex in common. We are here concerned with the problem of constructing a matching with a maximum number of elements.

One of the first problems of this type is that of *distinct representatives*: given a finite set S and a family of subsets $(S_1, S_2, \cdots, S_k)$, find a k-tuple of elements $(a_1, a_2, \cdots, a_k)$ such that $a_i \in S_i$ $(i = 1, 2, \cdots, k)$ and $i \neq j$ implies that $a_i \neq a_j$; such a k-tuple is called a *system of distinct representatives* of the family of subsets. This problem has been considered by D. König and by P. Hall; it reduces to the matching problem if we draw a bi-chromatic graph whose vertices are, on the one side, the elements of S, and on the other, k arbitrary vertices $x_1, x_2, \cdots, x_k$; we shall join an element a of S to vertex x_i if a belongs to the set S_i. If the maximum matching of this graph uses all x_i, it will define a system of distinct representatives. The following well-known example illustrates this problem: in a co-ed college, each girl has m boy-friends, and each boy has m girl-friends; is it possible for every girl to dance simultaneously with one of her boy-friends, and every boy with one of his girl-friends? The answer is yes, as can easily be seen by means of graph theory.

Another matching problem was that of the Battle of Britain: given a group of pilots and a number of planes, each requiring two pilots; for various reasons (language, abilities, etc.) certain pilots may not be paired off together. Determine the maximum number of planes which can fly together at the same time. If we draw the graph whose vertices represent the pilots and whose edges represent the compatibilities, we must now find a maximum matching.

It is easy to see that a maximum matching of a graph G is a maximum internally stable set for an auxiliary graph whose vertices are the edges of G; nevertheless, the matching problem is simpler and results in some very precise theorems. Let us say that a graph G is *simple* if it consists of two disjoint sets X and Y and a multivalued function Γ mapping X into Y.

THEOREM 1 (P. HALL, improved by O. ORE). *If G is a simple graph, then the number of vertices in X that a maximum matching must leave out is exactly*:

$$\delta_0 = \max_{A \subset X} [n(A) - n(\Gamma A)].$$

This theorem has many applications in pure mathematics, particularly in algebra (Dilworth) and in the theory of doubly stochastic matrices (Birkhoff). The number of different maximum matchings has been studied by M. Hall.

THEOREM 2 (W. TUTTE, improved by C. BERGE). *Given a maximum matching* **V**, *the number of vertices not incident with an edge in V is equal to the number* ξ *defined in the Theorem at the end of* **3**.

Given a matching **V**, we shall say here that the edges of **V** are *strong* and that the edges of **U**$-$**V** are *weak.* An *alternating chain* is a chain which does not use the same edge twice and is such that for any two successive edges one is strong and the other is weak. A vertex x which is not adjacent to a strong edge is said to be *neutral.* If there exists an alternating chain going from one neutral point to another, the matching **V** will not be maximum, since by reversing all the strong and weak edges of this chain we shall obtain a matching greater than **V**. To elaborate further:

THEOREM 3 (BERGE). *A matching* **V** *is maximum if and only if there does not exist an alternating chain connecting a neutral point to another neutral point.*

7. Hamiltonian paths. Given a graph $G=(X, \Gamma)$, a path passing through each vertex once and only once is called a *Hamiltonian path*; a circuit passing through each vertex once and only once is a *Hamiltonian circuit. A factor* of graph $G=(X, \Gamma)$ is a partial subgraph (X, Δ) such that $n(\Delta x)=1$ for all $x \in X$.

A Hamiltonian circuit is a factor, but the converse is not necessarily true: a factor can also consist of several circuits without any vertex in common. In the unoriented theory one can similarly define a *Hamiltonian chain*, a *Hamiltonian cycle* and a *semi-factor*; for instance, a semi-factor is a partial subgraph in which each vertex is of degree 2. A graph without any semi-factor may possess a factor, and vice-versa; in Fig. 5 the first graph possesses one factor and no semi-factors and the second graph possesses one semi-factor and no factors.

FIG. 5

The problem of dividing a graph into semi-factors was first discussed by Petersen and stemmed from another problem in pure mathematics considered by Hilbert. The problem of finding a Hamiltonian chain appears very frequently: many mathematicians have been concerned with the problem of moving a knight on a chess board so that it occupies each square once and only once. In Operational Research, the problem of the Hamiltonian path occurs when we have a number of tasks to perform, certain of which must take precedence over others, and when we try to determine the order of performance.

THEOREM 1 (REDEI). *If in a graph* $G=(X, \Gamma)$, *every pair of vertices is joined by an arc in at least one direction, then there exists a Hamiltonian path.*

One can deduce from this that if in a tournament each player is matched with all the others, it is possible to arrange them in such an order that each one is better than the following one.

THEOREM 2 (DIRAC). *If for a graph $G=(X, \Gamma)$, without considering its orientation, $n(\Gamma x-\{x\})\geqq\frac{1}{2}n(X)$ for all $x\in X$, then there exists a Hamiltonian cycle.*

THEOREM 3. *A necessary and sufficient condition for a graph (X, Γ) to possess a factor is that $n(S)\leqq n(\Gamma S)$ for all $S\subset X$.*

In fact, the determination of a factor reduces to the matching problem in the following way: given a graph $G=(X, \Gamma)$, with $X=\{x_1, x_2, \cdots, x_n\}$, let us construct an auxiliary graph G' with two sets of n vertices $\{y_1, y_2, \cdots, y_n\}$, and $\{z_1, z_2, \cdots, z_n\}$. The points y_i and z_i represent two copies of the ith vertex x_i of G, and we draw the edge $[y_j, z_k]$ if an arc of G goes from x_j to x_k. If the maximum matching of G' contains n edges, it defines a factor in graph G.

Very often, the best way to construct a Hamiltonian circuit is to construct *all* the factors, and in order to do this, the Boolean operations are of great assistance.

THEOREM 4 (PETERSEN, BAEBLER). *Given a graph G and a number $k>0$ such that every vertex of the graph is adjacent to exactly k edges; if k is even, the graph G possesses a semi-factor; if k is uneven, and if for every subset S, with $S\neq X$, $S\neq\emptyset$, the number of edges linking S to $X-S$ is greater than or equal to 2, then the graph possesses a semi-factor.*

References

1. D. König, Theorie der endlichen und unendlichen Graphen, Ak. Verlag., Leipzig, 1936.
2. C. Berge, Théorie des graphes et ses applications, Dunod, Paris 1958.
3. ———, The theory of graphs and its applications, Methuen (London) and Wiley (New York), 1962.
4. F. Harary and R. Z. Norman, Graph Theory as a Mathematical Model in Social Science, University of Michigan, Ann Arbor, 1953.

RETRACTIONS ONTO SPHERES

R. H. BING, The University of Wisconsin and The Institute for Advanced Study

Suppose that S is a 2-sphere in E^3. It is known (Theorem 8-88 of [1]) that S separates E^3 into exactly two components and is the common boundary of each. We call the unbounded component Ext S and the bounded component Int S.

In case S is round, it is obvious that there is a retraction of $S+\text{Ext } S$ onto S. One which comes quickly to mind sends a point p of $S+\text{Ext } S$ to the point where the segment from p to the center of Int S intersects S. If the sphere is

wild, however, as the Alexander horned sphere illustrated on page 176 of [1], it is not so obvious that there is such a retraction. It is the purpose of this note to show that there is.

If X is a subset of Y, a retraction r of Y onto X is a continuous transformation of Y onto X such that r is the identity on X. A set X is called an AR (absolute retract) if whenever X is embedded as a closed set in a metric space Z, there is a retraction of Z onto X. It is called an ANR (absolute neighborhood retract) if whenever it is embedded as a closed set in a metric space Z, there is an open subset U of Z containing X and a retraction of U onto X. It is known (Theorem 2-36 of [1]) that S is an ANR.

Although Theorems 2–4 may be too deep for the beginner, Theorem 1 should be accessible to those acquainted with such topology texts as [1].

THEOREM 1. *If S is a 2-sphere embedded in E^3, there is a retraction of $S+\text{Ext } S$ onto S.*

Proof. We would like to provide an elementary proof that effectively describes the retraction so that one can visualize the retraction and have some idea as to where points are sent.

Let N be a neighborhood of S such that there is a retraction r of $\overline{N}$ onto S. That there is such a retraction follows from the fact that S is an ANR. Perhaps this is the weak link in our effort to describe effectively the retraction but if one considers how one proves that S is an ANR from the Tietze extension theorem, one may be able to see where r sends points. In fact, since we are working in E^3 rather than in an abstract normal space, perhaps the reader can concoct an easier proof of the version of the Tietze extension theorem we need than that given in Section 2-7 of [1]. Such an easy proof is given in [3].

We suppose that $\overline{N}$ is a bounded polyhedron and that $\text{Ext } S-\overline{N}$ is connected. If this is not already the case, we could adjust the N considered in the previous paragraph to bring it about by boring holes in that N and whittling it down to size. We make no effort to keep $\overline{N}$ from having handles and in general it will.

Let C be a large cube whose interior contains $\overline{N}$. There is a retraction of $E^3-\text{Int } C$ onto $\text{Bd } C$ so if we can find a retraction of $C-\text{Int } S$ onto S, the truth of Theorem 1 follows.

Consider a triangulation of $C-N$. Let $s_1, s_2, \cdots, s_n$ be the closed 3-simplexes (tetrahedra) of this triangulation. We suppose that the s's are ordered so that s_n has a 2-face on $\text{Bd } C$, s_{n-1} has a 2-face on $\text{Bd } C+s_n$, s_{n-2} has a 2-face on $\text{Bd } C+s_n+s_{n-1}, \cdots$, and s_1 has a face on $\text{Bd } C+s_n+s_{n-1}+\cdots+s_2$. Note that if N_0 denotes $\overline{N}$ and N_i denotes $N_0+s_1+s_2+\cdots+s_i$, then $\text{Ext } S-N_i$ is connected.

Since $\text{Ext } S-N_{i-1}$ is connected, s_i has a 2-face which does not belong to N_{i-1}. Let f_i be a particular such face of s_i, and g_i be a retraction of s_i onto $\text{Bd } s_i-\text{Int} f_i$.

We now extend r to a retraction taking N_1 onto S. We do this starting at low dimensional faces of s_1 and proceeding one at a time until only f_1 is left. That we can do this follows from Lemma 6-42 of [1]. Denote the extended retraction by r'. Once r has been extended to take Bd $s_i - \text{Int} f_i$ onto S, we define $r_1 = r$ on N_0, $r_1 = r'g_1$ on s_1. We continue extending r in this fashion so that r_2 takes N_2 into S, r_3 takes N_3 onto S, $\cdots$, and r_n takes N_n onto S.

We note that we did not use any properties of 2-spheres in E^3 that are not equally true for $(n-1)$-spheres in E^n so we have the following version of Theorem 1.

THEOREM 2. *For each $(n-1)$-sphere S in E^n and each point p of* Int S, *there is a retraction of $E^n - \{p\}$ onto S.*

Theorem 2 can be generalized. Daniel R. McMillan made the interesting observation that we did not use the fact that the space into which S was embedded was E^n and that we could have used a triangulated n-manifold-with-boundary M instead of E^n. The component U of $M-S$, whose closure is to be retracted onto S, would need to have a point removed before retracting unless $U \cdot \text{Bd } M \neq 0$ or $\overline{U}$ is noncompact. By working a bit harder we could even remove the hypothesis that M can be triangulated. In place of M we could even use a pseudo-manifold.

A set is called i-connected if each continuous transformation of the boundary of an $(i+1)$-cell into X can be extended to take the $(i+1)$-cell into X. Note that we did not use in the proof of Theorem 2 the fact that S is an $(n-1)$-sphere but merely that it is an i-connected $(i=0, 1, \cdots, n-2)$ ANR. Hence we have the following

THEOREM 3. *Suppose that X is a closed subset of a compact n-manifold M such that X is an i-connected $(i=0, 1, \cdots, n-2)$ ANR. Then for each point p of a component U of $M-X$ there is a retraction of $U+X-\{p\}$ onto X.*

Proof. Let N be an open subset of M containing X such that there is a retraction r_0 of $\overline{N}$ onto X and $B_1, B_2, \cdots, B_m$ be a finite number of topological n-cells in U such that $U \subset \overline{N} + B_1 + B_2 + \cdots + B_m$.

Let g_1 be a map of Bd B_1 into X that agrees with r_0 on $\overline{N} \cdot \text{Bd } B_1$. To prove that there is such a g_1 we argue as follows: Regard X as embedded in a Hilbert cube H. Since X is an ANR there is an open subset N_1 of H containing X and a retraction g' of N_1 onto X. It follows from the Tietze extension Theorem that there is a map g'' of Bd B_1 into H that agrees with r_0 on $\overline{N} \cdot \text{Bd } B_1$. Let T be a triangulation of Bd B_1 of such small mesh that each simplex of T that intersects $\overline{N} \cdot \text{Bd } B_1$ lies in $g''^{-1}(N_1)$. Let $g_1 = g'g''$ on each simplex of T in $g''^{-1}(N_1)$. Then g_1 may be extended to the other simplexes of T one at a time by starting at those of low dimension and using the fact that X is i-connected $(i=0, 1, \cdots, n-2)$.

Let p_1 be a point of Int B_1 and g_1' be a retraction of $B_1 - \{p_1\}$ onto Bd B_1. Let r_1 be the retraction of $(X+U\cdot\overline{N}+B_1)-\{p_1\}$ onto X which is $g_1'g_1$ on $B_1-\{p_1\}$ and r_0 elsewhere.

Continuing as suggested in the last paragraph, we let p_i be a point of Int B_i and get a retraction r_2 of $(X+U\cdot\overline{N}+B_1+B_2)-(\{p_1\}+\{p_2\})$ onto X, $\cdots$, and a retraction r_m of

$$(X + U\cdot\overline{N} + B_1 + B_2 + \cdots + B_m) - (\{p_1\} + \{p_2\} + \cdots + \{p_m\})$$
$$= X + U - (\{p_1\} + \{p_2\} + \cdots + \{p_m\})$$

onto X.

Let $q_1, q_2, \cdots, q_j$ be a sequence of points in U such that q_j is the p of the statement of Theorem 3, each p_i appears in the sequence, and each adjacent pair q_i, q_{i+1} lies on the interior of an n-simplex K_i in U. By modifying r_m in Int K_1 we obtain a retraction r_{m+1} of $X+U-(\{q_2\}+\{q_3\}+\cdots+\{q_j\})$ onto X. Eliminating the q's one at a time and changing the r's we arrive at a retraction r_{m+j-1} of

$$X + U - \{q_j\} = X + U - \{p\} \text{ onto } X.$$

We can avoid using the hypothesis that M is compact by using a locally finite collection of B's instead of a finite collection. If X is not compact we would embed the one-point-compactification of X in H to get the g's. In fact, if $\overline{U}$ is not compact, we would not even need to remove a point p from U before retracting. However, we shall not pursue this further.

If S is a wild 2-sphere in E^3, $S+\text{Int } S$ is called a *crumpled cube.* M. K. Fort, Jr. showed me a proof using the Side Approximation Theorem that a crumpled cube is an AR. The result generalizes as follows.

THEOREM 4. *If S is an $(n-1)$-sphere in E^n, $S+$*Int *S is an AR.*

This result follows from the fact that $S+\text{Int } S$ is the retract of a closed n-cell as shown in Theorem 2, an n-cell is an AR (Theorem 2-34 of [1]), and any retract of an AR is an AR.

An alternate proof of Theorem 4 can be obtained by studying the homotopy connectedness properties of $S+\text{Int } S$ and applying the results on page 289 of [2].

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References

1. John G. Hocking and Gail S. Young, Topology, Addison-Wesley, Reading, Mass., 1961.
2. Kasimir Kuratowski, Topologie, vol. 2, 3rd ed., Warsaw, 1961.
3. R. H. Bing, Spheres in E^3, this MONTHLY, 71 (1964) 353–364.

INVERSE RELATIONS AND COMBINATORIAL IDENTITIES

JOHN RIORDAN, Bell Telephone Laboratories, Murray Hill, N. J.

1. Introduction. The inverse relations considered here are typified by

$$
\begin{aligned}
y^n &= (x+1)^n = \sum \binom{n}{k} x^k, \qquad\qquad n = 0, 1, \cdots \\
x^n &= (y-1)^n = \sum (-1)^{n+k} \binom{n}{k} y^k
\end{aligned}
\tag{1}
$$

or in a form more suggestive of the Jacobian injunction "always invert," by

$$
\begin{aligned}
a_n(x) &= (1+x)^n = \sum \binom{n}{k} x^k \\
x^n &= \sum (-1)^{n+k} \binom{n}{k} a_k(x).
\end{aligned}
\tag{1a}
$$

Such relations occur frequently in combinatorial analysis in a variety of contexts. Each pair is associated with an identity, such as, in the present instance,

$$
\delta_{nm} = \sum (-1)^{k+m} \binom{n}{k} \binom{k}{m}
\tag{2}
$$

with δ_{nm} the Kronecker delta. (The sum, here as above, is taken over the full range of nonzero values of the summand, with the convention that $\binom{n}{k} \equiv 0$, $k<0$, and need not be indicated.) As will appear, these orthogonal combinatorial identities have wider implication than the associated pairs of relations from which they proceed. In particular, they imply other pairs of relations and other identities, and thus provide a guide line through the forest of these prolific entities. Unfortunately the guide is weak since what emerges is the usual embarrassment of riches, with open paths in many directions.

The object of this paper is to assemble a variety of old and new results on the subjects of the title. A study of the relations in equations (1), which despite appearances are worth extended attention, sets the stage for Stirling numbers, for relations associated with Legendre and Chebyshev polynomials, and for other results.

2. The simplest inverse relations. It is convenient to begin with the pair of relations of equations (1) or (1a). Equation (2) follows from substitution of either into the other and equating coefficients of powers of the variable x or y. Since it is an orthogonality on the coefficients it follows that (1) and (1a) may be replaced by

$$
a_n = \sum \binom{n}{k} b_k, \qquad b_n = \sum (-1)^{n+k} \binom{n}{k} a_k.
\tag{1b}
$$

Moreover, sign factors may be moved at pleasure; a more symmetrical form is

$$a_n = \sum (-1)^k \binom{n}{k} b_k, \qquad b_n = \sum (-1)^k \binom{n}{k} a_k. \tag{1c}$$

Further, the orthogonality may be taken as associated with matrix multiplication; the coefficients of the relations are associated with a matrix and its inverse. Thus equations (1b) are associated with matrices $B = \{b_{ij}\}$ and $B^{-1} = \{\beta_{ij}\}$, where

$$b_{ij} = \binom{i}{j}, \qquad \beta_{ij} = (-1)^{i+j}\binom{i}{j};$$

in fact, by (2), $BB^{-1} = B^{-1}B = I$, with I the identity matrix. Both matrices are of triangular form, of finite or infinite extent. Thus the familiar arrays of coefficients

$$\begin{array}{cccc} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \\ & \cdots & & \end{array} \qquad\qquad \begin{array}{cccc} 1 & & & \\ -1 & 1 & & \\ 1 & -2 & 1 & \\ -1 & 3 & -3 & 1 \\ & \cdots & & \end{array}$$

ordinarily not regarded as mathematical objects become so by adding the brackets or braces indicating matrices.

Next, $\delta_{nm} = \delta_{mn}$, so (2) is the same as

$$\delta_{nm} = \sum (-1)^{m+k} \binom{k}{n}\binom{m}{k} \tag{2a}$$

implying

$$a_n = \sum \binom{k}{n} b_k, \qquad b_n = \sum (-1)^{n+k} \binom{k}{n} a_k \tag{3}$$

the relations connecting binomial moments a_n, and probabilities b_n, and usually not related to (1). The arrays of coefficients are now

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & \cdots \\ & 1 & 2 & 3 & \\ & & 1 & 3 & \\ & & & 1 & \end{array} \qquad\qquad \begin{array}{ccccc} 1 & -1 & 1 & -1 & \cdots \\ & 1 & -2 & 3 & \\ & & 1 & -3 & \\ & & & 1 & \end{array}$$

The triangle of coefficients has been rotated.

Next,

$$\delta_{nm} = \delta_{n+p,m+p} = \sum (-1)^{n+k} \binom{n+p}{k+p} \binom{k+p}{m+p} \tag{2b}$$

which implies

$$a_n = \sum \binom{n+p}{k+p} b_k, \qquad b_n = \sum (-1)^{n+k} \binom{n+p}{k+p} a_k. \tag{4}$$

Equations (4) have been used by L. Carlitz in [2]. It is worth noticing that the arrays of these coefficients are those for $p=0$ with the first p columns removed; thus for $p=1$, they are

$$\begin{array}{llll} 1 & & & \\ 2 & 1 & & \\ 3 & 3 & 1 & \\ 4 & 6 & 4 & 1 \\ & \cdots & & \end{array} \qquad \begin{array}{rrrr} 1 & & & \\ -2 & 1 & & \\ 3 & -3 & 1 & \\ -4 & 6 & -4 & 1 \\ & \cdots & & \end{array}$$

These several points show the orthogonality condition, equation (2), to be prolific in consequences. Another variation follows from the matrix equation: $BB^{-1}=I$, which implies $(BB^{-1})^p = B^pB^{-p}=I$. Writing $B^p = \{b_{ij}(p)\}$, we find that

$$b_{ij}(p) = \sum b_{ik}(p-1)\binom{k}{j} = p^{i+j}\binom{i}{j}, \qquad p = \pm 1, \pm 2, \cdots.$$

Hence no essentially new pair of inverse relations appears.

Turn now to numbers introduced by I. Lah [6]; they are defined by

$$(-x)_n = \sum L_{nk}(x)_k = (-x)(-x-1)\cdots(-x-n+1),$$

whose inverse is $(x)_n = \sum L_{nk}(-x)_k$ so that $\delta_{nm} = \sum L_{nk}L_{km}$. But (see problem 16 of Chapter 2 of [8])

$$L_{nk} = (-1)^n \frac{n!}{k!}\binom{n-1}{k-1}$$

and

$$\begin{aligned} \delta_{nm} &= \sum (-1)^{n+k} \frac{n!}{k!}\binom{n-1}{k-1} \frac{k!}{m!}\binom{k-1}{m-1} \\ &= \frac{n!}{m!} \sum (-1)^{n-1+k} \binom{n-1}{k}\binom{k}{m-1} \end{aligned}$$

or, using (2) $\delta_{nm} = (n!/m!)\delta_{n-1,m-1}$. This points the way to other modifications of (2), yielding more inverse relations; these are given $\delta_{nm} = \delta_{nm}F(n, m)$ if $F(n, n) = 1$, $F(n, m) \neq \infty$ for the range of n and m in question.

3. Stirling numbers. The Stirling numbers of first kind, $s(n, k)$, and second kind, $S(n, k)$, are usually defined by the inverse relations

$$(x)_n = \sum s(n, k)x^k, \qquad x^n = \sum S(n, k)(x)_k \tag{5}$$

with $(x)_n = x(x-1) \cdots (x-n+1)$. Hence, as is well known,

$$\delta_{nm} = \sum s(n, k)S(k, m) = \sum S(n, k)s(k, m) \tag{6}$$

and hence, as above,

$$a_n = \sum s(n, k)b_k, \qquad b_n = \sum S(n, k)a_k. \tag{5a}$$

All the parallel implications of (2) hold equally. The pair of relations, similar to (3),

$$a_n = \sum s(k, n)b_k, \qquad b_n = \sum S(k, n)a_k,$$

suggest a kind of moment, unfortunately nonexistent in probability and statistics. The numbers associated with the powers of matrices have been studied, however, by E. T. Bell in [1].

One example of further possibilities now open is as follows. Take the basic pair as

$$a_n(x) = \sum S(n, k)x^k, \qquad x^n = \sum s(n, k)a_k(x) \tag{5b}$$

and write

$$a_n(x; 1) - a_{n-1}(x; 1) = a_n(x) \tag{7}$$

so that

$$\begin{aligned} a_n(x; 1) &= a_n(x) + a_{n-1}(x) + \cdots + a_0(x) = \sum_{k=0}^{n} x^k \sum_{m=0}^{n} S(m, k) \\ x^n &= \sum s(n, k)[a_k(x; 1) - a_{k-1}(x; 1)] \\ &= \sum [s(n, k) - s(n, k+1)]a_k(x; 1) \end{aligned} \tag{8}$$

is a new pair. Since, with a prime denoting a derivative, it follows from $S(n, k) = S(n-1, k-1) + kS(n-1, k)$ that

$$a_n(x) = xa_{n-1}(x) + xa'_{n-1}(x), \qquad n = 1, 2, \cdots$$

it follows by (7) that

$$a_n(x; 1) = 1 + xa_{n-1}(x; 1) + xa'_{n-1}(x; 1), \qquad n = 1, 2, \cdots \tag{9}$$

which implies a simple recurrence relation for the coefficients.

This is readily generalized by writing

$$a_n(x; j) - a_{n-1}(x; j) = a_n(x; j-1), \qquad j = 1, 2, \cdots.$$

Then it is found similarly that $a_0(x; j) = 1$,

$$a_n(x;j) = \binom{n+j-1}{n} + xa_{n-1}(x;j) + xa_{n-1}'(x;j), \qquad n = 1, 2, \cdots. \tag{10}$$

Thus if

$$a_n(x;j) = \sum_{k=0} a_{nk}(j)x^k, \qquad j = 1, 2, \cdots \tag{11}$$

with $a_{n0}(j) = \binom{n+j-1}{n}$, $a_{00}(j) = 1$, $a_{nk}(j) = ka_{n-1,k}(j) + a_{n-1,k-1}(j)$, then

$$x^n = \sum_{k=0} b_{nk}(j)a_k(x;j) \qquad j = 1, 2, \cdots, \tag{12}$$

where

$$b_{nk}(j) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} s(n, k+i).$$

Similar results follow from the introduction of polynomials

$$\begin{aligned} \alpha_n(x;1) + \alpha_{n-1}(x;1) &= a_n(x) \\ \alpha_n(x;j) + \alpha_{n-1}(x;j) &= \alpha_n(x;j-1), \qquad j = 2, 3, \cdots. \end{aligned} \tag{13}$$

Indeed

$$\alpha_n(x;j) = (-1)^n \binom{n+j-1}{n} + x\alpha_{n-1}(x;j) + x\alpha_n'(x;j) \tag{14}$$

while $\beta_{nk}(j) = \sum_{i=0}^{j} \binom{j}{i} s(n, k+i)$, where

$$x^n = \sum_{k=0} \beta_{nk}(j)\alpha_k(x;j). \tag{15}$$

Similar developments may be obtained for polynomials associated with $b_n(x) = \sum s(n, k)x^k$.

4. Chebyshev polynomials. The Chebyshev polynomials $T_n(x) = \cos n\theta$, $\theta = \cos^{-1}x$, are associated with a pair of inverse relations which may be written as follows

$$a_n = \sum \binom{n}{k} b_{n-2k}, \qquad b_n = \sum (-1)^k \frac{n}{n-k}\binom{n-k}{k} a_{n-2k}. \tag{16}$$

It is assumed that both a_n and b_n are null for negative indices. The orthogonality they imply is

$$\begin{aligned} \delta_{m0} &= \sum (-1)^{m+j} \binom{n}{j} \frac{n-2j}{n-m-j} \binom{n-m-j}{m-j} \\ &= \sum (-1)^j \binom{n-2j}{m-j} \frac{n}{n-j} \binom{n-j}{j}. \end{aligned} \tag{17}$$

Indeed, if $a_n = \sum_{2k \leq n} a_{nk} b_{n-2k}$, $b_n = \sum_{2k \leq n} b_{nk} a_{n-2k}$ then

$$\delta_{m0} = \sum a_{nj} b_{n-2j,m-j} = \sum a_{n-2j,m-j} b_{nj}$$

with j not greater than the smaller of m and $n/2$.

The first half of (17) has been proved simply and directly by H. W. Gould [4]. A direct proof of (16) may be given as follows. Write

$$b_n(x) = \sum_{2k \leq n} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}. \tag{18}$$

Then $b_0(x) = 1$, $b_1(x) = x = xb_0(x)$, $b_2(x) = x^2 - 2$. For $n = 3, 4, \cdots$, it is easy to show that

$$b_n(x) = xb_{n-1}(x) - b_{n-2}(x), \qquad n = 3, 4, \cdots \tag{19}$$

since

$$\frac{n}{n-k}\binom{n-k}{k} = \binom{n-k}{k} + \binom{n-k-1}{k-1}$$

or

$$xb_n(x) = b_{n+1}(x) + b_{n-1}(x), \qquad n = 2, 3, \cdots \tag{19a}$$

while $xb_1(x) = b_2(x) + 2b_0(x)$, $xb_0(x) = b_1(x)$. If $x^n = \sum a_{nk} b_{n-2k}(x)$ then equation (19a) and its initial modifications imply

$$\begin{aligned} a_{nk} &= a_{n-1,k} + a_{n-1,k-1}, \qquad k < [n/2] \\ a_{2n,n} &= 2a_{2n-1,n-1} \\ a_{2n+1,n} &= a_{2n,n} + a_{2n,n-1} \end{aligned}$$

with brackets indicating integral part. These are the familiar recurrences for binomial coefficients and, along with boundary conditions, they prove (16).

It is worth noting that the alternative procedure of working from $a_n(x) = \sum_{2k \leq n} \binom{n}{k} x^{n-2k}$ is not so simple.

The "rotated" form of (16) is

$$a_n = \sum_{k=0} \binom{n+2k}{k} b_{n+2k}, \qquad b_n = \sum_{k=0} (-1)^k \frac{n+2k}{n+k} \binom{n+k}{k} a_{n+2k}. \tag{20}$$

If new polynomials $b_n(x; 1)$ are defined by

$$b_n(x; 1) - b_{n-2}(x; 1) = b_n(x) \tag{21}$$

with $b_n(x)$ the polynomial defined above (equation (18)), then

$$b_0(x; 1) = b_0(x) = 1,\ b_1(x; 1) = b_1(x) = x,\ b_2(x; 1) = b_2(x) + b_0(x) = x^2 - 1$$

and, since

$$\frac{n}{n-k}\binom{n-k}{k} = \binom{n-k}{k} + \binom{n-k-1}{k-1},$$
(22)
$$b_n(x;1) = \sum_{2k\leq n} (-1)^k \binom{n-k}{k} x^{n-2k}$$

and

(23)
$$\begin{aligned} x^n &= \sum \binom{n}{k} b_{n-2k}(x) = \sum \binom{n}{k} [b_{n-2k}(x;1) - b_{n-2-2k}(x;1)] \\ &= \sum \left[\binom{n}{k} - \binom{n}{k-1}\right] b_{n-2k}(x;1). \end{aligned}$$

The polynomial $b_n(x;1)$ is the Chebyshev polynomial $U_n(x/2)$ where $U_n(x) = \sin(n+1)\theta/\sin\theta$, $\cos\theta = x$. Extension of (21) to

$$b_n(x;j) - b_{n-2}(x;j) = b_n(x;j-1), \qquad j = 2, 3, \cdots$$

leads to nothing interesting. On the other hand, $\beta_n(x;1) + \beta_{n-2}(x;1) = b_n(x)$ yields the pair

(24)
$$a_n = \sum \binom{n+1}{k} b_{n-2k}, \quad b_n = \sum (-1)^k \frac{n+1}{n+1-k}\binom{n+1-k}{k} a_{n-2k},$$

whose orthogonality relation is just (17) with n replaced by $n+1$. Thus (24) may be generalized to

(25)
$$\begin{aligned} a_n &= \sum \binom{n+p}{k} b_{n-2k} \\ b_n &= \sum (-1)^k \frac{n+p}{n+p-k}\binom{n+p-k}{k} a_{n-2k}, \qquad p = 0, 1, 2, \cdots. \end{aligned}$$

Returning to (17), rewritten as

(17a)
$$\begin{aligned} \delta_{m0} &= \sum (-1)^{m+j}\binom{n}{j}\left[\binom{n-m-j}{m-j} + \binom{n-m-j-1}{m-j-1}\right] \\ &= \sum (-1)^j \binom{n-2j}{m-j}\left[\binom{n-j}{j} + \binom{n-j-1}{j-1}\right] \end{aligned}$$

two identities may be screened out, namely

(26)
$$1 = \sum_{j=0}^{m} (-1)^{m+j}\binom{n}{j}\binom{n-m-j}{m-j}$$

(27)
$$\binom{2m-1}{m} = \sum_{j=0} (-1)^{m+j+1}\binom{n}{j}\binom{n-m-j-1}{m-j-1}.$$

The Chebyshev inverse relations given in (16) are an instance of the following, due to H. W. Gould [4]

$$\tag{28} \begin{aligned} f(a) &= \sum_{k=0} \binom{a}{k} F(a + bk - k) \\ F(a) &= \sum_{k=0} (-1)^k \frac{a}{a+bk} \binom{a+bk}{k} f(a + bk - k). \end{aligned}$$

For $b = -2$, in present notation (28) may be written

$$a_n = \sum_{3k \leq n} \binom{n}{k} b_{n-3k}, \qquad b_n = \sum_{3k \leq n} (-1)^k \frac{n}{n-2k} \binom{n-2k}{k} a_{n-3k}$$

and the corresponding orthogonal relation may be written

$$\begin{aligned} \delta_{m0} &= \sum_{j=0}^{m} (-1)^{m+j} \binom{n}{j} \left[\binom{n-2m-j}{m-j} + 2 \binom{n-2m-j-1}{m-j-1} \right] \\ &= \sum_{j=0} (-1)^j \binom{n-3j}{m-j} \left[\binom{n-2j}{j} + 2 \binom{n-2j-1}{j-1} \right]. \end{aligned}$$

Then, if $f_{nm} = \sum_{j=0}^{m} (-1)^{m+j} \binom{n}{j} \binom{n-2m-j}{m-j}$ with $f_{n0} = 1, f_{n1} = 2$, it is found by recurrence that

$$f_{nm} = f_{n-1,m} = f_{0m} = (-1)^m \binom{-2m}{m} = \binom{3m-1}{m}.$$

Hence,

$$\tag{29} \begin{aligned} \binom{3m-1}{m} &= \sum_{0}^{m} (-1)^{m+j} \binom{n}{j} \binom{n-2m-j}{m-j} \\ &= 2 \sum_{0}^{m-1} (-1)^{m+j+1} \binom{n}{j} \binom{n-2m-j-1}{m-j-1}. \end{aligned}$$

It is worth noting that the first form of (29) may also be written

$$\frac{2}{3} \binom{3m}{m} = \sum_{0}^{m} (-1)^{m+j} \binom{3m}{m} \binom{m}{j} \binom{n}{3m} \binom{n-j}{2m}^{-1}$$

or

$$\frac{2}{3} \binom{n}{3m}^{-1} = \sum_{0}^{m} (-1)^{m+j} \binom{m}{j} \binom{n-j}{2m}^{-1}$$

an instance of the protean character of binomial identities.

5. Associated Legendre polynomials. If $P_n(x)$ is a Legendre polynomial, the associated polynomial in question is

$$q_n(x) = (1 - x)^n P_n\left(\frac{1 + x}{1 - x}\right) = \sum \binom{n}{k}^2 x^k. \tag{30}$$

What is its inverse? More precisely, if

$$x^n = \sum (-1)^{n+k} \beta_{nk} q_k(x) \tag{31}$$

what are the coefficients β_{nk}?

Of the many recurrences for $q_n(x)$, the following (cf. problem 15 of Chapter 7 of [8]) is apt for present purposes:

$$q_n'(x) + x q_n''(x) = n^2 q_{n-1}(x)$$

with primes denoting derivatives. By (31)

$$nx^{n-1} + (n - 1)x^{n-1} = \sum (-1)^{n+k}\beta_{nk}[q_k'(x) + xq_k''(x)] = \sum (-1)^{n+k}\beta_{nk}k^2 q_{k-1}(x)$$

or $\sum (-1)^{n-1+k}\beta_{n-1,k}n^2 q_k(x) = \sum (-1)^{n+k}\beta_{nk}k^2 q_{k-1}(x)$. Hence,

$$\beta_{nk} = (n/k)^2\beta_{n-1,k-1} = \binom{n}{k}^2 \beta_{n-k,0} = \binom{n}{k}^2 \beta_{n-k}$$

with the last a definition. Using this in (31) yields

$$x^n = \sum (-1)^{n+k}\binom{n}{k}^2 \beta_{n-k} q_k(x). \tag{31a}$$

Then $\beta_0 = 1$ and since $q_k(0) = 1$

$$0 = \sum (-1)^{n+k}\binom{n}{k}^2 \beta_{n-k}, \tag{32}$$

a recurrence which may be taken as a definition of the numbers. The first few values are

n	0	1	2	3	4	5	6	7
β_n	1	1	3	19	211	3651	90921	3091513

It is tempting to suppose that (32) may be replaced by some simpler linear or quasi-linear recurrence like

$$\sum_{j=0}^{k} A_j(n)\beta_{n+j} = 0,$$

where the $A_j(n)$ are polynomials in n, but Professor Carlitz has proved (private communication) that the latter is impossible. The relations $q_{2n+1}(-1) = 0$ and $q_{2n}(-1) = (-1)^n\binom{2n}{n}$ lead, however, to

$$1 = \sum (-1)^k \beta_{n-2k}\binom{n}{2k}^2\binom{2k}{k}. \tag{33}$$

The numbers β_n appear in other notations in L. Carlitz [3], where they arise in the expansion

$$\frac{1}{J_0(2\sqrt{z})} = \sum_{n=0} \beta_n \frac{z^n}{n!^2}$$

with $J_0(z)$ the Bessel function. Carlitz also gives the inverse relations

$$w_n(x) = \sum \binom{n}{k}^2 \beta_k(-x)^{n-k}, \qquad x^n = \sum (-1)^k \binom{n}{k}^2 w_k(x) \tag{34}$$

which however have the same orthogonality as (30) and (31a), namely

$$\delta_{nm} = \sum (-1)^{k+m} \binom{n}{k}^2 \binom{k}{m}^2 \beta_{k-m} = \sum (-1)^{k+m} \binom{n}{k}^2 \binom{k}{m}^2 \beta_{n-k}. \tag{35}$$

To find a pair of inverses associated with Legendre polynomials involving only binomial coefficients, consider $r_n(x) = (1+x)^n q_n(x/(1+x)) = P_n(1+2x)$. Then

$$\begin{aligned} r_n(x) &= \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} x^k = \sum_{k=0}^{n} \binom{n+k}{2k} s_k(x), \\ s_k(x) &= \binom{2k}{k} x^k. \end{aligned} \tag{36}$$

For simplicity, this may be examined for the related function

$$\rho_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k} x^k. \tag{37}$$

The recurrence for $\rho_n(x)$ is found to be

$$\rho_n(x) = (2 + x)\rho_{n-1}(x) - \rho_{n-2}(x), \qquad n = 2, 3, \cdots$$

while $\rho_1(x) = (2+x)\rho_0(x) - 1 = 1 + x$. Using these as before, it is found that

$$x^n = \sum (-1)^{n+k} \frac{2k+1}{n+k+1} \binom{2n}{n+k} \rho_k(x) \tag{38}$$

which is the inverse of (37).

6. Generating functions. Exponential generating functions lead directly to a number of inverse relations with binomial coefficients. The simplest pair, Eq. (1b), is equivalent to $\exp xa = \exp x(b+1)$, $a^n \equiv a_n$, $b^n \equiv b_n$, $\exp x(a-1) = \exp xb$, with a, b umbral or Blissard variables. More generally the inverse relations

$$a_n = \sum \binom{n}{k} c_{n-k} b_k, \qquad b_n = \sum \binom{n}{k} \gamma_{n-k} a_k \tag{39}$$

are equivalent to $\exp xa = \exp(b + c)$, $\exp xb = \exp(a + \gamma)$, and it is necessary that $\exp(c+\gamma) = 1$. Of course, a, b, c, γ are all umbral.

As a first example, take $\exp xc = (e^t - 1)t^{-1}$ so that $c^n \equiv c_n = (n+1)^{-1}$. Then $\exp x\gamma = t(e^t - 1)^{-1} = \exp Bt$ with B_n a Bernoulli number (in the even suffix notation). Then

$$(40) \qquad a_n = \sum \binom{n}{k} (n - k + 1)^{-1} b_k, \qquad b_n = \sum \binom{n}{k} B_{n-k} a_k$$

are inverse relations.

Next, consider $2 \exp xa = \exp x(b+1) + \exp x(b-1) = (e^x + e^{-x}) \exp xb$. Then $\exp x\gamma = 2(e^x + e^{-x})^{-1} = \exp xE$ with E_n a Euler number ($E_{2n+1} = 0$), and

$$(41) \qquad a_n = \sum \binom{n}{2k} b_{n-2k}, \qquad b_n = \sum \binom{n}{2k} E_{2k} a_{n-2k}$$

is an inverse pair. Its "rotated" form is

$$(42) \qquad a_n = \sum \binom{n+2j}{2j} b_{n+2j}, \qquad b_n = \sum \binom{n+2j}{2j} E_{2j} a_{n+2j}.$$

For sums having only odd binomial coefficients the generating function relation is $\exp xa = \frac{1}{2}(e^x - e^{-x}) \exp xb = (1/2x)(e^x - e^{-x})x \exp xb$ and

$$\exp x\gamma = \frac{2x}{e^x - e^{-x}} = \exp xd,$$

the last in a notation convenient for present purposes. The numbers d_n apparently have no patronymic. The inverse relations are

$$(43) \qquad a_n = \sum \binom{n}{2k+1} b_{n-2k-1}, \qquad nb_{n-1} = \sum \binom{n}{2k} d_{2k} a_{n-2k}.$$

Now turn to an instance of the Lagrange theorem in the form

$$(44) \qquad f(z) = f(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} D^{n-1}[f'(x)e^{nx}]_{x=0}$$

with $w = ze^{-z}$, $D = d/dx$, and the prime denoting a derivative. Then, if $f(z) = \exp zb$, $b^n \equiv b_n$

$$a_n = D^{n-1}[f'(x)e^{nx}]_{x=0} = b(b+n)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} n^{n-1-k} b_{k+1} = \sum_{k=0}^{n} \binom{n}{k} n^{n-1-k} k b_k.$$

But, directly from (44),

$$\exp zb \equiv b_0 + \sum_{n=1}^{\infty} \frac{z^n e^{-nz}}{n!} a_n,$$

which implies $b_n = \sum (-1)^{n+k}\binom{n}{k} k^{n-k} a_k$. The inverse pair

$$(45) \qquad a_n = \sum \binom{n}{k} n^{n-1-k} k b_k, \qquad b_n = \sum (-1)^{n+k} \binom{n}{k} k^{n-k} a_k$$

is an instance of the Abel inverses given by H. W. Gould [5]. Some examples of its use are as follows.

Take $f(z) = (1-z)^{-1}$, so that $b_n = n!$. But, with $S(n, k)$ a Stirling number of the second kind, Δ the difference operator,

$$n! = n! S(n, n) = \Delta^n 0^n = \sum (-1)^{n+k} \binom{n}{k} k^{n-k} k^k$$

and by (45)

$$(46) \qquad n^n = \sum \binom{n}{k} n^{n-1-k} k \cdot k!$$

a relation appearing in [9].

Next, take $f(z) = e^{-z}(1-x)^{-1} = \exp zD$, with $D_n = \Delta^n 0!$, a displacement number (=subfactorial). Then $f'(z) = ze^{-z}(1-z)^{-2}$ and

$$a_n = \sum \binom{n-1}{k} D^k[x(1-x)^{-2}] D^{n-1-k}(e^{(n-1)x})$$

with both derivatives evaluated at $x = 0$. Thus

$$a_n = \sum \binom{n-1}{k} k \cdot k!(n-1)^{n-1-k} = (n-1)^n$$

the last by use of (46). Hence, by (45)

$$(47) \quad (n-1)^n = \sum \binom{n}{k} n^{n-1-k} k D_k, \qquad D_n = \sum (-1)^{n+k} \binom{n}{k} k^{n-k} (k-1)^k.$$

The first of these appeared in [9], the second is due to H. J. Ryser [10].

Next take $f(z) = \exp xz$, so that $b_n = x^n$; then (45) becomes

$$(48) \quad a_n \equiv a_n(x) = \sum \binom{n}{k} n^{n-1-k} k x^k, \qquad x^n = \sum (-1)^{n+k} \binom{n}{k} k^{n-k} a_k(x).$$

These are actually relations for enumerating cycle-free mappings or labeled forests of rooted trees; the coefficient of x^k in $a_n(x)$ is the number of forests with n labeled points and k rooted trees, which is to say that $a_n(x)$ is the enumerator of labeled rooted forests with n labeled points by number of rooted trees. The reader may be reminded that $R(y)$, the enumerator of rooted trees with all points labeled by number of points, satisfies the equation $ze^{-z} = y$ with $z = R(y)$, and if

$$a(x, y) = \sum_{n=1}^{\infty} a_n(x)y^n/n!$$

then $a(x, y) = \exp xR(y)$. Now

$$a(1, y) = \sum a_n(1)y^n/n! = \exp R(y) = y^{-1}R(y) = \sum_1^{\infty} (n+1)^{n-1}y^n/n!.$$

Hence by (48)

$$(49)\quad (n+1)^{n-1} = \sum \binom{n}{k} n^{n-1-k}k, \qquad 1 = \sum (-1)^{n+k}\binom{n}{k} k^{n-k}(k+1)^{k-1}.$$

The corresponding enumeration for labeled forests of (free) trees goes as follows. First, if $A_n(x)$ is the enumerator of forests of trees with n labeled points by number of trees, then

$$(50)\quad \begin{aligned} A(x, y) &= \sum_{n=1} A_n(x)y^n/n! = \exp x[R(y) - R^2(y)/2] \\ &= \exp x(z - z^2/2), \qquad z = R(y), \qquad ze^{-z} = y \end{aligned}$$

and if $\exp x(z - z^2/2) = \sum B_n(x)z^n/n!$ then

$$(51)\quad B_n(x) = \sum_{k=0} (-1)^k \frac{n!}{2^k k!(n-2k)!} x^{n-k}.$$

Thus by (46)

$$(52)\quad A_n(x) = \sum \binom{n}{k} n^{n-1-k}kB_k(x), \qquad B_n(x) = \sum (-1)^{n+k}\binom{n}{k} k^{n-k}A_k(x).$$

The first of equations (52) is equivalent to a result of Alfred Rényi [7]; the second, its inverse, seems to be new. The result of Rényi just mentioned, in present notation, is as follows. Write

$$A_n(x) = \sum A_{nj}x^j, \qquad B_n(x) = \sum B_{nj}x^j;$$

then by the first of (52), and by (51)

$$(53)\quad A_{nj} = \sum \binom{n}{k} n^{n-1-k}kB_{kj} = \frac{1}{j!}\sum_{k=0}^{j} (-\tfrac{1}{2})^k \binom{n-1}{k+j-1} n^{n-j-k}(j+k)!$$

Thus $A_{n1} = n^{n-2}$ (the number of labeled trees with n points),

$$A_{n2} = n^{n-4}(n-1)(n+6)/2,\ A_{n3} = n^{n-6}(n-1)(n-2)(n^2+13n+60)/8.$$

On the other hand, the second half of (52), or

$$A_{n+1}(x) = x\sum \binom{n}{k} T_{k+1}A_{n-k}(x), \qquad T_k = k^{k-2}$$

leads to

$$A_{nn} = 1$$

$$A_{n,n-1} = \binom{n}{2}$$

$$A_{n,n-2} = 3\binom{n+1}{4}$$

$$A_{n,n-3} = 15\binom{n+2}{6} + \binom{n}{4}$$

$$A_{n,n-6} = 105\binom{n+3}{8} + 15\binom{n+1}{6} + 5\binom{n}{5}$$

but there seems to be no simple general form.

References

1. E. T. Bell, Generalized Stirling transforms of sequences, Amer. J. Math., **62** (1940) 717–724.

2. L. Carlitz, On the product of two Laguerre polynomials, J. London Math. Soc., **36** (1961) 399–402.

3. ———, The coefficients of the reciprocal of $J_0(x)$, Arch. Math., **6** (1955) 121–127.

4. H. W. Gould, A new convolution formula and some new orthogonal relations for inversion of series, Duke Math. J., **29** (1962) 393–404.

5. ———, A series transformation for finding convolution identities, Duke Math. J., **28** (1961) 193–202.

6. I. Lah, Eine neue Art von Zahlen, ihre Eigenschaften und Anwendung in der mathematischen Statistik, Mitteilungsbl. Math. Statist., **7** (1955) 203–212.

7. Alfred Rényi, Some remarks on the theory of trees, Publ. Math. Inst. Hung. Acad. Sci., **4** (1959) 73–85.

8. John Riordan, An introduction to combinatorial analysis, Wiley, New York, 1958.

9. ———, Enumeration of linear graphs for mappings of finite sets, Ann. Math. Statist., **33** (1962) 178–185.

10. Herbert John Ryser, Combinatorial mathematics, Carus Math. Monograph No. 14, New York, 1963.

THE NUMBER OF PARTITIONS OF A SET

GIAN-CARLO ROTA, Massachusetts Institute of Technology

Let S be a finite nonempty set with n elements. A *partition* of S is a family of disjoint subsets of S called "blocks" whose union is S. The number B_n of distinct partitions of S has been the object of several arithmetical and combinatorial investigations. The earliest occurrence in print of these numbers has never been traced; as expected, the numbers have been attributed to Euler, but an explicit reference to Euler has not been given, and Bell [7] doubts that it can

be found in Euler's work. The properties of these numbers are periodically being rediscovered, as recently as 1962 (cf. [13]). Following Eric Temple Bell, we shall call them the *exponential numbers*. Bell [4, 5, 6, 7], used the notation ϵ_n; on the other hand, Jacques Touchard [29 and 30] used a_n to celebrate the birth of his daughter Ann; Becker and Riordan [3] used B_n in honor of Bell. We shall follow their choice.

A great many problems of enumeration can be interpreted as counting the number of partitions of a finite set; for example, the number of rhyme schemes for n verses, the number of ways of distributing n distinct things into n boxes (empty boxes permitted), the number of equivalence relations among n elements (cf. [8]), the number of decompositions of an integer into coprime factors when n distinct primes are concerned (cf. Bell [7]), the number of permutations of n elements with ordered cycles (cf. Riordan [27], page 77 ff.), the number of Borel fields over a set of n elements (cf. Binet and Szekeres [8]), etc., etc. Exponential numbers occur frequently in probability, and their theory is closely related to that of the Poisson-Charlier polynomials (see below).

Several explicit expressions for the exponential numbers are known, and can be found in [2, 3, 5, 6, 10, 13, 14, 15, 16, 22, 25, 29, 30, 32]. One of the simplest ways of describing the sequence B_n is by its exponential generating function

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{e^x-1}, \tag{1}$$

where we have set $B_0 = 1$ by convention. All known explicit formulas, however, except the one we shall derive, rely to a greater or lesser degree upon direct enumerations leading to nonimmediate recursions for the sequence B_n.

In this note we shall give a new formula for the exponential numbers (formula (4) below) which differs from the previous ones in that it relies least upon direct counting arguments, and which hinges instead upon some elementary considerations of a "functorial" nature. It is the author's conviction that formula (4), which we derive below, is the natural description of the exponential numbers. The basic idea is a general one, and can be applied to a variety of other combinatorial investigations. We shall see that it easily leads to quick derivations of the properties of the B_n.

Consider an auxiliary finite set U having u elements, $u > 0$. We shall examine the structure of the set U^S of *functions* with domain S, a set with n elements, and range a subset of U. The basic fact is that there are u^n distinct such functions, as is evidenced by the most elementary of counting arguments. We shall now examine this set of functions in greater detail.

To every function $f: S \to U$ there is naturally associated a partition π of the set S, called the *kernel* of f, defined as follows. Two elements a and b of S are to belong to the same block of π, if and only if $f(a) = f(b)$.

How many distinct functions are there with a given kernel π? This question is easily answered. Indeed, let $N(\pi)$ denote the number of distinct blocks of the partition π. A function having kernel π must take distinct values on distinct

blocks of π. Thus, such a function takes altogether $N(\pi)$ distinct values, and the number of distinct such functions equals the number of *one-to-one* functions from a set of $N(\pi)$ elements to the set U. Again, it is well known that such a number is $u(u-1)\cdots(u-N(\pi)+1)=(u)_{N(\pi)}$, and this expression is called the *factorial power* of the number u, with exponent $N(\pi)$.

Now, every function has a unique kernel. Therefore we have the following identity, valid for all integers $u>0$:

$$\sum_{\pi} (u)_{N(\pi)} = u^n, \tag{2}$$

where the sum on the left ranges over all partitions of π the set S.

We now come to the main idea. Let V be the vector space over the reals consisting of all polynomials in the single variable u. Any sequence of polynomials of degrees $0, 1, 2, \cdots$, is a basis for this vector space, in particular, the sequence $(u)_0=1$, $(u)_1$, $(u)_2$, $(u)_3$, $\cdots$. Since a linear functional L on V is uniquely determined by assigning the values it takes on an arbitrary basis, there exists a unique linear functional L on V such that

$$L(1) = 1, \qquad L((u)_k) = 1, \qquad k = 1, 2, 3, \cdots.$$

Applying L to both sides of (2) we obtain

$$\sum_{\pi} L((u)_{N(\pi)}) = L(u^n); \tag{3}$$

but, by the definition of L, the left side simplifies to a sum of as many ones as there are partitions of the set S. In other words, (3) simplifies to

$$B_n = L(u^n). \tag{4}$$

This formula is the explicit expression for the exponential numbers which we wanted to establish. Let us see now how it can be used to derive the other properties of the exponential numbers.

We begin by deriving the recursion formula for the numbers B_n,

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{5}$$

Now, since $u(u-1)_n=(u)_{n+1}$, we have $L(u(u-1)_n)=1=L((u)_n)$. Since the polynomials 1, $(u)_n$ for $n=2, 3, \cdots$ form a basis for the vector space V, it follows from the linearity of L, that

$$L(up(u-1)) = L(p(u)) \tag{6}$$

for every polynomial p. In particular, for $p(u)=(u+1)^n$ we obtain

$$L(u^{n+1}) = L((u+1)^n),$$

but this is precisely formula (5), as we wanted to show.

Note that identity (6) for all polynomials p together with the initial condi-

tion $L(1)=1$ completely characterizes the linear functional L, as defined by (4), since the argument by which we have established (6) is reversible. We shall now use this fact in establishing the generating function (1) for the exponential numbers. To this end, let $g_n/n!$ be the nth coefficient in the Taylor series expansion of e^{e^x-1}:

$$\sum_{n=0}^{\infty} \frac{g_n}{n!} x^n = e^{e^x-1}.$$

There exists a unique linear functional M on V such that $M(u^n)=g_n$, and it will suffice to prove that $L=M$, to conclude that $g_n=B_n$. Now,

$$e^{e^x-1} = M(e^{xu}),$$

where $M(e^{xu})$ is *defined* as

$$\sum_{n=0}^{\infty} \frac{M(u^n)}{n!} x^n.$$

Differentiating, we get

$$e^x e^{e^x-1} = M\left(\frac{d}{dx} e^{xu}\right) = M(ue^{xu}), \tag{7}$$

whence $M(e^{x(u+1)})=M(ue^{xu})$. Expanding the functions $e^{x(u+1)}$ and e^{xu} into Taylor series in the variable x and comparing terms, we obtain $M((u+1)^n)=M(u^{n+1})$.

But, since the polynomials u^n form a basis for V, this implies at once property (6). Hence $M=L$.

Note that differentiating under M, as we have done in (7), does not require any continuity properties of the functional M: it is "purely formal."

There is another, more amusing derivation of the generating function directly from (4), which goes as follows:

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{L(u^n)}{n!} x^n = L(e^{ux}).$$

Now, set $e^x=1+v$, and expand $(1+v)^u$ by the binomial theorem:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n &= L((1+v)^u) = L\left(\sum_{n=0}^{\infty} \frac{(u)_n}{n!} v^n\right) = \sum_{n=0}^{\infty} \frac{L((u)_n)}{n!} v^n \\ &= e^v = e^{e^x-1}, \quad \text{q.e.d.}\end{aligned}$$

In this derivation, it may at first seem puzzling (as suggested by R. D. Schafer) that infinite sums have been commuted with L, without discussing any continuity properties of L. The puzzle is solved as soon as it is noticed that all appearances of the symbol L can be completely eliminated, and the whole derivation amounts to the proof of an infinite sequence of identities relating the coefficients of two Taylor series. The use of L is just a speedy way of establishing these identities.

Next, we shall establish the remarkable formula of Dobinski [14]:

$$B_{n+1} = \frac{1}{e}\left(1^n + \frac{2^n}{1!} + \frac{3^n}{2!} + \frac{4^n}{3!} + \cdots\right). \tag{8}$$

We begin by noticing that the exponential series $e = \sum_{k=0}^{\infty} 1/k!$ can be trivially rewritten as $e = \sum_{k=0}^{\infty} (k)_n/k!$, where n is any nonnegative integer. In view of the definition of the linear functional L, this gives

$$L((u)_n) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{(k)_n}{k!}.$$

Using again the fact that the polynomials $(u)_n$ form a basis for the vector space V, and that the functional L is linear, we infer at once that

$$L(p(u)) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{p(k)}{k!} \tag{9}$$

for any polynomial p. Dobinski's formula now follows by setting $p(u) = u^n$.

Dobinski's formula is particularly suited to the computation of B_n for large n, by an application of the Euler-Maclaurin summation formula (cf. [16] and [25]).

Identity (9) establishes an important property of the linear functional L, namely, that it is *positive definite* on the half-line $[0, \infty)$. We can therefore define a sequence of *orthogonal polynomials* relative to L, and the properties of classical systems of orthogonal polynomials (cf. Szegö [28]) will apply to this set. Such a set of polynomials, we shall now prove, is

$$h_n(u) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (u)_{n-k}, \tag{10}$$

where we use Touchard's notation h_n from [30].

We first note that (6) can be rewritten in more enlightening form by using operator notation. Let $E: p(u) \to p(u+1)$ be the shift operator, let $D: p(u) \to p'(u)$ be the derivative, and let $V: f(x) \to f(1)$ be the *linear* functional consisting in evaluating a function at $x = 1$. Then (6) can be rewritten for any integer $k \geq 0$, by iteration, as

$$L(E^k p(u)) = L(p(u) V D^k x^u),$$

where we have used the fact that $(u)_k = VD^k x^u$. It follows from linearity if g is any polynomial, that

$$L(g(E)p(u)) = L(p(u)Vg(D)x^u).$$

Now set $g(x) = (1-x)^j$, giving $g(E) = (-1)^j \Delta^j$, the iterated difference operator. For this choice of g, we have evidently $Vg(D)x^u = (-1)^j h_j(x)$. Set $p(u) = h_n(u)$, and obtain

$$(-1)^{j+n}L(\Delta^j h_n(u)) = L(h_n(u)h_j(u)).$$

If $j>n$, $\Delta^j h_n$ vanishes identically, proving the orthogonality of the polynomials, and if $j=n$, we get $L(h_n(u)^2)=n!$, which gives the normalizing factors.

The polynomials h_n are the special case of the Poisson-Charlier polynomials (cf. Szegö [28], p. 34) obtained by setting $a=1$, in Szegö's notation. As remarked by Touchard [30], they are particularly useful for computation of the exponential numbers by recursion. Formulas for the first seven polynomials are given by Touchard [30].

These examples suffice to give an idea of the use of formula (4), and to support the contention that this formula gives the natural definition of the exponential numbers. Formula (4) has been suggested by the Blissard calculus techniques so useful in enumerative analysis, (cf. Riordan [27], Ch. 2 Section 4); by the systematic use of linear functionals we can give a rigorous foundation to this calculus, as well as extend its uses in some directions. We hope to implement these contentions in a future publication.

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The following bibliography contains all publications known to the author which study the exponential numbers. He will greatly appreciate any suggestions of omitted works.

References

1. C. A. Aitken, Edinburgh Math. Notes, 28 (1933) 18–33.

2. F. Anderegg, Problem 129, this MONTHLY, 8 (1901) 54.

3. H. W. Becker and John Riordan, The arithmetic of Bell and Stirling numbers, Amer. J. Math., 70 (1934) 385–394.

4. Eric Temple Bell, Exponential polynomials, Ann. of Math., 35 (1934) 258–277.

5. ———, Exponential numbers, Trans. Amer. Math. Soc., 41 (1934) 411–419.

6. ———, Exponential numbers, this MONTHLY, 41 (1934) 411–419.

7. ———, The iterated exponential integers, Ann. of Math., 39 (1938) 539–557.

8. E. F. Binet and G. Szekeres, On Borel fields over finite sets, Ann. Math. Stat., 29 (1957) 494–498.

9. Garrett Birkhoff, Lattice theory, Amer. Math. Soc., 1948, rev. ed.

10. Ugo Broggi, Rendiconti dell' Istituto Lombardo di Scienze e Lettere, 2nd series, 66 (1933) 196–202.

11. E. Catalan, Note sur une équation aux différences finies, J. Math. Pures Appl., 3 (1838) 508–516.

12. Ernesto Cesaro, Nouvelles Annales des Mathématiques, 4 (1885) 39.

13. Martin Cohn, Shiman Even, Karl Menger, Jr., and Philip K. Hooper, On the number of partitionings of a set of n distinct objects, this MONTHLY, 69 (1962) 782–785.

14. G. Dobinski, Grunert's Archiv, 61 (1877) 333–336.

15. Maurice d'Ocagne, Sur une classe de nombres remarquables, Amer. J. Math., 9 (1886) 353–380.

16. Leo F. Epstein, A function related to the series e^{e^x}, J. Math. Phys., 18 (1939) 153–173.

17. P. Epstein, Archiv der Mathematik und Physik, 8 (1904–05) 329–330.

18. I. M. H. Etherington, Nonassociative powers and a functional equation, Math. Gaz., 21 (1937) 36–39.

19. Glover, Tables of applied mathematics, Ann Arbor, 1923.
20. A. Krug, Archiv der Mathematik und Physik, 9 (1905) 189–191.
21. E. Lucas, Théorie des nombres, vol. 1, Gauthier Villars, Paris, 1891.
22. N. S. Mendelsohn, Problem 4340, this MONTHLY, 56 (1949) 187.
23. ———, Applications of combinatorial formulae to generalizations of Wilson's theorem, Canad. J. Math.,1 (1947) 328–336.
24. Silvio Minetola, Principii di Analisi Combinatoria, Giornale di Matematiche, 45 (1907) 333–366.
25. Leo Moser and Max Wyman, An asymptotic formula for the Bell numbers, Trans. Roy. Soc. Canada, Sect. III, 49 (1955) 49–54.
26. E. Netto, Lehrbuch der Kombinatorik, Teubner, Leipzig, 1901.
27. John Riordan, An introduction to combinatorial analysis, Wiley, New York, 1958.
28. Gabor Szegö, Orthogonal polynomials, Revised edition, Amer. Math. Soc., 1959.
29. Jacques Touchard, Propriétés arithmétiques de certains nombres récurrents, Ann. Soc. Sci., Bruxelles, A 53 (1933) 21–31.
30. ———, Nombres exponentiels et nombres de Bernoulli, Canad. J. Math., 8 (1956) 305–320.
31. William Allen Whitworth, Choice and chance, Cambridge, Deighton, Bell and Co., 1901 (reprinted by Stechert).
32. G. T. Williams, Numbers generated by the function e^{e^x-1}, this MONTHLY, 52 (1945) 323–327.
Additional references are to be found in the master's thesis of F. Finlayson, University of Alberta, 1955.

ON THE SPANS OF DERIVATIVES OF POLYNOMIALS

RAPHAEL M. ROBINSON, University of California, Berkeley

1. Introduction. By the span of a polynomial all of whose roots are real, we shall mean the difference between its largest and smallest roots. We are interested in the following problem: *If the span of a polynomial $f(x)$ of degree n with real roots is given, how is the span of its k-th derivative maximized?* It will be sufficient to consider polynomials $f(x)=x^n+\cdots$ all of whose roots lie in the interval $-1\leqq x\leqq 1$. Then all of the roots of $f^{(k)}(x)$ lie in the same interval, and we try to maximize the difference between its largest and smallest roots.

We shall suppose throughout that $k\leqq n-2$, so that $f^{(k)}(x)$ will have more than one root. On the other hand, we see that the problem is trivial if $n\geqq 2k+2$. For in this case, we may put $k+1$ roots of $f(x)$ at each end point $x=\pm 1$. Then $f^{(k)}(x)$ will have a root at each end point, and will therefore have span 2. If $n>2k+2$, some of the roots of $f(x)$ are arbitrary, whereas for $n=2k+2$ the span of $f^{(k)}(x)$ is maximized only for $f(x)=(x-1)^{k+1}(x+1)^{k+1}$.

Thus the nontrivial cases of the problem are those with $k+2\leqq n\leqq 2k+1$. We shall show in Section 2 that in these cases *the span of $f^{(k)}(x)$ can be maximized only when all of the roots of $f(x)$ are at the end points $x=\pm 1$.*

It remains to decide how many roots of $f(x)$ should lie at each end. The obvious conjecture is that if $n=2m$ then $f(x)$ should have m roots at each end point $x=\pm 1$, and that if $n=2m+1$ then there should be m roots at one end and $m+1$ at the other. The evidence for this conjecture is overwhelming, but I have not found a complete proof. In 1961, I verified the conjecture numerically for $n\leqq 25$ and all $k\leqq n-2$, using the IBM 704 computer at the Computer Center of the University of California, Berkeley. In every case, the spans of the derivatives increased to a maximum as the multiplicities of the roots of $f(x)$ at the two end points became more nearly equal. Also, a proof in some special cases is given in Section 4. I hope that this note will stimulate someone to give a complete proof of the conjecture.

2. The main result. Suppose that all but one of the roots of $f(x)$ are given, in the interval $-1\leqq x\leqq 1$, and consider where the last root a should be chosen in this interval to maximize the span of $f^{(k)}(x)$. Thus

$$f(x) = (x - a)g(x),$$

where the roots of $g(x)$ are given, and so

$$f^{(k)}(x) = (x - a)g^{(k)}(x) + kg^{(k-1)}(x).$$

If we introduce the auxiliary function

$$\phi(x) = x + \frac{kg^{(k-1)}(x)}{g^{(k)}(x)},$$

we see that the $n-k$ roots of $f^{(k)}(x)$ are just the roots of $\phi(x)=a$, together with the common roots of $g^{(k)}(x)$ and $g^{(k-1)}(x)$. The latter roots are independent of a. The common roots of $g^{(k)}(x)$ and $g^{(k-1)}(x)$ are of course the multiple roots of $g^{(k-1)}(x)$. Since all the roots of $g(x)$ are real, we see that these are just the roots of $g(x)$ which have multiplicity greater than k.

It is shown in Section 3 that unless $\phi(x)$ reduces to the form $\phi(x)=Ax+C$, then the largest root of $\phi(x)=a$ is an increasing strictly convex function of a, whereas the smallest root is increasing and strictly concave.

Since the roots of $f^{(k)}(x)$ are just the roots of $\phi(x)=a$ together perhaps with some roots independent of a, we see that in all cases the largest root u of $f^{(k)}(x)$ is a convex function of a, and the smallest root v is concave, in the wide sense. The span $u-v$ of $f^{(k)}(x)$ is therefore a convex function of a. Hence for $-1\leqq a\leqq 1$, the maximum value of the span can certainly be attained for $a=\pm 1$.

Indeed, the maximum span can be attained only at one or both of these points, unless the span is independent of a. This can happen only if both u and v are linear functions of a. Since u and v cannot both be solutions of $Ax+C=a$, the only possibility is that both u and v are independent of a, which means that u and v are roots of $g(x)$ of multiplicity greater than k. Putting $a=1$ shows that we must have $u=1$; similarly, $a=-1$ yields $v=-1$. Thus $g(x)$ has more than k roots at each end point. In particular, $n\geqq 2k+2$, so that we have a trivial case.

Applying this argument to each root of $f(x)$ in turn shows that, except in the trivial cases, the span of $f(x)$ can be maximized only when all of the roots of $f(x)$ are at the end points $x=\pm 1$.

3. The auxiliary function. For the function $\phi(x)$ introduced in Section 2, the equation $\phi(x)=a$ has equally many roots for all values of a, and we wish to study the dependence of these roots on a. Notice that the only poles of $\phi(x)$ are at ∞ and at those roots of $g^{(k)}(x)$ which are not roots of $g^{(k-1)}(x)$. Such a root must be a simple root of $g^{(k)}(x)$, since the roots of $g^{(k-1)}(x)$ are all real. It follows that all the poles of $\phi(x)$ are simple.

We now compute the principal parts of $\phi(x)$ at the poles. Near ∞ we have

$$\phi(x) = x + \frac{k(bx^{n-k} + \cdots)}{(n-k)bx^{n-k-1} + \cdots} = \frac{n}{n-k}x + \cdots .$$

If c is a root of $g^{(k)}(x)$ which is not a root of $g^{(k-1)}(x)$, then near $x=c$ we have

$$\phi(x) = \frac{kg^{(k-1)}(c) + \cdots}{g^{(k+1)}(c)(x-c) + \cdots} = \frac{g^{(k-1)}(c)}{g^{(k+1)}(c)}\,\frac{k}{x-c} + \cdots .$$

The residue is seen to be negative. Indeed, since $h(x)=g^{(k-1)}(x)$ is a polynomial which has only real roots, we see that it can have only positive maxima and negative minima. Thus $h(x)$ and $h''(x)$ always have opposite signs at the roots of $h'(x)$ which are not roots of $h(x)$. In other words, $g^{(k-1)}(c)$ and $g^{(k+1)}(c)$ must have opposite signs.

It follows that $\phi(x)$ has the form

$$\phi(x) = Ax - \sum_{r=1}^{s} \frac{B_r}{x-c_r} + C,$$

where $A>0$ and $B_r>0$ for $r=1, \cdots, s$. If $s=0$, then $\phi(x)=Ax+C$. Now suppose that $s>0$, and assume $c_1<c_2< \cdots <c_s$. We have

$$\phi'(x) = A + \sum_{r=1}^{s} \frac{B_r}{(x-c_r)^2}, \qquad \phi''(x) = -2\sum_{r=1}^{s} \frac{B_r}{(x-c_r)^3}.$$

Thus $\phi'(x)>0$, where defined, and hence $\phi(x)$ increases from $-\infty$ to ∞ in each of the intervals $(-\infty, c_1)$, (c_1, c_2), $\cdots$, (c_{s-1}, c_s), (c_s, ∞). Hence $\phi(x)$ assumes every value $s+1$ times. Also, notice that $\phi''(x)<0$ for $x>c_s$ and $\phi''(x)>0$ for $x<c_1$. Thus the function $\phi(x)$ is strictly concave for $x>c_s$ and strictly convex for $x<c_1$. It follows that the largest root of $\phi(x)=a$ is an increasing strictly convex function of a, and that the smallest root is increasing and strictly concave.

4. Some special cases. We have shown that in the nontrivial cases, where $k+2\leqq n\leqq 2k+1$, the span of $f^{(k)}(x)$ can be maximized only for

$$f(x) = (x-1)^p(x+1)^q,$$

where $p+q=n$. We would like to show that p and q should be taken as nearly equal as possible.

We start with a remark that applies to both the trivial and nontrivial cases. For $f(x)$ of the above form, the largest root of $f^{(k)}(x)$ increases from -1 to 1 as p increases from 0 to $k+1$, and remains equal to 1 for larger p. Also, the smallest root of $f^{(k)}(x)$ remains equal to -1 until p reaches $n-k-1$, and then increases to 1 as p increases to n. These results follow readily from Section 2 by moving the roots one at a time continuously from -1 to 1.

Thus for the trivial cases $n \geqq 2k+2$, the span increases with p until p reaches $k+1$, remains at 2 until p reaches $n-k-1$, and then decreases again. For the nontrivial cases $k+2 \leqq n \leqq 2k+1$, the span increases until p reaches $n-k-1$ and decreases beyond $k+1$, but its behaviour for $n-k-1 \leqq p \leqq k+1$ is uncertain, since both the largest and smallest roots of $f^{(k)}(x)$ increase. However, the case $n=2k+1$ is completely resolved: the span is maximized when $f(x)$ has k roots at one end and $k+1$ at the other.

For two other cases, $k=n-2$ and $k=n-3$, the desired result can be obtained by calculations which are elementary but somewhat lengthy. We give a brief summary of the conclusions. Considering the function $f(x)$ displayed above, expanding, differentiating, and making the substitutions

$$\lambda = \frac{p-q}{n}, \qquad t = x - \lambda,$$

we find that

$$\frac{2f^{(n-2)}(x)}{n!} = t^2 - \frac{1-\lambda^2}{n-1},$$

$$\frac{6f^{(n-3)}(x)}{n!} = t^3 - 3\,\frac{1-\lambda^2}{n-1}\,t + 4\,\frac{\lambda(1-\lambda^2)}{(n-1)(n-2)}\,.$$

Looking first at the quadratic, we see that

$$[\text{Span of } f^{(n-2)}(x)] = 2\left(\frac{1-\lambda^2}{n-1}\right)^{1/2},$$

which is larger the nearer λ is to 0. The maximum is attained for $\lambda=0$ when n is even, and for $\lambda=\pm 1/n$ when n is odd. Hence, dropping the condition that the roots of $f(x)$ lie in the interval $-1 \leqq x \leqq 1$, we see that

$$[\text{Maximum Span of } f^{(n-2)}(x)] = \begin{cases} \dfrac{S}{(n-1)^{1/2}} & \text{if } n \text{ is even,} \\[2ex] \dfrac{S(n+1)^{1/2}}{n} & \text{if } n \text{ is odd,} \end{cases}$$

for the class of polynomials $f(x)$ of degree n and span S.

Turning now to the cubic, we see that it has three real roots for $-(n-2)/n \leqq \lambda \leqq (n-2)/n$. Using the trigonometric solution, it is not hard to show that

$$[\text{Span of } f^{(n-3)}(x)] = 2\left(\frac{3(1-\lambda^2)}{n-1}\right)^{1/2} \cos\left[\frac{1}{3}\arcsin\left\{\frac{2\lambda}{n-2}\left(\frac{n-1}{1-\lambda^2}\right)^{1/2}\right\}\right]$$

for $-(n-2)/n \leqq \lambda \leqq (n-2)/n$. The only possible values of λ which we have excluded are ± 1, and in these cases the span is 0. Thus the span is larger the closer λ is to 0, and so

$$[\text{Maximum Span of } f^{(n-3)}(x)] = \begin{cases} S\left(\dfrac{3}{n-1}\right)^{1/2} & \text{if } n \text{ is even,} \\ \dfrac{S[3(n+1)]^{1/2}}{n} \cos\left[\dfrac{1}{3}\arcsin\dfrac{2}{(n-2)(n+1)^{1/2}}\right] & \text{if } n \text{ is odd,} \end{cases}$$

for the class of polynomials of degree n and span S.

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ASSIGNMENT OF NUMBERS TO VERTICES

JOHN H. LINDSEY II, California Institute of Technology and Harvard University

In a recent paper [1], L. Harper discussed the following problem. Given a cube in n dimensions, how shall the integers from 0 to 2^n-1 be assigned to the vertices of this n-cube in such a way as to minimize the sum, over all neighboring pairs of vertices, of the absolute value of the difference of the numbers assigned to each vertex? He proved that one can do no better than to consider the n-cube as the set of 2^n n-tuples of 0's and 1's, and assign to the vertex labelled by an n-tuple the integer whose binary expansion is that n-tuple (although there are other assignments that are as good). This problem arose from the following problem in coding theory: assign n-tuples of zero and one to the numbers from 0 to 2^n-1 in such a way as to minimize the average absolute numerical error made due to all single errors which arise in transmission of the binary words. Using symbols from a *k-letter alphabet* instead of from a *two letter alphabet*, the problem becomes one of minimizing the sum of the absolute values of the differences of the numbers assigned to all pairs of neighboring vertices. Two vertices are called *neighboring* if they agree in all but one coordinate. Here the numbers from 0 to k^n-1 are to be assigned. The proper generalizations of Harper's results are proved in this paper. We shall also reprove Harper's theorem.

THEOREM. *Let S be a finite set*: $S\colon=\{x_1,\cdots,x_m\}$. *Let f be a function valued* 0 *or* 1 *on pairs of objects of S with the properties* $f(x,y)=f(y,x)$, $f(x,x)=0$. *Let g be a function on S such that we have the set equality*: $\{0,\cdots,m-1\}=\{g(x_1),g(x_2),\cdots,g(x_m)\}$. *Then*

$$\sum_{i<j} f(x_i,x_j)\,|g(x_i)-g(x_j)| = \sum_{k=0}^{m-1}\sum_{\substack{g(x)\leq k\\ g(y)>k\\ x,y\in S}} f(x,y).$$

Proof. We define another function

$$h_k(a,b):=\begin{Bmatrix}1 & \text{if } a\leqq k \text{ and } b>k\\ 0 & \text{otherwise}\end{Bmatrix}.$$

$$\begin{aligned}\sum_{k=0}^{m-1}\sum_{\substack{g(x)\leq k\\ g(y)>k}} f(x,y) &= \sum_{k=0}^{m-1}\sum_{x,y\in S} h_k(g(x),g(y))f(x,y)\\ &= \sum_{x,y\in S} f(x,y)\sum_{k=0}^{m-1} h_k(g(x),g(y)) \qquad (\text{since } h_k(g(x),g(x))=0)\\ &= \sum_{x\neq y} f(x,y)\sum_{k=0}^{m-1} h_k(g(x),g(y)) \qquad (\text{since } f(x,y)=f(y,x))\\ &= \sum_{\substack{\text{unordered}\\ \text{pairs }(x,y)\\ x\neq y}} f(x,y)\sum_{k=0}^{m-1}[h_k(g(x),g(y))+h_k(g(y),g(x))]\\ &= \sum_{\substack{\text{unordered}\\ \text{pairs }(x,y)\\ x\neq y}} f(x,y)T(x,y),\end{aligned}$$

where

$$T(x,y)=\sum_{k=0}^{m-1}[h_k(g(x),g(y))+h_k(g(y),g(x))].$$

Consider the case $g(x)<g(y)$. Then

$$\begin{aligned}\sum_{k=0}^{m-1} h_k(g(x),g(y)) &= (\text{the number of } k\text{'s such that } g(x)\leqq k<g(y))\\ &= g(y)-g(x).\end{aligned}$$

Similarly, $\sum_{k=0}^{m-1} h_k(g(y),g(x))=0$ since for no k is $g(y)\leqq k<g(x)$. Thus $T(x,y)=g(y)-g(x)$ if $g(x)<g(y)$. Since $T(x,y)$ is symmetric in x and y, $T(x,y)=T(y,x)=g(x)-g(y)$, if $g(x)<g(y)$. Thus, $T(x,y)=|g(y)-g(x)|$ if $g(x)\neq g(y)\Rightarrow T(x,y)=|g(y)-g(x)|$ if $x\neq y$, since g is one to one.

This gives us the result of the theorem.

COROLLARY. *If the hypothesis of the theorem holds, and if in addition, for each* $x \in S$, $\sum_{y \in S} f(x, y) = C$, *where* C *does not depend upon* x, *then*:

$$\sum_{i<j} f(x_i, x_j) \mid g(x_i) - g(x_j) \mid = \sum_{k=0}^{m-1} \sum_{\substack{g(x) \leq k \\ g(y) > k}} f(x, y)$$

$$= \sum_{k=0}^{m-1} \left(\sum_{\substack{g(x) \leq k \\ y \in S}} f(x, y) - \sum_{\substack{g(x) \leq k \\ g(y) \leq k}} f(x, y) \right)$$

$$= \sum_{k=0}^{m-1} (1 + k)C - \sum_{k=0}^{m-1} \sum_{\substack{g(x) \leq k \\ g(y) \leq k}} f(x, y)$$

$$= \frac{C(m)(m+1)}{2} - 2 \sum_{k=0}^{m-1} \sum_{\substack{\text{unordered} \\ \text{pairs } (x,y) \\ x \neq y \\ g(x) \leq k \\ g(y) \leq k}} f(x, y)$$

$$= C' - 2 \sum_{k=0}^{m-1} C_k,$$

where

$$C' = \frac{C(m)(m+1)}{2}, \quad \textit{and} \quad C_k = \sum_{\substack{\text{unordered} \\ \text{pairs } (x,y) \\ g(x) \leq k \\ g(y) \leq k \\ x \neq y}} f(x, y).$$

Now we return to our problem. S is the set of vertices $(a_1, \cdots, a_n)$ $a_i = 0, 1, \cdots, l_i$. We see that $f((a_1, \cdots, a_n), (b_1, \cdots, b_n)) = 1$ if and only if $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ differ in exactly one component. The corollary holds since $\sum_y f(x, y) = \sum_{i=1}^{n} l_i$. If $f(x, y) = 1$ we call x and y *neighbors*. Also $g(x)$ is the number we assign to the vertex x.

If we consider g the process of numbering the vertices in the order $0, 1, \cdots, m-1$ (i.e., labeling a vertex 0, then labeling another vertex $1, \cdots$) then the quantity C_k we call the *connectedness* of the set of numbered (labeled) vertices after the first $k+1$ vertices have been numbered $0, \cdots, k$. If our assignment g maximizes C_k for $k = 0, \cdots, m-1$, that is,

$$C_k^* \geq \sum_{\substack{\text{unordered} \\ \text{pairs } (x,y) \\ x \in S' \\ y \in S'}} f(x, y)$$

for any set S' of $k+1$ points, then $\sum_{i<j} f(x_i, x_j)\,|g(x_i)-g(x_j)|$ is minimized. I shall find assignments g which maximize C_k for $k=0, \cdots, m-1$ in the problem I am considering.

As a first lemma, we do the case of two dimensions; the general case is made to depend on this case. It is convenient and even necessary in the induction hypothesis contained in the proof below to prove the result for rectangles.

LEMMA. *In a rectangle with m rows and n columns of lattice points with $m \leqq n$, we minimize the sum of the absolute values of the differences of the numbers assigned to neighboring vertices by numbering the "first" row completely (in arbitrary fashion), then the "second" row, etc. Which row is "first," which is "second," . . . is chosen in arbitrary fashion. These are the only assignments that minimize the sum of absolute values of differences (with the understanding that in the case $m=n$ we include the additional assignments from the symmetry of rotating the square by* 90°).

We prove this by Harper's argument, showing that the above method of numbering maximizes connectedness after N lattice points have been numbered, $N=0, \cdots, mn$. We induce first on n, then on m, the cases $n=1$, and the case $m=1$ being trivial.

We may move all the points in $S=\{$lattice points which have been numbered (assume S maximizes connectedness)$\}$ as far to the left and top as possible. This does *not decrease connectedness* (e.g. moving points to the left keeps row connectedness the same and maximizes connectedness between rows as now the connectedness between two rows is the minimum of the number of points in S in each of the two rows).

If the top row is filled, we proceed by induction on the $(m-1)$ by n rectangle left, since the top row contributes $\frac{1}{2}n(n-1)+(N-n)$ to the total connectedness regardless of where the remaining $N-n$ points in S are placed. Since we moved points as far to the top as possible, the first row has more points in S than any other row. Suppose $n_1=$the number of points in $(S\cap(\text{the first row}))$. Then all N numbered points must lie in an m by n_1 rectangle. In particular all N points lie in the m by $(n-1)$ rectangle excluding the last column, since we may assume $n_1<n$. If $n=m$ we use the induction on m for the same n: $n^1=m$, $m^1=n-1$, i.e., $n^1=n$, $m^1<m$. This says we maximize connectedness in the remaining m by $(n-1)$ figure by numbering by columns, which process is symmetric to the numbering by rows in the original m by m square.

If $n>m$, then $n-1\geqq m$ and we maximize connectedness by numbering row by row in the m by $n-1$ figure. Thus without decreasing connectedness we change S to the Figure 1, where x's mark the numbered points, $q=$the number of rows which contain numbered points, $n_q=$the number of points which are numbered in the last numbered row; then $n_q\geqq 1$.

If $n_q\leqq q-1$, we move the n_q numbered points in the last numbered row to the first $q-1$ rows in the last column. The original contribution to connectedness of the n_q points was $\frac{1}{2}(n_q)(n_q-1)+n_q(q-1)$. Their contribution in their

changed position is $\frac{1}{2}n_q(n_q-1)+n_q(n-1)$, which is a *greater* contribution since $n>m\geqq q$.

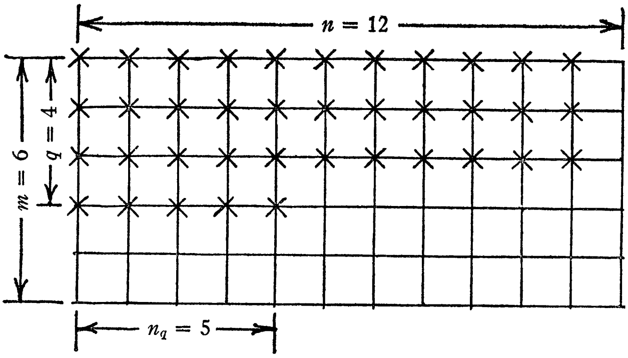

FIG. 1. Partially numbered rectangle

If $n_q>q-1$, we move $(q-1)$ of the points in the last row to the first $q-1$ rows in the last column. This changes their contribution to connectedness from $\frac{1}{2}(q-1)(q-2)+(q-1)(q-1)+(q-1)(n_q-q+1)$ to $\frac{1}{2}(q-1)(q-2)+(q-1)(n-1)$, which is at least as large, since $n-1\geqq n_q$. The new position is a case already covered since the first row is completely numbered.

Thus we have shown that row by row numbering maximizes connectedness at every stage. I now show this is the *only* numbering that does this within a permutation of the rows (except, of course, for column by column in the case $m=n$). In such a numbering the first n points numbered must occupy a row (or possibly a column in case $m=n$), since in this way each point has $(n-1)$ connections, and this is the *only* way any point can have as many as $(n-1)$ connections. Assume the first $i\cdot n$ numbered points must occupy i rows to maximize connectedness. Then the next n numbered points must maximize connectedness in the remaining $(m-i)$ by n rectangle since the first in points contribute connectedness $\frac{1}{2}i(n)(n-1)+\frac{1}{2}n(i)(i-1)+(i)(n)$ to the connectedness of the $(i+1)n$ points, *regardless* of where the last n of the $(i+1)n$ points are placed. Thus, the last n of the $(i+1)n$ points must occupy a row, and the first $(i+1)n$ points occupy $(i+1)$ rows. Hence, a numbering which maximizes connectedness at every stage must occupy i rows (or columns if $m=n$) with the first $i(n)$ numbered points, and must, therefore, be row by row (or possibly column by column if $m=n$) numbering.

We are now ready to prove the theorem of this paper.

THEOREM 1. *The class of assignments of numbers to vertices of a rectangular parallelepiped of lattice points* $(a_1, a_2, \cdots, a_n)$ *in n-dimensions of sides* l_i, $l_1\geqq l_2\cdots\geqq l_n$, *where* $0\leqq a_i\leqq l_i$, $1\leqq i\leqq n$, *which minimize the sum of the absolute values of the difference of the numbers assigned to neighboring vertices (two vertices*

agreeing in all but one coordinate being called neighbors) is exhausted by assignments of the following type: for any fixed i such that $l_i = l_n$, we number the points $(a_1, \cdots, a_n)$ such that $a_i = \pi(1)$; the order in which these points are numbered is governed by induction on n. Then we number the points $(a_1, \cdots, a_n)$ such that $a_i = \pi(2)$, etc., where π is a permutation of $0, 1, \cdots, l_i$.

Proof. The case $n=2$ is the lemma. Thus, we assume $n \geqq 3$. Consider the lattice $\{(a_1, \cdots, a_n) \mid a_i = 0, 1, \cdots, l_i, i = 1, \cdots, n\}$; $l_1 \geqq l_2 \cdots \geqq l_n$. Let S = set of $(a_1, \cdots, a_n)$ which have been numbered. Assume S maximizes connectedness.

If we consider a_i = constant we have an $(n-1)$-dimensional figure, where, by induction on n, we achieve maximum connectedness by numbering the $(n-2)$ dimensional hyperplanes of the hyperplane a_i = constant one by one; that is, we first number the longest row in the hyperplane a_i = constant, then the largest rectangle, then the largest rectangular parallelepiped, etc. Recursively, this numbering of the hyperplane a_i = constant is described as follows:

$$(a_1, \cdots, a_n) \in S,\ b_i = a_i,\ b_n = a_n,\ b_{n-1} = a_{n-1}, \cdots, b_{j+1} = a_{j+1},$$
$$b_j < a_j\ (b_1, \cdots, b_n) \in S \text{ for } j = 1, \cdots, n\ (j = n : b_i = a_i,\ b_n < a_n).$$

Changing the original numbering of the hyperplane a_i = constant to the "rectangular" numbering described for each of $a_i = 0, \cdots, l_i$ (we shall denote this operation by R_i) maximizes connectedness in each hyperplane a_i = constant. It also maximizes connectedness between the hyperplanes $a_i = c_1$ and $a_i = c_2$, since after the operation R_i the connectedness between these two hyperplanes is the minimum over $c = c_1, c_2$ of ($S \cap$ the hyperplane $a_i = c$), which is the most it could possibly be, still keeping the number of points in ($S \cap$ hyperplane (a_i = constant $= c$)) the same for $c = 0, \cdots, l_i$. Hence the operation R_i on S *does not decrease connectedness.*

Define $\sum_i = \sum_{X \in S}$ (ith component of X). We now apply the operations $R_1, R_2, R_3, \cdots, R_n; R_1, R_2, R_3, \cdots, R_n; R_1, R_2, \cdots$ to S in that order, one after the other. Each operation does not decrease connectedness. $\sum_n$ never increases under the operations $R_1, \cdots, R_n$, since R_n leaves the points of S in the same nth dimension hyperplanes (hyperplanes: nth component = constant). And the operations R_i move points to the lowest possible nth dimension hyperplane keeping them in the same ith dimension hyperplane. Since $\sum_n$ has only a finite number of possible values, then after some number of operations, $\sum_n$ *never changes.* This means that the R_i operations no longer shift position to lower nth dimension hyperplanes; thus, the first step in each R_i operations becomes to move points to the lowest possible $(n-1)$th dimension hyperplane keeping them in the same nth and ith hyperplanes. Therefore, after the stage where $\sum_n$ never changes, $\sum_{n-1}$ *never increases.* Then after more operations $\sum_{n-1}$ never changes. After this point, $\sum_{n-2}$ never increases. Finally $\sum_n, \cdots, \sum_1$ never change, which means that we leave every point in the same nth dimension

hyperplane, . . . , and the same 1st dimension hyperplane. Thus, we do not move points. Therefore, after a finite number of operations, we have rectangularized S in each hyperplane $a_i=0, \cdots, l_i$ $(i=1, \cdots, n)$.

Hence, at this point we may assume

$$(a_1, \cdots, a_n) \in S, b_i = a_i, b_n = a_n, \cdots, b_{j+1} = a_{j+1}, b_j < a_j \Rightarrow (b_1, \cdots, b_n) \in S \quad \text{for } i = 1, \cdots, n;\ j = 1, \cdots, n.$$

Let $(\alpha_1, \cdots, \alpha_n)$ be the point in S with the highest nth component, highest $(n-1)$th component (of points with the same nth component), etc. If $\alpha_n \geqq 2$, $(\alpha_1, \cdots, \alpha_n) \in S \Rightarrow (\alpha_1, l_2, \cdots, l_{n-1}, \alpha_n-1) \in S \Rightarrow$(since $n \geqq 3$)$(l_1, \cdots, l_{n-1}, \alpha_n-2) \in S \Rightarrow (a_1, \cdots, a_{n-1}, \alpha_n-2) \in S$, $a_i=0, \cdots, l_i$; $i=1, \cdots, n-1$.

Therefore, (a point of S of nth component $=\alpha_n$)$\Rightarrow$all points of nth component (α_n-2) are in S. But $(\alpha_1, \cdots, \alpha_n) \in S \Rightarrow (\alpha_1, \cdots, \alpha_{n-1}, i) \in S$, $i=2, \cdots, \alpha_n$ $\Rightarrow$any point $(a_1, \cdots, a_n)$ with $a_n \leqq \alpha_n-2$ is in S.

If $\alpha_n=0$, our configuration is in $(n-1)$ dimensions and is covered by induction. If $\alpha_n \geqq 1$, $(\alpha_1, l_1, \cdots, l_{n-1}, \alpha_n-1) \in S \Rightarrow ((b_1, \cdots, b_{n-1}, \alpha_n-1) \in S$ if $b_{n-1} < l_{n-1}$ or $b_{n-1}=l_{n-1}$, $b_{n-2} < l_{n-2}$, etc.). Therefore, the only points with nth component α_n-1 which need *not* necessarily be in S are $(i, l_2, \cdots, l_{n-1}, \alpha_n-1)$ with $i=\alpha_1+1, \cdots, l_1$.

If for some $i=1, \cdots, n-1$ we have $\alpha_i=l_i$, then $(l_1, l_2, \cdots, l_{n-1}, \alpha_n-1) \in S$, so that hyperplanes: nth component $0, 1, \cdots, \alpha_n-1$, are *completely full*. Since the hyperplane: nth component $=\alpha_n$ is rectangularized and it contains the rest of the points of S, S is rectangularized in n dimensions, which we were trying to show maximized connectedness.

If $\alpha_i < l_i$ for all $i=1, \cdots, n-1$, we may convert the point $(\alpha_1, \cdots, \alpha_n)$ which contributes connectedness $\alpha_1+ \cdots +\alpha_n$ to the point $(\beta_1, l_2, l_3, \cdots, l_{n-1}, \alpha_n-1)$, where $\beta_1=1+\max\{x \mid (x, l_2, \cdots, l_{n-1}, \alpha_n-1) \in S\}$. (Note that if $\beta_1=1+l_1$, the hyperplane: nth component $=\alpha_n-1$ is completely full and we are through as in the preceding paragraph.) Now $(\alpha_1, l_2, \cdots, l_{n-1}, \alpha_n-1) \in S \Rightarrow \beta_1 \geqq 1+\alpha_1$. Thus $(\beta_1, l_2, \cdots, l_{n-1}, \alpha_n-1)$ adds connectedness $\beta_1+l_2+ \cdots +l_{n-1}+\alpha_n-1 \geqq \alpha_1+l_2+ \cdots +l_{n-1}+\alpha_n > \alpha_1+\alpha_2+ \cdots +\alpha_n$. Hence, we have *increased* connectedness, a contradiction. This proves that no method of numbering is *better than* hyperplane by hyperplane numbering.

To show that numbering hyperplane by hyperplane is not merely as good as but actually *better than* any other method, we just have to show that the only way to maximize connectedness in numbering $l_1 l_2 \cdots l_{n-1}$ points is to number an entire hyperplane. For, after showing this we may use the same induction agreement used with the two-dimensional rectangle.

Suppose S is a configuration of $l_1 \cdots l_{n-1}$ points which maximizes connectedness, but is *not* a hyperplane. We have already shown that after applying a finite series of the operations $R_1, \cdots, R_n$; $R_1, \cdots$, then S is "rectangularized"; in this case S becomes the hyperplane nth component $=0$. Thus, at some stage of the operations S changes from a nonhyperplane to a hyperplane. Thus, R_i

turns a nonhyperplane configuration S which maximizes connectedness into a hyperplane S' with the same connectedness (S' is not orthogonal to the i axis, since R_i moves points). Thus R_i does not increase connectedness in the $(n-1)$ dimensional hyperplane H_C: ith component = constant, C. But after R_i we have that $S' \cap H_C$ is an $(n-2)$ dimensional hyperplane of $l_1 \cdots l_{i-1}l_{i+1} \cdots l_{n-1}$ points. By induction on n, this is the only way to maximize connectedness in H_C, so that before R_i, $S \cap H_C$ was an $(n-2)$-dimensional hyperplane. The only way to maximize connectedness is that S is to line up these l_i $(n-2)$-dimensional hyperplanes. But this is an $(n-1)$-dimensional hyperplane, contradicting the supposition that S is not a hyperplane. This completes the proof of Theorem 1.

By the theorem, we readily show that the number of ways we can number the lattice $\{(a_1, \cdots, a_n) \mid a_j = 0, \cdots, m-1; j-1, \cdots, n\}$ to minimize the sum of absolute errors of pairs of neighboring vertices is

$$(m!)^{(m^n-1)/(m-1)} \prod_{j=0}^{n-1} (n-j)^{(m^j)}.$$

Also, the minimum sum obtained is $(1/6)(m^n-1)m^n(m+1)$. And as in [1], the average sum over *all* assignments is $(1/6)n(m-1)(m)^n(m^n+1)$.

Similarly, we generalize Harper's *maximization* theorem.

THEOREM 2. *The notation as in Theorem* 1, *the sum of the absolute differences is maximized by the following process: in the case of the square we take any n mutually disjoint (no* 1*'s in the same position) permutation matrices. We number the n points (where* 1 *appears in the* 1*st permutation matrix) randomly. Then we number the n points (where* 1 *appears in the* 2*nd permutation matrix) randomly, etc. We do the same in n dimensions, except that here "ones of the permutation matrix" are replaced by "a configuration of m^{n-1} points of which any two agree in at most $(n-2)$ components."*

Proof. We find such a configuration by taking an $(n-1)$-dimensional such configuration in the first hyperplane, an $(n-1)$-dimensional such configuration which has no-points in common with the first hyperplane configuration in the second hyperplane, etc. Thus an n-dimensional configuration consists of m mutually disjoint $(n-1)$-dimensional configurations. Thus, we have no connectedness after the first m^{n-1} points have been numbered, $k(k-1)n \cdot m^{(n-1)}/2$ connections after the first $k(m)^{n-1}$ points have been numbered. This *minimizes* connectedness at each stage of the numbering and hence achieves a maximum.

Remarks. There are $[(m^{(n-1)})!]^m x$ (the number of Latin n-dimensional hyperchessboards of side m) such numberings. Here the number of "Latin chessboards" means the number of mappings f of $\{(a_1, \cdots, a_n) \mid a_i = 0, \cdots, m-1; i = 1, \cdots, n\}$ onto $\{0, \cdots, m-1\}$, with the condition: $f((a_1, \cdots, a_n)) = f((a_1, a_2, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n))$ implies $a_i = b$, for $i = 1, \cdots, n$. We assign the number $(m^{n-1})f((a_1, \cdots, a_n)) + g((a_1, \cdots, a_n))$ to the point $(a_1, \cdots, a_n)$, where the only condition on the function g is the set equality

$\{g((a_1, \cdots, a_n)) | f((a_1, \cdots, a_n)) = i\} = \{0, 1, \cdots, m^{(n-1)} - 1\}$ when $i = 0, \cdots, m-1$. Note that there is at least one Latin chessboard, e.g.: $f((a_1, \cdots, a_n)) \equiv a_1 + a_2 + \cdots + a_n \pmod{m}$. It can be easily shown that we minimize connectedness parallel to the i-axis by distributing the points of S as evenly as possible among the m^{n-1} pencils parallel to the i-axis. The assignments $h((a_1, \cdots, a_n)) = m^{n-1} f((a_1, \cdots, a_n)) + g((a_1, \cdots, a_n))$ are exactly those assignments which do this for all $i = 1, \cdots, n$.

The maximum value attained is $(1/6)n(m^2-1)m^{2n-1}$. Thus the minimum assignment is $1/n$th the average assignment, asymptotically as m or n gets large, whereas the average assignment is $(m/m+1)$th of the maximum assignment as m gets large.

Reference

1. Lawrence Harper, Jr., Optimal assignments of numbers to vertices, J. Soc. Indust. Appl. Math., to appear.

This paper was one of two winners of the first E. T. Bell prize for undergraduate research in mathematics at the California Institute of Technology. The problem arose at the Jet Propulsion Laboratory, which the Institute operates with support from the National Aeronautics and Space Administration.

Editorial Note. In this MONTHLY 70 (1963) 706–711, A. A. Blank asked whether $\pi/8$ may be the minimal area of a star-shaped domain within which a unit segment can be turned through 360°. C. S. Ogilvy has called attention to a demonstration by R. J. Walker (Pi Mu Epsilon Journal, 1 (1952) 275) that a unit segment can be turned through 360° in a five-pointed star with area approximately three quarters of $\pi/8$.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables 46, Florida

A VISUAL DISPLAY OF SOME PROPERTIES OF THE DISTRIBUTION OF PRIMES

M. L. STEIN, S. M. ULAM, AND M. B. WELLS, University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico

Suppose we number the lattice points in the plane by a single sequence, e.g. Fig. 1 by starting at (0, 0) and proceeding counterclockwise in a spiral so that $(0, 0) \to 1$, $(1, 0) \to 2$, $(1, 1) \to 3$, $(0, 1) \to 4$, $(-1, 1) \to 5$, $(-1, 0) \to 6$, $(-1, -1) \to 7$, $(0, -1) \to 8$, $(1, -1) \to 9$, $(2, -1) \to 10$, $(2, 0) \to 11$, $(2, 1) \to 12$, $(2, 2) \to 13$, etc.

Consider the set P of those lattice points whose single index becomes a prime. Under our correspondence, points of P located on straight lines have indices which ultimately consist of values of a quadratic form. This is easily seen because the third differences between neighboring points on a straight line are 0 and after a finite number of indices which vary linearly have been passed, the progression becomes truly quadratic.

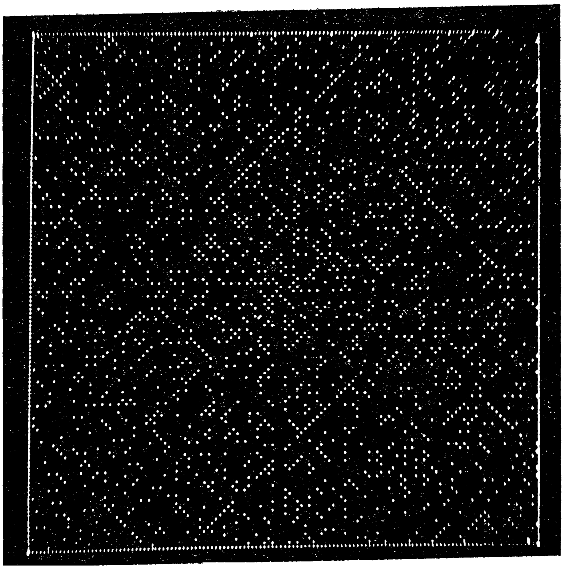

FIG. 1

The set P appears to exhibit a strongly nonrandom appearance (i.e. a different appearance from randomly chosen sets whose densities are like those of primes; that is, asymptotically $\log n/n$). This is due, of course, to the fact that some lines corresponding to quadratic forms which are factorable are devoid of primes; some other quadratic forms are rich in primes. A glance at a picture showing the set P reveals many such lines. It is a property of the visual brain which allows one to discover such lines at once and also notice many other peculiarities of distribution of points in two dimensions. In a visualization of a one-dimensional sequence this is not so much the case. (Perhaps an acoustic interpretation would be more suggestive?)

In addition to the well-known Euler form: y^2+y+41, one could observe instantly many other prime-rich forms. One line rather prominent in Fig. 1 in the lower half of the picture has numbers of the form $4x^2+170x+1847$; as pointed out by the referee, this is reducible into Euler's form by putting $y=2x+42$. (Under the enumeration above, the horizontal, vertical or diagonal straight lines correspond to quadratic forms, which have a leading term $4x^2$.) We have tried

other "Peano numberings" of the lattice points. For example, (Fig. 2) for lattice points in the positive quadrant:

$$(0, 0) \to 1, \ (0, 1) \to 2, \ (1, 1) \to 3, \ (1, 0) \to 4$$
$$(0, 2) \to 5, \ (1, 2) \to 6; \ (2, 2) \to 7, \ (2, 1) \to 8, \ (2, 0) \to 9, \text{ etc.}$$

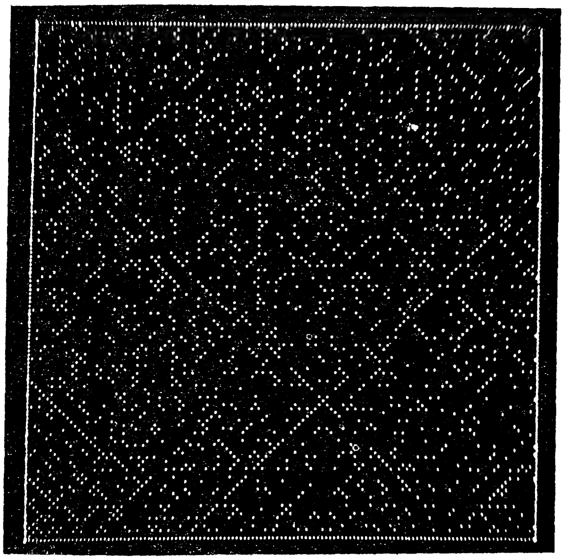

FIG. 2

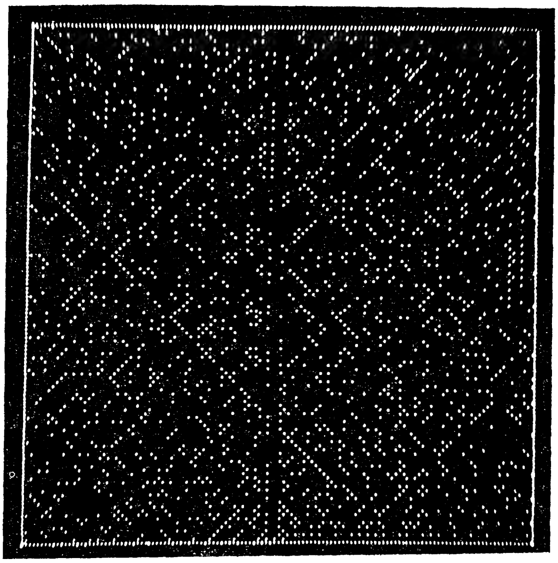

FIG. 3

The successive points on the principal lines will ultimately have coordinates given by quadratic forms with leading term $1 \cdot x^2$. Or, (Fig. 3) for points in the upper half plane:

$$(0, 0) \to 1, \ (-1, 0) \to 2, \ (-1, 1) \to 3, \ (0, 1) \to 4$$

$$(1, 1) \to 5, \ (1, 0) \to 6, \ (-2, 0) \to 7, \ (-2, 1) \to 8$$

$$(-2, 2) \to 9, \ (-1, 2) \to 10, \ (0, 2) \to 11, \ (1, 2) \to 12$$

$$(2, 2) \to 13, \ (2,1) \to 14, \ (2, 0) \to 15 \text{ and so on.}$$

There, the principal straight lines correspond to forms with the term $2x^2$.

The obvious questions, e.g.: Is the distribution of points of P asymptotically symmetric in every angle from the origin? Are there lines containing infinitely many primes? What is the asymptotic density of points on lines (e.g. are there pairs of nonfactorable lines with different asymptotic densities)? etc. seem to be hardly answerable with the present knowledge of the distribution of primes.

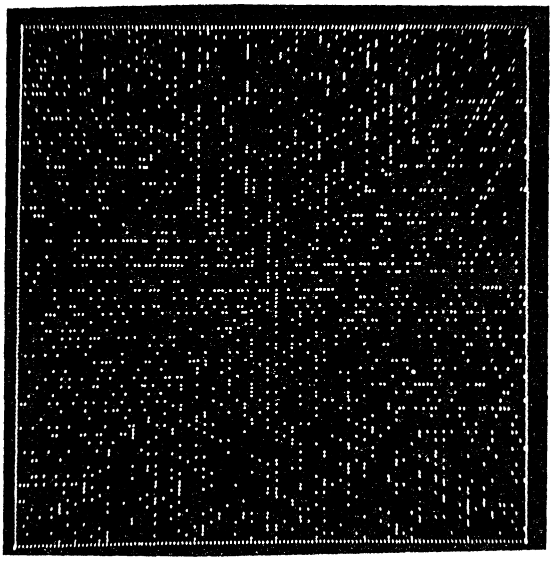

FIG. 4

We have observed many nonfactorable but, so to say, "almost factorable" lines, i.e. lines extremely poor in primes. We should add, as a curiosity, that as we displayed similarly the set L of lucky numbers (see [1]), in (Fig. 4) (the numbering of lattice points in this case is by a "discontinuous spiral," i.e. as in (Fig. 3) but going through both half planes); it appears again that there is a great deal of "structure"; in particular, some of the principal lines are manifest. This is much more surprising, of course, since there is no obvious multiplicative property of the set of luckies and no relation which is rigorous between the divisibility of quadratic forms and the definition of the sieve determining the lucky numbers.

The first observation of the properties of the P set was made on a few hundred points by hand. On the electronic computing machine "Maniac II" in Los Alamos we have been able to use a scope attached to the machine, which can display up to 65,000 points obtained as a result of calculation. This is then photographed and our pictures show a few of the results. We have magnetic tapes containing tables of primes up to ninety million. After discovering the quadratic forms which seem to be rich in primes up to $n=100{,}000$ or so, we then investigated primes up to ten million for such forms (see [2]). A few of the statistics are given below:

For primes in the Euler form $n=x^2+x+41$ we found the ratio r of these to all numbers of this form n up to 10,000,000 to be $r=.475 \cdots$.

(1) For the form $n=4x^2+170x+1847$ there are 727 primes in the first 1560 for numbers of this form $(1\leqq n\leqq 10{,}000{,}000)$ $(r=.466)$.

(2) For $n=4x^2+4x+59$ yields $r=.437 \cdots$.

(3) For $n=2x^2+4x+117$ (the "rare" form) $r=.050 \cdots$.
(A reason for rarity is that for no prime $p<29$ is divisibility by p excluded)

(4) The quadratic form $n=x^2+x+1$ is rich in "luckies." Up to numbers $n\cong 300{,}000$ $r=.29 \cdots$.

This work was performed under the auspices of the U. S. Atomic Energy Commission.

References

1. V. Gardiner, R. Lazarus, N. Metropolis, and S. Ulam, On certain sequences of integers defined by sieves, Math. Mag., 29 (1956) 117–122.

2. Cf. Beeger, Nieuw Arch. Wisk., 20 (1939) 50.

ON THE MEAN VALUES OF INTEGRAL FUNCTIONS AND THEIR DERIVATIVES DEFINED BY DIRICHLET SERIES

J. S. GUPTA, Indian Institute of Technology, Kanpur, India

1. Consider the Dirichlet series $f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, where $\lambda_{n+1}>\lambda_n$, $\lambda_1\geqq 0$, $\lim_{n\to\infty}\lambda_n=\infty$, $s=\sigma+it$ and

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n}=0. \tag{1.1}$$

Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. Let $\sigma_c=\infty$ then σ_a will also be infinite, since according to a known result ([1], p. 4) a Dirichlet series which satisfies (1.1) has its abscissa of convergence equal to its abscissa of absolute convergence and therefore $f(s)$ represents an integral function.

We define the mean values of $f(s)$ as

$$I_r(\sigma)=I_r(\sigma,f)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^r\,dt, \qquad (r>0),$$

and that of $f^{(m)}(s)$, the mth derivative of $f(s)$, as

$$I_r(\sigma, f^{(m)}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f^{(m)}(\sigma+it)|^r dt.$$

A few properties of $I_r(\sigma)$ for $r=2$ are investigated in this paper.

2. THEOREM 1. *$I_2(\sigma)$ increases steadily with σ and* $\log I_2(\sigma)$ *is a convex function of σ.*

Proof. We have

$$I_2(\sigma) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(s)|^2 dt.$$

Now

$$\begin{aligned} |f(s)|^2 &= \sum_{m=1}^{\infty} a_m e^{(\sigma+it)\lambda_m} \sum_{n=1}^{\infty} \bar{a}_n e^{(\sigma-it)\lambda_n} \\ &= \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n} + \sum_{m\neq n}\sum a_m \bar{a}_n e^{\sigma(\lambda_m+\lambda_n)+it(\lambda_m-\lambda_n)}, \end{aligned}$$

the series on the right being absolutely convergent and uniformly convergent in any finite t-range. Hence we may integrate term by term and obtain

$$\frac{1}{2T}\int_{-T}^{T} |f(s)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n} + \sum_{m\neq n}\sum a_m \bar{a}_n e^{\sigma(\lambda_m+\lambda_n)} \frac{2\sin T(\lambda_m-\lambda_n)}{2T(\lambda_m-\lambda_n)}.$$

The factor involving T is bounded for all T, m and n so that the double series converges uniformly with respect to T, and each term tends to zero as $T\to\infty$. Hence the sum tends to zero as T tends to infinity and thus we get

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(s)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n}. \tag{2.2}$$

The fact that $I_2(\sigma)$ is steadily increasing is then obvious from (2.2). To prove convexity of $\log I_2(\sigma)$, we have

$$\frac{\partial^2}{\partial\sigma^2}(\log I_2(\sigma)) = \frac{I_2(\sigma)I_2''(\sigma) - (I_2'(\sigma))^2}{I_2^2(\sigma)}$$

and by Schwarz's inequality

$$\begin{aligned} (I_2'(\sigma))^2 = \left(\sum_{n=1}^{\infty} 2\lambda_n |a_n|^2 e^{2\sigma\lambda_n}\right)^2 &\leqq \left(\sum |a_n|^2 e^{2\sigma\lambda_n}\right)\left(\sum |a_n|^2 4\lambda_n^2 e^{2\sigma\lambda_n}\right) \\ &= I_2(\sigma)I_2''(\sigma). \end{aligned}$$

Hence the result.

THEOREM 2. *If $I_2(\sigma, f')$ is the mean value of $f'(s)$, the first derivative of $f(s)$, then for $\sigma>0$,*

$$I_2(\sigma,f') \geqq \frac{1}{2^2}\left(\frac{\log I_2(\sigma)}{\sigma}\right)^2 I_2(\sigma). \tag{2.3}$$

Proof. We have

$$\begin{aligned}
I_2(\sigma,f') &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f'(\sigma+it)|^2dt \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|\lim_{\epsilon\to 0}\frac{f(\sigma+it)-f(\sigma(1-\epsilon)+it)}{\epsilon\sigma}\right|^2 dt \\
&\geqq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\lim_{\epsilon\to 0}\left\{\frac{|f(\sigma+it)|-|f(\sigma(1-\epsilon)+it)|}{\epsilon\sigma}\right\}^2 dt
\end{aligned}$$

$$\begin{aligned}
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\lim_{\epsilon\to 0} \\
&\quad\cdot\left\{\frac{|f(\sigma+it)|^2+|f(\sigma(1-\epsilon)+it)|^2-2|f(\sigma+it)|\,|f(\sigma(1-\epsilon)+it)|}{\epsilon^2\sigma^2}\right\}dt.
\end{aligned} \tag{2.4}$$

Now, from Schwarz's inequality, we have

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}2|f(\sigma+it)|\,|f(\sigma(1-\epsilon)+it)|\,dt \\
&\qquad\leqq \frac{1}{2T}2\left\{\int_{-T}^{T}|f(\sigma+it)|^2dt\int_{-T}^{T}|f(\sigma(1-\epsilon)+it)|^2dt\right\}^{1/2}.
\end{aligned}$$

Using this inequality in (2.4) and taking the limit outside the integral, which is justified since all the integrals are uniformly convergent, we get

$$I_2(\sigma,f') \geqq \lim_{\epsilon\to 0}\left\{\frac{(I_2(\sigma))^{1/2}-(I_2(\sigma(1-\epsilon)))^{1/2}}{\epsilon\sigma}\right\}^2.$$

Let

$$\phi(\sigma) = \frac{\log I_2(\sigma)}{\sigma};$$

then $\phi(\sigma)$ is an increasing function of σ. Therefore

$$\begin{aligned}
I_2(\sigma,f') &\geqq \lim_{\epsilon\to 0}\left\{\frac{e^{\sigma\phi(\sigma)/2}-e^{(\sigma-\epsilon\sigma)\phi(\sigma)/2}}{\epsilon\sigma}\right\}^2 = \left\{\tfrac{1}{2}\phi(\sigma)e^{\sigma\phi(\sigma)/2}\right\}^2 \\
&= \frac{1}{2^2}\left(\frac{\log I_2(\sigma)}{\sigma}\right)^2 I_2(\sigma).
\end{aligned}$$

Hence the result.

THEOREM 3. *If* $f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *be an integral function of order* $\rho(0<\rho<\infty)$, *lower order* λ, *type* τ *and lower type* ν, *then*

$$\text{(i)}\quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log I_2(\sigma)}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

$$\text{(ii)}\quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log I_2(\sigma)}{e^{\rho\sigma}} = \begin{matrix}2\tau\\ 2\nu\end{matrix}.$$

Proof. We have

$$I_2(\sigma) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^2 dt \leqq \{M(\sigma)\}^2, \tag{2.5}$$

where $M(\sigma)$ is the l.u.b. of $|f(\sigma+it)|$, $(-\infty<t<\infty)$, when σ is constant less than σ_a.

Also, from (2.2),

$$I_2(\sigma) = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n} \geqq \{\mu(\sigma)\}^2, \tag{2.6}$$

where $\mu(\sigma)=|a_{N(\sigma)}|e^{\sigma\lambda_{N(\sigma)}}$ is the maximum term of rank $N(\sigma)$, for Re $(s)=\sigma$, in the series for $f(s)$.

We get, from (2.5) and (2.6),

$$\{\mu(\sigma)\}^2 \leqq I_2(\sigma) \leqq \{M(\sigma)\}^2. \tag{2.7}$$

Since for functions of finite order $\log\mu(\sigma)\sim\log M(\sigma)$, it follows, from (2.7), that

$$\log\{I_2(\sigma)\}^{1/2} \sim \log M(\sigma). \tag{2.8}$$

The results in (i) and (ii) now follow easily from (2.8) since

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

and

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \begin{matrix}\tau\\ \nu\end{matrix}.$$

I am grateful to Dr. R. S. L. Srivastava for his helpful suggestions in the preparation of this paper.

Reference

1. V. Bernstein, Leçons sur les progrès récents de la théorie des séries de Dirichlet, Gauthier Villars, Paris, 1933.

A FORMULA FOR THE DERIVATIVES OF TCHEBYCHEF POLYNOMIALS OF THE SECOND KIND

E. J. SCOTT, University of Illinois, Urbana

Recently, in [1], a formula for the derivatives of Legendre polynomials was derived. It is the purpose of this note to obtain an analogous formula for the derivatives of Tchebychef polynomials of the second kind by making use of properties of the Gegenbauer polynomials. The formula for derivatives of Legendre polynomials and partition formulas for $P_n^{(k+1/2)}(x)$ and $P_n^{(k+1)}(x)$ $(k=0, 1, 2, \cdots)$, in terms of Legendre and Tchebychef polynomials of the second kind, fall out as a consequence.

When $\alpha=\beta$ the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is termed an ultraspherical or Gegenbauer polynomial and defined as follows (see [2]):

$$P_n^{(\lambda)}(x) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+\frac{1}{2})} P_n^{(\lambda-1/2,\lambda-1/2)}(x), \qquad \lambda > -\tfrac{1}{2}. \tag{1}$$

If $\lambda=\frac{1}{2}$, $P_n^{(1/2)}(x)=P_n(x)$, the Legendre polynomials, whereas $\lambda=1$ yields $P_n^{(1)}(x)=U_n(x)$, the Tchebychef polynomials of the second kind. When $\lambda=0$, the polynomial $P_n^{(\lambda)}(x)$ vanishes identically for $n\geqq 1$ and equals 1 for $n=0$.

A generating function for the Gegenbauer polynomials is

$$\sum_{n=0}^{\infty} t^n P_n^{(\lambda)}(x) = (1-2xt+t^2)^{-\lambda}. \tag{2}$$

In addition, we have (see [2])

$$\frac{d}{dx} P_{n+1}^{(\lambda)}(x) = 2\lambda P_n^{(\lambda+1)}(x). \tag{3}$$

A direct consequence of (3) is the formula

$$\frac{d^k}{dx^k} P_{n+k}^{(\lambda)}(x) = 2^k\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1)P_n^{(\lambda+k)}(x). \tag{4}$$

We examine two special, but important cases. Consider $P_n^{(\lambda+k)}(x)$ for $\lambda=\frac{1}{2}$. In view of (2) and the generating function for the Legendre polynomials, we have

$$\sum_{n=0}^{\infty} t^n P_n^{(k+1/2)}(x) = [(1-2xt+t^2)^{-1/2}]^{2k+1} = \left[\sum_{r=0}^{\infty} t^r P_r(x)\right]^{2k+1}.$$

Equating the coefficients of t^n gives the expression

$$P_n^{(k+1/2)}(x) = \sum_{i_1+i_2+\cdots+i_{2k+1}=n} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2k+1}}(x), \tag{5}$$

where the ordered set of indices $i_1, i_2, \cdots, i_{2k+1}$ assume the values of all permutations of all sets of $2k+1$ nonnegative integers having a sum equal to n.

Thus for $\lambda = \frac{1}{2}$, equation (4) becomes

$$(6) \quad \frac{d^k}{dx^k} P_{n+k}(x) = 1\cdot 3\cdot 5 \cdots (2k-1) \sum_{i_1+i_2+\cdots+i_{2k+1}=n} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2k+1}}(x),$$

which is the formula derived in [1].

When $\lambda = 1$, we have similarly

$$\sum_{n=0}^{\infty} t^n P_n^{(k+1)}(x) = [(1-2xt+t^2)^{-1}]^{k+1} = \left[\sum_{r=0}^{\infty} t^r U_r(x)\right]^{k+1},$$

where $U_r(x)$ is the Tchebychef polynomial of the second kind. Equating the coefficients of t^n, we obtain

$$(7) \qquad P_n^{(k+1)}(x) = \sum_{i_1+i_2+\cdots+i_{k+1}=n} U_{i_1}(x)U_{i_2}(x)\cdots U_{i_{k+1}}(x).$$

Here, $i_1, i_2, \cdots, i_{k+1}$ constitute an ordered set of indices which take on the values of all permutations of all sets of $k+1$ nonnegative integers whose sum is n. In this case, therefore, equation (4) becomes

$$(8) \qquad \frac{d^k}{dx^k} U_{n+k}(x) = 2^k k! \sum_{i_1+i_2+\cdots+i_{k+1}=n} U_{i_1}(x)U_{i_2}(x)\cdots U_{i_{k+1}}(x).$$

References

1. Mary L. Boas, A formula for the derivative of Legendre polynomials, this MONTHLY, 70 (1963) 643–4. (See also a proof by B. S. Popov in Mitrinović, D. S., Zbornik matematičkih problema, 3rd ed., Belgrade, vol. I (1962) 435–6; this reference was supplied to the editor by Prof. Mitrinović.)

2. Gabor Szegö, Orthogonal Polynomials, Amer. Math. Soc., New York, 1939.

EXTENSIONS OF LINEAR TRANSFORMATIONS

HERBERT A. GINDLER, University of Pittsburgh

Let X and Y be normed linear spaces with completions $\hat{X}$ and $\hat{Y}$ respectively. We denote by $[X, Y]$ the class of all linear transformations A for which the domain of A is all of X, the range of A is a subset of Y and A is continuous. If $A \in [X, Y]$, we study the relationship between the state of A, as defined by A. E. Taylor [1; p. 235], and the state of $\hat{A}$, the unique continuous extension of A to the completions of X and Y. All the implications in this respect are stated in Theorem 1 and examples are given to show that no further results can be obtained without additional hypotheses.

THEOREM 1. *Let X and Y be normed linear spaces with completions $\hat{X}$ and $\hat{Y}$ respectively. Suppose $A \in [X, Y]$ and let $\hat{A}$ in $[\hat{X}, \hat{Y}]$ be the unique continuous extension of A to all of $\hat{X}$. Then*

(a) *$R(A)$, the range of A, is dense in Y if and only if $R(\hat{A})$, the range of $\hat{A}$, is dense in $\hat{Y}$;*

(b) A^{-1} *exists and is continuous if and only if* $\hat{A}^{-1}$ *exists and is continuous*;
(c) *if* A^{-1} *does not exist*, $\hat{A}^{-1}$ *does not exist*;
(d) *if* $\hat{A}^{-1}$ *exists and is continuous*, $R(\hat{A})$ *is closed*;
(e) *if* $\hat{A}^{-1}$ *exists and* $R(\hat{A})$ *is closed*, $\hat{A}^{-1}$ *is continuous.*

Proof. The proof of part (a) of the theorem is essentially topological. First we observe that a subset of Y is dense in Y if and only if it is also dense in $\hat{Y}$.

Now suppose that $R(A)$ is dense in Y. Then, since $R(\hat{A})\supset R(A)$ and $R(A)$ is dense in $\hat{Y}$, it follows that $R(\hat{A})$ is also dense in $\hat{Y}$. Conversely, suppose that $R(\hat{A})$ is dense in $\hat{Y}$. Since every point in $R(\hat{A})$ is obtained as a limit of elements of $R(A)$, it follows that $R(A)$ is dense in $\hat{Y}$. By the observation of the previous paragraph, $R(A)$ is dense in Y. This proves part (a) of the theorem.

If A^{-1} exists and is continuous, then there exists $m>0$ such that $\|Ax\|/\|x\| \geqq m$ for all nonzero vectors $x\in X$. If $\hat{x}\in\hat{X}$ then, by the definition of $\hat{X}$, there exists a sequence $\{x_n\}\subset X$ with $\lim_{n\to\infty} x_n=\hat{x}$, and $\lim_{n\to\infty} Ax_n=\hat{A}\hat{x}$ by the definition of $\hat{A}$. Hence, $\|\hat{A}\hat{x}\|/\|\hat{x}\|=\lim_{n\to\infty}\|Ax_n\|/\|x_n\|\geqq m$ for each nonzero vector $\hat{x}\in\hat{X}$. This implies that $\hat{A}^{-1}$ exists and is continuous. Conversely, if $\hat{A}^{-1}$ exists and is continuous, then, since A is a restriction of $\hat{A}$, it follows that A^{-1} exists and is continuous.

Part (c) of the theorem follows from the fact that $\hat{A}$ extends A while parts (d) and (e) follow readily from well-known properties of bounded linear operators on Banach spaces. [1; Theorems 4.2-E and 4.2-H].

The theorem just proved can be used to make a state diagram for A and $\hat{A}$ of the theorem.

To define the *state* of a linear transformation A, the range $R(A)$ is classified according to the following possibilities:

I. $R(A)=Y$,
II. $R(A)$ dense in Y but $R(A)\neq Y$,
III. $R(A)$ not dense in Y;

while the inverse of A is classified according as:

1. A has a continuous inverse,
2. A has a discontinuous inverse,
3. A has no inverse.

All possible combinations consisting of a roman numeral together with an arabic numeral for a subscript constitute the 9 possible states for the operator A. Thus we write $A\in\mathrm{III}_1$, A is classified III_1, or the state of A is III_1, whenever the range of A is not dense in Y but A has a continuous inverse. Similar statements can be made using the other states as well.

If A and B are linear transformations, the *state of the pair* (A, B) is the ordered pair of states of A and B, respectively.

The state diagram is a device conceived by A. E. Taylor for displaying systematically the relationships between the states of two operators. The diagram consists of nine columns—one for each of the nine possible states for A. Similarly, each row is labeled on the left by one of the nine states for $\hat{A}$. A letter—*a*, *b*, *c*, *d*

or e—is placed at the intersection of a row and column if that pair of states cannot occur for A and $\hat{A}$ by virtue of the designated part of the Theorem 1. For example, the letter "a" is placed in the 7th column (labeled III_1) and 4th row (labeled II_3) because part (a) of Theorem 1 shows that A cannot have the property that its range is not dense in Y if the range of $\hat{A}$ is dense in $\hat{Y}$. Thus, each square in the state diagram represents a state of the pair $(A, \hat{A})$. (See Taylor [1, p. 236].) Examples will be given to show that each of the blank squares represents a pair of states which can actually occur. Hence, one can determine which pairs of states can occur by examining the appropriate square in the diagram.

THE STATE DIAGRAM FOR A IN $[X, Y]$ AND $\hat{A}$, ITS CONTINUOUS EXTENSION TO THE COMPLETIONS OF X AND Y.

$\hat{A}$ \ A	I_1	I_2	I_3	II_1	II_2	II_3	III_1	III_2	III_3
III_3	a	a	a	a	a	a	b		
III_2	a	a	a	a	a	a	b		c
III_1	a	a	a	a	a	a		b	b
II_3	b			b			a	a	a
II_2	b		c	b		c	a	a	a
II_1	d	b	b	d	b	b	a	a	a
I_3	b			b			a	a	a
I_2	b	e	c	b	e	c	a	a	a
I_1		b	b		b	b	a	a	a

Examples will now be given to show that all the blank squares in the state diagram represent states which can actually occur.

There are seven blank squares for which A and $\hat{A}$ are in the same state. Examples for these situations can be obtained as follows. Let X and Y be any two Banach spaces and let A in $[X, Y]$ be in the desired state. Then $X=\hat{X}$, $Y=\hat{Y}$ and $A=\hat{A}$ so that the states for A and $\hat{A}$ are the same. Of course, states $(\mathrm{I}_2, \mathrm{I}_2)$ and $(\mathrm{II}_1, \mathrm{II}_1)$ cannot be obtained in this way because of Theorem 1 (d) and (e).

To see that $(\mathrm{I}_2, \mathrm{II}_2)$ can occur, suppose that $\hat{X}$ and $\hat{Y}$ are any two Banach spaces and suppose $\hat{A}$ is any element of $[\hat{X}, \hat{Y}]$ which is classified II_2. Now choose $Y=R(\hat{A})$ and A in $[\hat{X}, Y]$ with $A=\hat{A}$. Clearly A and $\hat{A}$ have the desired states. In a completely analogous fashion $(\mathrm{I}_3, \mathrm{II}_3)$ can be obtained.

By choosing any Banach space Y, letting X be a dense subspace not equal to Y, and letting $Ax=x$ for all $x\in X$, one gets $A\in[X, Y]$ and A is classified II_1 while $\hat{A}$ is the identity on Y; so $\hat{A}$ is classified I_1.

To obtain the remaining examples we make use of the sequence spaces.

Let X be the subspace of (c_0) for which $\xi_k\neq 0$ for at most finitely many different values of k. X is a dense subspace of (c_0). If $x=\{\xi_k\}\in X$ and if $y=\{\eta_k\}\in Y$, where Y may be X, (c_0) or l_∞, define $Ax=y$ by $\eta_k=\xi_k-2\xi_{k+1}$ for $k=1, 2, \cdots$. In order to show that $R(\hat{A})=(c_0)$, we solve the equations $\eta_k=\xi_k-2\xi_{k+1}$ for ξ_{k+1} obtaining

$$\xi_{k+1}=\frac{1}{2^k}\xi_1-\left(\frac{1}{2^k}\eta_1+\frac{1}{2^{k-1}}\eta_2+\cdots+\tfrac{1}{2}\eta_k\right). \tag{1}$$

If $\{\eta_n\}\in(c_0)$ and if $\epsilon>0$, then there exists a positive integer N such that $|\eta_n|<(\epsilon/2)$ for all $n>N$. Letting $M=\sup_{1\leq n\leq N}|\eta_n|$, we see that

$$\begin{aligned}|\xi_{k+1}| &\leq \left|\frac{1}{2^k}\xi_1\right|+\left|\frac{1}{2^k}\eta_1+\frac{1}{2^{k-1}}\eta_2+\cdots+\frac{1}{2^{k-N+1}}\eta_N\right| \\ &\quad+\left|\frac{1}{2^{k-N}}\eta_{N+1}+\cdots+\frac{1}{2}\eta_k\right| \\ &<\frac{1}{2^k}|\xi_1|+\frac{1}{2^k}(2^{N-1}M)N+\frac{\epsilon}{2}\cdot\end{aligned}$$

Take $\xi_1=0$ and choose K so large that $1/2^K<\epsilon/2^NMN$. Then, for all $k\geqq K$,

$$|\xi_{k+1}|<\frac{1}{2^k}2^{N-1}NM+\frac{\epsilon}{2}<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

We conclude that, for $\xi_1=0$ and ξ_{k+1} given by equation (1), the sequence $\{\xi_k\}\in(c_0)$ provided only that the sequence $\{\eta_k\}\in(c_0)$. This shows that $R(\hat{A})\supset(c_0)$. The reverse inclusion follows directly from the definition of A. Also, from (1) with $\eta_k=0$ for all k, we see that $\hat{A}x=0$ if and only if

$$x=\xi_1\left(1,\frac{1}{2},\cdots,\frac{1}{2^k},\cdots\right)$$

where ξ_1 is arbitrary. If $\xi_1\neq 0$, these vectors do not belong to the domain of A, so that A^{-1} exists. But A^{-1} cannot be continuous in view of part (b) of Theorem 1. Moreover, if $\{\eta_k\}\in X$, there exists a positive integer N such that $\eta_k=0$ if $k>N$. Letting $\xi_1=\eta_1+2\eta_2+\cdots+2^{N-1}\eta_N$ in (1), we see that $\xi_{k+1}=0$ if $k>N$, so that $\{\xi_k\}\in X$. These facts permit us to produce three more examples. If $Y=X$, we have $\hat{Y}=\hat{X}=(c_0)$ and the state for A is I_2 while the state for $\hat{A}$ is I_3. If $Y=(c_0)$, the state for $\hat{A}$ remains unchanged but the state for A is II_2. Choosing $Y=l_\infty$ provides an example of $(\mathrm{III}_2, \mathrm{III}_3)$.

To obtain an example of the pair of states $(\mathrm{II}_3, \mathrm{I}_3)$ let X be the subspace of

l_2 for which $\xi_k \neq 0$ for at most finitely many values of k, and let $Y = l_2$. Define $Ax = y$ by $\eta_k = \xi_{k+1}$ for $k = 1, 2, \cdots$, where $x = \{\xi_k\} \in X$ and $y = \{\eta_k\} \in l_2$. If x is any vector for which $\xi_k = 0$ if $k \neq 1$, then $Ax = 0$. Also, the range of A is X. Hence, A is classified II_3. One readily verifies that $\hat{A}$ is classified I_3.

Let X and Y be as in the preceding paragraph and define $Ax = y$ by

$$\eta_k = \frac{2^k - 1}{2^{2k-1}} \xi_1 + 2^{1-k}\xi_2 + 2^{1-k}\xi_3 + \cdots + 2^{1-k}\xi_{k+1}$$

for $k = 1, 2, \cdots$. We have $\hat{X} = l_2$ and $\hat{A}$ has the same form as A except $\hat{A}$ acts on l_2. Setting $\eta_k = 0$ for all k, one sees that $\hat{A}x = 0$ implies $\xi_k = -2^{1-k}\xi_1$ for $k \geqq 2$ and ξ_1 arbitrary. Since these vectors are not in X but are in l_2, it follows that A^{-1} exists but $\hat{A}^{-1}$ does not. To see that the range of A, hence the range of $\hat{A}$, is dense in l_2, let u_k in l_2 be the vector $(0, \cdots, 0, 1, 0, \cdots, 0)$ (with 1 in the kth place) and let $x_k = (0, \cdots, 0, 2^{k-1}, -2^{k-1}, 0, \cdots, 0)$ where the nonzero entries are in the $(k+1)$st and $(k+2)$nd places. We have $Ax_k = u_k$ for $k = 1, 2, \cdots$, from which it follows that $R(A)$ is dense in l_2. It can also be shown that $y = \{\eta_k\}$ with $\eta_{2k-1} = 0$, $\eta_{2k} = 2^{1-k}$ for $k = 1, 2, \cdots$ is not in the range of $\hat{A}$. These results combined with part (b) of Theorem 3.1 show that A is classified II_2 while $\hat{A}$ is classified II_3. By choosing Y to be the range of A one obtains the states $(\mathrm{I}_2, \mathrm{II}_3)$. This completes the promised set of examples.

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Reference

1. A. E. Taylor, Introduction to functional analysis, Wiley, New York, 1958.

A THREE-POINT PROPERTY IN STRAIGHT LINE SPACES

RAYMOND W. FREESE, St. Louis University

Let S denote a metric space in which each pair of points is joined by a unique metric segment. We define a subset M of S as possessing the double isosceles property provided that two connecting segments of every isosceles triple belong to M, Marr and Stamey ([1]) have shown the following

THEOREM. *Let M be a closed, connected subset of a Euclidean or hyperbolic space. If M has the double isosceles property, then M is convex.*

The purpose of this note is to prove that this theorem is valid in a slightly wider class of spaces, namely straight line spaces (i.e., finitely compact, convex, externally convex metric spaces in which the linearity of two of the four triples of a quadruple implies the linearity of the remaining two). In the following, the background of [1] and [2] are assumed.

To show the corresponding version of the above theorem, we need the following result.

LEMMA. *Let M be a closed and connected subset of a straight line space with x, y two arbitrary but distinct points of M. Let L denote the set of all points t of the space such that $xt = ty$, where xt denotes the distance of x, t. If M has the double isosceles property, then given any point z common to L and M such that $S(x, z)$ and $S(y, z)$ are both contained in M, either there exists a z' of M such that $xz' = yz' < xz = yz$ or the segment joining x, y lies in M.*

Proof. Let z be any point common to L and M, z not in $S(x, y)$. For each point p of $S(x, z)$ we may assume $py \geqq px$, for in the contrary case there exists a point p' of $S(p, x)$ distinct from p such that $p'y = p'x$ which proves the lemma. Similarly it may be assumed for points q of $S(y, z)$ that $qx \geqq qy$.

Let s denote a point of $S(z, y)$ with $z \neq s \neq y$ and hence $s \neq x$. Then $zs < zx$. Again, since segments are connected sets and the metric is continuous, there exists a point w of $S(x, z)$ such that $xw = ws$. Since M has the double isosceles property, two of the segments joining x, w, s lie in M. Hence $S(x, s)$ is contained in M or $S(w, s)$ is contained in M. If $S(w, s)$ is contained in M, then since $sx \geqq sy$ and $wy \geqq wx$, there exists a point u of $S(w, s)$ and hence of M such that $ux = uy$ and $ws = wu + us$. Then $ws < wz + zs$ follows from metric betweenness transitivities of the space and hence $wu < wz$ or $us < sz$. However, $wu < wz$ implies $xu < xz$ while $us < sz$ implies $yu < yz$. Accordingly $xu = yu < xz = yz$. If $S(x, s)$ is contained in M, the same result follows with x replacing w, which completes the proof.

It is then clear that, since the space is finitely compact, extending the method of proof used in [1] yields the following theorem:

THEOREM. *Let M be a closed, connected subset of a straight line space. If M has the double isosceles property, then M is convex.*

References

1. J. M. Marr and W. L. Stamey, A three-point property, this MONTHLY, 69 (1962) 22–25.
2. L. M. Blumenthal, Theory and applications of distance geometry, Oxford, 1953.

A DIFFERENTIAL EQUATION

P. R. P. RAO, Indian School of Mines, Dhanbad, India.

The differential equation

$$y'(x) = \sum_{i=0}^{n} f_i(x) y^i(x), \tag{1}$$

where n is any positive integer, is a very general one, special cases of which are the linear equation, Bernoulli's equation, Riccati's equation, and Abel's equation of the first kind. We present a transformation below, which under certain conditions (see (4)) reduces (1) to an equation in which the variables are separable. We also show that certain other ordinary nonlinear differential equations of the first order can be transformed to the form typified by (1) by suitable

transformations on the dependent variable y alone. It is assumed that f_{n-1} and f_n are twice differentiable, that f_i $(i=0, 1, \cdots, n-2)$ are differentiable, and that $f_n>0$.

Substituting $y=vu-f_{n-1}/nf_n$, we obtain from (1) for $n\geqq 4$

$$f_n^{n-1}vu' = U(x) + uJ(x) + \sum_{r=2}^{n-2} f_n^r v^r G_r(x)u^r + f_n^n v^n u^n, \tag{2}$$

where, we have put

$$\begin{aligned} U(x) &= \sum_{j=0}^{n} (-1)^j n^{-j} f_{n-1}^j f_j f_n^{n-1-j} + (f_n^{n-3}/n)(f_n f_{n-1}' - f_{n-1}f_n'), \\ J(x) &= \left(v\sum_{j=1}^{n} (-1)^{j-1} jn^{1-j} f_j f_{n-1}^{j-1} f_n^{n-j}\right) - f_n^{n-1}v', \\ G_r(x) &= \sum_{j=r}^{n} (-1)^{j-r} n^{r-j} \binom{j}{r} f_j f_{n-1}^{j-r} f_n^{n-j-1}. \end{aligned} \tag{3}$$

Assuming that $U\neq 0$, on the interval on which $U>0$, we define v by $f_n^n v^n = U$, and obtain

$$\begin{aligned} U^{1-1/n}J(x) &= \left[f_n' f_n^{n-3} + \sum_{j=1}^{n} (-1)^{j-1} jn^{1-j} f_j f_{n-1}^{j-1} f_n^{n-j-1}\right]U - 1/nf_n^{n-2}U', \\ f_n^r v^r G_r &= U^{r/n}G_r. \end{aligned} \tag{3.1}$$

Hence, if $n-1$ constants, C_r exist, such that

$$\begin{aligned} &1/nf_n^{n-2}U' - \left(f_n^{n-3}f_n' + \sum_{j=1}^{n} (-1)^{j-1} jn^{1-j} f_j f_{n-1}^{j-1} f_n^{n-j-1}\right)U = C_1 U^{2-1/n}, \\ &G_r U^{r/n-1} = C_r, \qquad r = 2, 3, \cdots, n-2, \end{aligned} \tag{4}$$

then we obtain from (2) the following equation in which the variables are separable:

$$u' = (f_n^{2-n}U^{1-1/n})(1 - C_1 u + C_2 u^2 + \cdots + C_{n-2}u^{n-2} + u^n). \tag{5}$$

We may note that (2) is valid for $n=2, 3$, if the third term on its right side is taken equal to zero, and for $n=1$ if the last two terms on its right side are taken equal to zero. Thus, for $n=2, 3$, U and C_1 are given by the first equation of (3) and the first equation of (4) respectively, and we need to take $C_2, C_3, \cdots$ each equal to zero to arrive at (5). When $n=1$, the first equation of (3) gives $U(=(f_0/f_1)')$, and in this case it is convenient to take $v=\exp(\int f_1 dx)$, and hence $C_1, C_2 \cdots$ each equal to zero to arrive at the separable form $u'=(1/v)(f_0/f_1)'$.

The reduction of (1) to (5) shows that under certain conditions, the transformation $y=vu+F$ reduces an ordinary nonlinear differential equation of the

first order to one in which the variables are separable, provided that the nonlinearity of the differential equation is due only to the presence of y^n with $n>1$. But the nonlinearity of an equation $y'=f(x, y)$ may be due also to the presence of y^{-n} or $y^{\pm 1/n}$, where n is a positive integer, and $1/n$ is + or − consistently. What follows below indicates that, through suitable substitutions made on y, these nonlinearities can be transformed to the earlier one:

Case (i): If the equation under study is

$$y'(x) = f_0(x) + f_1(x)y(x) + f_2(x)y^{-1}(x) + \cdots + f_{n+1}(x)y^{-n}(x), \tag{6}$$

where n is any positive integer, $y=1/w$ transforms (6) to

$$-w' = f_1 w + f_0 w^2 + f_2 w^3 + \cdots + f_{n+1} w^{n+2}, \tag{6.1}$$

which is typified by (1).

Case (ii): If the equation under study is

$$y'(x) = f_0(x) + f_1(x)y(x) + f_2(x)y^{1/2}(x) + \cdots + f_n(x)y^{1/n}(x), \tag{7}$$

the substitution $y=w^L$, where L is the least common multiple of $2, 3, \cdots, n$, transforms (7) to

$$Lw' = f_1 w + \sum_{r=2}^{n} f_r w^{1-L+L/r} + f_0 w^{1-L}, \tag{7.1}$$

an equation typified by (6), which can be transformed to the form (1) by $w=1/u$.

If, on the other hand, the given equation is

$$y'(x) = f_0(x) + f_1(x)y(x) + f_2(x)y^{-1/2}(x) + \cdots + f_n(x)y^{-1/n}(x), \tag{8}$$

then $y=w^{-L}$ transforms (8) to

$$-Lw' = f_1 w + f_0 w^{1+L} + \sum_{r=2}^{n} f_r w^{1+L+L/r}, \tag{8.1}$$

which is typified by (1).

Another case of interest is provided by the equation

$$y'(x) = f_0(x) + f_1(x)y(x) + \sum_{i=2}^{n} f_i(x)y^{a_{i-1}/n}(x), \tag{9}$$

where the a's are each less than n, a positive integer. Substituting $y=u^{-n}$, we obtain from (9)

$$-nu' = f_1 u + f_0 u^{n+1} + \sum_{i=2}^{n} f_i u^{n+1-a_{i-1}}, \tag{9.1}$$

which is typified by (1). We conclude by citing two examples:

Example (a). Let $y'=f_0+f_1y+\sum_{i=2}^{n} f_i y^{b_i}$, where $b_i=p_i/q_i$, p_i, q_i are integers prime to each other, the q_i are all positive, the p_i are all positive or all negative,

and $|p_i| < q_i$. Substituting $y = w^L$, where $L = q_1 q_2 q_3 \cdots q_n$, we obtain from the given equation

$$Lw' = f_1 w + f_0 w^{1-L} + f_2 w^{1+c_2-L} + \cdots + f_n w^{1+c_n-L},$$

where $c_i = Lb_i$. Now $c_i + 1 - L$ are all negative integers, and so is $1-L$. Hence this equation is typified by (6), and therefore can be transformed to the form (1). For instance, if $y' = f_0 + f_1 y + f_2 y^{2/3} + f_3 y^{3/4}$, the equation in w is $12w' = f_1 w + f_3 w^{-2} + f_2 w^{-3} + f_0 w^{-11}$, which by the substitution $w = 1/t$, becomes $-12t' = f_1 t + f_3 t^4 + f_2 t^5 + f_0 t^{13}$.

Example (b). Let $y' = f_0 + f_1 y + f_2 y^{p/q} + f_3 y^{-r/s}$, where p/q, r/s are proper fractions, and p, q, r, s are positive integers. The substitution $y = u^{-qs}$ transforms the given equation to $-qsu' = f_1 u + f_2 u^{1+s(q-p)} + f_3 u^{1+q(s-r)} + f_0 u^{1+qs}$, which is typified by (1).

The author is grateful to the referee for his comments.

ON THE METRIZABILITY OF INVERTIBLE SPACES

WILLIAM J. GRAY, University of Alabama

DEFINITION. *A topological space (X, t) is invertible iff $U \in t, U \neq \varnothing \Rightarrow$ there exists an onto homeomorphism $h\colon X \to X$ such that $h(X-U) \subset U$; h is an inverting homeomorphism for U.*

In [1] (where invertible spaces were introduced) and [2] it is shown, among other things, that if there exists $U \in t$, $U \neq \varnothing$, with property P, where P is a separation property T_i, $i = 0, 1, 2$, normality, or regularity, then X has property P. Here we show that if P is metrizability, then X is metrizable.

THEOREM. *If (X, t) is an invertible space containing an open metrizable subset $U \neq \varnothing$, then X is metrizable.*

Proof. Since U is T_1 and regular (T_3), so is X. Choose $x \in U$. By regularity we have V, W open in X such that $x \in V \subset \overline{V} \subset W \subset \overline{W} \subset U$. Let h be an inverting homeomorphism for V and $\beta = \bigcup_{i=1}^{\infty} B_i$ be a σ-discrete base for U, where each B_i is a discrete family of open subsets of U (see [3]). (An element of β is an open set which belongs to at least one of the B_i). Define

$$C_i = \{B \cap V;\ B \in B_i\} \qquad \gamma = \bigcup_{i=1}^{\infty} C_i$$

$$D_i = \{B \cap W;\ B \in B_i\} \qquad \delta = \bigcup_{i=1}^{\infty} D_i$$

$$A_i = \{h^{-1}(C) \cap X - \overline{V};\ C \in C_i\} \qquad \alpha = \bigcup_{i=1}^{\infty} [A_i \cup D_i]$$

γ and δ are σ-discrete bases for V and W, respectively. We plan to show that α is a σ-locally finite base for (X, t).

To show that α is a base, take $x \in S$, S open in X. If $x \in X-\bar{V}$, $h(x) \in h(S) \cap V$. We have $C \in \gamma$ such that $h(x) \in C \subset h(S) \cap V$. Then $x \in h^{-1}(C) \subset S \cap h^{-1}(V) \subset S$. Hence $x \in h^{-1}(C) \cap X-\bar{V} \subset h^{-1}(C) \subset S$. If $x \in \bar{V} \subset W$, $x \in S \cap W \Rightarrow$ for some $D \in \delta$, $x \in D \subset S \cap W \subset S$.

To show that α is a σ-locally finite base, take $x \in X$. We consider cases.

Case 1. If $x \in X-W$, we have $h(x) \in V$, and $S \subset V$ open, $h(x) \in S$ such that $S \cap C \neq \varnothing$ for at most one $C \in C_i$ for each i. Then $x \in h^{-1}(S)$, and $h^{-1}(S) \cap h^{-1}(C) \neq \varnothing$ for at most one $C \in C_i$, for each i. If $x \notin \bar{W}$, $h^{-1}(S) \cap X-\bar{W}$ can intersect at most one member of $A_i \cup D_i$. If $x \in \bar{W}$, choose an open set $T \subset U$, $x \in T$, $T \cap B \neq \varnothing$ for at most one $B \in B_i$ for each i. Then $h^{-1}(S) \cap T$ intersects at most two members of $A_i \cup D_i$, for each i.

Case 2. If $x \in V$, we have $S \subset W$ open, $x \in S$, $S \cap D \neq \varnothing$ for at most one $D \in D_i$ for each i. Then $x \in S \cap V$, and the last set can intersect at most one member of $A_i \cup D_i$ for each i.

Case 3. If $x \in W-V$, $h(x) \in V \Rightarrow$ we have S open, $h(x) \in S$ such that $S \cap C \neq \varnothing$ for at most one $C \in C_i$ for each i. Then $x \in h^{-1}(S)$ and $h^{-1}(S) \cap h^{-1}(C) \neq \varnothing$ for at most one $C \in C_i$ for each i. Furthermore, we have $T \subset W$ open, $x \in T$, and $T \cap D \neq \varnothing$ for at most one $D \in D_i$. Then $h^{-1}(S) \cap T$ intersects no more than two members of $A_i \cup D_i$ for each i.

Since X has a σ-locally finite base, and is T_3, X is metrizable.

References

1. P. H. Doyle and J. G. Hocking, Invertible spaces, this Monthly, 68 (1961) 959–965.
2. Norman Levine, Some remarks on invertible spaces, this Monthly, 70 (1963) 181–183.
3. J. L. Kelley, General Topology, Van Nostrand, New York, 1955.

ON A GENERAL AREA-PERIMETER RELATION FOR TWO-DIMENSIONAL LATTICES

Joseph Hammer, University of Sydney

Edward A. Bender has proved in [1] that any convex domain D whose area $A(D)$ is greater than its half perimeter $S(D)$, that is

$$A(D) > S(D), \tag{1}$$

contains a lattice point.

In this note we shall prove a more general theorem.

Theorem. *If*

$$A(D) > rS(D), \tag{2}$$

where r is any positive integer, then D contains r lattice points.

To prove this, we shall use the following lemma.

Lemma. *Any convex domain D whose area $A(D)$ is greater than n-times its half perimeter $S(D)$, where n is any positive integer, can be divided into n convex domains, such that each area is greater than the corresponding half perimeter.*

We prove it by induction. It is trivial for $n=1$. Supposing it to be true until $n=k$, we show it is true for $n=k+1$. By the inductive hypothesis we have

$$A(D) > (k+1)S(D). \tag{3}$$

Let us cut the domain into two parts (Fig. 1), with areas $A_1(D) = A(D)/(k+1)$ and $A_2(D) = kA(D)/(k+1)$. From (3)

$$A_1(D) = \frac{1}{k+1} A(D) > S(D) \tag{4}$$

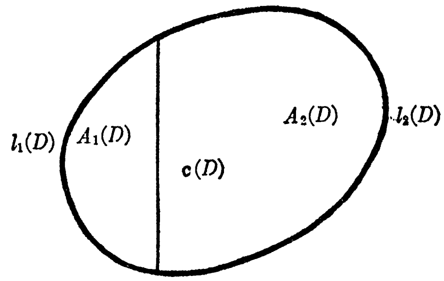


FIG. 1

But (Fig. 1)

$$S(D) > \tfrac{1}{2}[l_1(D) + c(D)] = S_1(D) \tag{5}$$

since the domain is convex. Therefore (4) becomes $A_1(D) > S_1(D)$. Similarly, by (5)

$$A_2(D) > kS(D) > k\tfrac{1}{2}[l_2(D) + c(D)] = kS_2(D).$$

But, by supposition, $A_2(D)$ can be divided into k parts in the required way. Thus our statement is true for any n. By applying this lemma to Bender's theorem we prove our theorem.

Reference

1. Edward A. Bender, Area-perimeter relations for two-dimensional lattices, this MONTHLY, 69 (1962) 742–744.

OPAQUE SETS OF DEGREE α

R. E. D. JONES, University of Wichita and The Boeing Company

A set of points in the closed unit square of the Cartesian plane is said to be *opaque* if the set casts a shadow in every direction. Thus if we write $Q = \{(x, y): 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1\}$, a subset A of Q is opaque if and only if every straight line L satisfying $L \cap Q \neq \emptyset$ also satisfies $L \cap A \neq \emptyset$.

The set consisting of the union of the two diagonals of Q is opaque and has

linear measure $2\sqrt{2}$. The following example shows that there are linearly measurable opaque sets with linear measure less than $2\sqrt{2}$.

Example. Let $f(P)$ be the sum of the distances from P to each of the points $(0, 1)$, $(1, 1)$ and $(1, 0)$. The minimum value of this function is

$$f(\overline{P}) = \frac{\sqrt{2} + \sqrt{6}}{2} \cdot$$

The set consisting of the three closed line segments connecting $\overline{P}$ and $(0, 1)$, $\overline{P}$ and $(1, 1)$, $\overline{P}$ and $(1, 0)$ and the closed line segment connecting $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ is opaque and has linear measure $\sqrt{2}+\frac{1}{2}\sqrt{6}$.

In his thesis the author has shown that the linear measure of a linearly measurable opaque set must be at least two. This improves the previously known bound of $\pi/2$ (cf. [1]). It is not known whether or not there exists a linearly measurable opaque set with linear measure μ such that $2 \leqq \mu < \sqrt{2}+\sqrt{6}/2$.

Three properties of opaque sets which follow easily from the definition are:

1). If A is an opaque set, then the derived set of A, $A' = \{p : p \text{ is a limit point of } A\}$, is an opaque set.

2). If A is a closed opaque set then there is a set $B \subset A$ such that B is a perfect opaque set.

3). If A_n are nested closed opaque sets, then $\bigcap_n A_n$ is an opaque set.

Properties 2) and 3) do not necessarily hold if the set A in 2) is not closed and the sets A_n in 3) are not closed.

We will write Q^o for the interior of the closed unit square, that is, $Q^o = \{(x, y) : 0 < x < 1, 0 < y < 1\}$. If B is any set or ordinal number, $|B|$ will denote its cardinal number.

If $A \subset Q$ and α is a nonzero cardinal number, we say that A is *opaque of degree* α in case (i) A is an opaque set, and (ii) if L is any line which intersects Q^o, then $|L \cap A| = \alpha$.

The sets $Q - Q^o$ and Q are opaque sets of degree two and c respectively. It is clear that there are no opaque sets of degree one or of degree greater than c. The following theorem establishes the existence of opaque sets of degree α for any cardinal number α between two and c.

THEOREM. *For any cardinal number α, such that $2 \leqq \alpha \leqq c$, there exists an opaque set, A, of degree α.*

Proof. The set A will be defined by transfinite induction. There are c lines which intersect Q^o; well-order the set of these lines to form a transfinite sequence, $L_1, L_2, \cdots, L_\gamma, \cdots$, with the additional property that

$$\left| \bigcup_{\beta<\gamma} L_\beta \right| < c$$

for any line L_γ of the sequence. Suppose $A_\beta \subset L_\beta \cap Q$ has been defined for all $\beta < \gamma$ such that L_β contains exactly α points of $\bigcup_{\delta \leq \beta} A_\delta$ and at most α points of

$\bigcup_{\beta<\gamma} A_\beta$ are collinear. If L_γ contains exactly α points of $\bigcup_{\beta<\gamma} A_\beta$, let $A_\gamma=\emptyset$. If L_γ contains η points of $\bigcup_{\beta<\gamma} A_\beta$, with $\eta<\alpha$, then let A_γ be a subset of $L_\gamma \cap Q$ such that

$$\left| L_\gamma \cap \left(\bigcup_{\beta\leq\gamma} A_\beta \right) \right| = \alpha,$$

and such that at most α points of $\bigcup_{\beta\leq\gamma} A_\beta$ are collinear. Then the set

$$A = \bigcup_{\beta\in B} A_\beta, \qquad B = \{1, 2, \cdots, \gamma, \cdots\}$$

is an opaque set of degree α.

To complete the proof we need only show that for any $\gamma\in B$, $|\gamma|<c$, there exist points in $L_\gamma\cap Q$ which satisfy the conditions of the construction.

First suppose $\alpha=n$ is a positive integer greater than or equal to two. Let ν be the cardinality of all possible n-tuples, whose entries are elements of $\bigcup_{\beta<\gamma} A_\beta$. Then ν is greater than or equal to the cardinality of the set of distinct lines, each containing n points of $\bigcup_{\beta<\gamma} A_\beta$. From the ordering of the sequence

$$L_1, L_2, \cdots, L_\gamma, \cdots, \left| \bigcup_{\beta<\gamma} A_\beta \right| \leqq |\gamma|\, n < c.$$

If $\left|\bigcup_{\beta<\gamma} A_\beta\right|$ is not finite then $\nu=\left|\bigcup_{\beta<\gamma} A_\beta\right|^n=\left|\bigcup_{\beta<\gamma} A_\beta\right|<c$ while if $\left|\bigcup_{\beta<\gamma} A_\beta\right|$ is finite so is ν. Hence at most ν points of $L_\gamma\cap Q$ will not satisfy the conditions of the construction; therefore the construction is valid for any finite α.

If α is an infinite cardinal number, $\alpha<c$, we may choose any α points of $L_\gamma\cap Q$ for A_γ. Since if $\left|L\cap(\bigcup_{\beta<\gamma} A_\beta)\right|\leqq\alpha$, then $\left|L\cap(\bigcup_{\beta\leqq\gamma} A_\beta)\right|\leqq\alpha$.

Reference

1. F. Bagemihl, Some opaque subsets of a square, Mich. Math. J., 6 (1959) 99–103.

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland.

TOTAL VARIATION AND UNIFORM CONVERGENCE

DAVID A. SÁNCHEZ, University of Chicago

As is well known, given a real valued function $f(x)$, $a\leqq x\leqq b$, the total variation $V_a^b(f)$ of the function is a finitely additive interval function on the set of all intervals belonging to the domain of the function. Less well-known is the fact that the total variation is a lower semicontinuous functional; i.e., given $f_n(x)$,

$n=1, 2, \cdots$, converging pointwise to $f(x)$ for $a \leqq x \leqq b$, then

$$\bigvee_a^b (f) \leqq \liminf_{n\to\infty} \bigvee_a^b (f_n).$$

For a proof of this and other properties of the total variation see [1].

By restricting ourselves to the class consisting of functions $f(x)$, $-\infty < x < \infty$, of bounded variation, we obtain a similar result. Suppose that $f_n(x)$, $n=1, 2, \cdots$, converge pointwise to $f(x)$, $-\infty < x < \infty$. Then since $\bigvee_{-\infty}^{\infty}(f) < \infty$, given an arbitrary $\epsilon > 0$ we may choose an interval $[-k, k]$, $k=k(\epsilon)>0$, such that $\bigvee_{-\infty}^{\infty}(f) - \epsilon \leqq \bigvee_{-k}^{k}(f)$. The desired lower semicontinuity follows from the previous result.

Suppose now that we consider the class K consisting of functions $f(x)$, $-\infty < x < \infty$, of bounded variation. Then we shall distinguish two types of convergence in K:

1) Uniform convergence on $-\infty < x < \infty$.
2) Uniform convergence on every compact subset of $-\infty < x < \infty$, but nonuniform convergence on $-\infty < x < \infty$.

The following examples show the possible behavior of the total variation with respect to these two modes of convergence.

Example 1. Let $f_n(x) = x - n$ if $n \leqq x \leqq n + 1/2n$, $f_n(x) = -x + n$ if $n + 1/2n \leqq x \leqq n + 1/n$, $f_n(x) = 0$ otherwise, $n = 1, 2, \cdots$, and let $f(x) = 0$, $-\infty < x < \infty$. The convergence is of type 1), and furthermore

$$\bigvee_{-\infty}^{\infty} (f) = \lim_{n\to\infty} \bigvee_{-\infty}^{\infty} (f_n).$$

Hence in this case the total variation is convergent.

Example 2. Let $f_n(x) = x - 2^{-n}k$ if $n + 2^{-n}k \leqq x \leqq n + 2^{-n}k + 2^{-n-1}$, $f_n(x) = -x + 2^{-n}(k+1)$ if $n + 2^{-n}k + 2^{-n-1} \leqq x \leqq n + 2^{-n}(k+1)$, $f_n(x) = 0$ otherwise, $k = 0, 1, \cdots, 2^n - 1$, $n = 1, 2, \cdots$, and let $f(x) = 0$, $-\infty < x < \infty$. The convergence is of type 1), and

$$\bigvee_{-\infty}^{\infty} (f) = 0 < \liminf_{n\to\infty} \bigvee_{-\infty}^{\infty} (f_n) = 1.$$

Example 3. Let $f_n(x) = 0$ if $x \leqq n$, $f_n(x) = x - n$ if $n \leqq x \leqq n+1$, $f_n(x) = 1$ if $n + 1 \leqq x$, $n = 1, 2, \cdots$, and let $f(x) = 0$, $-\infty < x < \infty$. The convergence is of type 2), and

$$\bigvee_{-\infty}^{\infty} (f) = 0 < \liminf_{n\to\infty} \bigvee_{-\infty}^{\infty} (f_n) = 1.$$

We will now show that convergence of type 2) in K, and convergence of the total variation imply that the convergence is of type 1).

THEOREM. *Let* $f_n(x)$, $n=1, 2, \cdots$, *be of bounded variation in* $-\infty<x<\infty$. *Suppose that the* $f_n(x)$ *converge uniformly on every compact set to* $f(x)$, *of bounded variation in* $-\infty<x<\infty$, *and in addition* $\lim_{n\to\infty} V_{-\infty}^{\infty}(f_n)=V_{-\infty}^{\infty}(f)$. *Then the* $f_n(x)$ *converge uniformly to* $f(x)$ *on* $-\infty<x<\infty$.

Proof. Since $f(x)$ is of bounded variation, we may choose $k=k(\epsilon)>0$ so large that $V_{|x|\geqq k}(f)<\epsilon$ for arbitrary $\epsilon>0$. By convergence of the total variation we see that $|V_{-\infty}^{\infty}(f_n)-V_{-\infty}^{\infty}(f)|<\epsilon$ if n is large enough. By lower semicontinuity of the total variation we know that $-\epsilon<V_{-k}^{k}(f_n)-V_{-k}^{k}(f)$ for n large enough, and it follows that there exists a positive integer N_1 such that $n>N_1$ implies $V_{|x|\geqq k}(f_n)<4\epsilon$.

Since the convergence is uniform on $-k\leqq x\leqq k$, we know that there exists a positive integer N_2 such that $n>N_2$ and $x\in[-k, k]$ implies $|f_n(x)-f(x)|<\epsilon$. Then for $n>N=\max(N_1, N_2)$ and $x\in[k, \infty)$ we have $|f_n(x)-f(x)|<6\epsilon$, and a similar result holds for $x\in(-\infty, -k]$. But $\epsilon>0$ was arbitrary, and k depends only on $f(x)$, so we may assert the convergence is uniform on $-\infty<x<\infty$.

COROLLARY 1. *The conclusion of the theorem holds if the condition: "Given arbitrary* $\epsilon>0$, *there exists a positive integer* M *and a constant* $k>0$ *such that* $V_{|x|\geqq k}(f_n)<\epsilon$ for $n>M$" *is substituted for the condition* "$\lim_{n\to\infty} V_{-\infty}^{\infty}(f_n)=V_{-\infty}^{\infty}(f)$."

COROLLARY 2. *The conclusion of the theorem holds if the condition* "$\lim_{n\to\infty} V_{-\infty}^{\infty}(f_n-f)=0$" *is substituted for the condition:* "$\lim_{n\to\infty} V_{-\infty}^{\infty}(f_n)=V_{-\infty}^{\infty}(f)$."

Proof. Since $V_{-\infty}^{\infty}(f_n)-V_{-\infty}^{\infty}(f)\leqq V_{-\infty}^{\infty}(f_n-f)$, and the right hand expression can be made as small as desired for n large enough, it follows that $\limsup_{n\to\infty} V_{-\infty}^{\infty}(f_n)\leqq V_{-\infty}^{\infty}(f)$. But this implies that $\lim_{n\to\infty} V_{-\infty}^{\infty}(f_n)=V_{-\infty}^{\infty}(f)$ and the desired result follows.

Reference

1. L. Cesari, Variation, Multiplicity, and Semicontinuity, this MONTHLY, 65 (1958) 317–332.

A GENERALIZED INEQUALITY

NATHAN ALTSHILLER COURT, University of Oklahoma

1. THEOREM. *If* α, β *are two angles such that* $0<\alpha<\beta\leqq\pi/2$, *we have*

$$\frac{\sin\alpha}{\sin\beta}>\frac{\alpha}{\beta}. \tag{1}$$

Proof. On the x-axis OX mark the points A, B, X, so that $OA=\alpha$, $OB=\beta$, $OX=\pi/2$; draw the perpendiculars $AC=\sin\alpha$, $BD=\sin\beta$, $XL=1$, on the same side of OX (Fig. 1).

If I is the point of intersection of the lines AC, OD, we have

$$AI:BD=OA:OB, \quad\text{or}\quad AI:\sin\beta=\alpha:\beta. \tag{2}$$

Now the points O, C, D, L, lie on the curve $y=\sin x$, which in the first quadrant

is convex; hence the point I lies between A and C, and therefore $AC > AI$. Thus if in (2) we replace AI by $AC = \sin \alpha$, we obtain (1).

Observe that the formula (1), put in the form

$$\frac{\sin \alpha}{\alpha} > \frac{\sin \beta}{\beta} \tag{3}$$

shows that $f(\alpha) = \sin \alpha / \alpha$ is a *diminishing* function in the first quadrant.

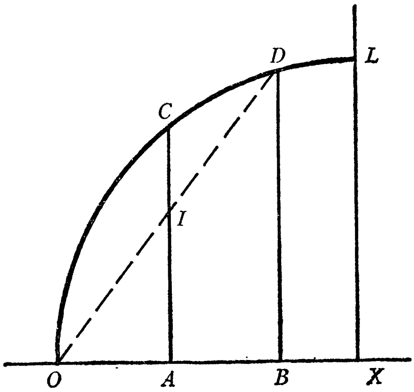


FIG. 1

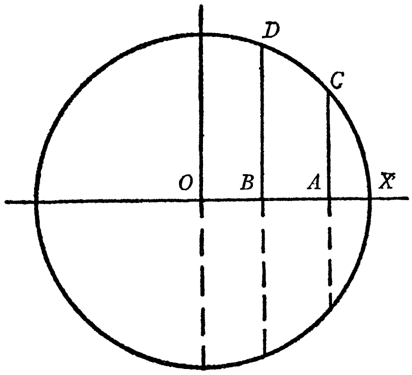


FIG. 2

2. A special case. If in (3) we put $\beta = \pi/2$, the formula becomes

$$\frac{\sin \alpha}{\alpha} > \frac{2}{\pi} \cdot \tag{4}$$

This formula finds applications in the theory of functions. See, for instance, E. C. Titchmarsh, *Theory of Functions*, Oxford, 1932, p. 7. Einar Hille, *Analytic Functions*, Ginn and Co., New York, 1958, vol. 1, p. 248.

3. A geometrical inequality. The inequality (1) remains valid if its four terms are multiplied by 2. The terms $2 \sin \alpha$, $2 \sin \beta$ may be interpreted as being the chords of the arcs 2α, 2β, respectively, in the same circle.

Geometrically this amounts to adjoining to the first quadrant of the Fig. 2 its reflection in the x-axis. We have the following result:

Given two unequal chords of a circle, the value of the ratio of the shorter chord to the longer is greater than the value of the ratio of the minor arcs subtended respectively by the given chords.

The proposition remains valid if one of the two chords is a diameter of the given circle.

Furthermore, the proposition is applicable to two chords of a sphere or two chords of two equal spheres, and the minor arcs subtended by the chords on the great circles passing through those chords.

ELEMENTARY PROOFS FOR THE EQUIVALENCE OF FERMAT'S PRINCIPLE AND SNELL'S LAW

MICHAEL GOLOMB, Purdue University

In a recent SMSG monograph by Beckenbach and Bellman [1] Fermat's minimum time principle for the path of a reflected ray of light is verified by means of an elementary inequality, and also by the well-known geometric construction ascribed to Heron. For the path of a refracted ray the authors state that verification of the minimum time principle does not seem to be obtainable in a similar fashion and that recourse must be made to the differential calculus. Below we give an arithmetic and a geometric proof for the law of refraction which are of a similar character to those given in [1] for the law of reflection.

Suppose that a plane F divides two homogeneous media M_1 and M_2 of a different optical density so that in M_1 light rays travel with velocity v_1 and in M_2 they travel with velocity v_2. Fermat's principle says that the path of a ray from a point P_1 in M_1 to a point P_2 in M_2 is such that it consumes less time than any other path connecting P_1 and P_2. It is clear that only those paths need to be considered which lie in a plane G perpendicular to F and which consist of two line segments P_1Q, QP_2 with $Q\in F\cap G$. Make $F\cap G$ the x axis of a Cartesian coordinate system and let (x_1, y_1), (x_2, y_2), $(x, 0)$ be the coordinates of P_1, P_2, Q, respectively, with $x_1<x_2$, $y_1>0$, $y_2<0$. The problem is to minimize the function

$$t(x) = \frac{1}{v_1}\sqrt{\{(x-x_1)^2+y_1^2\}} + \frac{1}{v_2}\sqrt{\{(x_2-x)^2+y_2^2\}}.$$

Let θ_1, θ_2 be any real numbers. Then, by Cauchy's inequality,

$$\pm(x_i - x)\sin\theta_i \pm y_i\cos\theta_i \leqq \sqrt{\{(x_i-x)^2+y_i^2\}} \qquad (i = 1, 2).$$

Therefore

$$t(x) \geqq \frac{(x-x_1)\sin\theta_1 + y_1\cos\theta_1}{v_1} + \frac{(x_2-x)\sin\theta_2 - y_2\cos\theta_2}{v_2}. \tag{1}$$

If we now require that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} \tag{2}$$

then the lower bound in (1) is independent of x, and it is attained for $x=x^*$, i.e. $t(x^*)$ is the minimum time for any path, if equality holds in (1) for $x=x^*$. This is the case if and only if

$$\begin{aligned} \sin\theta_1 &= \frac{x-x_1}{\sqrt{\{(x-x_1)^2+y_1^2\}}}, & \cos\theta_1 &= \frac{y_1}{\sqrt{\{(x-x_1)^2+y_1^2\}}} \\ \sin\theta_2 &= \frac{x_2-x}{\sqrt{\{(x_2-x)^2+y_2^2\}}}, & \cos\theta_2 &= \frac{-y_2}{\sqrt{\{(x_2-x)^2+y_2^2\}}}. \end{aligned} \tag{3}$$

It is seen that as x varies in (3) from x_1 to x_2, $\sin\theta_1$ increases monotonically from 0 to $u_1=(x_2-x_1)/\sqrt{\{(x_2-x_1)^2+y_1^2\}}$ and $\sin\theta_2$ decreases monotonically from $u_2=(x_2-x_1)/\sqrt{\{(x_2-x_1)^2+y_2^2\}}$ to 0. Thus there is exactly one set of values $x=x^*$, $\theta_1=\theta_1^*$, $\theta_2=\theta_2^*$, with $x_1<x^*<x_2$, $0<\theta_1^*<\arcsin u_1$, and $0<\theta_2^*<\arcsin u_2$, for which (2) and (3) are satisfied. If Q_* denotes the point $(x^*, 0)$ then by (3) θ_i^* is the angle between P_iQ_* and the normal to F $(i=1, 2)$, and (2) expresses Snell's law of refraction.

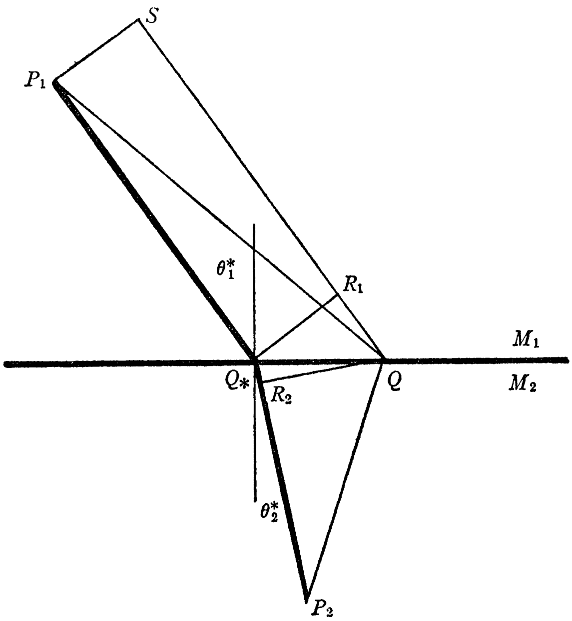


FIG. 1

The following geometric proof goes back to Huygens [2]. Referring to the Figure (with $v_1>v_2$ and $x>x^*$), let SQ be parallel to P_1Q_*, SP_1 and R_1Q_* perpendicular to P_1Q_*, and R_2Q perpendicular to P_2Q_*. Then

$$\frac{R_1Q}{R_2Q_*}=\frac{\sin\theta_1^*}{\sin\theta_2^*}=\frac{v_1}{v_2}.$$

Hence, if $|AB|$ denotes the time to traverse the segment AB,

$$|R_1Q| = |R_2Q_*|$$

and

$$\begin{aligned}|P_1Q_*| + |Q_*P_2| &= |SR_1| + |R_1Q| - |R_2Q_*| + |Q_*P_2|\\ &= |SQ| + |R_2P_2|\\ &\leqq |P_1Q| + |QP_2|.\end{aligned}$$

Equality holds here if and only if $Q=Q_*$. In a similar way one obtains this inequality for $x<x^*$. Thus the minimum time property of the path $P_1Q_*\cup Q_*P_2$ is proved.

References

1. E. Beckenbach and R. Bellman, An Introduction to Inequalities, New Mathematical Library, Random House, New York, 1961.
2. C. Huygens, Treatise on Light, translated by S. P. Thompson, Dover, New York, 1962.

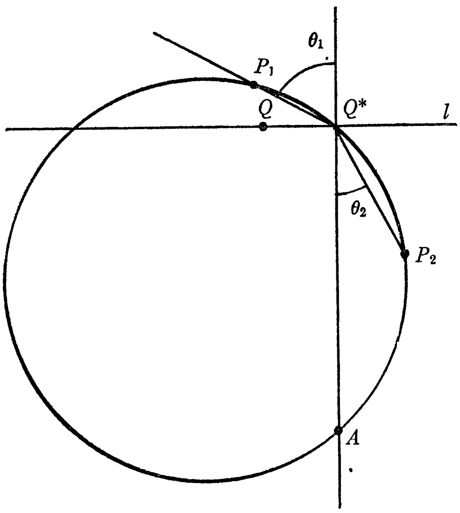
A GEOMETRIC PROOF OF THE EQUIVALENCE OF FERMAT'S PRINCIPLE AND SNELL'S LAW

DANIEL PEDOE, Purdue University

Let l be the line separating the media, and let $P_1Q^*P_2$ be the actual path of the ray according to Snell's Law, where $\sin\theta_1/\sin\theta_2=v_1/v_2$. We wish to show that for any other point Q on the line

$$P_1Q/v_1 + P_2Q/v_2 > P_1Q^*/v_1 + P_2Q^*/v_2,$$

so that the time taken for traversing the actual path is a minimum. Draw the circle through the points P_1, Q^* and P_2, and let the perpendicular to l through Q^* intersect this circle again at the point A. Then we note that $AP_1=2R\sin\theta_1$, and $AP_2=2R\sin\theta_2$, where R is the radius of the circle $P_1Q^*P_2$. Therefore $AP_1/AP_2=v_1/v_2$; that is, $AP_1=k/v_2$ and $AP_2=k/v_1$, where k is a constant.



Applying the theorem of Ptolemy to the four concyclic points P_1, Q^*, P_2 and A, we have the equality

$$P_1P_2\cdot AQ^* = P_1Q^*\cdot AP_2 + P_2Q^*\cdot AP_1,$$

whereas if Q is any point $\neq Q^*$ on the line l, the extension of the Ptolemy theorem which arises naturally by inverting the triangle inequality [1] gives the inequality

$$P_1P_2\cdot AQ < P_1Q\cdot AP_2 + P_2Q\cdot AP_1.$$

If we substitute for AP_1 and AP_2, we obtain the equality

$$k(P_1Q^*/v_1 + P_2Q^*/v_2) = P_1P_2\cdot AQ^*,$$

and the inequality

$$k(P_1Q/v_1 + P_2Q/v_2) > P_1P_2\cdot AQ.$$

Hence the Fermat principle of minimum time is established for a ray which satisfies Snell's Law, since $AQ > AQ^*$.

Reference

1. D. Pedoe, Circles, Pergamon, London, 1957.

ON THE RIEMANN INTEGRAL IN TWO DIMENSIONS

SOLOMON MARCUS, University of Bucharest

A subdivision of an interval I consists of a finite number of intervals having at most edges or vertices in common, whose union is I.

In dealing with Riemann type of integrals on a two dimensional compact interval I, the norm of a subdivision σ of I is available in at least two manners: (a) If $|J|_A$ is the area of the interval J, then the norm $|\sigma|_A$ is the maximum $|J|_A$ for any J of σ; (b) if $|J|_S$ = maximum side length of J, then the norm $|\sigma|_S$ is the maximum of $|J|_S$ for all J of σ. A limit taken as $|\sigma|_S\to 0$ is weaker than a limit as $|\sigma|_A\to 0$, since any statement true for all σ such that $|\sigma|_A < d$ will be true for all σ such that $|\sigma|_S < \sqrt{d}$ (see [1], pp. 102–103).

It is well known that the norm customary in the theory of the Riemann integral in two dimensions is the norm $|\sigma|_S$, but no explanation of this fact is given. The avoidance of $|\sigma|_A$ can be justified as follows. If the norm $|\sigma|_A$ is used, then the theorem "*A continuous function on I is Riemann integrable on I*" will not be true. To prove this, let I be the unit square and let σ_n be the subdivision formed by n equidistant parallels to the x-axis. For instance, σ_3 is constituted by the segments (x, y) with $0\leqq x\leqq 1$ and $y=0$, $1/3$, $2/3$, 1. Consider now the function $f(x, y)=x$. If σ_n consists of the intervals $J_1, J_2, \cdots, J_i, \cdots, J_n$, then, by choosing $(\xi_i, \eta_i)\in J_i$ $(1\leqq i\leqq n)$, we obtain the Riemann sum

$$S_{\sigma_n} = \sum_{i=1}^{n} f(\xi_i, \eta_i)\cdot |J_i|_A = \sum_{i=1}^{n} \xi_i |J_i|_A.$$

Now consider a number ξ such that $0<\xi<1$ and, for each $i\leqq n$ and for each

n, put $\xi_i=\xi$. The corresponding Riemann sums are then equal to $\xi\cdot|I|_A$; we have

$$\lim_{n\to\infty} |\sigma_n|_A = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{S}_{\sigma_n} = \xi\cdot |I|_A.$$

But the result is a function of ξ; thus, f is not integrable on I.

If, instead of $|\sigma|_A$, the norm $|\sigma|_S$ is used, then the property of uniform continuity yields Riemann integrability immediately. On the other hand, for the subdivisions σ_n considered above we have, for each n, $|\sigma_n|_S=1$; thus, $|\sigma_n|_S\not\to 0$ as $n\to\infty$.

It would be interesting to make a systematic investigation of the integral defined with the norm $|\sigma|_A$.

No essential modification arises when, instead of the norm $|\sigma|_S$, we use the norm $|\sigma|_D=$the maximum of $|J|_D$ for all J of σ, where $|J|_D$ is the diameter of J. These two norms can be considered as equivalent, since $|\sigma_n|_S\to 0$ implies $|\sigma_n|_D\to 0$ and conversely.

Reference

1. T. H. Hildebrandt, Introduction to the Theory of Integration, Academic Press, New York, London, 1963.

ANOTHER APPROACH TO THE ALTERNATING SUBGROUP OF THE SYMMETRIC GROUP

CLIFFORD E. WEIL, Princeton University

Let σ be a permutation of $\{1, \cdots, n\}$, and define $\phi(\sigma)$ by

$$\phi(\sigma) = \prod_{1\leq i<j\leq n} \frac{\sigma(j) - \sigma(i)}{|\sigma(j) - \sigma(i)|}.$$

The number $\phi(\sigma)$ may also be expressed in the following ways:

1) if τ is any permutation of $\{1, \cdots, n\}$, then

$$\phi(\sigma) = \prod_{\tau(i)<\tau(j)} \frac{\sigma(\tau(j)) - \sigma(\tau(i))}{|\sigma(\tau(j)) - \sigma(\tau(i))|}$$

2) if k is the number of times that $1\leqq i<j\leqq n$ implies $\sigma(j)<\sigma(i)$, then

$$\phi(\sigma) = (-1)^k.$$

From 2) it is seen that ϕ maps S_n—the group of all permutations of $\{1, \cdots, n\}$ —into the set $\{1, -1\}$. We show now that ϕ is a homomorphism into the multiplicative group $\{1, -1\}$. If σ and τ are in S_n, and if $\phi(\tau)=(-1)^k$, then

$$\phi(\sigma\circ\tau)=\prod_{1\leqq i<j\leqq n}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{|\sigma(\tau(j))-\sigma(\tau(i))|}$$
$$=\left[\prod_{\substack{\tau(i)<\tau(j)\\ i<j}}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{|\sigma(\tau(j))-\sigma(\tau(i))|}\right]\left[\prod_{\substack{\tau(j)<\tau(i)\\ i<j}}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{|\sigma(\tau(j))-\sigma(\tau(i))|}\right]$$
$$=\left[\prod_{\substack{\tau(i)<\tau(j)\\ i<j}}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{|\sigma(\tau(j))-\sigma(\tau(i))|}\right]\left[\prod_{\substack{\tau(j)<\tau(i)\\ i<j}}\frac{\sigma(\tau(i))-\sigma(\tau(j))}{|\sigma(\tau(i))-\sigma(\tau(j))|}\right](-1)^k$$
$$=\left[\prod_{\substack{\tau(i)<\tau(j)\\ i<j}}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{|\sigma(\tau(j))-\sigma(\tau(i))|}\right]\left[\prod_{\substack{\tau(i)<\tau(j)\\ j<i}}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{|\sigma(\tau(j))-\sigma(\tau(i))|}\right]\phi(\tau)$$
$$=\prod_{\tau(i)<\tau(j)}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{|\sigma(\tau(j))-\sigma(\tau(i))|}\phi(\tau)$$
$$=\phi(\sigma)\phi(\tau).$$

The alternating subgroup A_n of S_n is the kernel of the homomorphism. Thus A_n is a normal subgroup. The permutations in A_n are called even; the others are called odd. That is, if σ is even, then $\phi(\sigma)=1$, and if σ is odd, $\phi(\sigma)=-1$. Since ϕ is a homomorphism, the composition of two even permutations is even, of two odd permutations is even, and of an odd permutation and an even permutation is odd. For $n\geqq 2$ it is easy to construct a permutation σ for which $\phi(\sigma)=-1$. From properties of homomorphisms, the number of even permutations is half the number of permutations in S_n.

CONTOUR INTEGRATION FOR RATIONAL FUNCTIONS

ERWIN JUST AND NORMAN SCHAUMBERGER, Bronx Community College of The City University of New York

Let $P(Z)$ and $Q(Z)$ be polynomials (in the complex variable z) of degrees m and n respectively with lead coefficients a and b respectively. If C is a simple closed path containing the poles of $P(z)/Q(z)$ in its interior, it is known that

$$\int_C [P(z)/Q(z)]dz = 2\pi ia/b \qquad \text{if } n-m=1,$$

and

$$\int_C [P(z)/Q(z)]dz = 0 \qquad \text{if } n-m\geqq 2,$$

(where the integral is taken along C in the positive sense). The purpose of this note is to present a simple proof of these assertions.

It is assumed that $P(z)$ and $Q(z)$ have no zeros in common. If the zeros z_i of $Q(z)$ occur with multiplicity j_i $(i=1, 2, \cdots, r)$, then the decomposition of

$P(z)/Q(z)$ into partial fractions yields

$$P(z)/Q(z) = \sum_{i=1}^{r} \sum_{k=1}^{j_i} A_{ik}/(z - z_i)^k$$

from which it follows that

$$\lim_{|z| \to \infty} zP(z)/Q(z) = \sum_{i=1}^{r} A_{i1},$$

where it is to be observed that A_{i1} is the residue of $P(z)/Q(z)$ with respect to the pole z_i. It follows that if $n \geqq m+2$, then

$$\int_C [P(z)/Q(z)]dz = 2\pi i \sum_{i=1}^{r} A_{i1} = 2\pi i \lim_{|z| \to \infty} zP(z)/Q(z) = 0.$$

On the other hand, if $n = m+1$, then

$$\int_C [P(z)/Q(z)]dz = 2\pi i \sum_{i=1}^{r} A_{i1} = 2\pi i \lim_{|z| \to \infty} zP(z)/Q(z) = 2\pi i a/b.$$

It may be noted, further, that if $m \geqq n$ then $P(z)/Q(z) = F(z) + R(z)/Q(z)$, where $F(z)$ is a polynomial and $R(z)$ is a polynomial of degree $n-1$. Since $F(z)$ has no poles $\int_C [P(z)/Q(z)]dz = \int_C [R(z)/Q(z)]dz$, and the latter integral has already been considered.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

MATHEMATICS FOR THE ELEMENTARY EDUCATION MAJOR AT THE UNIVERSITY OF MARYLAND

HELEN L. GARSTENS, University of Maryland

Every student of the University of Maryland preparing to teach at the elementary level fulfills a requirement of eight hours of mathematics. These eight hours constitute a one year course (four hours per semester) and are taken in addition to the customarily provided methods course in arithmetic. The course was conceived, prepared and written by the staff of the University of Maryland Mathematics Project. The whole activity received the fullest cooperation from the College of Education and the Department of Mathematics of the University, in addition to the support of the National Science Foundation.

The philosophy underlying the course does not provide simply for the presentation and memorization of a large mass of material to familiarize the student with the mathematics beginning to appear in the elementary texts of today. To be sure, it is hoped that each student will reach a level of competence such that he will be able to read with discrimination present-day texts. This level of competence is meant to be achieved, however, by a better understanding of the foundations of the mathematics he will be expected to teach, by fostering an appreciation of how mathematics is developed, by providing experiences in developing deductive mathematical systems, by creating a classroom atmosphere in which the student participates in the process of determining the direction which the investigations will take, by presenting logical techniques which yield clues for establishing proofs. The mathematics around which these goals are draped is taken from the study of number theory and elements of geometry, since these studies underlie computation and measurement which form the essential body of mathematical knowledge pursued in grades K-6.

Two texts have been prepared for the course: *Mathematics for the Elementary School Teacher*, Book I (which deals primarily with the development of number systems), and Book II (which considers topics in geometry). The chapter headings for the texts follow:

Book I. I. Induction and Deduction; II. The System of Natural Numbers; III. Mathematical Systems; IV. Groups and Fields; V. The System of Integers; VI. The System of Rational Numbers; VII. Topics in Number Theory.

Book II. I. Sets of Points; II. Additional Topics in Logic; III. Congruence; IV. Linear and Angle Measure; V. Area and Volume; VI. Similarity; VII. Graphs on a Plane.

No listing of chapter headings can possibly indicate anything more than the skeleton on which the flesh of the course is shaped. It is not the goal of the Maryland course to develop creative mathematicians, but, rather, to prepare teachers who will be able to identify mathematical potential among their pupils, and will provide an atmosphere in which this potential can flourish. It is hoped that the teachers who have experienced this year of mathematics will not require their pupils to memorize a myriad of unrelated rules, but, rather, seek patterns and make generalizations which will facilitate achievement of the target activities for K-6.

In line with this attitude, the students are not presented with a series of proofs which they are called upon to reproduce periodically. Rather, the stress is placed on "letting the student in" on how one goes about searching for clues to establish a conjecture, surmised as a result of experimentation or intuition. The little example of a deductive system, based on the Peano Axioms, used to develop certain properties of the system of natural numbers is a stunning experience for a beginning student not highly motivated in the study of mathematics. Almost one hundred per cent of the students who enter the course think it is "natural" for $2+7$ to be equal to $7+2$. They would not be so ready to ascribe the property of commutativity under addition to any two irrational numbers.

In this situation, they would insist on some sort of demonstration of why this should happen. They are unaware, however, of the structure underlying the system of natural numbers and attribute all manner of properties to these numbers on the basis of "common sense." Unfortunately, their "common sense" also leads them to the "obvious" conclusion that there are half as many odd numbers as there are natural numbers!

Once the student realizes how productive an arithmetic one can engage in, when one may use a number system whose structure is that of a field, then motivation is high for the subsequent development of the system of integers and that of rationals. Since an integer is defined as an equivalence class of ordered pairs of natural numbers, the students are not sure whether the system will have an additive identity, or, if it exists, what form it will take. Instead of stating the theorem: "The additive identity for the system of integers is (n, n)," and following this with a series of steps in a synthetic proof, the student is encouraged to guess, try out his guess, make intelligent changes as a result of experimentation, or, possibly, assume the existence of the additive identity and attempt algebraically to find its form. Once a conjecture is made as to the possible form of the additive identity if it exists, the student proceeds to an analysis which provides clues for the steps required to make a deductive proof. The length or detail of the analysis depends upon the needs of the individual student. A reconsideration of the steps taken provides the clues for writing the proof.

Similarly, deductive systems are developed in the geometry as often as the material permits for this level of student. Many of the facts studied in Book II are familiar to those students who have taken a course in geometry in the secondary school. Frequently the facts were accepted as a result of single experiments, or because of their intuitive appeal. Theorems for which proofs were given became the gospel. Of the hundreds of students who have taken the Maryland course, none realized the nature and function of the axioms. The axioms were accepted as "true" because they seemed to fit in with the students' experiences in the physical world. The possibility that a preliminary definition might be modified if this modification led to significant generalization, is a thought which is at first shocking to the student. To dispel this untrue characterization of the rigidity of mathematics is one of the goals of the course in geometry.

The topics selected for the course in geometry are those which are basic to the understanding of direct and indirect measurement. Since the study of measurement in the Maryland course is based upon the concept of congruence, the topic of congruence is developed without dependence upon the notion of a measure. A congruence is a one-to-one correspondence characterized by a set of axioms. Every effort is made to keep the language consistent. Thus, there is a distinction between the statements that two line segments are congruent, and two line segments are equal.

The notion that measurement is approximate is developed so that the student realizes that the approximation is inherent in the nature of measurement,

and does not depend upon the fact that the observer may not have 20-20 vision! The difference between the measure of a triangle (its perimeter), and the measure of a triangular region (its area), is a distinction new to the students. The failure to differentiate between a triangle and its region results in having the students define the perimeter as "the distance around the triangle."

The geometry course is one which is newly developed and with which the staff of the University of Maryland Mathematics Project has had only two years of experience. Its content is therefore not as definitive as that of Book I with which there has been six years of experimentation. Many more units were written than it has been found possible to teach in the time allotted. Consequently, it is hoped that a third text will eventually be written.

No formal evaluation of the effects of the study of this course upon the teaching of mathematics in the elementary level has been made. There is, however, the enthusiasm of the students, their successful completion of the course, and the gratifying reports of the professors supervising the student teachers.

CHARACTERISTICS AND SERVICE LOADS OF MATHEMATICS AND SCIENCE TEACHERS

DONALD J. DESSART, University of Tennessee

Concurrent with numerous efforts to improve the secondary school mathematics and science curriculums and the subject matter backgrounds of teachers comes a timely report on characteristics and teaching loads of science and mathematics teachers. (See [1].) This survey conducted by the American Association for the Advancement of Science and the National Association of State Directors of Teacher Education and Certification, with the support of the National Science Foundation, revealed that much effort is still needed to attain standards of teacher preparation recommended by the Cooperative Committee on the Teaching of Science and Mathematics.

In the area of mathematics teacher preparation, the Cooperative Committee recommended 30 undergraduate and 15 graduate semester hours of college work. The survey showed that nearly four out of ten mathematics teachers in grades 7–8 had less than 9 semester hours of college mathematics, and that teachers with this minimal training were teaching 34 per cent of the 7th and 8th grade mathematics classes. In each of the three other categories ((a) 9–17 hours; (b) 18–29 hours; and (c) 30 or more hours) were two teachers out of ten.

The situation in grades 9–12 was much better. At this level nearly four out of ten mathematics teachers had 30 or more semester hours of college credit, which was considerably better than the situation in physics where about one out of ten teachers had comparable training. At the lowest level of training, less than 9 hours, fell about one out of ten of the mathematics teachers and nearly three out of ten physics teachers.

The sample studied in the survey was selected to represent all teachers of science and mathematics in public and private secondary schools of the United States during the spring of 1961. The U. S. Registry of Junior and Senior High School Science and Mathematics Teaching Personnel of the National Science Teachers Association was used to obtain a sample of 3957 teachers, about 3 per cent of the Registry. Of this sample, 3012 returned questionnaires. The number of mathematics teachers was 1782.

Findings for the entire sample of science and mathematics teachers revealed that:

1. Salaries ranged from below \$3000 to over \$10,000 with the median salary falling in the \$5000–\$5449 category.

2. The average workweek was 45 hours in which 23 were spent in teaching; 17 were devoted to preparation for teaching, paper grading, study hall supervision, athletics, dramatics, and band; and 5 were dissipated in bus duty and lunch room supervision.

3. About one out of five teachers had participated in at least one NSF institute.

4. Seventy per cent of classes were taught by full time teachers of the particular science or mathematics subject; 24 per cent by teachers who gave most of their time to the particular subject; and only six per cent were taught by teachers who gave most of their time to subjects unrelated to science or mathematics.

An examination of the statistics which applied to mathematics teachers in particular showed that:

a. Sixty-three per cent of teachers who taught mathematics but no other science were men, and seventy-five per cent of teachers who taught sciences as well as mathematics were men.

b. Teachers with only one or two classes in a particular subject were assumed to be devoting most of their teaching time to other classes. In mathematics only 12 per cent of the classes were taught by such teachers, whereas in physics this figure was 63 per cent. The statistics for chemistry, biology, and general science were 45, 30, and 25 per cent, respectively.

c. Of all the secondary school mathematics teachers the largest percentage, 59 per cent, taught mathematics at only the 9th to 12th grade levels, 24 per cent taught mathematics at only the 7th and 8th grade levels, and the smallest percentage, 16 per cent, taught mathematics at both levels.

d. Of all 7th and 8th grade mathematics classes, 34 per cent were taught by teachers who had less than 9 semester hours of college mathematics, 19 per cent by teachers who had 9–17 hours, 26 per cent by teachers who had 18–29 hours, and 21 per cent by teachers who had 30 or more hours.

e. Of all 9–12th grade mathematics classes, 11 per cent were taught by teachers who had less than 9 semester hours of college mathematics, 12 per cent by teachers who had 9–17 hours, 32 per cent by teachers who had 18–29 hours, and 45 per cent by teachers who had 30 or more hours.

f. In grades 9–12, almost half of the teachers of mathematics, chemistry, or physics received their bachelor's degrees since 1950, while about two-thirds of the teachers of mathematics 7–8 or biology received their degrees since 1950.

A very encouraging set of statistics revealed that among teachers who received their bachelor's degrees before 1950, 18 per cent of the mathematics 7–8 teachers and 34 per cent of the mathematics 9–12 teachers had taken mathematics courses since 1950. Among the teachers with bachelor's degrees taken during 1950–61, 20 per cent of the mathematics 7–8 teachers and 34 per cent of the mathematics 9–12 teachers had taken courses in their subjects since receiving their degrees. Undoubtedly, the National Science Foundation Summer Institute Program and Academic Year Institute Program had contributed to this commendable situation.

In addition to the continuation of National Science Foundation programs to maintain and improve the quality of teachers in the secondary school, there should be an increased effort to insure that prospective secondary school teachers receive adequate preparation in mathematics before beginning their teaching careers. For many secondary school teachers the decision to teach mathematics is often made after the completion of bachelor's degree work, rather than before, with the result that their mathematics work is done as inservice or summer study. Such a situation would seem to indicate that colleges and universities need to expend considerably more effort in attracting and encouraging high quality students to mathematics teaching as a career, while these people are still undergraduates.

Reference

1. Secondary School Science and Mathematics Teachers: Characteristics and Service Loads, U. S. Government Printing Office, Washington, D. C., 1963, 35 cents.

CAREER CHOICES

"The Future Teacher," a publication of the National Commission on Teacher Education and Professional Standards reports for the second year the career choices of 2620 finalists in the 1963 National Honor Society Scholarship Contest. The 1963 contest was the sixth consecutive contest in which teaching outranked all other professions in the eyes of the superior high school seniors. Science was the only field which interested more boys than did teaching. It was the second choice for girls. Percentage of career choices were as follows: teaching, 22%; science, 18%; medicine, 11%; engineering, 10%; mathematics, 8%; government, 5%; law, 5%; communications, 3%.

Other choices indicated that 10% were undecided and smaller numbers designated arts, business, ministry, psychology, social work, nursing, sociology, and dentistry.

SCIENCE TALENT SEARCH

On January 29, 1964 Science Service announced the names of the forty most prominent student scientists from the nation's high school seniors, as determined in the 23rd annual Science Talent Search. The forty students were chosen from among 3141 completely qualified entrants. There were ten girls and thirty boys from thirty-eight schools and thirty-six communities in seventeen states. The two schools with two winners were in New York at Forest Hills High School, and Erasmus Hall High School in Brooklyn. Requirements included taking the science aptitude examinations and writing a research report on an individual science project. School records and faculty recommendations were also taken into consideration.

Among the forty winners nine chose projects in mathematics. There was a tenth with a project in computer mathematics. Still another winner indicates that she wants to become a teacher of mathematics or science when her education is complete. Her topic for the science talent research was in the field of bacteriology. A copy of the honors and winners list of the Science Talent Search may be obtained by sending a stamped (ten cents) large envelope, self-addressed, to Science Service, 1719 N Street, N.W., Washington, D. C. 20036.

PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (other than proposers') should be sent to Professor Starke.

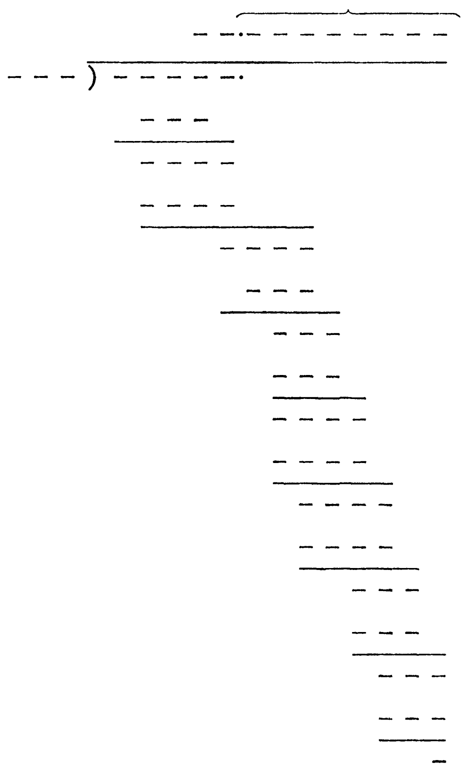
ELEMENTARY PROBLEMS

All solutions of Elementary Problems should be sent to A. E. Livingston, Dept. of Math., University of Alberta, Edmonton, Alberta, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before August 31, 1964.

E 1691. *Proposed by the late Harry Langman, Clarkson College of Technology*

In the following skeleton long division each dash represents a digit and the eight-digit block under the brace represents a repetend. Given that the dividend

is a multiple of three, reconstruct the division and show that it is unique.



E 1692. *Proposed by Roy Feinman, Rutgers University*

Let A and B be sets containing m and n elements respectively, with $1 \leqq m \leqq n$. How many relations are there from A to B whose domain is A and whose range is B?

E 1693. *Proposed by F. J. Servedio, Iona College, New Rochelle, N. Y.*

Given three concurrent circles such that their other three points of intersection are collinear, prove that their centers are concyclic with the point of concurrency.

E 1694. *Proposed by Omar Khayyam, Jr., University of California at Berkeley*

Let R, r, P, p respectively represent the circumradius, inradius, perimeter, and perimeter of the orthic triangle of a given acute triangle. Prove that $P/p = R/r$.

E 1695. *Proposed by J. B. Reynolds, Sugar Run, Pa.*

If the trisectors of the exterior angles of a triangle are drawn so that those adjacent to each side intersect, prove that the intersections are vertices of an equilateral triangle.

E 1696. *Proposed by D. I. A. Cohen, Princeton University, and Ralph Greenberg, University of Pennsylvania*

Prove that a polynomial $P(x)$ which assumes real values for real x and imaginary values for imaginary x must be linear.

E 1697. *Proposed by Harley Flanders, Purdue University*

Let $2^r \leqq n < 2^{r+1}$ and $0 \leqq k \leqq n-2^r$. Show that the binomial coefficients $\binom{n}{k}$ and $\binom{n-2^r}{k}$ have the same parity.

E 1698. *Proposed by D. L. Silverman, National Security Agency, Fort Meade, Md.*

Solve the equation $984+\sum_{k=1}^{10} p_k^2 = \prod_{k=1}^{10} p_k$, where the p_k are primes.

E 1699. *Proposed by Azriel Rosenfeld, Yeshiva University*

Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be permutations of $1, \cdots, n$. Prove that: (1) $a_1+b_1, \cdots, a_n+b_n$ can be incongruent modulo n if and only if n is odd, (2) $a_1b_1, \cdots, a_nb_n$ can never be incongruent modulo n if $n>2$.

E 1700. *Proposed by Jon Petersen, Saskatchewan Power Corporation, Regina, Saskatchewan*

Given that $T_0=1$, $T_{n+1}=\sum_{k=1}^{n} T_kT_{n-k}$. Express T_n as a function of n.

SOLUTIONS

Explanation of the Magistrate's Decision

E 1611 [1963, 757]. *Proposed by Peter Ungar, Courant Institute, New York University*

Magistrate: How do you know he had been speeding?

Officer: His car skidded 30 ft. with all four wheels locked, going up a 30° slope. I later made a test on the road in front of this courthouse, which is paved with the same material. I slammed on the brakes of the defendant's car at 60 mph and it skidded to a halt also in exactly 30 ft. But this road is level. Obviously he had greatly exceeded the speed limit of 60 mph going up the hill.

Magistrate (after consulting his slide rule): The charge of speeding is dismissed.

Explain the magistrate's decision.

I. *Solution by A. M. Glicksman, High School of Science, Bronx, New York.* If a car of weight W, coefficient of friction u, and initial speed v_0 skids a distance d along a level road, then $uWd = Wv_0^2/2g$, and $u=v_0^2/2gd$. Now if this car skids the same distance d up a like road of incline θ, its initial speed v is determined by

$$Wv^2/2g = (uW\cos\theta + W\sin\theta)d.$$

Solving for v and eliminating u we obtain $v=(v_0^2\cos\theta+2gd\sin\theta)^{1/2}$. Substitut-

ing $v_0=88$ ft/sec, $\theta=30°$, $g=32.2$ ft/sec^2, $d=30$ ft, we find $v\approx 87.6$ ft/sec $=59.7$ mi/hr, which is below the speed limit.

II. *Solution by Anton Glaser and Ralph Rush, Ogontz Campus of The Pennsylvania State University.* If the defendant's speed was $(88+x)$ ft/sec, then

$$(88 + x)^2 = (88)^2(\sqrt{3}/2) + 30g.$$

Since x is *negative* for $g<33$, the defendant was *not* speeding.

III. *Solution by Rory Thompson, San Diego State College.* To stop in 30 ft. from 60 mi/hr, the officer would have withstood a deceleration of at least (88 ft/sec)^2/(60 ft) $=129$ ft/sec$^2\approx 4$ g's. However, no coefficient of sliding friction is anywhere near 4, to say nothing of the effect on the driver. Therefore the officer was lying.

IV. *Solution by an unnamed student.* Politics.

Also solved by A. N. Aheart, Merrill Barnebey, Walter Bluger, Brother T. Brendon, Jay Burch, A. C. Claus, D. I. A. Cohen and S. P. Cohen (jointly), R. J. Eckert and G. A. Marxman (jointly), Michael Fried, Michael Goldberg, A. G. Grace, Jr., R. J. Herbert and D. T. Kexel and P. J. Welsh (jointly), R. H. Hines, Jr., J. E. Homer, Jr., R. A. Katz, R. N. Kesarwani, E. E. Kinerk, J. F. Leetch, Charles Lewis, D. C. B. Marsh, Morris Morduchow, P. R. Nolan, J. M. Pasachoff, Jon Petersen, Stanton Philipp, V. S. Poythress, E. H. Primoff, Anatol Rapoport, Peter Saltz, Perry Scheinok, John Shaw, Larry Shields, Kenneth Siler, D. L. Silverman, W. M. Stone, G. O. Temple, Simon Vatriquant, W. K. Viertel, and the proposer.

The Mistaken Professor

E 1612 [1963, 758]. *Proposed by C. S. Ogilvy, Hamilton College*

The exam question was: If $xy=4$, find d^2y/dx^2 and d^2x/dy^2. The student found d^2y/dx^2 correctly; he then wrote down its reciprocal and labeled that d^2x/dy^2—and it was right. The professor claimed that this was just luck, and that it could not happen with any other function. Was the professor right?

Solution by E. S. Eby, U. S. Navy Underwater Sound Laboratory, New London, Connecticut. The condition $(d^2y/dx^2)(d^2x/dy^2)=1$ leads to the differential equation $(d^2y/dx^2)^2+(dy/dx)^3=0$, whose general solution is $(x+a)(y+b)=4$. Except for a translation, the professor was right.

Also solved by A. N. Aheart, Frank Alden and Ann Bowlus (jointly), Merrill Barnebey, E. R. Barnes, Jerry Bebernes, P. L. Chessin, R. J. Cormier, R. J. Egbert, J. M. Elkin, J. A. Faucher, Michael Goldberg, F. L. Griffin, R. H. Hines, Jr., Erwin Just and Norman Schaumberger (jointly), R. A. Katz, T. S. Keck, Max Klicker, J. D. E. Konhauser, G. F. Lowerre, Robert Maas, D. C. B. Marsh, Morris Morduchow, W. I. Nissen, Jr., P. R. Nolan, M. J. Pascual, Richard Pavelle, Jon Petersen, Stanton Philipp, V. S. Poythress, E. H. Primoff, Lawrence Ringenberg, Alan Rosenberg, Lester Rubenfeld, M. S. R. K. Sastry, Perry Scheinok, D. L. Silverman, W. M. Stone, R. M. Summerville, D. T. Teodoro, H. W. Vayo, Andy Vince, Raymond Whitney, K. C. Williams, K. L. Yocom, and the proposer.

This is essentially Problem E 1046 [1953, 480], where it is pointed out that, in general, d^ny/dx^n and d^nx/dy^n are reciprocals if $(x+a)(y+b)=(n!)^{2/(n-1)}$.

Traveling Around a Circuit

E 1613 [1963, 758]. *Proposed by F. M. Sioson, University of Hawaii*

Towns T_i $(i=1, 2, \cdots, n)$ are cyclically connected by a one-way road, thus forming an n-gon. There are n means of transportation, $m_1, m_2, \cdots, m_n$, available, and under the ith means of transportation a mile is traveled in a_i minutes. If one starts from town T_j and goes around the circuit using the different means of transportation $m_1, m_2, \cdots, m_n$ consecutively, changing transportations only after each town, the circuit is completed in $\sum_{i=1}^{n} a_i - 2a_j$ hours. Find the length of the circuit.

Solution by E. L. Magnuson, HRB-Singer, Inc., State College, Pa. Let d_i be the distance from town T_i to the next town. A start from every town provides a system of n linear equations in the n unknowns $d_1, d_2, \cdots, d_n$. In matrix notation these equations can be written as

$$\begin{bmatrix} a_1 & a_2 \cdots a_n \\ a_n & a_1 \cdots a_{n-1} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_2 & a_3 \cdots a_1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ d_n \end{bmatrix} = \begin{bmatrix} 60(A - 2a_1) \\ 60(A - 2a_2) \\ \cdot \\ 60(A - 2a_n) \end{bmatrix},$$

where $A = \sum a_i$. The sum of all rows reveals that the length of the circuit, $\sum d_i$, is equal to $60(n-2)$ miles.

Also solved by D. A. Breault, D. I. A. Cohen, Michael Fried, Michael Goldberg, R. J. Herbert and D. T. Kefel and P. J. Welsh (jointly), Stephen Hoffman, Erwin Just and Norman Schaumberger (jointly), Kenneth Kramer, Loren C. Larsen, D. C. B. Marsh, Stephen Montague, J. W. Moon, P. N. Muller, P. R. Nolan, F. D. Parker, Jon Petersen, Stanton Philipp, E. H. Primoff, Perry Scheinok, Rory Thompson, Simon Vatriquant, Gary Venter, Andy Vince, Julius Vogel, K. S. Williams, K. L. Yocom, and the proposer.

The proposer pointed out that the special case $n=3$ of this problem appears in *College Algebra* by F. D. Perez and V. A. Tan (Manila: Macaraig Pub. Co., 1948).

A Projection of a Conic

E 1614 [1963, 758]. *Proposed by R. T. Hood, Franklin College, Franklin, Indiana*

A plane is intersected by a right circular cone and its axis. The resulting conic is projected onto a plane perpendicular to the axis. Show that the axis passes through a focus of the projected conic.

Solution by the proposer. Using both cartesian and polar coordinates, the equations of the cone and plane may be taken as $z=mr$ and $z=ax+b$. The projection of the conic on the xy-plane is then $mr=ax+b$. Converting fully to polar coordinates we have

$$r = b/(m - a\cos\theta),$$

which is a conic with focus at the origin. This conic is an ellipse, parabola, or hyperbola according as $a<m$, $a=m$, or $a>m$.

Also solved by Walter Bluger, W. G. Brady, Michael Goldberg, Loren C. Larson, D. C. B. Marsh, C. S. Ogilvy, Jon Petersen, Perry Scheinok, and Simon Vatriquant.

All solutions were analytic except that of Ogilvy.

An Isometric Mapping

E 1615 [1963, 758]. *Proposed by M. W. Pownall, Colgate University*

Define the distance between two integers i and j to be $d(i, j)=|i-j|$. Show that any one-to-one mapping of the set of integers onto itself which preserves the relation $d(i, j) \leqq k$, for some fixed positive integer k, is an isometry.

Solution by the proposer. An interval of length k in the set of integers may be characterized as a maximal collection in which each pair has distance $\leqq k$. The integers i, j are consecutive if and only if there exist two intervals I_1, I_2, each of length k, such that $I_1 \cap I_2=\{i, j\}$. Since the given mapping must permute the intervals of length k, it must preserve consecutiveness, and the theorem follows.

Also solved by D. I. A. Cohen, W. D. Jackson, Loren C. Larsen, Robert Maas, D. C. B. Marsh, Jon Petersen, D. L. Silverman, Andy Vince, and Raymond Whitney.

A Triangle Inequality Involving the Altitudes

E 1616 [1963, 758]. *Proposed by Leonard Carlitz, Duke University*

Show that in an acute triangle, $h_1+h_2+h_3 \leqq 3(R+r)$, where the h_i are the altitudes, R the circumradius, and r the inradius, and show that the equality sign holds only in the case of an equilateral triangle.

I. *Solution by A. N. Aheart, West Virginia State College.* Denote the angle bisectors of the triangle by t_1, t_2, t_3. Clearly $h_1+h_2+h_3 \leqq t_1+t_2+t_3$, with equality if and only if the triangle is equilateral. Now in Problem E 1573 [1964, 93] it is shown that $t_1+t_2+t_3$ never exceeds three times the sum d of the (signed) distances of the circumcenter from the three sides of the triangle, with equality if and only if the triangle is equilateral. But Carnot's Theorem (see Altshiller-Court, *College Geometry*, 2nd ed., p. 83) states that $d=R+r$. The desired result now follows.

II. *Solution by W. J. Blundon, Memorial University of Newfoundland.* Let H be the orthocenter of the triangle $A_1A_2A_3$ and let H_1, H_2, H_3 be the feet of the altitudes through A_1, A_2, A_3 respectively. It is well known (see Johnson, *Modern Geometry*, p. 191) that $\sum A_iH=2(R+r)$. Since the triangle is acute, H is an interior point and we may apply the Erdös-Mordell Theorem to the point H, obtaining $\sum A_iH \geqq 2\sum HH_i$, with equality only for an equilateral triangle. Thus

$$\sum h_i=\sum A_iH+\sum HH_i \leqq (3/2) \sum A_iH=3(R+r),$$

with equality if and only if the triangle is equilateral.

Also solved by J. W. Baldwin, Leon Bankoff, Michael Goldberg, Franz Leuenberger, Andrzej Makowski, D. C. B. Marsh, M. Perisastri, Stanton Philipp, P. D. Thomas, and the proposer.

Leuenberger had earlier (Einige Dreieckungsleichungen, *Elemente der Mathematik*, 13 (1958), 121–6)proved that $\sum h_i \leqq 9R/2$. Since $2r \leqq R$, the inequality of problem E 1616 is stronger.

A Quadrilateral Inequality

E 1617 [1963, 758]. *Proposed by J. I. Nassar, Socony Mobil Oil Company, Inc., Paulsboro, New Jersey*

Let A, B, C, D be any four points in the plane, and let PQ be the line segment joining the midpoints of BC and AD. Show that $|AB-CD| \leqq 2PQ \leqq AB+CD$, where one of the equalities holds if and only if AB is parallel to (or collinear with) CD.

I. *Solution by H. L. Chow, New York University.* Join P, Q to M, the midpoint of AC. Then $2PM=AB$, $2QM=CD$. Therefore

$$AB + CD = 2(PM + QM) \geqq 2PQ,$$
$$|AB - CD| = 2|PM - QM| \leqq 2PQ.$$

One of the equalities holds if and only if M falls on PQ, that is, if and only if AB is parallel to CD.

II. *Solution by M. J. Pascual, Watervliet Arsenal, New York.* Denoting the vector from A to B by $\overline{AB}$ we find that

$$\overline{PQ} = \overline{AB} + (\overline{DA} + \overline{BC})/2,$$

and, by a different route, we find

$$\overline{PQ} = \overline{DC} + (\overline{AD} + \overline{CB})/2.$$

Hence $\overline{PQ}=(\overline{AB}-\overline{CD})/2$, and since

$$|\overline{AB}| - |\overline{CD}| \leqq |\overline{AB} - \overline{CD}| \leqq |\overline{AB}| + |\overline{CD}|,$$

with equality holding if and only if $\overline{AB}$ and $\overline{CD}$ either coincide or are parallel, the result follows. Since the above proof does not depend upon the points being coplanar, this restriction is unnecessary.

III. *Solution by Stanton Philipp, Seal Beach, California.* Let A, B, C, D be represented by the complex numbers a, b, c, d in the complex plane. Then the statement to be proved reduces to

$$||a - b| - |c - d|| \leqq |(a - b) - (c - d)| \leqq |a - b| + |c - d|,$$

with one equality if and only if AB is parallel to or collinear with CD. But this follows immediately from the triangle inequality for complex numbers and its known corollaries.

Also solved by W. M. Angel, W. R. Boland, Leonard Carlitz, D. I. A. Cohen, Ragnar Dybvik, Michael Fried, Michael Goldberg, Stephen Hoffman, J. E. Jean, Jr., W. H. Jurney, Kenneth Kramer, Loren C. Larson, C. D. B. Marsh, P. R. Nolan, Jon Petersen, V. S. Poythress, Perry Scheinok, Rory Thompson, Simon Vatriquant, Andy Vince, Charles Wexler, and the proposer.

Integers Less Than n and Prime to n

E 1618 [1963, 759]. *Proposed by Ralph Greenberg, University of Pennsylvania*

Find all integers n such that the $\phi(n)$ integers less than n and prime to n are: (a) relatively prime in pairs, (b) in arithmetic progression.

Solution by D. C. B. Marsh, Colorado School of Mines. (a) Let p be the smallest prime not dividing n. Then n satisfies the conditions of the problem if and only if $p^2 \nless n$. If $2\nmid n$, then $n=1$ (trivially) or 3; if $2|n$ but $3\nmid n$, then $n=2$ (trivially), 4, or 8; if $(2)(3)|n$ but $5\nmid n$, then $n=6$, 12, 18, or 24; if $(2)(3)(5)|n$ but $7\nmid n$, then $n=30$. Bertrand's conjecture may be used to show that no other n exists. Therefore $n=1$, 2, 3, 4, 6, 8, 12, 18, 24, 30.

(b) With $1\leqq n$ and $(1, n)=1$, we consider only arithmetic progressions of the form 1, $1+d$, $1+2d$, $\cdots$ and obtain in a manner similar to that used in (a): $n=1$, 6, any odd prime, any positive integral power of 2.

Also solved by K. F. Bailie, Leonard Carlitz, D. I. A. Cohen, D. M. Danvers, Michael Fried, Jon Petersen, Stanton Philipp, N. R. G. Rao, H. J. Ricardo, D. L. Silverman, A. M. Vaidya, and the proposer.

A Matrix Similar to Its Negative

E 1619 [1963, 759]. *Proposed by H. Kestelman, University College, London*

Let f be any function with integer values and A any n by n matrix whose (r, s)th element is zero whenever $f(r)+f(s)$ is even. Prove that A and $-A$ are similar.

Solution by D. C. B. Marsh, Colorado School of Mines. Consider the diagonal n by n matrix P with $p_{jj}=-1$ if $f(j)$ is even and $p_{jj}=+1$ if $f(j)$ is odd; $P=P^{-1}$. For $A=(a_{rs})$, $P^{-1}AP=(p_{rr}a_{rs}p_{ss})$. Now if $f(r)$ and $f(s)$ are of opposite parity, $p_{rr}p_{ss}=-1$ and $p_{rr}a_{rs}p_{ss}=-a_{rs}$; while if $f(r)$ and $f(s)$ are of like parity, $a_{rs}=0$ and $p_{rr}a_{rs}p_{ss}=0=-a_{rs}$. Thus $P^{-1}AP=-A$, proving that A and $-A$ are similar.

Also solved by Leonard Carlitz, Loren C. Larson, E. L. Magnuson, Jon Petersen, and the proposer.

An Insoluble Diophantine Equation

E 1620 [1963, 759]. *Proposed by D. L. Silverman, Beverly Hills, California*

Solve $x^3=4y(xy+z^2)$ in nonzero integers x, y, z.

Solution by W. J. Blundon, Memorial University of Newfoundland. Putting $2xy+z^2=n$ and eliminating y we have $x^4+z^4=n^2$, which has no solution in nonzero integers (see Niven and Zuckerman, *An Introduction to the Theory of Numbers*, John Wiley and Sons, Inc., pp. 100–2).

Also solved by Joseph Arkin, J. W. Baldwin, Leonard Carlitz, M. L. Chachere, J. A. H. Hunter, J. E. Jean, Jr., Erwin Just and Norman Schaumberger (jointly), Frank Kocher, R. D. Leitch, Viktors Linis, E. L. Magnuson, D. C. B. Marsh, W. I. Nissen, Jr., Jon Petersen, Stanton Philipp, N. R. G. Rao, C. S. Stuckey, G. C. Thompson, Simon Vatriquant, Andy Vince, Ron Wilder, K. L. Yocom, Aleksandras Zujus, and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before November 30, 1964.

5200. *Proposed by T. J. Head, Iowa State University*

If a group G is the set theoretic union of a family of proper normal subgroups each two of which have only the identity in common, then G is abelian. (Abelian groups which are such unions were described by J. W. Young in: *On the partitions of a group and the resulting classification*, Bull. Amer. Math. Soc., 33 (1927), 453–461.)

5201. *Proposed by A. Wilansky, Lehigh University*

On p. 181 of Banach's book, an isomorphism between c and c_0 is given: these are the spaces of convergent and null sequences. Show that no such isomorphism (continuous or not) can be given by a matrix map $\{x_n\}\in c\rightarrow\{\sum_k a_{nk}x_k\}\in c_0$.

5202. *Proposed by David R. Hayes, Duke University*

Let K be a field, let n be a positive integer and let $\{h_{ij,k}: 1\leqq i, j, k\leqq n\}$ be a set of n^3 elements of K. Show that the system of n^2 equations

$$X_iX_j = \sum_{k=1}^{n} h_{ij,k}X_k \qquad 1 \leqq i, j \leqq n$$

has at most $n+1$ solutions in K.

5203. *Proposed by D. J. Newman, Yeshiva University and L. A. Shepp, Bell Telephone Laboratories*

Is $\sum_{n=1}^{\infty} (1/n) \sin (x/n)$ a bounded function of x on the whole line?

5204. *Proposed by Ralph Greenberg, University of Pennsylvania*

Let a be an integer >1, and let $q_1, q_2, \cdots$ be a sequence of integers satisfying $q_{n+1}>2q_n$ for $n=1, 2, \cdots$. Prove that $\prod_{n=1}^{\infty} (1+1/a^{q_n})$ is irrational.

5205. *Proposed by James Duemmel, Malmstrom Air Force Base, Montana*

For a nontrivial linear space X (over the real or the complex field) with algebraic dual X', let $T(X, M')$ denote the weak topology generated on X by the subspace M' of X'. Is there a pair (X, M') such that $T(X, M')$ is the discrete topology?

5206. *Proposed by Robert Spira, Duke University*

Show that $\sum_{m=2}^{n}(-1)^m\binom{n}{m}\log m$ is an increasing function of n.

5207. *Proposed by H. S. Shapiro, New York University, and D. J. Newman, Yeshiva University*

Let $P(z)$, $Q(z)$ be polynomials with complex coefficients and degree not exceeding n. Show that

$$\operatorname*{Max}_{|z|\leqq 1}\left|z^n - \frac{P(z)}{Q(z)}\right| \geqq 1.$$

5208. *Proposed by L. Carlitz, Duke University*

Let $\tau(n)$ denote the number of divisors of n and let $k\geqq 1$. Show that

$$\sum_{n=1}^{\infty}\frac{(\tau(n))^k}{n^s} = \zeta(s)\prod_{p} A_k(p^{-s}),$$

where the product is taken over all primes p, and $A_k(x)$ is defined by means of $A_0(x)=1$, $(A(x)+1)^k = xA^k(x)$, where, after expansion $A^k(x)$ is replaced by $A_k(x)$.

SOLUTIONS OF ADVANCED PROBLEMS

Local Compactness under Open Mapping

5056 [1962, 926; 1963, 1017]. *Proposed by C. W. Kohls and M. E. Mahowald, Syracuse University*

In the book *Topology* by J. G. Hocking and G. S. Young, a space is defined to be locally compact if each point belongs to an open set whose closure is compact. On page 72, it is asserted that local compactness is invariant under open mappings. Give an example to show that this need not be the case if the spaces involved are not Hausdorff spaces.

III. *Solution by P. S. Schnare, Louisiana State University in New Orleans.* While the previous solutions do satisfy the problem as stated, they are not in fact counterexamples of the theorem given by Hocking and Young. Overlooked is the convention (*Topology*, p. 13) that "mapping" means "continuous function." The functions in the published solutions are not continuous. The following example avoids this defect.

Let $X=\bigcup_{n=1}^{\infty} A_n$, where $A_n=\{(n, 1), \cdots, (n, n)\}$, with basis consisting of sets B such that for some n, $B\subset A_n$ and $(n, 1)\in B$. X is locally compact (since for each n, A_n is an open-closed compact set) and T_0. Let $Y=N$ with topology $\mathfrak{I}=\{A\subset Y: A=\emptyset \text{ or } 1\in A\}$. Y is T_0 but not locally compact, since the closure of every (nonempty) open subset of Y is the whole space, which is not compact. However, the function $f: X\to Y$ defined by $f(n, m)=m$ is continuous, open and surjective. (In fact f is a local homeomorphism of X onto Y.)

Exponential Iterates

5098 [1963, 445]. *Proposed by Richard M. Dudley, University of California, Berkeley*

Let $\exp_1(z) = \exp(z)$ and $\exp_{n+1}(z) = \exp(\exp_n(z))$, $n = 1, 2, \cdots$. Show that there exists a nonconstant entire function f of one complex variable such that for each positive integer n there exists an entire function f_n such that $f(z) = \exp_n(f_n(z))$.

I. *Solution by the proposer.* We define entire functions $f_{mn}(z)$, $n = 1, 2, \cdots, m = 1, 2, \cdots, n$, with $f_{mn}(z) = \exp(f_{m+1,n}(z))$, $m = 1, \cdots, n-1$, as follows. Let $f_{11}(z) = \exp(z)$. Given $f_{mk}(z)$ for $m \leqq k \leqq n$ with $|f_{1k}(0) - f_{1k}(\frac{1}{2})| > 1$ for $1 \leqq k \leqq n$, choose a positive integer b_n large enough so that $f_{nn}(z) + 2\pi b_n i \neq 0$ for $|z| \leqq n$. Using the power series for any branch of $\log(f_{nn}(z) + 2\pi b_n i)$ valid for $|z| \leqq n$, we can find a polynomial $P_n(z)$ such that

$$|\exp(P_n(z)) - f_{nn}(z) - 2\pi b_n i|$$

is so small for $|z| \leqq n$ that

$$|\exp_r(P_n(z)) - f_{n+1-r,n}(z)| < 1/2^n$$

for $|z| \leqq n$, $r = 2, \cdots, n$, and $|\exp_n(P_n(0)) - \exp_n(P_n(\frac{1}{2}))| > 1$. Set $f_{n+1,n+1}(z) = P_n(z)$ and $f_{m,n+1}(z) = \exp_{n+1-m}(P_n(z))$ for $m < n+1$.

Now, for fixed m, the sequence $\{f_{mn}(z)\}_{m \leqq n}$ converges uniformly on every disk; its limit f_m is therefore an entire function. By continuity, we have $f_m(z) = \exp(f_{m+1}(z))$, $m = 1, 2, \cdots$, and $|f_1(0) - f_1(\frac{1}{2})| > 1$. Thus none of the f_m is constant and hence any f_m satisfies our requirements since $f_m(z) = \exp_n(f_{n+m}(z))$, for all positive integers m and n.

II. *Solution by I. N. Baker, Imperial College of Science and Technology, London, England.* Such functions arise in the theory of iterations in connection with functional equations studied by Schröder (Mathematische Annalen 1870), Koenigs (Annales de l'École Normale Supérieure (3) Suppl. 1884), Poincaré (Sur une classe nouvelle de transcendantes uniformes, Oeuvres, vol. 4) and others. In H. Kneser, Reele analytische Lösungen der Gleichung $\phi(\phi(x)) = e^x$ und verwandte Funktional-gleichungen, J. reine u. angew. Math., 187 (1950), 56–67, we find the function $f(z)$, denoted $\chi^{-1}(z)$ by Kneser, with the properties: (i) $f(z)$ is entire, (ii) $f(0) = c$, where $c = 0.3181 \cdots + i(1.3372 \cdots)$ is a solution of $e^c = c$, (iii) $f'(0) = 1$, (iv) $f(cz) = \exp\{f(z)\}$. It follows that $f(z) = \exp f(z/c) = \exp_n\{f(z/c^n)\}$, so that $f(z)$ solves our problem with $f_n(z) = f(z/c^n)$. The growth of $f(z)$ is faster than that of any $\exp_n(z)$. The property (iv) may be used to define "fractional iterates" of $\exp(z)$, e.g. $\exp_{1/2}(z) = f\{c^{1/2}f^{-1}(z)\}$, at least in the neighborhood of c, although these fractional iterates are never entire functions [Baker, Math. Annalen, 129 (1955), 174–180, and Math. Zeitschrift, 69 (1958) 147].

Also solved by George M. Bergman.

Solutions of an Old Equation

5101 [1963, 571]. *Proposed by K. Mahler, The University, Manchester, England*

If m and n are algebraic numbers satisfying $m^n = n^m$, $mn(n-1)(m-1)(m-n) \neq 0$, prove that two coprime rational integers h and k exist such that

$$m = (h/k)^{k/(h-k)}, \qquad n = (k/h)^{h/(k-h)}, \qquad hk(h-k) \neq 0.$$

If, in addition, m and n are algebraic integers, show that h or k may be chosen equal to 1.

Solution by Robert Breusch, Amherst College. The equation $m = n^{m/n}$ with m, n, and therefore m/n algebraic, $n \neq 0$, $n \neq 1$, implies, by Gelfond's theorem, that m/n is rational. Thus $m/n = k/h$, where k and h are nonzero rational integers, $k \neq h$, $(k, h) = 1$. We may assume that at least one of h, k is positive.

It follows that $m/n = n^{m/n-1}$, $k/h = n^{(k-h)/h}$, thus

$$n = (k/h)^{h/(k-h)}, \qquad m = (k/h)n = (h/k)^{k/(h-k)}.$$

Assume that $k-h > 0$ (and thus that $k > 0$). If m and n are algebraic integers, then $n^{k-h} = (k/h)^h$ is also an algebraic integer, and thus a rational integer. If $h > 0$, this means necessarily that $h = 1$. If $h < 0$, this means that h/k is a rational integer and thus that $k = 1$.

Also solved by Emil Grosswald, Alvin Hausner and Solomon Hurwitz, Oswald Wyler, and the proposer.

Editorial Note. For Gelfond's theorem, see LeVeque, *Topics in Number Theory,* v. II, p. 198. For a related discussion see A. Hausner, *Algebraic number fields and $m^n = n^m$*, this MONTHLY, 68 (1961) 856–862. Makowski cites a similar problem solved in *Wiskundige Opgaven met de Oplossingen,* 20. no. 5 (1959), 30–31.

Roots of sin $x = x$

5102 [1936, 571]. *Proposed by R. P. Boas, Jr., and W. R. Mann, Northwestern University*

The equation $\sin x = x$ clearly has no real roots other than $x = 0$. Does it have any complex roots?

I. *Solution by J. J. Roseman, New York University.* Picard's theorem gives an easy affirmative answer and, in fact, provides the following stronger result: The function $f(z) = z - \sin z$ takes every complex value an infinite number of times.

For, suppose there exists a complex number, α, which is taken at most a finite number of times by $f(z)$. Then, by Picard's theorem, $\alpha + 2\pi$ is taken an infinite number of times. Let $\{\zeta_n\}$ be an infinite sequence of distinct numbers such that $f(\zeta_n) = \zeta_n - \sin \zeta_n = \alpha + 2\pi$. Set $\zeta_n = 2\pi + w_n$. Then, $2\pi + w_n - \sin w_n = \alpha + 2\pi$ or $f(w_n) = w_n - \sin w_n = \alpha$. Thus, $f(z)$ takes α an infinite number of times in contradiction of the hypothesis. The proof is complete.

II. *Solution by R. P. Boas, Jr., Northwestern University.* The equation has an infinity of complex roots. For, $f(z)=z^{-1}\sin z-1$ is an even entire function of order 1; hence $f(z^{\frac{1}{2}})$ is an entire function of order $\frac{1}{2}$. As an entire function of non-integral order, it has an infinity of zeros.

III. *Solution by L. Carlitz, Duke University.* Let $P(x)$ be any polynomial (of degree n) and assume that the function $\phi(x)=P(x)-\sin x$ has only a finite number of zeros. Since $\phi(x)$ is an entire function of order 1, it follows from Hadamard's theorem that $\phi(x)=F(x)\cdot e^{Ax}$, where $F(x)$ is a polynomial and A is a constant. Differentiating $n+1$ times we get

$$\sin x = G_1(x)\cdot e^{Ax} \quad \text{or} \quad \cos x = G_2(x)\cdot e^{Ax},$$

where G_1, G_2 are polynomials. Since the left member has infinitely many zeros while the right member has at most a finite number, we have a contradiction. It follows that the equation $\sin x=P(x)$ has infinitely many complex zeros.

Also solved by George Bergman, W. H. Bonney, J. J. Bowers, S. Chowla, John H. E. Cohn' M. S. Demos, D. Ž. Djoković, H. E. Fettis, G. J. Giaccai, Ralph Greenberg, Emil Grosswald' J. Koekoek, L. J. Lange, LeRoy A. MacColl, K. W. Miller, J. S. Muldowney, C. Stanley Ogilvy' C. D. Olds, H. A. D. Paris, Walter Penney, J. D. Pryce, S. L. Segal, Andreas Thuswaldner, James S. W. Wong, and J. A. Zilber.

Editorial Note. G. H. Hardy [Mess. of Math. V.XXXI (1902), 161–165] solved the problem and gave the approximation

$$x\sim\pm(2n+\tfrac{1}{2})\pi+i\cdot\log\,(4n+1)\pi,\ n=1,2,\cdots.$$

The first ten complex solutions were calculated by Hillman and Salzer [Phil. Mag., 7 (34), 575 (1943), 5–49] to six decimals. These results are also listed in the Applied Math Series #37., *Tables of Functions and on Zeros of Functions*, of the National Bureau of Standards (November 1954).

A more general result is proved by L. S. Pontryagin, *On zeros of some transcendental functions*, Akad. Nauk SSSR., Ser. Mat. 6, (1942) 115–134, (Amer. Math. Soc. Transl. 1 (1955) 95–110).

Matrix Iteration with Polynomial Elements

5103 [1963, 572]. *Proposed by D. S. Mitrinović, Belgrade, Yugoslavia*

Determine integers a, b, c, d such that the matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^k = \begin{pmatrix} A(k) & B(k) \\ C(k) & D(k) \end{pmatrix}$$

holds for all positive integral k if A, B, C, D are polynomials in k. Can the result be generalized for matrices of order n?

Solution by T. I. Seidman, Boeing Research Laboratories, Seattle, Washington. We will find all quadruples of integers $\{a, b, c, d\}$ such that there are polynomials $\{A(k), B(k), C(k), D(k)\}$ for which the stated matrix equation holds for $k=1, 2, \cdots$.

Generalizing immediately to the nth order case (and ignoring for the moment the requirement that the entries be integers), let $M=((a_{ij}))$ be an $n\times n$ matrix and P a matrix such that $J=PMP^{-1}$ is in Jordan canonical form. If $M^k=((A_{ij}(k)))$ where the A_{ij} are polynomials then $J^k=(PMP^{-1})^k=PM^kP^{-1}$ also

has polynomial entries. Considering any block B of J with diagonal terms λ we have a corresponding block B^k of J^k with diagonal terms λ^k, so λ must be 0 or 1 to obtain a polynomial in k. Computation shows that, conversely, if $\lambda=0$, 1 then the entries in B^k are polynomials in k; therefore a matrix has the stated property if and only if all its eigenvalues are zero or one. Its characteristic equation then has the form

$$0 = (\lambda - 1)^r\lambda^{n-r} = \sum_{j=0}^{r} (-1)^j \binom{n}{j} \lambda^{n-j}$$

for some $r=0, \cdots, n$. Writing the characteristic equation in terms of the entries $\{a_{ij}\}$ and equating coefficients (taking $r=0, \cdots, n$ in turn) gives $n+1$ sets of conditions each of which must then be solved as Diophantine equations.

In particular, in the 2×2 case, with $r=2$, the conditions become $a+d=2$, $ad-bc=1$ and the set of integral solutions is given by

$$\begin{pmatrix} 1+uvw & -uv^2 \\ uw^2 & 1-uvw \end{pmatrix}$$

where u, v, w are arbitrary integers. Here $A(k)=(a-1)k+1$, $B(k)=bk$, $C(k)=ck$, $D(k)=(d-1)k+1$. The case $r=0$ leads to nilpotent matrices while $r=1$ is also trivial, giving matrices that are idempotent.

Also solved by J. L. Brenner, L. Carlitz, G. A. Heuer, J. D. Pryce, J. E. Shockley, R. Sibson, Jr., and the proposer.

Editorial Note. Carlitz (omitting the trivial case where M itself is nilpotent) gives the result in the form: The stated condition is satisfied if and only if M is of the form $M=I+N$ where I is the identity matrix and N is nilpotent.

Subsets in a Lattice

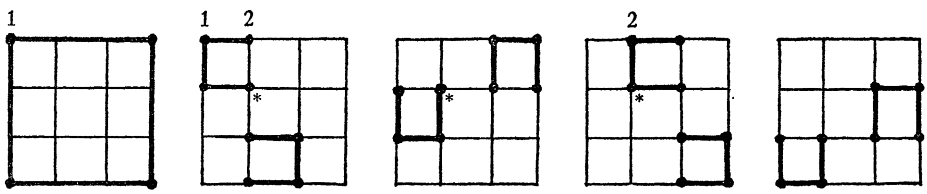
5104 [1963, 572]. *Proposed by Paul Sally, Jr., Boston College*

Consider the lattice L of points in E_n having integer valued coordinates. We define a *line* in L to be a line in E_n which contains points of L and is parallel to one of the coordinate axes. A set $S\subset L$ is called *admissible* if it is bounded, and if every line which intersects S contains at least two points of S. A set $N\subset L$ is called *null* if every line which intersects N contains an even number of points of N. Clearly every null set is an admissible set.

Prove (or disprove): Every nonempty admissible set contains a nonempty subset which is a null set.

Solution by George Bergman, Harvard University. For $n=2$ the assertion is true. For let us pick any $p_0\in S$, and then, for each $i>0$, pick p_i distinct from p_{i-1} but having the same first coordinate as the latter if i is odd, the same second coordinate if i is even. Eventually we must pick a $p_n=p_m$, $m<n$. Assume p_n the first point to duplicate a preceding point. It is easy to show that if m and n are of the same parity, $p_m, p_{m+1}, \cdots, p_{n-1}$ form a null set, while if the parities are different, $p_{m+1}, \cdots, p_{n-1}$ do.

For $n=3$ the assertion is false. The set S represented in the figure by successive cross-sections is a counterexample. In the diagram, whenever two points



in the same plane of cross-section are the only points of S on some line, the segment between them is darkened. When this is the case, any admissible subset of S containing one must contain the other.

We can similarly imagine the two points marked 1 as joined by a link perpendicular to the planes of cross-section, and each of the points of S in the first section as linked in the same way to one point above. Similarly, the two points marked 2, and the three pairs of points in analogous positions are linked.

The result is that S is entirely connected, and hence is a minimal admissible set. But it is not itself null, for the line through the points marked * has three points of S.

Counterexamples for any $n>3$ can be constructed from this one.

Also solved by the proposer whose counterexample (for $n=3$) involves 24 points. He conjectures that this is the minimal number of points in an admissible set which does not contain a nonempty null subset.

Semi-simple Unital Module

5105 [1963, 572]. *Proposed by Harley Flanders, Purdue University*

Let A be a ring with unit element and M a unital A-module. Suppose that for each submodule N satisfying $0<N$, there is a submodule P such that $M=N+P$, $P<M$. Is M then semi-simple? (See Bourbaki, *Algèbre*, Ch. 8, p. 36, exercise 4.)

Solution by E. R. Gentile, Buenos Aires, Argentina. The statement is not true without an additional requirement. Indeed, under the given condition we prove that M is semi-simple if and only if M is Artinian.

Let M be Artinian and let N, $0<N$, be a submodule. Choose then a minimal submodule P of M with the property $M=N+P$. If $0<N\cap P$, there is a submodule $H<M$ such that $N\cap P+H=M$. We have $P=P\cap M=P\cap(N\cap P+H)$ $=N\cap P+P\cap H$. Consequently $M=N+P=N+N\cap P+P\cap H=N+P\cap H$. The minimality of P implies (since $P\cap H\leqq P<M$) that $P\cap H=P$, whence $P\subset H$. It follows that $M=N\cap P+H=H$, a contradiction. Therefore $0=N\cap P$; from this, the semi-simplicity of M is clear. The converse is trivial.

Also solved by M. J. Greenberg, and by E. J. Taft.

Intersection of Algebraic Curves

5106 [1963, 572]. *Proposed by Fred Suvorov, Princeton, N. J.*

Let S be the projective plane over the complex numbers. Show that any finite set of points in S can be obtained as the complete intersection of two (possibly reducible) algebraic curves.

Solution by the proposer. Let $P_0, \cdots, P_n$ be a finite set of distinct points in the projective plane S. The case $n=0$ is trivial, so we suppose $n \geqq 1$. Choose homogeneous coordinates (x, y, z) so that P_0 is the point $(0, 1, 0)$ and the line $z=0$ contains none of the points $P_1, \cdots, P_n$.

Then $P_1, \cdots, P_n$ lie in the affine plane $z=1$ with affine coordinates (x, y). Let $a_1, \cdots, a_r$ be the set of distinct abscissas of the points $P_1, \cdots, P_n$. It is well known that one may pass a curve $y=f(x)$, where f is a polynomial, through a finite number of points with distinct abscissas. Thus we can find a finite set of polynomials $f_1(x), \cdots, f_s(x)$, (where s is the largest number of points P_k having the same abscissa) such that the points $(a_i, f_j(a_i))$, for $i=1, \cdots, r$, $j=1, \cdots, s$ are precisely the points $P_1, \cdots, P_n$, although some P_k may occur several times. Make each f_j of degree at least 2.

Now if C_1, C_2 are the closures in S of the affine curves defined by

$$\prod_{j=1}^{s} (y - f_j(x)) = 0; \qquad \prod_{i=1}^{r} (x - a_i) = 0,$$

then $C_1 \cap C_2$ is precisely the set of points $P_0, \cdots, P_n$.

Automorphisms of a Cyclic Group

5107 [1963, 572]. *Proposed by Andrew Zachariou, Athens Greece*

Find the conditions under which the group of automorphisms of a cyclic group of finite order is itself a cyclic group.

Solution by John Stout, senior, Manhattan College. Let the order of the cyclic group be m. In additive notation, write the elements of the group as $1, 2, 3, \cdots, m \equiv 0 \pmod m$. A given endomorphism is uniquely determined by the map of 1: $1 \to a$, $2 \to 2a$, etc., and will be an automorphism iff ("if and only if") $ja \equiv ka \pmod m$ implies $j \equiv k \pmod m$, i.e. a is relatively prime to m. Thus the group of automorphisms can be represented as the multiplicative group modulo m of the $\phi(m)$ positive integers less than and relatively prime to m. Now this group will be cyclic iff it contains a generator element α such that the set $\alpha, \alpha^2, \cdots, \alpha^{\phi(m)} \equiv 1 \pmod m$ contains all the elements of the automorphism group, and this will be true iff $\alpha^{j-k} \equiv 1 \pmod m$ implies $j \equiv k \pmod{\phi(m)}$, i.e. α is a primitive root of m. Finally, there exists a primitive root mod m iff m has one of the values: $1, 2, 4, p^\beta, 2p^\beta$ where p is an odd prime and β is a positive integer.

Also solved by George Bergman, W. H. Bonney, Robert Bowen, L. Carlitz, John H. E. Cohn, M. E. Dieckman, D. Ž. Djoković, Bruce Erickson, J. S. Frame, R. W. Gilmer, Jr., M. L. Glasser,

A. R. Krishna, Ruth J. Little, A. E. Livingston, C. R. MacCluer, H. A. D. Paris, W. C. Waterhouse, and Oswald Wyler.

Editorial Note. Paris notes that a suggestion of the solution is given in Burnside, *Theory of Groups*, section 88, p. 114.

Banach Spaces with Common Basis

5108 [1963, 572]. *Proposed by Albert Wilansky, Lehigh University*

Give an example of two Banach spaces X, Y such that the set $X \cap Y$ contains a sequence B which is a basis for each space but not a basis for $X \cap Y$ as a normed space with the norm $p_X + p_Y$; p_X, p_Y being the norms of X, Y.

Solution by J. D. Pryce, Newcastle, England. The following example relies on the fact that any infinite dimensional separable Banach space has a Hamel base of cardinality c.

Let X be (l^2) and Y_0 be (l^1). These are separable Banach spaces with a basis. Let (x_n), (y_n) denote the usual basis elements in X, Y_0, respectively, and let M, N be the subspaces consisting of (finite) linear combinations of the basis elements. By an argument involving Zorn's lemma or equivalent, one can show there exists a linear (but not necessarily topological) isomorphism of Y_0 onto X, such that y_n maps to x_n for each n. Let Y be X, furnished with the norm carried over from Y_0 by the isomorphism. Now N maps onto M, and for all u in M, $p_X(u) \leqq p_Y(u)$, for u is of form $\lambda_1 \cdot x_1 + \cdots + \lambda_n \cdot x_n$ and

$$p_X(u) = \left(\sum |\lambda_i|^2\right)^{1/2} \leqq \sum |\lambda_i| = p_Y(u).$$

It follows that each sequence in M which is Cauchy (p_Y) is Cauchy (p_X). Now each z in X is uniquely expressible in the form

$$z = \sum_{r=1}^{\infty} \lambda_r \cdot x_r \quad \text{(with respect to } p_Y).$$

This, since its partial sums are a Cauchy sequence in M, is also convergent with respect to p_X to an element y. (Neither z nor y in general lying in M.) Now we cannot have $y = z$ for every z, else the closed graph theorem would show that p_X and p_Y were equivalent, hence (l^2) and (l^1) would be linearly homeomorphic, which is not so.

Accordingly, consider a z for which $y \neq z$; it is easy to see that no infinite sum of basis elements can converge to z in the norm $p_X + p_Y$.

Editorial Note. For a sufficient condition that the basis be a basis for $X \cap Y$ see A. Wilansky, *Functional Analysis*, Blaisdell Press, 1964, Section 11.4, problem 40.

Cycle Structure of an Infinite Permutation

5109 [1963, 572]. *Proposed by M. S. Klamkin, State University of New York at Buffalo*

Given the infinite permutation

$$P \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \cdots \\ 1 & 2 & 4 & 3 & 5 & 7 & 6 & 8 & 10 & 12 \cdots \end{pmatrix}$$

where the second row is formed by taking in order from the natural numbers, 1 odd, 2 even, 3 odd, $\cdots$, $2n$ even, $2n+1$ odd, $\cdots$. What is the cycle structure of this permutation?

Solution by George Bergman, Harvard University. Let I_n designate the set of n integers $\{i \mid \frac{1}{2}n(n-1) < i \leqq \frac{1}{2}n(n+1)\}$. Examination of the given permutation shows that it acts on I_n by the law: $i \to 2i - u_n$, where $u_n = \frac{1}{2}n^2$ if n is even, $u_n = \frac{1}{2}(n^2+1)$ if n is odd. The "pivot" of this action is u_n; u_n is fixed, numbers of I_n less than u_n are decreased, numbers of I_n greater than u_n are increased.

But we see that even the greatest integer in I_n is not increased as far as u_{n+1}, and even the least integer in I_{n+1} is not decreased as far as u_n; hence the interval $J_n = \{i \mid u_n \leqq i < u_{n+1}\}$ is sent into itself. This J_n contains $2[n/2]+1$ elements. Let us represent them by the integers 0 through $2[n/2]$, writing j for $u_n + j$. Then the action of our permutation is: $j \to 2j$ for $j \leqq [n/2]$, $j \to 2j - 2[n/2] - 1$ otherwise. In other words, the elements of J_n are permuted exactly as the residue classes (mod $2[n/2]+1$) are permuted under multiplication by 2.

The nature of the permutation is as follows: for each divisor d of $2[n/2]+1$, the elements $i = u_n + j$ of J_n such that $(2[n/2]+1, j) = d$, form a cycle of order $f((2[n/2]+1)/d)$, where $f(k)$ is the least m such that $k \mid 2^m - 1$. This number-theoretic function is described in standard texts. For example, let $n = 15$, $J_n = \{i \mid 113 \leqq i < 128\}$, represented by $\{j \mid 0 \leqq j < 15\}$. The permutation for these integers is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 \end{pmatrix}.$$

The cycles are given by:

$d = 1$: $(1\ 2\ 4\ 8)$, $(7\ 14\ 13\ 11) \sim (114\ 115\ 117\ 121)$, $(120, 127, 126, 124)$

$d = 3$: $(3\ 6\ 12\ 9) \sim (116, 119, 125, 122)$

$d = 5$: $(5\ 10) \sim (118\ 123)$

d 15: $(0) \sim (113)$, fixed.

f takes on every integral value (for $f(2^m - 1) = m$); therefore all cycles are finite, and there are infinitely many cycles of every finite order.

Also solved by L. Carlitz, Donald Liss, P. Catherine Varga, and Oswald Wyler.

Improper Integral

5110 [1963, 573]. *Proposed by L. Carlitz, Duke University*

Show that

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left(\frac{\cos\theta}{x\cos\theta + i\sin\theta}\right)^{\nu} d\theta = \begin{cases} \pi(x+1)^{-\nu} & (|1-x| < 1) \\ \pi(x-1)^{-\nu} & (|1+x| < 1), \end{cases}$$

where $R(\nu) > -1$.

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute, Hartford. We prove the extended result that the given integral equals $\pi(x+1)^{-\nu}$ if $R\{x\}>0$ and $\pi(x-1)^{-\nu}$ if $R\{x\}<0$. To do this let $w=\cos\theta/(x\cos\theta+i\sin\theta)$. Then the given integral equals

$$\oint_C \frac{iw^\nu dw}{(1-x^2)w^2+2xw-1}$$

where C is the circle $|w-1/2x|=1/2|x|$ traversed in the clockwise or counterclockwise directions according as $R\{x\}>0$ or $R\{x\}<0$, respectively. (Actually, this integral is improper at $w=0$ corresponding to $\theta=\pm\frac{1}{2}\pi$ in the given integral. However, no difficulty is involved as can be seen by indenting C so that $w=0$ lies outside C and applying appropriate limiting procedures together with the restriction $R\{\nu\}>-1$.)

The above integrand has simple poles at $w=1/(x+1)$ and $1/(x-1)$, respectively. If $R\{x\}>0$ only the former lies inside C while if $R\{x\}<0$ only the latter lies inside C. Then applying the residue theorem we obtain the above stated result.

Also solved by C. M. Becker, D. Ž. Djoković, M. L. Glasser, Emil Grosswald, J. Koekoek, R. E. Shafer, Oswald Wyler, and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, Carleton College and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. Programmed Materials: K. O. May, Carleton College, Northfield, Minn. Films: E. P. Vance, Oberlin College, Oberlin, Ohio.

Lie Algebras. By Nathan Jacobson. Interscience Publishers, Wiley, New York, 1962. ix+331 pp., $10.50.

Since Hochschild (Bull. Amer. Math. Soc., 69 (1963) 37–39) has given such a thorough review of the material in Jacobson's book, I shall not discuss its substance at any length, but shall confine myself instead to the role the book should play both in our graduate education and as a source of learning the subject on the part of a professional mathematician.

As for the subject matter, suffice it to say that Jacobson gives a very complete discussion of the structure theory, representation theory and automorphisms of Lie algebras, with the emphasis almost totally on the split case. One finds a wealth of material covering the usual topics and, very often, a wide range of unusual ones.

Let us return to the question of the impact of the book. To my mind the book fulfills its two purposes—education of the beginner and the experienced one—to perfection. A person knowing nothing about Lie algebras when he picks up the book will, by the time he finishes it, have been brought close to the current research areas in the purely algebraic parts of the theory. (Very little is said about Lie groups or the interrelation of Lie algebras with analysis.) The development is thorough, in many places novel, devoid of twists and gimmicks, and should offer very little difficulty to anyone well-acquainted with linear transformations. It is the only source that I know which starts the reader at absolute scratch, takes him in a coherent, motivated way through Lie algebras, and finally drops him off at a point where he can speak with some intelligence and authority on the subject. The book should be ideally suited for a second year graduate course; in fact it was used as a text at the University of Chicago with what seems to have been great success.

If the book has any drawbacks—and which book does not?—the most serious, in my opinion (and what to others may be one of its great virtues) is that there is a certain uniformity and sameness in the approach to many of the theorems. Also there probably should be more stress laid on the innumerable high spots enjoyed by the subject.

Professor Jacobson deserves the thanks of all of us for having made the beautiful subject matter of Lie algebras so readily available to a wide public, both to the sophisticated old-timer and to the budding young student.

I. N. HERSTEIN, University of Chicago

An Introduction to the Theory of Stationary Random Functions. By A. M. Yaglom. Translated from a revised version by Richard A. Silverman. Prentice-Hall, 1962. xiii+235 pp. $7.95.

A probabilistic process is stationary if it is time dependent but impervious to a simultaneous identical shift of the time parameter in all random variables constituting the process. The importance of stationary processes in general probability is comparable, e.g., to states of equilibrium in general thermodynamics. Stationary processes are studied intensively because they are accessible to detailed treatment by Fourier Analysis and, consequently, can be applied readily to problems in physics, engineering and information theory. The book is a readable monograph which will appeal to students and readers of various interests. Also, a brief but important appendix by D. B. Lowdenslager reports on "Some recent developments."

S. BOCHNER, Princeton University

The Calculus, a Genetic Approach. By Otto Toeplitz. University of Chicago Press, 1963. xiv+192 pp. $6.50.

The German edition of this book, edited by Koethe, was published by Springer in 1949, nine years after the death of its author. Toeplitz, a victim of

Nazi persecution, had written the notes for the work over a period of years, starting in the mid-twenties. It was his aim to make the basic concepts of the calculus come to life by showing how and why they arose, and how they changed over a period of two and a half millennia. The appearance of this book on the American scene is fortunate; it can serve as a welcome antidote to many of the introductory texts published in recent years. The student reared exclusively on these texts must gain the impression that calculus is an antiseptically wrapped package, "untouched by human hands" (or minds), logically very pure, and, like very pure food, somewhat dull. This book is neither dull, nor logically pure. It is easy to point out gaps and inconsistencies: the limit of a sequence is fully defined, the limit of a function not at all; differentials, essentially undefined, are used without inhibition; the function-of-a-function rule gets a relatively sophisticated treatment (even though the existence of the final derivative is assumed without proof), while the derivatives of inverse functions are disposed of with a reference to Leibniz; in the definition of the definite integral, only values of the function at the endpoints of the subintervals are considered; theorem 1 on page 64 might imply the existence of $\int_0^1 dx/(1-x)$; etc. But all of this is relatively immaterial. In fact, Toeplitz probably intended to write the first volume of this work at the level of rigor customary and possible in the late 18th century, with just a few excursions into more modern reasoning. (The projected second volume did not materialize.)

The English version of the book contains four chapters. Chapter I, The Nature of the Infinite Process, is in the reviewer's opinion, the best part of the book. It is a superb account of the development of the concept of a real number, beginning with the Greek discovery of incommensurables, and culminating in a discussion of infinite sequences. Chapter II, The Definite Integral, is also very well written. Particularly interesting are the section on Archimedes' squaring of the parabola, and the excellent critique of Cavalieri's reasoning. Chapter III, Differential And Integral Calculus, is less inspired. Most of it is a somewhat unsatisfactory exposition of standard material on differentiation and integration techniques. Exceptions are the fascinating sections on Napier, and on the Leibniz-Newton controversy. Chapter IV, Application To Problems Of Motion, starts with a slightly confusing section on "velocity and acceleration." Then it leads, with detours over vibrational motion, to a concise and very readable treatment of planetary motion.

There are about sixty exercises, many of them only loosely connected with the text, almost all of them interesting and challenging; some of them are really quite beyond the scope of the text. The translation by Luise Lange is on the whole faithful to the original German. It is probably unavoidable that some of the flavor of the original should get lost in any translation; but the reviewer regrets that for instance, on page 2, the powerfully descriptive "Abgrund des Unendlichen" ("abyss of the infinite") appears as the pedestrian "antinomy of the infinite." The English edition is somewhat marred by a considerable number

of nontrivial misprints, and by a crowding of formulas into too little space. The book can be highly recommended for independent study, or as supplementary reading suggested to classes in introductory (or even advanced) calculus.

ROBERT BREUSCH, Amherst College

Operational Calculus and Generalized Functions. By A. Erdélyi. Holt, Rinehart and Winston, New York, 1962. viii+103 pp. $2.75.

This book is one of the volumes in the publisher's Athena Series (under the editorship of Edwin Hewitt) and gives a brief introduction to the theory of convolution quotients as developed by the Polish mathematician Jan Mikusiński.

It should be well-known by now that operational calculus based on the widely taught Laplace transformation theory is insufficient to take care of the delta "function" or other so-called "impulse functions" which cannot really be called functions in any usual sense of the word. Since Heaviside introduced his operational methods at the turn of the century, a number of attempts have been made to devise an operational calculus which would be satisfactory and, especially, which would allow some rigorous justification for obviously valid results found by Heaviside's ingenious methods.

Mikusiński has made a notable contribution in his theory and it is most appropriate to have this new and brief introduction to the theory. There can be, however, no substitute for Mikusiński's own masterpiece *Operational Calculus* (1953) published in an English translation by Pergamon in 1959, but Professor Erdélyi's book is an admirable introduction to the simpler parts of the theory. Generally speaking, the chapters in Erdélyi's book follow the order of topics in roughly the first half of Mikusiński's book, except that the extended material on electrical circuits, networks, statics of beams, and other such topics has been omitted, or condensed.

The book is well-written and has a convenient index of notations. In a future edition the word "sifting" should appear in the index. The reviewer found only a few minor typographical errors. There are 126 problems with answers, hints, or further references for three-fourths of them. The problems vary from manipulatory exercises to worthwhile theorem-proving. It is all quite adequate for an introductory graduate course.

A glowing account of Mikusiński's work was given by Hans Freudenthal in his paper *Operators—from Heaviside to Mikusiński*, Simon Stevin, 33 (1956) 3–19 (Dutch). A thoughtful review by Jacob Korevaar, Math. Rev., 22 (1961) #5854, will indicate the still controversial nature of the contending theories of operational calculus. Indeed, expounders of the Mikusiński theory proudly say that it is time for mathematicians to drive out the Laplace transformation. Because of all of this it is very pleasing to have an inexpensive and brief introduction to the Mikusiński theory.

H. W. GOULD, West Virginia University

The Language of Mathematics. By Frank Land. Doubleday and Company, Inc., Garden City, New York, 1963. 264 pp. $4.95.

This is an American edition of a book originally published in England in 1960. Though not quite as pretty as the English edition, it represents a fine piece of book making, with over 300 elegantly executed illustrations—many in two colors. The preface gives an explanation of the British coinage system and its conversion to the American system. The author, who is a Professor of Education at the University of Hull, writes with delightful charm and real insight into the esthetics and the pedagogy of elementary mathematics. The book seems to have two excellent uses: (1) for the general nonmathematical reader who would like to learn pleasantly about arithmetic, mensuration, elementary algebra and geometry, and statistics; (2) for the teacher of high school mathematics who is looking for enrichment material to supplement class presentations. The interesting underlying history of the various topics is meritoriously handled. There are a few simple exercises scattered through the book.

HOWARD EVES, University of Maine

Mathematics: The Man-made Universe. By S. K. Stein. W. H. Freeman and Company, San Francisco, 1963. xiii+316 pp. $6.50.

We have here an excellent "introduction to the spirit of mathematics," ideal for either the student or the general reader of mathematics, and a good book for a college course in mathematics appreciation, a teachers institute, or source material for lectures and for mathematics clubs. The material, even when very familiar, is handled in a lively way; and, though based on quite meager prerequisites, it does not avoid getting to the meat of the matter. The exercise lists at the ends of the seventeen chapters are copious, well chosen, and often challenging. Some readers may feel that the material too strongly favors work with numbers and pays too little attention to other areas of mathematics.

HOWARD EVES, University of Maine

Matrix Algebra for Social Scientists. By Paul Horst. Holt, Rinehart, Winston, New York, 1963. xxi+517 pp. $10.00.

Mathematicians, and all others who are troubled by incorrect statements, muddled thought, and maddening tedium, will do well to shun this book. The author takes more than 300 pages to do very little more than introduce the student to the multiplication of matrices and to an Everest of terminology. Professor (of psychology) Horst claims that he has adopted this "leisurely pace" and departed from "the traditional approach of the mathematician" because that approach "has negatively conditioned many students to the study of mathematics." Readers of this MONTHLY may further be interested to learn that " . . . mathematicians not infrequently become confused when they attempt to use the transpose in the solution of problems," that "the multiplication of supermatrices is simple in principle, but in practical applications even competent mathematicians are easily confused," and so on and on.

The last 200 pages are devoted to topics the author feels to be especially useful in the social sciences, and more particularly in multiple regression analysis, factor analysis, and analysis of variance: orthogonality, rank, special decompositions, generalized inverses and linear equations. Computational directions, with machine solutions in mind, are given in painstaking detail. Some of this might possibly be helpful to a social scientist.

I. H. ROSE, Hunter College

Mathematical Models in the Social Sciences. By John G. Kemeny and J. Laurie Snell. Ginn and Company, Boston, 1962. vii+145 pp. $6.00.

This elegantly written, slim, but highly compact volume is to be applauded. It is refreshing to find mathematical authors not unwilling to devote a little space to illuminating discussion of the mathematics they are about to develop. After an excellent introductory chapter on methodology, the authors proceed in the following eight chapters to treat problems of preference rankings, ecology, market stability, conformity, stabilization of money flow, population growth, service time, organization theory and optimal scheduling. The mathematical tools employed include axiomatic techniques, excerpts from real variable and differential equation theory, Markov chains, probability theory, queueing theory, graph theory, and dynamic programming.

Little or no knowledge of the social sciences involved is expected or needed, but the authors presuppose a mathematical background of a year of calculus and a semester of "finite" mathematics, and suggest this text and background as a (two year) minimal undergraduate course for social scientists desiring to do theoretical work. Let us, however, be realistic. Students with this limited background would have to be exceptionally talented mathematically to do anything worthwhile with the text. Even students with more maturity and appreciable prior knowledge of the mathematical tools listed above had better be a cut above the mediocre if they are to benefit from the book.

Indeed, this, to me, is where the value of the book to mathematics majors lies: In an honors or seminar course, or in a mathematics club project, or in an elective course for properly chosen students, it would serve admirably to introduce the student to bold, imaginative, original, creative mathematics—an introduction we generally postpone far too long. The exercises are stimulating, and include larger-scale projects that will afford even the best student ample opportunity to exercise his ability and originality to the utmost.

I. H. ROSE, Hunter College

Theorie der analytischen Funktionen einer komplexen Veränderlichen. By Heinrich Behnke and Friedrich Sommer. 2nd Ed. Springer Verlag, Berlin, 1962. xii+603 pp. DM 74.00.

The first edition (1955) of this work was not reviewed in this MONTHLY, so that it is appropriate to acknowledge the appearance of the revision. The authors have given a comprehensive treatment of the classical theory of func-

tions of a complex variable in their first three chapters. The remaining three chapters deal with conformal mapping, with concrete Riemann surfaces as carriers of analytic functions, and with functions and differentials on compact Riemann surfaces. The noncompact case is treated in a final section.

Frequently, throughout the volume, the authors cite relevant literature containing extensions as well as substantiation of the quoted results. Advantage is taken of many opportunities to give brief introductions to related fields—vector spaces, elementary lattice theory, etc. The topology of surfaces is dealt with in an appendix to Chapter IV, and the treatment consists in a clear but intuitive discussion of the classification of compact orientable surfaces.

Behnke and Sommer emphasize expansion theorems. For example, the Weierstrass and Mittag-Leffler developments, in the third chapter, are a bit more general than ordinarily found in texts: the region of existence of a holomorphic or meromorphic function is an arbitrary region of the plane, and the prescribed zeros or poles are subject merely to the restriction that these points shall not cluster in the interior. That such a region is precisely the domain of holomorphy or meromorphy of some (suitable) function is also proved. These results, as well as the Runge theorem, are then extended in the last chapter to arbitrary noncompact Riemann surfaces.

The second edition contains some clarifications over the earlier one. In particular, the topology of plane sets is treated in the more general setting of metric spaces, the theory of normal families has been revised, and the introduction of Riemann surfaces has been rewritten. Additional minor improvements have also been made. Worth imitating is the notation "$\check{f}$" for the inverse function to f; this was already a feature of the first edition.

There seem to be few misprints, and these appear trivial. The last two chapters provide an excellent motivational basis for the abstract treatment of Riemann surfaces customary currently. However, I missed a reference to "Riemann Surfaces" by L. V. Ahlfors and L. Sario (Princeton, 1960). (Some works that appeared later than that book were included in the bibliographic notes to the second edition.)

The book gives a fresh and up-to-date account of its field. While it has no exercises, there is much that will challenge the critical reader. The work will be of value to every mathematician with an interest in complex function theory.

ERNEST C. SCHLESINGER, Connecticut College

Readings in Mathematical Programming. By S. Vajda. Wiley, New York, 1962. viii+130 pp. $4.25.

This volume is the second edition of *Readings in Linear Programming*, Wiley, 1958. The new title reflects recent developments in the fields which have become generally associated with linear programming. The first edition contained three chapters on theory and twenty-one chapters each devoted to a small example problem. The best known examples (Caterer Problem, Trim Loss Problem, etc.) are all included. The additions to the new edition are three chapters on integer

programming and two each on dynamic programming and quadratic programming. Solutions to the few problems in the volume are now included as well.

The reviewer has found the first edition useful as required independent reading in linear programming courses. The additional material is helpful but most students will not be able to assimilate it independently. The pure integer programming problem formulated in Chapter XXV requires only five equations in the formulation of the model rather than the ten which the author uses. Neither does the author caution that the mixed integer algorithm illustrated in Chapter XXVI need not terminate in a finite number of steps.

The author should have extended himself somewhat more in revising the bibliography; there are only eight additions and a number of relevant texts are not cited. In spite of this fact, the beginner who is interested in applications will find this a useful book to consult.

ROBERT L. GRAVES, University of Chicago

Generalized Analytic Functions. By I. N. Vekua. Moscow, 1959. Translation. Addison-Wesley, Reading, Mass., 1962. xxix+668 pp. $14.75.

A generalized analytic function is a complex-valued function $w(z)$ which satisfies an equation of the form (1) $Lw \equiv \partial w/\partial \bar{z} + A(z)w + B(z)\bar{w} = 0$. This definition is equivalent to that of a pseudo-analytic function (of the first kind) given by L. Bers. Except for Lecture Notes by Bers (Theory of Pseudo-analytic Functions, New York University, 1952), the present book is, as far as I know, the only one in the mathematical literature dealing with generalized analytic functions (GAF). It contains a wealth of material, mostly known but partially new. The book consists of two parts. Part I develops the mathematical theory of GAF. In Chapter I a theory of integral operators is developed. In Chapter II a positive quadratic form in two variables is reduced to a canonical form $\Lambda(dx^2+dy^2)$, and this is later used to show that any first order elliptic system of two equations (in two independent variables) can be reduced to the form $Lw = F$. Thus the theory of GAF can be used in studying this elliptic system. Chapter III develops for GAF a general theory similar to that of the classical theory of analytic functions (the Cauchy integral formula, Taylor and Laurent expansions, etc.). Finally, Chapter IV deals with boundary value problems for GAF, the boundary condition being of the form $\operatorname{Re}[\lambda(z)w(z)] = \gamma(z)$. Part II gives applications to problems of the infinitesimal bending of surfaces (Chapter V) and the membrane theory of shells (Chapter VI).

One of the most important characteristics of the book is that the assumptions made on the coefficients of the equations are very weak (thus, in (1), A and B are only assumed to belong to L^p, $p > 2$). This in part is responsible for the rather heavy nature of the analysis which underlies the material of the book. These weak assumptions are not only motivated by the mathematical interest but also by physical applications.

The richness of material in the book should make it an excellent reference book, but a reader who is interested only in getting a brief and concise idea

about the basic facts of GAF may find himself lost in the bulk of material especially since 129 pages of hard analysis precede the introduction of the GAF.

A knowledge of Lebesgue integration and the elements of functional analysis is presupposed.

AVNER FRIEDMAN, Northwestern University

Advances in Computers, Vol. 3. Edited by Francis L. Alt and Morris Rubinoff. Academic Press, New York, 1962. xiii+361 pp. $12.00.

This book contains several long articles which describe some important current applications of analog and digital computers. The authors have culled the pertinent recent publications and have provided the reader with an evaluation of the more significant developments.

Here, briefly, are some of the highlights.

1. *The Computation of Satellite Orbit Trajectories* by Samuel D. Conte.

In conjunction with treating several methods for the determination of orbits, the author has an excellent discussion of the use of numerical computations with these methods.

2. *Multiprogramming* by E. F. Codd.

Multiprogramming is the concurrent use of a digital computer on two or more problems. The author presents the subject matter in a clear concise manner, carrying the reader from a discussion of the basic concepts to the hardware implementation. Problems of scheduling, storage allocation and queueing are discussed with emphasis on realistic and optimal techniques for programming several systems like the IBM 7030 "Stretch," RM400, Gamma 60, and Atlas.

The article can be read by people with no prior knowledge of the subject, but can also broaden the understanding of those who have some awareness of multiprogramming.

3. *Recent Developments in Nonlinear Programming* by Philip Wolfe.

At the present time there is extensive activity by several investigators who are applying both analog and digital computers to the following problem, usually called the optimization problem. Given a real valued function of several parameters with equality or inequality constraints, find the values of the parameters for which the function is an extremum (maximum or minimum). To perform this task the computer must be programmed so that each change of parameters, subject to the constraints, increases (decreases) the function of which a maximum (minimum) is being sought.

This is a simple programming procedure. The primary difficulty comes because we cannot be certain when we find an extremum if it is the largest maximum or smallest minimum. Related questions are (1) "How many extrema are there?" and (2) "Where should we look in the parameter space for them?" These questions remain largely unanswered at the present time.

The author of this paper discusses an important subclass of the general optimization problem, namely that in which both the function and constraints are expressed as nonlinear algebraic equations. He has used a geometric approach to assist the reader in understanding each method. I commend the author for the clarity and conciseness of his discussion.

4. *Alternating Direction Implicit Methods* by Garrett Birkhoff, Richard S. Varga, and David Young.

This paper describes some techniques which may be used to solve elliptic and parabolic partial differential equations. Unfortunately, I must apologize to the authors and reader because my lack of experience in this field prevents me from assessing properly the value of these methods. Because of the stature of the authors, however, I would suggest that this article is probably useful to interested readers.

5. *Combined Analog-Digital Techniques in Simulation* by Harold K. Skramstad.

Of the two devices the analog computer is faster, more easily programmed, and provides solutions in a form more rapidly analyzed by engineers. The digital computer is more accurate and can be programmed to perform more mathematical operations. A hybrid computer is one which combines analog and digital operations and provides a versatility in solving problems not found in the analog or digital computer alone.

There is increasing activity in the development of hybrid computers. My own opinion is that this will mark the most significant advance in scientific computation in the present decade. In particular I see this computer spurring work in the optimization of systems.

The author describes with thoroughness and clarity the several techniques for hybridization which have already been pursued with success. His paper is essential reading for those who would like to know the structure of the computer likely to be in use five years hence.

6. *Information Technology and the Law* by Reed C. Lawlor.

This paper describes how the computer can be used for (1) documentation of particular legal points and (2) estimation of the probability of a favorable decision, based upon pertinent factors in a case. Property 1 is a cataloguing operation which is routine in a conceptual sense. Property 2 requires a priori weighting of factors and is strongly dependent upon the judgment of the programmer.

The article makes for entertaining and easy reading.

In summary this book should be read by anyone interested in current developments in computers. Except for the last one on law, however, the articles require effort on the part of the reader.

LEON LEVINE, Scientific Data Systems

Écrits et Mémoires Mathématiques d'Évariste Galois. Edition critique intégrale de ses manuscrits et publications. Edited by R. Vourgne et J. P Azra. Gauthier-Villars, Paris, 1962. xxii+541 pp. $16.50.

It is a strange fact that until now there has been no complete and critical edition of the manuscripts, studies, fragments and correspondence of such a giant of modern mathematics as Évariste Galois. Part of the reason may be found in the peculiar prudery of some biographers who serve more as censors than historians. This is probably the reason why Galois' fuming statements against the French Academy of Science, written in the Sainte-Pélagie prison, did not appear in print until a couple of years ago.

"This book includes all we have left of Évariste Galois, memoirs, articles, studies, scratch notes and letters. We have tried to make it look a little like their author." (Foreword) The many facsimiles are valuable and interesting; they include excerpts from his scientific writings up till the letters from the last fateful night before the duel. But there are also amusing pages of doodling, drawings, impertinent remarks, and repetitions of his beloved's name, Stephanie. These pages are peculiarly reminiscent of the notes of Abel, greatly admired by Galois.

In short, this is a book of great value which should be included in every mathematical library.

OYSTEIN ORE, Yale University

Theory and Design of Digital Machines. By Thomas C. Bartee, Irwin L. Lebow, and Irving S. Reed. McGraw-Hill, New York, 1962. ix+322 pp. $11.50.

The authors have obviously written this book for use as a text to be used for senior or graduate level course work (as is mentioned in the inside jacket cover preface). I believe that because of the format in which this book is written and because of the authoritative material contained therein, it is equally useful as a self-teaching or reference text for mathematicians, programmers, and engineers in the field.

By defining the basic components and the fundamental algebraic techniques used in elementary computer design and the logical functions (gates) the authors establish the basic aspects of computer design in the early chapters of this book. In the later chapters a more exhaustive approach is presented for the expansion of the algebraic techniques for circuit design. Discussions of both the general purpose computer and the special purpose computer provide the reader with a look at actual computer system design requirements, especially of the computer control functions.

The authors have presented a systematic approach to the logical design of digital machines which seems fundamental both to the design engineer and to the mathematician-programmer.

LEON LEVINE, Scientific Data Systems

Introduction to Differentiable Manifolds. By Serge Lang. Interscience, New York, 1962. vi+126 pp. $7.00.

There are two aspects of this book that strike the reader immediately. The first is that the book begins with the basic definitions of categories, functors, and natural transformations. The second is that manifolds as here defined are locally homeomorphic to Banach spaces. We would like to comment on both of these aspects before turning to a discussion of the specific contents of the book.

Category theory is playing a role in modern mathematics analogous to that played by set theory fifty to a hundred years ago. Roughly speaking, set theory began as a descriptive theory. Then, as its universal applicability was realized, and as more powerful set theoretical methods developed, it became a prescriptive theory, until, by now, its prescriptives have become so intrinsic to mathematics that they tend to be accepted merely as descriptions.

Category theory is just beginning to enter into the prescriptive stage of this sequence. When it was invented some eighteen years ago by Eilenberg and MacLane (Trans. A. M. S. 58 (1945) 231–295), it was almost purely descriptive, and its usefulness seemed restricted to algebraic topology and a few special topics in algebra. However, about ten years ago, it was found that there were quite reasonable ways to introduce additional structure into general categories, principally through the use of related notions of universal mapping problems, adjoint functors and representable functors. These ideas have been most dramatically exploited by Grothendieck in a series of papers mainly related to algebraic geometry. The methods and prescriptions laid down by him seem, however, to be universally applicable, and one can expect that in the near future they will completely revolutionize the presentation of abstract mathematics, even of set theory itself. In the terminology of Kuhn (The Structure of Scientific Revolution, University of Chicago Press, 1962), there has been a change in the paradigms of mathematics.

Lang's book is the first in differential geometry to take advantage of these developments. One finds them in the universal mapping properties characterizing submanifolds (p. 19), kernels and cokernels of vector bundle morphisms (p. 43), and fibre products or pullbacks (p. 23 and p. 38). The only really significant use of functors is in the construction of associated bundles (p. 47). The use of category theory, then, is rather minimal but it is probably as much as—if not more than—contemporary differential geometric traffic will bear. A systematic study of differentiable manifolds from the standpoint of category theory will include a good deal more than is found in this slim volume. For example, what about (generalized) direct limits of manifolds?

We now turn to the second innovation of the book, that of considering locally Banachian manifolds. From this standpoint the book can be considered, in terminology and spirit, as a continuation of Chapters 8 and 10 of Dieudonné, Foundations of Modern Analysis (Academic Press, 1960). Chapter 8 is essentially the prerequisite for Lang and, in fact, both books are enriched by a concurrent reading.

Infinite dimensional manifolds were first formally considered by J. Eells (see references in Lang), and the observation that most of the elementary theory generalizes easily is due to him. He is also responsible for the only significant examples, function spaces, and the only deep theorem, a generalization of Alexander-Poincaré duality. These results probably provide a sufficient motivation for treating the infinite dimensional case *ab initio*, aside from the fact that such a treatment forces proofs to be as natural and coordinate free as possible—a desirable end in itself. Also, it is possible that an infinite dimensional manifold with a distinguished two-form (or perhaps a one-form) is the natural habitat of a quantum field theory.

Finally, we turn to the specific contents of the book. Chapter I deals with the necessary category and calculus preliminaries, the latter being closely related to Dieudonné, Chapters 7 and 8.

Chapter II gives the basic definitions of manifolds, differentiable morphisms, etc. The definition of the tangent space on page 20 is marred by a serious misprint. Presumably the defining sentence should read "The equivalence classes of such triples form a set called the tangent space of X at $x \cdots$." There is also an unfortunate set-theoretical quibble in that, as defined, the equivalence classes are proper classes and hence too large to be elements of anything. An alternative definition without this defect is given on page 64. The chapter concludes with a discussion of partitions of unity. It might be pointed out that the locally compact manifolds dealt with first are exactly the finite-dimensional manifolds. The most general case treated is that of locally Hilbertian manifolds. There is a brief appendix on manifolds with boundary.

Chapter III discusses vector bundles but not general differentiable fibre spaces. The fundamental construction introduced is that of the pullback of a bundle over a morphism. Exact sequences of bundles are discussed in a rather strange, noncategorical way, presumably because the category of Banach spaces does not have nice exactness properties. The only examples are the tangent bundle, the normal bundle of an immersed submanifold and bundles constructed from them by applying functors to get associated bundles. There is a misprint in the discussion of these on page 47 where $L(E, E')$ should read $L(E', E)$.

Chapter IV treats vector fields, the existence of flows, the exponential map and tubular neighborhoods. The main innovation is in the use of sprays, the exposition being credited to Palais. The reader will probably find it helpful to read Appendix 2 of Lang and Chapter 10 of Dieudonné while reading this chapter.

Chapter V discusses differential forms, the bracket of vector fields, the exterior derivative and the Poincaré lemma in a more or less standard way. The relation between the exterior derivative and the bracket operation is interestingly clarified in terms of differentiable functors.

Chapter VI proves the Frobenius theorems in full generality as in Dieudonné except that the use of vector bundles greatly simplifies the notation.

Chapter VII shows that Riemannian metrics exist on locally Hilbertian

manifolds. The orthogonal group of a Hilbert space is called the Hilbert group and it is shown that a Riemannian metric on a bundle is equivalent to a reduction of the group of the bundle to the Hilbert group. Finally there are two applications, one to straightening tubular neighborhoods and the other giving a nice derivation of the geodesic spray determined by a Riemannian metric.

One is brought to the threshold of differential geometry, but unfortunately, except for a short appendix on Hilbert spaces, the book ends here. It would have been nice to have a discussion of some examples of manifolds in this sense or a proof, for example, that the geodesics constructed from the geodesic spray have stationary length. The style of the book is as condensed as a research paper and it does not even contain problems hinting at all of the auxiliary results that should appear in a complete discussion of the subject, even within its own framework. It seems highly unlikely that anyone who is not already familiar with differential geometry will get much from reading this book. Nevertheless, the paradigms have changed and the subject will never be the same again. No course in differential geometry can afford to neglect the viewpoint and the material of this book and, in fact, with sufficient geometrical padding it could be the basis for a very interesting series of lectures.

JOHN W. GRAY, University of Illinois

For more recent significant results see R. S. Palais, Morse theory on Hilbert Manifolds, *Topology*, 2 (1963) 299–340.

BRIEF MENTION

The Fibonacci Quarterly. The official Journal of the Fibonacci Association, vol. 1, no. 1.
This new journal is "devoted to the study of integers with special properties." The editors hope for reader participation from mathematics teachers and students at all levels.

Deuxième Congrès de L'Association Française de Calcul et de Traitement de L'Information. A.F.C.A.L.T.I. Gauthier-Villars, Paris, 1962. 524 pp.

Analytic Geometry and Calculus, 2nd ed. By Lyman M. Kells. Prentice-Hall, Englewood Cliffs, N. J., 1963. 656 pp. $8.75.

Optical Character Recognition. By G. L. Fischer, Jr., D. K. Pollock, B. Radock, and M. E. Stevens. Spartan Books, Washington D. C., 1962. viii+412 pp. $10.00.

Tables of Normalized Associated Legendre Polynomials. By S. L. Belousov. Translated from the Russian by D. E. Brown. Vol. 18 in Mathematical tables series. Pergamon Press, New York, 1962. 379 pp. $20.00.

Philosophy of Science. Edited by Bernard Baumrin. The Delaware Seminar. Vol. 1, 1961–1962. Wiley, New York, 1963. 370 pp. $9.75.

Sixteen papers are grouped under the following headings: I. Two Basic Distinctions, II. Scientific Explanation and Prediction, III. Philosophical Aspects of the Foundation of Mathematics, IV. Philosophical Aspects of Biology, V. Philosophical Aspects of the Social Sciences, VI. Philosophical Aspects of Physics.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Associate Secretary, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor E. A. Cameron, University of North Carolina, represented the Association at the inauguration of Donald C. Dearborn as President of Catawba College on February 20.

Dr. Mariano Garcia, Jr., University of Puerto Rico, represented the Association at the installation of Lawrence C. Wanlass as First President of the College of the Virgin Islands on April 11.

Dean Earl Walden, New Mexico State University, represented the Association at a convocation and subsequent events inaugurating the Seventy-Fifth Anniversary of the University of New Mexico on February 25–28.

Dr. Abolghassem Ghaffari, National Bureau of Standards, Washington, D. C., has accepted a position as Aerospace Scientist with the National Aeronautics and Space Administration, Greenbelt, Maryland.

Professor S. A. Jennings, University of British Columbia, has been appointed Professor and Head of the Department of Mathematics at the University of Victoria.

Dr. Julius Lieblein, David Taylor Model Basin, Washington, D. C., has accepted a position as Mathematical Statistician in the Office of the Assistant Postmaster General for Finance, Post Office Department, Washington, D. C.

Professor Morris Marden, Chairman of the Mathematics Department, has been appointed University of Wisconsin-Milwaukee Distinguished Professor.

Mr. Winston Riley, III, CEIR, London, England, has been appointed Manager of management science at CEIR's research and computing center, Arlington, Virginia.

Professor C. F. Stephens, State University College of New York at Geneseo, has been appointed Acting Chairman of the Mathematics Department.

Associate Professor H. E. Taylor, Florida State University, Tallahassee, has been appointed Acting Chairman of the Mathematics Department.

Professor O. L. Dustheimer, Youngstown University, died on April 2, 1963. He was a charter member of the Association.

Professor Emeritus C. H. Lehmann, Cooper Union, died on December 31, 1963. He was a member of the Association for 46 years.

Professor F. H. Miller, Cooper Union, died on January 11, 1964. He was a member of the Association for 38 years.

UNIVERSITY OF MONTREAL—SÉMINAIRE DE MATHÉMATIQUES SUPÉRIEURES

Under the sponsorship of the North Atlantic Treaty Organization (NATO) and the Canadian Mathematical Congress, the third session of the University of Montreal SÉMINAIRE DE MATHÉMATIQUES SUPÉRIEURES will be held next summer from July 6 to August 14, 1964. The program will include five main courses: Professor Jean Dieudonné, Institut des Hautes Etudes Scientifiques, Paris, "Fondement de la géométrie algébrique moderne"; Professor Beno Eckmann, Ecole Polytechnique Fédérale, Zurich, "Homotopie et cohomologie"; Professor Peter Hilton, Cornell University, Ithaca, "Catégories non-abéliennes"; Professor Geoffrey Fox, Université de Montréal, "Intégra-

tion dans les groupes topologiques"; Professor Paulo Ribenboim, Queen's University, Kingston, "Théorie des valuations." Apart from these courses, the program will include a certain number of lectures given by guest speakers. Registrants may make application for financial assistance to cover travel and living expenses. To obtain full information and registration forms, write to: Department of Mathematics, University of Montreal, P. O. Box 6128, Montréal, Québec, Canada.

ARLINGTON STATE COLLEGE

A three-week conference on the teaching of relativity at the undergraduate level will be held between June 15 and July 3, 1964, at Arlington State College with financial support from the National Science Foundation. It is intended for college teachers interested in the basic ideas and principles of special and general relativity and their incorporation into physics, mathematics, and engineering courses. The program will consist of lectures, discussions, problems, projects and possibly some experimental work. Most of the lectures will be given by members or visiting members of the Southwest Center for Advanced Studies in Dallas, among them Michel Cahen, Wolfgang Rindler, Ivor Robinson and Joseph Weinberg. There will be stipends and travel allowances for the participants. A selected group of participants will act as discussion leaders and assistants at the conference. These will attend a one-week preliminary conference starting June 8.

Arlington State College is located in northern Texas, about halfway between Dallas and Fort Worth. All buildings involved are modern and fully air-conditioned' There are lakes and recreational areas nearby.

Further information should be requested immediately from the conference director, Dr. Jason Ellis, Department of Physics, Arlington State College, Arlington, Texas.

UNIVERSITY OF OKLAHOMA

The annual national conference of the Advanced Placement Program in Mathematics will be held at the University of Oklahoma on June 25–27. Further information may be obtained by writing to the Mathematics Department, University of Oklahoma, Norman, Oklahoma.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE NEW JERSEY SECTION

The eighth annual meeting of the New Jersey Section of the Mathematical Association of America was held at the Lawrenceville School, Lawrenceville, New Jersey, on November 2, 1963. Dr. Sheldon S. Myers, Chairman of the Section, presided at the morning and afternoon sessions. Ninety-seven persons attended the meeting, including fifty-six members of the Association.

At the business meeting, Professor Robert M. Walter of Douglass College, was elected Chairman of the Section (November 1964) and Professor Joshua Barlaz of Rutgers—The State University was elected Member-at-Large of the Executive Committee (November 1966). Reports were given by Miss Janet Dunning, Chairman of the High School Contest Committee; Dr. H. O. Pollak, Governor of the Section; Professor F. A. Varrichio, Secretary-Treasurer; and Professor John Reckzeh, Chairman of the Speaker's Bureau.

At the morning session the following papers were presented:

1. *Changes in Twentieth Century Algebraic Activity* by Professor Earl J. Taft, Rutgers—The State University, introduced by the Chairman.

Algebraic investigations up to the early part of the twentieth century centered around concrete questions in group theory, linear algebra, algebraic number theory and algebraic geometry. Then the emphasis shifted to the postulational development of abstract systems such as groups, integral domains and modules. During the last three or four decades, algebra has developed both inside and outside this classical framework. For example, homological algebra and universal algebra arose inside it, and topological algebra and categorical algebra outside. The development has been marked by connections of various algebraic disciplines with each other and with other mathematical disciplines which inspired them.

2. *Numerical Analysis vs. Mathematics* by Dr. R. W. Hamming, Bell Telephone Laboratories, Incorporated. (By invitation.)

The purpose of this talk is to show by examples that the standards, tastes, aims and objectives of Numerical Analysis are very different from those of Mathematics, and that mathematical rigor is often less relevant than choosing a reasonable model.

At the afternoon session the following paper was presented:

The Role of Calculus in Secondary School by Professor Albert A. Blank, New York University. (By invitation.)

The universal adoption of calculus as the capstone of the secondary curriculum is probably only a matter of time. When it is appreciated that the twelfth year student of mathematics is probably better motivated and more capable than the typical college freshman, much of the argument against the adoption of calculus loses its force. In many schools the elimination of the weak seventh and eighth grade traditional course provides the extra time needed for preparing the ground. We may expect great improvements in the training of teachers and the preparation of students for further work in mathematics and science to ensue.

As part of the program a discussion followed Professor Blank's paper. Doctor Marion G. Epstein of Educational Testing Service, Doctor R. W. Hamming of Bell Laboratories, Inc., and Mr. Peter J. Kiernan of the Lawrenceville School commented on Calculus in Secondary School.

F. A. VARRICHIO, *Secretary*

SPECIAL DECEMBER MEETING OF THE OHIO SECTION

A special meeting of the Ohio Section of the Mathematical Association of America was held at Denison University, Granville, Ohio, on Saturday, December 7, 1963. There were seventy-four persons registered in attendance, including sixty-eight members of the Association. Professor Charles E. Capel, Chairman of the Section presided at the morning session and Professor Robert Roberts, of the Program Committee presided in the afternoon.

No formal papers were presented at the sessions, which were devoted to general discussion on three subjects: 1) the freshman-level undergraduate curriculum, 2) teacher training and certification, and 3) revision of the by-laws of the Ohio Section. Professor David Lipsich, of The University of Cincinnati made the opening statement and led the discussion on the first subject, Professor Clarence Heinke of Capital University on the second and Professor Charles Capel of Miami University on the third.

FOSTER BROOKS, *Secretary*

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Summer Meeting at the University of Massachusetts, Amherst, Massachusetts on August 25, 26 and 27, 1964.

The Register will be conducted from 9:00 A.M. to 5:00 P.M. on each of these three days.

There is no charge for registration, either to job applicants or to employers, except when the late registration fee for employers is applicable. Provision will be made for anonymity of applicants upon request and upon payment of $3.00 to defray the cost involved in handling anonymous listings.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence, Rhode Island 02906 for application forms or for position description forms. These forms must be completed and returned to Providence not later than July 15, 1964, in order to be included in the listings at the Summer Meeting in Amherst. Position Description forms which arrive after this closing date, but before August 10 will be included in the register at the meeting for a late registration fee of $3.00. The printed listings will be available for distribution both during and after the meeting.

It is essential that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

PROPOSED AMENDMENT TO THE BY-LAWS OF THE MAA

Increasing costs of administering the Association have necessitated an increase in the annual dues of the Association from $5 to $6, as recommended by the Board of Governors at its meeting on January 24, 1964, at the University of Miami. The Association incurred a deficit in 1962. The surplus in 1963 was due to some nonrecurring credits, deferment of payment of several substantial bills to 1964 which would normally have been paid in 1963, and to no charge being made for distribution to the Association's membership of the COMBINED MEMBERSHIP LIST, which is distributed only every other year (such a charge occurred in 1962 and will again occur in 1964). The cost of administering the Association is expected to increase further by the need for engaging the services of a professional auditor and legal counsel and for providing for certain insurance benefits for clerical employees of the Association. While the Association receives overhead on its grants from NSF, this overhead is compensation only for the additional expenses incurred in administering these projects; it does not cover the increase in general Association activities brought about indirectly by these projects. The Association has also been asked to make certain unanticipated payments to NSF in connection with these NSF-sponsored programs. The budget for 1964 anticipates a deficit of $20,000.

Consequently, at the business meeting of the Association to be held at the University of Massachusetts in Amherst, Massachusetts, August 25, 1964, a motion will be made to amend Article VII, Section 2, as follows:

The annual dues of each ordinary member shall be six dollars ($6), including a subscription to the official journal.

HENRY L. ALDER, *Secretary*

THE CHAUVENET PRIZE

The Board of Governors at its meeting on January 24, 1964, at the University of Miami voted to adopt the following regulations governing the Association's Award of the Chauvenet Prize.

1. The Chauvenet Prize is to be awarded at the Annual Meeting in January of the Mathematical Association of America. The prize is to be $100, together with a certificate, and is to be awarded to a member of the Association for a noteworthy paper published in English, such as will come within the range of profitable reading for members of the Association. The purpose of the prize is to stimulate the writing of expository works by American scholars.

2. Ordinarily, the award is to be made for material published during the calendar year beginning January 1 two years prior to the time of the award. However, if deemed

APPOINTMENT OF MAA REPRESENTATIVES

The Board of Governors has approved a plan whereby *MAA Representatives* shall be appointed in all universities and colleges, including junior colleges, in the United States and Canada. Responsibility for appointment of these representatives has been assigned to the Section Chairmen. The Buffalo office of the Association has requested appointment of these representatives from each Section Chairman by June 15. He will send a list of all MAA Representatives appointed within his Section to the Buffalo office.

The procedure for selecting MAA Representatives in each Section is left to the officers of the Section. The following procedures are suggested:

a. Some Sections may find it suitable to have nominations made to the Section Chairman by the Section Nominating Committee or Section Membership Committee, others by assigning this responsibility to the Section officers.

b. If the chairman of the mathematics department is a member of the MAA, another suggestion is that the Section Chairman request him to nominate a representative for the institution, preferably someone other than himself. (The reasons for this should be obvious from the description of the duties of the representative as listed below. On the other hand, there may be cases where the department chairman would be the only appropriate person, obviously so if he is the only MAA member in the department.)

c. If the chairman of the mathematics department is not a member of the MAA, it is suggested that a direct approach be made to some MAA member, asking him to serve as MAA Representative. If no one in the department is an MAA member, every effort (including personal contact) should be made to remedy the situation.

The duties of the MAA Representatives are outlined as follows:

The primary purpose of the MAA Representative is to provide members and prospective members with current information on the activities of the Association and to secure new MAA members from both faculty and students. The methods used to achieve this are largely left to each representative, but it should be emphasized that the MAA is not interested in increasing its membership just for the sake of size and that no "high-pressure" membership campaigns are contemplated. It is known, however, that many persons are not members of the MAA who would benefit from membership and who, in turn, would benefit the MAA.

It is suggested that the MAA Representative make an annual personal invitation to the members of the staff to join the MAA, with special attention to new staff members. Students (both graduate and undergraduate) should be made aware of the opportunity to join the MAA through notices and announcements. Additional contacts can be made through publicizing sectional and national meetings and by assisting in arrangements to maximize attendance at Section meetings by, for example, arranging group transportation when needed.

It is suggested that MAA Representatives in each Section meet at the time of Section meetings. They might be addressed by the Sectional Governor on matters considered by the Board of Governors.

Every effort will be made by the Buffalo office and the Section officers to provide the MAA Representatives with information which might be of interest to them.

HENRY L. ALDER, *Secretary*

REPORT OF THE TREASURER FOR THE YEAR 1963

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1963. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes a copy of the complete report of the Treasurer may obtain one by writing to the Buffalo office of the Association.

Because of a change in the way in which NSF is handling grant payments, it is difficult to compare this report with those for previous years. In brief the assets of the Association have decreased during 1963, but the liabilities have also decreased by a corresponding amount.

The funds held by the Association for its own use and for publication purposes have all increased with one exception. The balance in the Dunkel Fund is smaller due to the cost of printing MAA Studies Volume II.

A donor who prefers to remain anonymous has established a new fund whose income is unrestricted but will probably be used for publication purposes. Several smaller contributions received during the year have been added to the General Fund.

	January 1, 1963	*December 31, 1963*
ASSETS OF THE ASSOCIATION		
M & T Trust Company, checking account	$225,097	$ 96,624
M & T Trust Company, special account	—	33,199
Savings accounts	17,179	—
Securities at market values	396,873	216,934
	$639,149	$346,757
FUNDS OF THE ASSOCIATION		
Current Fund	$ 734	$ 4,944
MATHEMATICS MAGAZINE	2,727	150
Carus Fund	49,479	57,204
Chace Fund	9,325	11,324
Houck Fund	8,436	9,040
Chauvenet Fund	2,876	2,974
Dunkel Fund	24,233	21,897
Anonymous Fund	—	4,237
General Fund	121,341	163,778
	$219,151	$275,548
Visiting Lecturers	65,051	19,236
Secondary School Lecturers	72,978	7,940
CUPM	272,793	3,796
High School Contests	874	4,620
Production of Films	215	690
Educational Media	8,087	—
Cooperative Summer Seminars	—	1,728
NSF grants	—	33,199
	$639,149	$346,757

CALENDAR OF FUTURE MEETINGS

Forty-fifth Summer Meeting, University of Massachusetts, Amherst, August 24–26, 1964.

Forty-eighth Annual Meeting, Denver-Hilton Hotel, Denver, Colorado, January 28–30, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Southern Illinois University, Carbondale, May 14–15, 1965.
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 19–20, 1965.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY, Rutgers, The State University, New Brunswick, November 7, 1964.
NORTHEASTERN, Worcester Polytechnic Institute, Worcester, Mass., Nov. 28, 1964.
NORTHERN CALIFORNIA, College of San Mateo, February 6, 1965.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Washington State University, Pullman, Washington, June 19, 1964.
PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 21, 1964.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN MATHEMATICAL SOCIETY, Pullman, WASHINGTON, June 20, 1964.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Maine, Orono, June 22–26, 1964.
ASSOCIATION FOR COMPUTING MACHINERY, Philadelphia, August 25–28, 1964.
CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Detroit, November 26–28, 1964.
INSTITUTE OF MATHEMATICAL STATISTICS, Berne, Switzerland, September 14–16, 1964.
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Seattle, Washington, July 1, 1964.
OPERATIONS RESEARCH SOCIETY OF AMERICA, Hotel Leamington, Minneapolis, October 7–9, 1964.

NEW AND RECENT TEXTS AND REFERENCES FROM MACMILLAN

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY, Vol. 71, No. 5, May, 1964, pp. 471–592

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JUNE-JULY 1964

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10053.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For nonmembers the subscription price is $8.00.

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BIRKHOFF'S AXIOMS FOR SPACE GEOMETRY

ROLAND BROSSARD, Université de Montréal

1. Introduction. The motivation for this study is a definition of euclidean geometry leaving open the possibility of extension to higher dimensional spaces, based on the intuitive ideas concerning the use of graduated rulers and protractors. In fact, the only essential change in the Birkhoff's presentation is a weakening in the protractor axiom; this weakening allows the geometry of three-dimensional space to be constructible.

The system of axioms is based on "coordinate functions"; they are intuitively conceived as "applications" of a long graduated ruler to the lines, and as "applications" of a protractor to the plane bundles of half-lines. Distance, angular measure, and the betweenness relation are defined in terms of coordinate functions. The axioms are formulated in such a way as to exhibit a certain symmetry between the properties of coordinate functions for the elements of the lines and the properties of coordinate functions for the elements of the bundles, the main difference consisting in the value-rings.

2. Primitive notions. *Points* are abstract undefined objects. Primitive terms are: point, line, coordinate function of a line, half-line, bundle of half-lines, and coordinate function of a bundle of half-lines.

3. Axioms on points, lines, and coordinates of the points of the lines. Certain subclasses of points are called *lines*. The axioms on the lines are:

L_1. *There exist at least two distinct points.*

L_2. *If A and B are two distinct points, then there exists one and only one line containing A and B.*

L_3. *There exist points not all on the same line.*

A set of points is said to be *collinear* if this set is a subset of a line. Two sets are *collinear* if the union of these two sets is collinear. Coordinate functions on the lines are introduced by the following axiom.

CL_1. *There exists associated with each line* L, *a nonempty class X of one-to-one mappings x of L onto the field R of real numbers. If x_i is a member of X and if x_j is any one-to-one mapping of* L *onto R, then x_j is a member of X if and only if for all $A \in$* L *and for all $B \in$* L.

$$|x_i(A) - x_i(B)| = |x_j(A) - x_j(B)|.$$

The elements of X are called *coordinate functions* of L. The *distance* between two points A and B, denoted AB (or BA) is defined to be the unique nonnegative number $|x(A) - x(B)|$ where x is an arbitrary member of X. The point B is *between* the points A and C if A, B, and C belong to the same line and either $x(A) < x(B) < x(C)$ or else $x(C) < x(B) < x(A)$. We shall now show that this betweenness relation is defined independently of the coordinate function considered on the line containing the points A, B, and C.

If B is between A and C with respect to a coordinate function x_i then

(3.1) $x_i(A) < x_i(B) < x_i(C)$ or else (3.2) $x_i(C) < x_i(B) < x_i(A)$.

Let x_j be an arbitrary coordinate function for the same line. Axiom CL_1 implies that

$$x_i(A) - x_i(\text{B}) = x_j(A) - x_j(B) \quad \text{or else} \tag{3.3}$$

$$x_i(A) - x_i(B) = x_j(B) - x_j(A), \tag{3.4}$$

$$x_i(A) - x_i(C) = x_j(A) - x_j(C) \quad \text{or else} \tag{3.5}$$

$$x_i(A) - x_i(C) = x_j(C) - x_j(A), \tag{3.6}$$

$$x_i(B) - x_i(C) = x_j(B) - x_j(C) \quad \text{or else} \tag{3.7}$$

$$x_i(B) - x_i(C) = x_j(C) - x_j(B). \tag{3.8}$$

If (3.1) is valid then all left members of (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) are negative and if (3.2) is valid the same left members are all positive. Equations (3.3), (3.5), and (3.7) are valid or else equations (3.4), (3.6), and (3.8) are valid because any two equations in a group implies the third one. Consequently with respect to x_j, B is also between A and C.

For collinear points A, B, C the point B is between the points A and C if and only if $AB+BC=AC$. If O and A are two distinct points of a line, we call *half-line OA* with *end-point O* the set of points P on that line such that O is not between A and P. In speaking of a half-line OA, the first element of the ordered couple (O, A) will always represent the end-point. If A and B are distinct points, the set of points containing A, B, and all the points between A and B is called a *segment*. The distance AB is called the *length* of the segment AB. A segment without its end-points is an *interval*.

M is the *mid-point* of the segment AB if M is an element of this segment and if $AM=MB$. If x is a coordinate function for the line AB then there exists on that line a point M defined by the relation

$$x(M) = \frac{x(A) + x(B)}{2}.$$

We can easily verify that M is between A and B (i.e., M belongs to the segment AB), and that $AM=MB$. Consequently every segment has a mid-point. There is only one mid-point because if M and M' are two mid-points of the segment AB, then $A\neq B$, M and M' belong to the line AB, $AM=MB$, $AM'=M'B$ and there exists a coordinate function x for the line AB such that

$$x(A) - x(M) = x(M) - x(B)$$

and

$$x(A) - x(M') = x(M') - x(B);$$

consequently $x(M')-x(M)=x(M)-x(M')$, $x(M)=x(M')$, and $M=M'$.

4. Axioms on bundles and coordinates of the half-lines of the bundles. Certain subclasses of the class of all half-lines with the same end-point are called *bundles*. The common endpoint O of the elements of a bundle is called the *vertex* of the bundle; the notation B_o will be used for a bundle of vertex O. An *angle* is an unordered couple of half-lines with the same end-point O; the point O is called the *vertex* of the angle, and the half-lines the *sides* of the angle. An angle is *straight* if the sides are distinct and collinear. The axiom on the bundle is:

B_1. *If l and m are two noncollinear half-lines with the same end-point O, then there exists one and only one bundle B_o containing these half-lines.*

The axiom on the coordinate functions of the bundles is:

CB_1. *There exists, associated with each bundle B_o, a nonempty class Φ of one-to-one mappings ϕ of B_o onto the equivalence classes of real numbers modulo 2π. If ϕ_i is a member of Φ and if ϕ_j is any one-to-one mapping of B_o onto the equivalence classes of real numbers modulo 2π, then ϕ_j is a member of Φ if and only if for all $l\in B_o$ and for all $m\in B_o$*

$$|\phi_i(l) - \phi_i(m)| \equiv |\phi_j(l) - \phi_j(m)|,$$

where $|\phi_i(l)-\phi_i(m)| \equiv |\phi_j(l)-\phi_j(m)|$ *stands for*

$$\phi_i(l) - \phi_i(m) = (\phi_j(l) - \phi_j(m)) \ (\text{modulo } 2\pi)$$

or

$$\phi_i(l) - \phi_i(m) = (\phi_j(m) - \phi_j(l)) \ (\text{modulo } 2\pi).$$

The elements of Φ are called *coordinate functions* of B_o. If x denotes a real number, we shall denote by $[x]$ the equivalence class modulo 2π containing x, and we shall denote by $\bar{y}$ the real number of the class $[y]$ such that $0\leqq\bar{y}<2\pi$. And $x\equiv y$ will mean $x=y$ (modulo 2π).

Let l, m be an angle belonging to a bundle B_o; the *measure* of the angle l, m with respect to B_o, denoted $\angle lm$, is the minimum of the two real numbers $\overline{\phi(l)-\phi(m)}$, $\overline{\phi(m)-\phi(l)}$ where ϕ is a coordinate function associated to the bundle B_o. $\angle AOB$ is independent of the coordinate function used to obtain it. If A, O, B are three distinct points, we write $\angle AOB$ for $\angle$(half-line OA)(half-line OB); the measure being calculated in a bundle B_o containing the half-lines OA and OB. We shall prove in the next section that the measure of an angle is independent of the bundle B_o in which this angle is embedded.

5. The continuity axiom and the similarity axiom.

CONTINUITY AXIOM. *If B_o is a bundle of vertex O, and if A, B are distinct nonvertex points of noncollinear half-lines of the bundle, then to every point P on the segment AB, there exists a half-line OC of B_o containing P such that*

$$[\angle AOP + \angle POB] = [\angle AOB].$$

Conversely if a half-line OC of the bundle B_o is such that $[\angle AOC + \angle COB] = [\angle AOB]$ *then there exists a point P belonging simultaneously to the halfline OC and to the segment AB.*

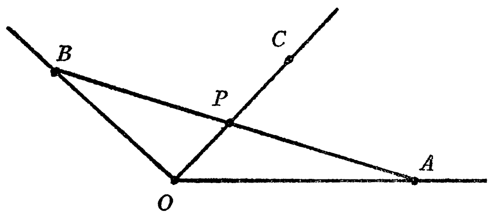

FIG. 1

For an angle defined by two noncollinear half-lines OA and OB, axiom B_1 implies that there exists only one bundle B_o containing this angle; therefore $\angle AOB$ is uniquely defined. If OA and OB coincide, the class of bundles containing the angle AOB may have more than one element, but in that case the measure of this angle is zero whatever the considered bundle of the class. We shall now show that the measure of a straight angle is π whatever the bundle in which this straight angle is embedded.

If an angle AOP has π for measure then the half-line OA is distinct from the half-line OP (otherwise $\angle AOP = 0$). If this angle AOP with measure π is not a straight angle then there exists a point B on the line AP, not on the line OA, such that P is between A and B (Fig. 1), and in the unique bundle B_o containing the half-lines OA, OB, OP the continuity axiom implies that $[\angle AOP + \angle POB] = [\angle AOB]$, that is to say,

$$[\pi] + [\angle POB] = [\angle AOB]. \tag{5.1}$$

But $\angle POB \neq 0$ for the points A, O, P are not collinear. Furthermore $\angle POB \neq \pi$, for if $\angle POB = \pi$ then from (5.1) we have $\angle AOB \equiv \pi + \angle POB \equiv \pi + \pi \equiv 2\pi \equiv 0$ and OA coincides with OB. Consequently (5.1) is contradictory and if an angle, with respect to a bundle containing this angle, has π for measure then this angle is a straight angle.

Let AOB be a straight angle and let B_o be an arbitrary bundle containing this angle. Then for an arbitrary admissible coordinate function ϕ for B_o we have either (i) $0 \leqq \overline{\phi(OA)} < \pi$ or else (ii) $\pi \leqq \overline{\phi(OA)} < 2\pi$. In case (i) let OC be the unique element of B_o such that $\phi(OC) = \phi(OA) + [\pi]$; then $\phi(OC) - \phi(OA) = \phi(OA) + [\pi] - \phi(OA) = [\pi]$, the angle AOC having, with respect to B_o, π for measure is a straight angle, and consequently the angle AOB, being equal to the angle AOC, also has π for measure. Similarly, in case (ii) let OD be the unique element of B_o such that $\phi(OD) = \phi(OA) - [\pi]$; then $\phi(OA) - \phi(OD) = \phi(OA) - (\phi(OA) - [\pi]) = [\pi]$, the angle AOD is straight, and the angle AOB has also π for measure.

This completes the proof that the measure of an angle is independent of the bundle in which it can be contained. Furthermore we have proved the following theorem concerning the measure of straight angles.

THEOREM 1. *The measure of an angle is* π *if and only if this angle is straight.*

We shall now consider properties of the bundles and their coordinate functions.

THEOREM 2. *If* $\phi \in \Phi$ *and if* ϕ_1 *is any mapping of* B_o *into the equivalence classes of real numbers modulo* 2π, *then* $\phi_1 \in \Phi$ *if and only if there exist* $\epsilon = [\pm 1]$, *and* θ *such that for every* $l \in B_o$, $\phi_1(l) = \epsilon\phi(l) + \theta$.

Proof. Sufficiency is obvious. To prove necessity suppose that $\phi_1 \in \Phi$. Because ϕ is one-to-one there exist half-lines m, n in B_o such that $\phi(m) = [0]$, $\phi(n) = [\pi]$. Let $\phi_1(m) = \theta$. Then

$$\phi_1(n) - \phi_1(m) = \pm (\phi(n) - \phi(m)) = \pm [\pi] = [\pi],$$

and so

$$\phi_1(n) = [\pi] + \theta. \tag{5.2}$$

For any $l \in B_o$ we have

$$\phi_1(l) - \phi_1(m) = \pm (\phi(l) - \phi(m)),$$

so

$$\phi_1(l) = \epsilon_l \phi(l) + \theta, \qquad \epsilon_l = [\pm 1].$$

Let $S = \{l \mid \epsilon_l = 1\}$, $T = \{l \mid \epsilon_l = -1\}$. We obviously have $m \in S \cap T$, and from (5.2), $n \in S \cap T$. Let $l' \in S$ and $l'' \in T$. Then $\phi_1(l') - \phi_1(l'') = \phi(l') + \theta + \phi(l'') - \theta = \phi(l') + \phi(l'')$. But we also have either

$$\phi_1(l') - \phi_1(l'') = \phi(l') - \phi(l'')$$

or

$$\phi_1(l') - \phi_1(l'') = - \phi(l') + \phi(l'').$$

In the first case we get $2\phi(l'') = [0]$, $\phi(l'') = [0]$ or $[\pi]$, $l'' = m$ or n, and $l'' \in S$; in the second case $l' \in T$. Hence, either S or T contains all the elements of B_o and ϵ_l is uniformly $[1]$ or $[-1]$. This proves the theorem.

The intuitive content is clear. Two coordinate functions for a bundle are related in such a way that one can be deduced from the other by a rotation of the protractor (graduated from 0 to 2π) or an inversion of the protractor followed by a rotation. The corresponding proposition for coordinate functions on the lines is also true; the proof is almost the same.

COROLLARY 1. *If* m *and* n *are two noncollinear half-lines of a bundle* B_o, *then there exists one and only one coordinate function* ϕ *such that* $\overline{\phi(m)} = 0$ *and* $\overline{\phi(n)} < \pi$.

Proof. Let $\phi_1 \in \Phi$. A necessary and sufficient condition for ϕ is that for some ϵ and θ

$$\phi(l) = \epsilon\phi_1(l) + \theta$$

with

$$\begin{aligned} [0] &= \epsilon\phi_1(m) + \theta, \\ \phi(n) &= \epsilon\phi_1(n) + \theta. \end{aligned} \tag{5.3}$$

Eliminating θ gives

$$\phi(n) = \epsilon(\phi_1(n) - \phi_1(m)).$$

Since m and n are noncollinear, $\phi_1(n) - \phi_1(m) \neq [0]$ or $[\pi]$, and hence there is a unique choice of $\epsilon = [\pm 1]$ such that $\phi_1(n) < \pi$. From (5.3), θ is then also uniquely determined.

We can here observe that the measurement of angles can be conceived intuitively as being obtained in the usual way, that is to say, application of a protractor (plain or half-disk) in such a way that one side of the angle coincides with the zero of the protractor and such that the other side corresponds to a number less than π (the measure of the angle).

COROLLARY 2. *If l, m, n are distinct elements of a bundle, and if l and n are collinear, then $\angle lm + \angle mn = \pi$.*

For if ϕ is the unique coordinate function such that $\overline{\phi(l)} = 0$ and $\overline{\phi(m)} < \pi$, then $\overline{\phi(n)} = \pi$ and $\angle lm + \angle mn = \pi$.

COROLLARY 3. *If l is a half-line of a bundle B_o and if α is a positive real number less than π and different from zero, then there exist two and only two distinct half-lines m, n such that $\angle lm = \angle ln = \alpha$.*

Proof. If ϕ is a coordinate function for B_o such that $\phi(l) = 0$ then there exist half-lines m and n such that $\phi(m) = -\phi(n) = [\alpha]$. Half-lines m and n have the required properties.

A *triangle* is an unordered set of three distinct points. The points are the *vertices* of the triangle. The three segments defined by the vertices of a triangle are the *sides* of the triangle. The three angles defined by the sides of a triangle are the *angles* of the triangle. In the context of triangles, for instance a triangle ABC, the measure of an angle, say angle ABC with vertex B, will be denoted $\angle B$ instead of $\angle ABC$. Two triangles are *similar* if the vertices can be labelled A, B, C and A', B', C' in such a way that

$$AB/A'B' = BC/B'C' = CA/C'A', \tag{a}$$

$$\angle A = \angle A', \qquad \angle B = \angle B', \qquad \angle C = \angle C'. \tag{b}$$

The constant ratio in (a) is called *factor of proportionality*. A triangle is *proper* if the vertices are noncollinear.

SIMILARITY AXIOM. *If two triangles ABC and $A'B'C'$ are such that $AB/A'B' = BC/B'C'$ and $\angle B = \angle B'$, then they are similar.*

6. Theorems on triangles.

THEOREM 4. *If two proper triangles ABC and $A'B'C'$ are such that $\angle A = \angle A'$ and $\angle B = \angle B'$, then they are similar.*

THEOREM 5. *If ABC is a proper triangle, then the sides AB and AC have equal length if and only if $\angle B = \angle C$.*

The proof of Theorem 4 and Theorem 5 can be found in [1].

LEMMA 1. *Let l be an element of a bundle B_o and let α, β be two numbers between zero and π. If m is a half-line of B_o such that $\angle lm = \alpha$, then there exists a half-line n of B_o with the following properties*: (a) $\angle ln = \beta$, (b) *for all points $A \in m$ and for all points $B \in n$ such that $A \neq B$ the segment AB has a point P in common with the unique line containing the half-line l.*

Proof. Let ϕ be a coordinate function such that $\phi(l) = [0]$ and $\phi(m) = [\alpha]$. Let n be a member of B_o such that $\phi(n) = [-\beta]$.

Case 1: $\alpha + \beta < \pi$. We have

$$(\phi(m) - \phi(l)) + (\phi(l) - \phi(n)) = \phi(m) - \phi(n),$$

i.e. $[\alpha] + (-([-\beta])) = [\alpha] - [-\beta]$, and $\angle lm + \angle ln = \angle mn$.

Case 2: $\alpha + \beta > \pi$. Let $\bar{l} \neq l$ be the half-line of B_o collinear with l; $\phi(\bar{l}) = [\pi]$. Then

$$(\phi(\bar{l}) - \phi(m)) + (\phi(n) - \phi(\bar{l})) = (\phi(n) - \phi(m)),$$

i.e. $([\pi] - [\alpha]) + (([-\beta]) - [\pi]) = [-\beta - \alpha] = [2\pi - (\alpha + \beta)]$, $[2\pi - (\alpha + \beta)] < \pi$, and $\angle \bar{l}m + \angle \bar{l}n = \angle mn$.

In both cases, the continuity axiom implies the desired result.

Case 3: If $\alpha + \beta = \pi$, then Theorem 1 implies that O is between A and B.

THEOREM 6. *If the triangles ABC and $A'B'C'$ are such that $AB/A'B' = BC/B'C' = CA/C'A'$, then they are similar.*

Proof. Case 1: A, B, and C are not collinear. Let $B_{A'}$ be a bundle of vertex A' containing the half-lines $A'B'$ and $A'C'$. Let l be a member of $B_{A'}$ such that (a) $\angle l(A'C') = \angle BAC$, (b) the segment $B'B''$, where B'' is on l and $A'B'' = A'B'$, has a point P in common with the line $A'C'$ (the existence of l is a consequence of Lemma 1). The triangles ABC and $A'B''C'$ are similar (similarity axiom). Consequently

$$AB/A'B'' = BC/B''C' = CA/C'A' = k = CA/C'A' = BC/B'C' = AB/A'B',$$

and $C'B' = C'B''$. The triangles $B'A'B''$ and $B'C'B''$ are isosceles and by Theorem 5 $\angle A'B'P = \angle A'B''P$, $\angle C'B'P = \angle C'B''P$.

If the collinear points A', P, C' are distinct, one of them is between the two

others, and the continuity axiom implies that $\angle A'B'C' = \angle A'B''C'$ (e.g. if P is between A' and C' then

$$[\angle A'B'P + \angle PB'C] = [\angle A'B'C'] \quad \text{and} \quad [\angle A'B''P + \angle PB''C] = [\angle A'B''C'],$$

so that $\angle A'B'C' = \angle A'B''C'$). If P coincides with A' or with B' the same is obviously true. This proves that the triangles ABC and $A'B'C'$ are similar. We observe that if A, B, and C are not collinear, then A', B', and C' are also not collinear.

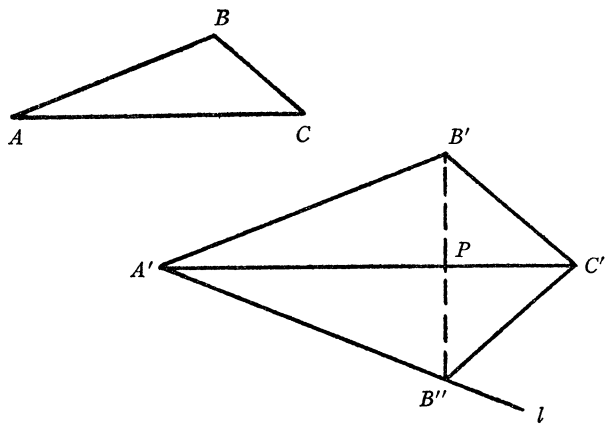

FIG. 2

Case 2: A, B, and C are collinear. In this case, A', B', and C' are also collinear. As A, B, and C are distinct, one of them, say B, is between the two others. It is sufficient to prove that the corresponding point B' is between A' and C'. Remembering that for collinear points A', B', C', the point B' is between the points A' and C' if and only if $A'B' + B'C' = C'A'$, we can see by the relations

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AB + BC}{A'B' + B'C'} = \frac{AC}{A'C'}$$

that if B' were not between A' and C' we would have a contradiction.

THEOREM 7. *The sum of the measures of the angles of a triangle is equal to* π.

Proof. We consider first the case where the triangle is a proper triangle with vertices A, B, and C. Let A', B', C' be the midpoints of the segments BC, CA, and AB respectively.

By multiple applications of the continuity axiom, we obtain:

1) A' between B and C implies the existence of a point E on line AA' and on line BB' between B and B';

2) B' between A and C implies the existence of a point E' on line BB' and on line AA' between A and A'; ABC being a proper triangle, $E = E'$;

3) A' between B and C implies the existence of a point D on line AA' and on line $C'B'$ between C' and B';

4) C' between A and B implies the existence of a point D' on line AE and on line $C'B'$ between A and E; again ABC being a proper triangle, $D=D'$. Furthermore E between A and A' and D between A and E imply that D is between A and A'. Then

$$[\angle AB'C' + \angle A'B'C'] = [\angle AB'A']$$

i.e., $\angle AB'C' + \angle A'B'C' = \angle AB'C'$ (for $\angle AB'C' < \pi$, and $\angle A'B'C' < \pi$), but $\angle AB'A' + \angle A'B'C = \pi$ and consequently $\angle AB'C' + \angle A'B'C' + \angle A'B'C = \pi$. The triangles ABC, $AB'C'$, $A'BC'$, and $A'B'C'$ being similar,

$$\angle AB'C' = \angle C, \qquad \angle A'B'C' = \angle B,$$

$$\angle A'B'C = \angle A, \qquad \angle A + \angle B + \angle C = \pi.$$

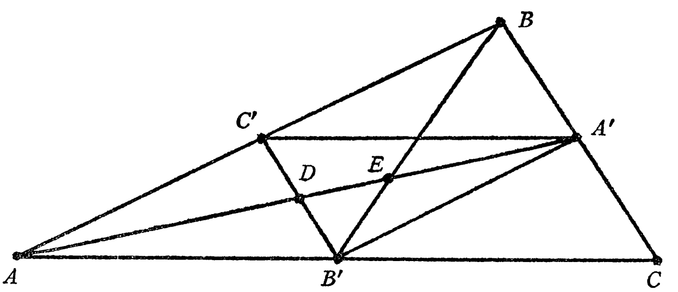

FIG. 3

To complete the proof, we observe that if the triangle is not proper, then one angle has for measure π and the two others have for measure zero.

As in [1] *congruence* of triangles is defined by similarity together with a factor of proportionality k equal to one.

Two distinct lines having a point in common determine six angles with nonnull measures. Two have π for measure and we can easily show that the four remaining ones form two sets, each set consisting of two distinct angles with the same measure. Two distinct lines having a point in common are said to be *perpendicular* if the four angles with measures different from zero and different from π have the same measure i.e., $\pi/2$.

LEMMA 2. *If L is a line and P is a point not on L, then there exists one and only one line containing P and perpendicular to L.*

Proof. Let A be an arbitrary point of L. There exists one and only one bundle of vertex A containing the lines L and AP. We shall denote this bundle by B_A. Let l' be the unique half-line of B_A, distinct from AP, such that $\angle ll' = \angle l(AP)$ where l is one of the half-lines with end-point A, determined by the line L and

the point A. There exists on l' a point P' such that $AP'=AP$. Lemma 1 implies that the segment PP' has a point I in common with the line L.

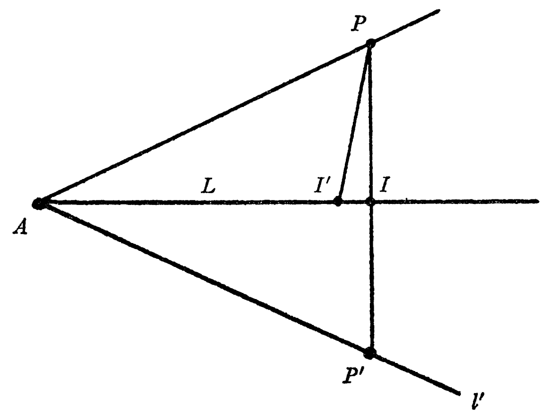

FIG. 4

If $I \neq A$, then the proper triangles API and $AP'I$ are congruent and $\angle AIP = \angle AIP' = \pi/2$ and PP' is perpendicular to L. If $I=A$, then PP' is also perpendicular to L. Furthermore, this perpendicular is unique because if there were to exist another perpendicular PI' to L the sum of the measures of the angles of the triangle PII' would be greater than π.

A triangle is *right-angled* if one of its angles has measure $\pi/2$.

LEMMA 3. *If a triangle ABC is right-angled at A, then the unique perpendicular from A to the line defined by the vertices B and C meets this line in a point D between B and C.*

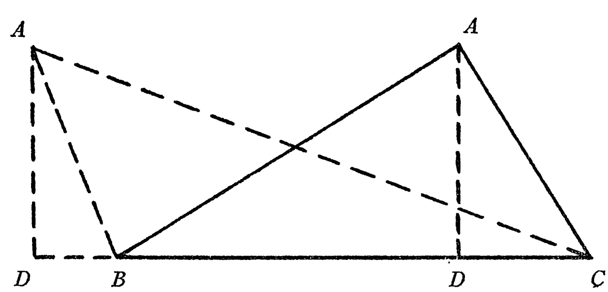

FIG. 5

Proof. If D coincides with B or with C, then the triangle would have two angles with measure $\pi/2$. If B were between D and C, we would have $[\angle DAB+\pi/2]=[\angle DAC]$, so that $\angle DAC > \pi/2$; the right-angled triangle DAC has then for sum of measures of its angles a number greater than π, which

is a contradiction. In the same way, C cannot be between B and D, and D is between B and C.

THEOREM 8. *If a triangle ABC is such that $\angle A = \pi/2$, then $(AB)^2 + (AC)^2 = (BC)^2$.*

Proof. Let AD be the unique perpendicular from A to BC (Fig. 5). The triangles CAD and CBA are similar (Theorem 4), and

$$CD/AC = AC/BC. \tag{6.1}$$

In the same way the triangles ABD and CBA are similar, and as D is between B and C, we have

$$(BC - CD)/AB = AB/BC. \tag{6.2}$$

The elimination of CD between (6.1) and (6.2) gives the desired relation.

COROLLARY 3. *If A, B, and C are distinct points, then*

$$AB + BC \geqq AC. \tag{6.3}$$

The equality holds if and only if the points A, B, C are collinear with B between A and C.

Proof. If A, B, and C are collinear, then $AB + BC \geqq AC$. The equality holds if and only if B is between A and C. If A, B, and C are not collinear, let B' be the point of intersection of the unique perpendicular from B to AC with AC. Then Theorem 8 implies $AB > AB'$, $BC > B'C$, and $AB + BC > AB' + B'C \geqq AC$ (B' being between A and C or not). If A, B, C were not collinear and if $AB + BC = AC$, then we would have a contradiction.

COROLLARY 4. *Distance on the set of all points is a metric on this set, that is to say, $AB = BA$, $AB + BC \geqq AC$, $AB \geqq 0$, and $AB = 0$ if and only if $A = B$.*

7. Parallel lines, and the concept of plane. A *plane* is defined to be the class of all points belonging to the half-lines of a bundle B_o; this class will be denoted by $\{B_o\}$.

THEOREM 9. *If two distinct points of a line are in a plane, then the whole line is in the plane.*

Proof. We know by the continuity axiom that if two distinct points P and B belong to a plane (Fig. 1), the points of the segment PB belong to the plane. If A is a point of the line PB not on the segment PB, then P is between A and B or B is between A and P; let P be between A and B. Then the half-lines OA and OB, where O is the vertex of the bundle B_o defining the plane, belong to a unique bundle $\overline{B}_o$, and the continuity axiom implies that the half-line OP is a member of $\overline{B}_o$. Then $\overline{B}_o = B_o$, and the line AB is in the plane $\{B_o\}$.

We shall now prove that a plane is uniquely determined by three noncollinear points. The proof will be preceded by two lemmas.

LEMMA 4. *If two bundles B_o and $B_{o'}$ are such that $O \neq O'$, if there exists a point A such that A is not on the line OO', and if $A, O, O' \in \{B_o\} \cap \{B_{o'}\}$, then the planes $\{B_o\}$ and $\{B_{o'}\}$ coincide.*

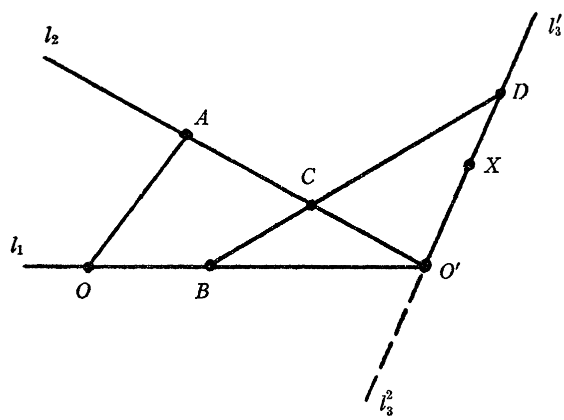

FIG. 6

Proof. Let X be a point of $\{B_{o'}\}$, i.e., a point of a half-line of $B_{o'}$. We want to prove that X also belongs to $\{B_o\}$. As the points of the lines OO', OA, and AO' belong to $\{B_o\}$, it is sufficient to show the existence of two distinct points of the line $O'X$ which belong to $\{B_o\}$. As O' already belongs to $\{B_o\}$, it remains to prove the existence of another point of $O'X$ belonging to $\{B_o\}$.

Let ϕ be the coordinate function for $B_{o'}$ such that $\phi(O'O) = \phi(l_1) = [0]$ and $\phi(O'A) = \phi(l_2) = [\alpha]$, $\alpha < \pi$. Then if we denote by l_3' and l_3^2 the distinct half-lines of $O'X$ with vertex O', $\overline{\phi(l_3')}$ or $\overline{\phi(l_3^2)}$ is between 0 and π (otherwise, X belongs to the line OO' and $X \in \{B_o\}$). Let $\phi(l_3') = [\beta]$, $\beta < \pi$, then either α is between 0 and β, or else β is between 0 and α (otherwise $\alpha = \beta$ so that X is on the line $O'A$ and $X \in \{B_o\}$). Let α be between 0 and β, then

$$([\beta] - [\alpha]) + ([\alpha] - [0]) = ([\beta] - [0]),$$

i.e.,

$$(\phi(l_3') - \phi(l_2)) + (\phi(l_2) - \phi(l_1)) = (\phi(l_3') - \phi(l_1)),$$

so that $[\angle l_3' l_2 + \angle l_1 l_2] = [\angle l_3' l_1]$. The continuity axiom implies that there exist points B, C, D on l_1, l_2, and l_3' respectively which are collinear. As $B, C \in \{B_o\}$, D is the required point, $X \in \{B_o\}$, and $\{B_{o'}\} \subseteq \{B_o\}$. Similarly if β is between 0 and α, $\{B_{o'}\} \subseteq \{B_o\}$. By symmetry $\{B_o\} \subseteq \{B_{o'}\}$.

If two lines of a plane have no point in common, then they are *parallel.*

LEMMA 5. *If B_o is a bundle, P is a point of $\{B_o\}$ different from O, and L is a line defined by a half-line of B_o not containing P, then there exists one and only one line in $\{B_o\}$ containing P and parallel to L.*

The proof of this lemma can be found in [1].

THEOREM 10. *Two planes coincide if and only if they have three noncollinear points in common.*

Proof. Let B_o and $B_{o'}$ be two bundles such that the three noncollinear points A, B, C belong to $\{B_o\} \cap \{B_{o'}\}$. We shall prove that if $X \in \{B_{o'}\}$, then $X \in \{B_o\}$. Suppose that $X \in \{B_{o'}\}$ so that X belongs to a half-line l_1 of $B_{o'}$. At least one of the three points A, B, C, say A, is not on the line l containing l_1. Let L be a line in $\{B_{o'}\}$ containing A and parallel to l. If B and C are not on L, then the lines AB and AC meet l in two distinct points U and V (Fig. 7); as U, $V \in \{B_o\}$, then $X \in \{B_o\}$. If B or C, say B, is on L then C is not on L and the lines AC and BC meet l in two distinct points V and W. V and W being in $\{B_o\}$, $X \in \{B_o\}$. In the same way $\{B_o\} \subseteq \{B_{o'}\}$, and $\{B_o\} = \{B_{o'}\}$.

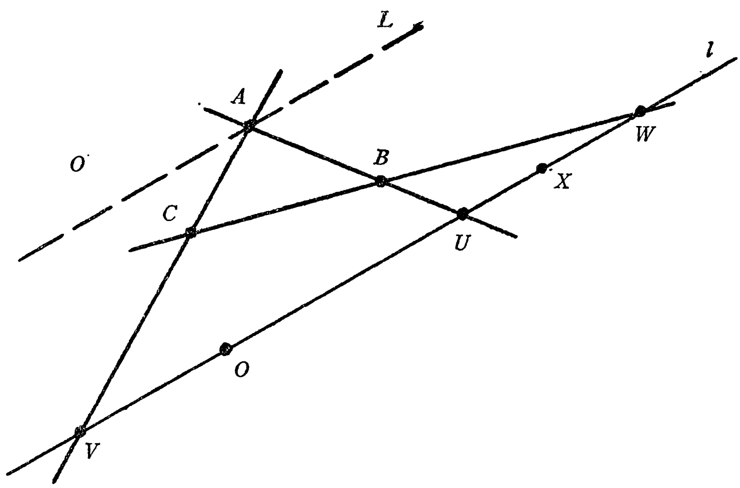

FIG. 7

The results of Lemmas 2 and 5 can now be formulated in a more general form as follows:

THEOREM 11. *In a plane, from a given point not on a given line there exists one and only one perpendicular to that line, and from a given point not on a given line, there exists one and only one parallel to that line.*

The following theorem is an immediate consequence of the continuity axiom.

THEOREM 12. *If three distinct points A, B, and C do not lie on the same line, and if D and E are two points such that C is between B and D, and E is between A and C, then there exists between A and B a point F such that D, E, F lie on the same line.*

8. 3-dimensional euclidean space. The following axiom is now added to the structure.

S. *There exists a point not on a given plane.*

The 3-dimensional euclidean space is introduced as defined by O. Veblen in [4]. A set of four noncoplanar points is called a *tetrahedron* whose *faces* are the interior of the triangles defined by the elements of the tetrahedron (the interior of a proper triangle ABC is defined to be the class of points P between X and Y, where X and Y belong to different intervals defined by the points A, B, and C). A *3-space* $ABCD$ is the set of all points collinear with any two points of the faces of the tetrahedron $ABCD$.

Except for the Axiom X, the remaining eleven axioms given in [4] are either a property of real numbers, a consequence of a definition given here, or one element of the following list of axioms and theorems: L_1, L_2, L_3, S, Theorem 11, and Theorem 12. The Axiom X says that all points belong to the same 3-space. We have then in [4] the proof of the following property concerning categoricity:

THEOREM 13. *If M_1 and M_2 are two models of a given 3-space, then they are isomorphic.*

I wish to acknowledge my indebtedness to the referee. I owe to him in particular the present formulation of Theorem 2 and its corollaries.

References

1. G. D. Birkhoff, A set of postulate for plane geometry based on scale and protractor, Ann. of Math., 33 (1932) 329–345; reprinted in George David Birkhoff, Collected mathematical papers, vol. III, Amer. Math. Soc., New York (1950), 365–381.
2. K. Borsuk and W. Szmielew, Foundations of Geometry, Amsterdam, 1960.
3. S. MacLane, Metric postulates for plane geometry, this MONTHLY, 66 (1959) 543–555.
4. Oswald Veblen, A system of axioms for geometry, Trans. Amer. Math. Soc., 5 (1904) 343–384.

ON THE INEQUALITY OF KANTOROVICH

E. F. BECKENBACH, University of California, Los Angeles

1. Introduction. For any positive numbers $(x) \equiv (x_1, x_2, \cdots, x_n)$, $n > 1$, and positive weights $(\alpha) \equiv (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\sum_{i=1}^{n} \alpha_i = 1$, we define the *mean of order* t, $-\infty \leqq t \leqq \infty$, of the numbers (x) with weights (α) by

$$M_t(x; \alpha) = \left(\sum_{i=1}^{n} \alpha_i x_i^t \right)^{1/t}$$

for t finite and $\neq 0$, and otherwise by

$$M_0(x;\,\alpha) \equiv \prod_{i=1}^{n} x_i^{\alpha_i},$$

$$M_{-\infty}(x;\,\alpha) \equiv \min_{i=1}^{n} x_i,$$

$$M_{\infty}(x;\,\alpha) \equiv \max_{i=1}^{n} x_i.$$

The familiar harmonic, geometric, and arithmetic means are included as the special cases $t=-1$, 0, and 1, respectively.

It is easy to show that $\lim_{t\to-\infty} M_t(x;\,\alpha)=M_{-\infty}(x;\,\alpha)$, $\lim_{t\to\infty} M_t(x;\,\alpha)=M_{\infty}(x;\,\alpha)$, and it readily follows from l'Hospital's rule that

$$\lim_{t\to 0} M_t(x;\,\alpha) = M_0(x;\,\alpha).$$

Thus $M_t(x;\,\alpha)$ is a continuous function of t in the closed interval $[-\infty,\,\infty]$.

If all the x_i are equal, $x_i=x_0$ for $i=1,\,2,\,\cdots,\,n$, then we have $M_t(x;\alpha)=x_0$ for all t; otherwise, as is well known [4, p. 17; 11, p. 26], $M_t(x;\alpha)$ is a strictly increasing function of t. In particular, this statement implies the classical inequality between the geometric mean $M_0(x;\,\alpha)$ and the arithmetic mean $M_1(x,\,\alpha)$, namely $M_0(x;\,\alpha) \leqq M_1(x,\,\alpha)$, with equality if and only if all the x_i are equal.

In the present paper we are concerned with lower and upper bounds of the ratio $M_s(x;\,\alpha)/M_r(x;\,\alpha)$, for arbitrary r and s, $-\infty \leqq r < s \leqq \infty$, and for positive x_i subject to certain additional constraints.

Lower bounds are discussed in Section 2. It follows from the monotonicity property of $M_t(x;\,\alpha)$, mentioned above, that we have

$$\frac{M_s(x;\,\alpha)}{M_r(x;\,\alpha)} \geqq 1 \quad \text{for } r < s,$$

with equality if and only if all the x_i are equal. We do not, however, assume the validity of this result; rather, it appears as a special case (the Corollary to Theorem 1) of a general inequality that is established below, in which some of the x_i are fixed, $x_i=c_i$, $i=1,\,2,\,\cdots,\,m$, and others are allowed to vary freely, $0<x_i<\infty$, $i=m+1,\,m+2,\,\cdots,\,n$. The proof of Theorem 1 implies a new and, in the author's estimation, illuminating derivation of the much-proved [5, p. 54] inequality between the geometric mean and the arithmetic mean.

In Section 3 we turn to a consideration of upper bounds. If the positive variables x_i are not further constrained, so that the ratio of the largest to the smallest x_i is not bounded, then neither is the ratio $M_s(x;\,\alpha)/M_r(x;\,\alpha)$ bounded. Accordingly, for given positive numbers A, B, with $0<A<B$, we consider variables x_i constrained by

$$0 < A \leqq x_i \leqq B, \qquad i = 1,\,2,\,\cdots,\,n. \tag{1}$$

This is a reasonable restriction in many applied problems.

For numbers x_i subject to the above constraint (1), and for $r=-1$, $s=1$, the inequality of Kantorovich [14, 15] gives an upper bound to the mean-value ratio, namely

$$\frac{M_1(x;\alpha)}{M_{-1}(x;\alpha)} \leqq \frac{(A+B)^2}{4AB}.$$

Numerous proofs of the inequality of Kantorovich have been given, and the inequality has been generalized in various ways (see the bibliography at the end of this paper). In particular, an upper bound, including that of Kantorovich as a special case, has been obtained by Cargo and Shisha [6] for arbitrary finite $r<s$.

If, now, some of the x_i are fixed, say $x_i=c_i$, $A\leqq c_i\leqq B$, $i=1, 2, \cdots, m$, and only the rest allowed to vary, then new upper bounds, usually more restrictive than those of Kantorovich and Cargo-Shisha, can be expected. Such upper bounds, including those of Kantorovich and Cargo-Shisha as special cases, are established in Section 3.

The methods of the present paper can be applied to yield analogous extensions of the inequalities of Hölder and Minkowski, as we shall show elsewhere.

2. Lower bound. Letting the symbol $M_t(c, x; \alpha)$ denote the mean of order t and weights (α) of the vector $(c_1, c_2, \cdots, c_m, x_{m+1}, x_{m+2}, \cdots, x_n)$, we now establish the following result.

THEOREM 1. *Let there be given positive weights* $(\alpha)\equiv(\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\sum_{i=1}^n \alpha_i=1$, $n>1$, *and positive numbers* $(c)\equiv(c_1, c_2, \cdots, c_m)$, $1\leqq m<n$. *For any positive numbers* $(x)\equiv(x_{m+1}, x_{m+2}, \cdots, x_n)$, *and any indices r and s*, $-\infty\leqq r<s\leqq\infty$, *we have*

$$\frac{M_s(c,x;\alpha)}{M_r(c,x;\alpha)} \geqq \frac{M_s(c,\bar{c};\alpha)}{M_r(c,\bar{c};\alpha)}, \tag{2}$$

where, except for the combination $r=-\infty$, $s=\infty$, *each component* $\bar{c}_j$, $j=m+1, m+2, \cdots, n$, *of* $(\bar{c})$ *is given by*

$$\bar{c}_j = \bar{c} = \begin{cases} \min\limits_{i=1}^{m} c_i, & r=-\infty,\ s \text{ finite}, \\ \left\{\dfrac{\sum_{i=1}^{m} \alpha_i c_i^s}{\sum_{i=1}^{m} \alpha_i c_i^r}\right\}^{1/(s-r)} & r \text{ and } s \text{ finite}, \\ \max\limits_{i=1}^{m} c_i, & r \text{ finite},\ s=\infty. \end{cases} \tag{3}$$

and where, for $r=-\infty$ *and* $s=\infty$, *each component* $\bar{c}_j$ *of* $(\bar{c})$ *is arbitrary,* $j=m+1, m+2, \cdots, n$, *subject only to*

$$\min_{i=1}^{m} c_i \leqq \bar{c}_j \leqq \max_{i=1}^{m} c_i. \tag{4}$$

The values $\bar{c}_j$ *given by* (3) *satisfy* (4). *Equality holds in* (2) *if and only if each* $x_j=\bar{c}, j=m+1, m+2, \cdots, n$, *except that for* $r=-\infty$ *and* $s=\infty$ *equality holds if and only if each* x_j *satisfies*

$$\min_{i=1}^{m} c_i \leqq x_j \leqq \max_{i=1}^{m} c_i, \qquad j=m+1, m+2, \cdots, n. \tag{5}$$

Proof. First, to show that the values $\bar{c}_j=\bar{c}$ given by (3) satisfy (4) for r and s finite (the other cases are immediate), let

$$a = \min_{i=1}^{m} c_i, \qquad b = \max_{i=1}^{m} c_i.$$

Then we have

$$\frac{\bar{c}}{a} = \left[\frac{\sum_{i=1}^{m} \alpha_i (c_i/a)^s}{\sum_{i=1}^{m} \alpha_i (c_i/a)^r}\right]^{1/(s-r)}.$$

Since $c_i/a \geqq 1$ and $s>r$, it follows that $(c_i/a)^{s-r} \geqq 1$, $(c_i/a)^s \geqq (c_i/a)^r$, whence we get $\bar{c}/a \geqq 1$, with equality if and only if $c_1=c_2=\cdots=c_m=a=\bar{c}$. Similarly, we obtain $\bar{c}/b \leqq 1$, again with equality if and only if all the c_i are equal, $i=1, 2, \cdots, m$.

For additional properties of the mean-value function $\bar{c}$, see [3, **8**].

Define $f(c, x; \alpha; r, s)$, or briefly $f(x)$, by

$$f(x) \equiv \frac{M_s(c, x; \alpha)}{M_r(c, x; \alpha)}. \tag{6}$$

For r and s finite and $rs \neq 0$, and for each j, $j=m+1, m+2, \cdots, n$, a computation (cf. [6]) gives

$$f_j(x) \equiv \frac{\partial}{\partial x_j} f(x) = P_j Q_j, \tag{7}$$

where

$$P_j \equiv \alpha_j x_j^{r-1} \left(\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{i=m+1}^{n} \alpha_i x_i^s \right)^{1/(s-1)} \left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r \right)^{-1/(r-1)}, \tag{8}$$

$$Q_j \equiv x_j^{s-r} \left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r \right) - \left(\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{i=m+1}^{n} \alpha_i x_i^s \right). \tag{9}$$

Since $P_j > 0$ for all positive (x), it follows from (7) and (9) that $f_j(x) = 0$ if and only if

$$x_j^{s-r} = \frac{\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{i=m+1}^{n} \alpha_i x_i^s}{\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r}. \tag{10}$$

Now the right-hand side of (10) is the same for all j, $j = m+1, m+2, \cdots, n$, and accordingly *any* solution $x_{m+1}, x_{m+2}, \cdots, x_n$ of the system of equations

$$f_{m+1}(x) = 0, \quad f_{m+2}(x) = 0, \cdots, f_n(x) = 0, \tag{11}$$

must have all its components equal, say $x_{m+1} = x_{m+2} = \cdots = x_n = \bar{x}$. From (10) we then obtain

$$\bar{x}^{s-r} = \frac{\sum_{i=1}^{m} \alpha_i c_i^s + \bar{x}^s \sum_{i=m+1}^{n} \alpha_i}{\sum_{i=1}^{m} \alpha_i c_i^r + \bar{x}^r \sum_{i=m+1}^{n} \alpha_i}.$$

Clearly, however, we have

$$\bar{x}^{s-r} = \frac{\bar{x}^s \sum_{i=m+1}^{n} \alpha_i}{\bar{x}^r \sum_{i=m+1}^{n} \alpha_i},$$

whence, from "proportion by division" [24], i.e., from the fact that

$$\frac{a}{b} = \frac{c}{d} \quad \text{implies} \quad \frac{a}{b} = \frac{c-a}{d-b}, \tag{12}$$

provided the denominators do not vanish, we obtain

$$\bar{x}^{s-r} = \frac{\sum_{i=1}^{m} \alpha_i c_i^s}{\sum_{i=1}^{m} \alpha_i c_i^r}.$$

Hence we have $\bar{x} = \bar{c}$, where $\bar{c}$ is given by (3), whence the system (11) has no solution other than

$$x_{m+1} = x_{m+2} = \cdots = x_n = \bar{c}. \tag{13}$$

That (13) does in fact furnish a solution to the system (11) follows, by (10), from "proportion by composition," wherein the minus signs in (12) are replaced by plus signs.

Geometrically speaking, we have now shown, for r and s finite and $rs\neq 0$, that there is precisely one horizontal tangent hyperplane to the hypersurface

$$S\colon y = f(x), \qquad 0 < x_i < \infty,$$

in the $(x_{m+1}, x_{m+2}, \cdots, x_n)$-space, and that the point of tangency is the point on S at which each $x_i = \bar{c}$.

We still have to show that there is a minimum value of y on S, namely the corresponding

$$\bar{y} = f(\bar{c}) = \frac{M_s(c, \bar{c}; \alpha)}{M_r(c, \bar{c}; \alpha)}\cdot$$

For any numbers A and B satisfying the inequalities

$$(14) \qquad\qquad 0 < A < \bar{c} < B,$$

consider the $(n-m)$-dimensional cube I_{AB} determined by the inequalities $A \leqq x_i \leqq B$, $i = m+1, m+2, \cdots, n$. Let ℓ be a ray, or half-line, extending perpendicularly from one of the coordinate hyperplanes and intersecting I_{AB}; thus ℓ is determined by

$$x_j > 0, \qquad x_i = x_{i0}, \qquad A \leqq x_{i0} \leqq B, \qquad\qquad i \neq j,$$

with x_j varying, $0 < x_j < \infty$, for some fixed j, $m+1 \leqq j \leqq n$, and with $x_i = x_{i0}$ fixed, $A \leqq x_{i0} \leqq B$, for all $i \neq j$.

Once more applying the principle of proportion by division, from (7)–(10) we see that on ℓ we have

$$f_j(x) = 0$$

at just one point, namely at the point where

$$x_j = x_{j0} = \left\{\frac{\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{\substack{i=m+1\\ i\neq j}}^{n} \alpha_i x_{i0}^s}{\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{\substack{i=m+1\\ i\neq j}}^{n} \alpha_i x_{i0}^r}\right\}^{1/(s-r)}.$$

We now show that x_{j0} satisfies the inequalities

$$(15) \qquad\qquad A < x_{j0} < B,$$

as follows. We have

$$(16) \qquad \frac{x_{j0}}{A} = \left[\frac{\sum_{i=1}^{m} \alpha_i (c_i/A)^s + \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i (x_{i0}/A)^s}{\sum_{i=1}^{m} \alpha_i (c_i/A)^r + \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i (x_{i0}/A)^r} \right]^{1/(s-r)} .$$

From (3) we get

$$\sum_{i=1}^{m} \alpha_i \left(\frac{c_i}{A}\right)^s = \left(\frac{\bar{c}}{A}\right)^{s-r} \sum_{i=1}^{m} \alpha_i \left(\frac{c_i}{A}\right)^r;$$

therefore, since $s-r>0$ and $0<A<\bar{c}$, we have

$$(17) \qquad \sum_{i=1}^{m} \alpha_i \left(\frac{c_i}{A}\right)^s > \sum_{i=1}^{m} \alpha_i \left(\frac{c_i}{A}\right)^r.$$

Since also $0<A \leqq x_{i0}$, $i \neq j$, it follows that $(x_{i0}/A)^{s-r} \geqq 1$, $i \neq j$, whence

$$(18) \qquad \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i \left(\frac{x_{i0}}{A}\right)^s \geqq \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i \left(\frac{x_{i0}}{A}\right)^r.$$

Hence, by (16), (17), and (18), we have $x_{j0}>A$. Similarly, we obtain $x_{j0}<B$.

The proof in the preceding paragraph could have been somewhat shortened if we had further constrained A and B to satisfy the inequalities

$$0 < A < \min_{i=1}^{m} c_i, \qquad B > \max_{i=1}^{m} c_i,$$

instead of merely the inequalities (14), and this actually would have been sufficient for our present purpose. The weaker hypothesis, however, gives a more precise description of the physical situation.

Now on ℓ we have

$$(19) \qquad Q = x_i^{s-r}\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i x_{i0}^r \right) - \left(\sum_{i=1}^{m} \alpha_i c_i^s + \sum_{\substack{i=m+1 \\ i \neq j}}^{n} \alpha_i x_{i0}^s \right),$$

which is a linear function of x_j^{s-r}. Since in (19) the coefficient of x_j^{s-r} is positive, and since $s-r>0$, it follows from (7), (8), and (19) that, on ℓ, f is a strictly decreasing function of x_j for $0<x_j<x_{j0}$, and a strictly increasing function of x_j for $x_{j0}<x_j<\infty$.

By (15), then, on the line segment

$$\ell_{AB} = \ell \cap I_{AB},$$

the function f assumes its maximum value only at one or both end points, and its minimum value only at an interior point, of ℓ_{AB}.

where

$$\max(c, x) \equiv \max_{i=1,\, j=m+1}^{m,n} (c_i, x_j)$$

Computations now yield

$$f_j = -\alpha_j x_j^{r-1} \max(c, x)\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r\right)^{-1/(r-1)}$$

for $x_j < \max(c, x)$, and

$$f_j = \left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{i=m+1}^{n} \alpha_i x_i^r\right)^{-1/(r-1)} \left(\sum_{i=1}^{m} \alpha_i c_i^r + \sum_{\substack{i=m+j \\ i\neq j}}^{n} \alpha_i x_i^r\right)$$

for $x_j = \max(c, x)$. Hence we have

$$f_j < 0 \qquad \text{for } x_j < \max(c, x),$$
$$f_j > 0 \qquad \text{for } x_j = \max(c, x),$$

so that f is a decreasing function of x_j for $x_j < \max(c, x)$, and an increasing function of x_j for $x_j = \max(x) > \max(c)$. Thus (2) holds in this case, with equality if and only if each

$$x_j = \max_{i=1}^{m} c_i, \qquad j = m+1, m+2, \cdots, n.$$

The case $r=0$, $s=\infty$, and the case $r=-\infty$, s finite, can be treated similarly.

Finally, for

$$f(c, x; \alpha; -\infty, \infty) \equiv \frac{\max(c, x)}{\min(c, x)},$$

we note from (4) that $\min(c, \bar{c}) = \min(c)$, $\max(c, \bar{c}) = \max(c)$. Accordingly, since $\min(c, x) \leqq \min(c)$, $\max(c, x) \geqq \max(c)$, with equality if and only if (5) holds, it follows that (2) is satisfied also for $r=-\infty$, $s=\infty$, with equality if and only if all the x_i satisfy (5).

COROLLARY. *For any positive numbers* $(x) \equiv (x_1, x_2, \cdots, x_n)$, $n > 1$, *and any positive weights* $(\alpha) \equiv (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\sum_{i=1}^{n} \alpha_i = 1$, *the mean-value function* $M_t(x; \alpha)$ *is a nondecreasing function of* t *for* $-\infty \leqq t \leqq \infty$, *and is strictly increasing unless all the* x_i *are equal.*

Proof. In Theorem 1, let $(c) \equiv (x_1)$ have just one member. Then by (3) and (4) we have, for any r and s, $-\infty \leqq r < s \leqq \infty$, $\bar{c}_j = \bar{c} = x_1$, $j = 2, 3, \cdots, n$, whence $M_r(c, \bar{c}; \alpha) = M_s(c, \bar{c}; \alpha) = x_1$. Substitution in (2) now yields

$$\frac{M_s(x; \alpha)}{M_r(x; \alpha)} \geqq \frac{x_1}{x_1} = 1,$$

or $M_s(x;\alpha) \geqq M_r(x;\alpha)$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.

3. Upper bound. Let $\sigma = 1 - \sum_{i=1}^m \alpha_i$, let α_0 be any value satisfying $0 \leqq \alpha_0 \leqq \sigma$, and let the symbol $M_t(c, A, B; \alpha, \alpha_0)$ denote the mean of order t and weights $(\alpha_1, \alpha_2, \cdots, \alpha_m, \sigma - \alpha_0, \alpha_0)$ of the vector $(c_1, c_2, \cdots, c_m, A, B)$.

It should be noted that here either one of the weights $\sigma - \alpha_0$ and α_0 might be 0. We nevertheless retain the definitions of $M_{-\infty}$ and M_∞ given on the first page of this paper, namely

$$M_{-\infty}(c, A, B; \alpha, \alpha_0) = \min(c, A, B), \quad M_\infty(c, A, B; \alpha, \alpha_0) = \max(c, A, B), \tag{21}$$

though now we have only

$$\lim_{t\to-\infty} M_t(c, A, B; \alpha, \alpha_0) \geqq M_{-\infty}(c, A, B; \alpha, \alpha_0),$$

$$\lim_{t\to\infty} M_t(c, A, B; \alpha, \alpha_0) \leqq M_\infty(c, A, B; \alpha, \alpha_0)$$

in place of the former equalities. The definitions (21) have been retained to make the statement of the following Theorem 2 simpler than it otherwise would be; further, with the definition (21) we have

$$\begin{aligned} \lim_{\substack{t\to-\infty \\ \sigma-\alpha_0\to 0^+}} M_t(c, A, B; \alpha, \alpha_0) &= M_{-\infty}(c, A, B; \alpha, \sigma), \\ \lim_{\substack{t\to\infty \\ \alpha_0\to 0^+}} M_t(c, A, B; \alpha, \alpha_0) &= M_\infty(c, A, B; \alpha, 0), \end{aligned} \tag{22}$$

and these are the limiting processes with which we are actually concerned.

The following result includes the inequalities of Kantorovich and Cargo-Shisha as special cases.

THEOREM 2. *Let there be given positive numbers A and B satisfying $0 < A < B < \infty$, positive weights $(\alpha) \equiv (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\sum_{i=1}^n \alpha_i = 1$, $n > 1$, and positive numbers $(c) \equiv (c_1, c_2, \cdots, c_m)$, $0 \leqq m < n$. For any positive numbers $(x) \equiv (x_{m+1}, x_{m+2}, \cdots, x_n)$ satisfying $A \leqq x_j \leqq B$, $j = m+1, m+2, \cdots, n$, and any indices r and s, $-\infty \leqq r < s \leqq \infty$, we have*

$$\frac{M_s(c, x; \alpha)}{M_r(c, x; \alpha)} \leqq \frac{M_s(c, A, B; \alpha, \alpha_0)}{M_r(c, A, B; \alpha, \alpha_0)}, \tag{23}$$

where

$$\alpha_0 = \begin{cases} 0 & \text{if } \theta < 0, \\ \theta & \text{if } 0 \leqq \theta \leqq \sigma, \\ \sigma & \text{if } \theta > \sigma, \end{cases} \tag{24}$$

with $\sigma = 1 - \sum_{i=1}^m \alpha_i$, and with θ given by

$$\theta = \frac{1}{s-r}\left[\frac{r\left(\sum_{i=1}^{m} \alpha_i c_i^r + \sigma A^r\right)}{B^r - A^r} - \frac{s\left(\sum_{i=1}^{m} \alpha_i c_i^s + \sigma A^s\right)}{B^s - A^s}\right] \tag{25}$$

for r and s finite, $rs \neq 0$, by respective limiting values of (25) *for $r=0$ or $s=0$ and the other finite, by $\theta=0$ for r finite and $s=\infty$, by $\theta=\sigma$ for $r=-\infty$ and s finite, and by $\theta=\sigma/2$ for $r=-\infty$ and $s=\infty$. For $m=0$, the value θ always satisfies the inequalities*

$$0 \leqq \theta \leqq \sigma. \tag{26}$$

Equality holds in (23), *for r and s finite, if and only if there is a subset*

$$(k_1, k_2, \cdots, k_p), \qquad 0 \leqq p \leqq n - m, \tag{27}$$

of $(m+1, m+2, \cdots, n)$ such that

$$\sum_{i=1}^{p} \alpha_{k_i} = \alpha_0,\ x_{k_i} = B \text{ for } i = 1, 2, \cdots, p, \text{ and } x_j = A \tag{28}$$

for all j in the complement of $(k_1, k_2, \cdots, k_p)$ with respect to $(m+1, m+2, \cdots, n)$; for $r=-\infty$ and s finite if and only if we have (28) *and*

$$\min_{i=1}^{m} (c_i) = A;$$

for r finite and $s=\infty$ if and only if we have (28) *and*

$$\max_{i=1}^{m} (c_i) = B;$$

and for $r=-\infty$, $s=\infty$ if and only if we have

$$\min_{i=1, j=m+1}^{m,n} (c_i, x_j) = A, \qquad \max_{i=1, j=m+1}^{m,n} (c_i, x_j) = B.$$

Proof. Let us note first that the observation (a) in Section 2 is a consequence merely of the fact that on 1 the function f first decreases as x_j increases from 0 to x_{j0}, and then increases as x_j increases from x_{j0} to ∞.

From this observation, we see that the function f takes on its maximum on I_{AB} only at certain vertices of I_{AB}. This is true in particular for the Kantorovich and Cargo-Shisha case $m=0$, as we see by considering (x) as the cartesian product of two nonnull factors, say (x_1) and $(x_2, x_3, \cdots, x_n)$; the maximum of f is attained only on the vertices of each of the two factors for any fixed determination of a point in the other factor, and hence only on the vertices of the cartesian product.

We accordingly consider the function $g(u; c, A, B; \alpha; r, s)$, or briefly $g(u)$, defined for r and s finite, $r<s$, $rs\neq 0$, by

$$g(u) \equiv \frac{\left[\sum_{i=1}^{m} \alpha_i c_i^s + uB^s + (\sigma - u)A^s\right]^{1/s}}{\left[\sum_{i=1}^{m} \alpha_i c_i^r + uB^r + (\sigma - u)A^r\right]^{1/r}}, \qquad 0 \leqq u \leqq \sigma, \tag{29}$$

and note that for some u_0, $0 \leqq u_0 \leqq \sigma$, we have

$$\max_{x \in I_{AB}} f(x) = g(u_0) \leqq \max_{0 \leqq u \leqq \sigma} g(u). \tag{30}$$

A computation yields $g'(u) = (u - \theta)N(u)$, where

$$N(u) = \frac{(r-s)(B^r - A^r)(B^s - A^s)\left[\sum_{i=1}^{m} \alpha_i c_i^s + uB^s + (\sigma - u)A^s\right]^{1/(s-1)}}{rs\left[\sum_{i=1}^{m} \alpha_i c_i^r + uB^r + (\sigma - u)A^r\right]^{1/(r+1)}}$$

and θ is given by (25). Clearly we have $N(u) < 0$ for $0 \leqq u \leqq \sigma$, so that

$$g'(u) > 0 \text{ for } u \text{ in } (u < \theta) \cap (0 \leqq u \leqq \sigma),$$
$$g'(u) < 0 \text{ for } u \text{ in } (u > \theta) \cap (0 \leqq u \leqq \sigma).$$

Hence, with α_0 given by (24), we have

$$\max_{0 \leqq u \leqq \sigma} g(u) = g(\alpha_0). \tag{31}$$

Accordingly, for r and s finite, $rs \neq 0$, (23) follows from (6), (30), and (31), with equality if and only if (28) holds.

In the special case $m = 0$, we have $\sigma = 1$, $g(0) = g(\sigma) = 1$, whence (26) follows from Rolle's theorem.

We note incidentally that if there is no subset (27) of $(m+1, m+2, \cdots, n)$ for which $\sum_{i=1}^{p} \alpha_{k_i} = \alpha_0$, then

$$\max_{x \in I_{AB}} \frac{M_s(c, x; \alpha)}{M_r(c, x; \alpha)}$$

is attained with $x_{k_i} = B$ either on a set with sum as little less than α_0 as possible or on a set with sum as little more than α_0 as possible; the two values must be computed and compared.

All the foregoing analysis holds also in the case $r = 0$ and s finite, $s > 0$, with limiting values as $r \to 0$ given by

$$g(u) = \frac{\left[\sum_{i=1}^{m} \alpha_i c_i^s + uB^s + (\sigma - u)A^s\right]^{1/s}}{B^u A^{\sigma - u} \prod_{i=1}^{m} c_i^{\alpha_i}}, \qquad 0 \leqq u \leqq \sigma,$$

11. G. H. Hardy, J. L. Littlewood, and G. Pólya, Inequalities, 2nd ed., Cambridge University Press, New York, 1952.

12. Peter Henrici, Two remarks on the Kantorovich inequality, this MONTHLY, 68 (1961) 904–906.

13. A. S. Householder and F. L. Bauer, On certain iterative methods for solving linear systems, Numer. Math., 2 (1960) 55–59.

14. L. V. Kantorovich, Funktsionalnii analyz i prikladnaya matematika, Uspehi Mat. Nauk, (N.S.) 3 (1948) 89–185.

15. ———, Functional analysis and applied mathematics, translated by C. D. Benster and edited by G. E. Forsythe, National Bureau of Standards, Report No. 1509, Washington, D. C., 1952, pp. 106–109.

16. J. Kurschak, Math. Phys. Lapok, 23 (1914) 378.

17. Marvin Marcus and A. H. Cayford, Further results on the Kantorovich inequality, Notices Amer. Math. Soc., 9 (1962) 300.

18. Marvin Marcus and N. A. Khan, Some generalizations of Kantorovich's inequality, Portugal. Math., 20 (1961) 33–38.

19. Morris Newman, Kantorovich's inequality, J. Res. Nat. Bur. Standards, 64B (1959) 33–34.

20. G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. 1, J. Springer Verlag, Berlin, 1925, pp. 57, 213–214.

21. A. H. Schopf, On the Kantorovich inequality, Numer. Math., 2 (1960) 344–346.

22. P. Schweitzer, Egy egyenlötlensig az aritmetikai középértékröl (An inequality concerning the arithmetic mean), Mat. Fyz. Lapok, 23 (1914) 257–261.

23. W. G. Strang, On the Kantorovich inequality, Proc. Amer. Math. Soc., 11 (1960) 468.

24. G. A. Wentworth, New school algebra, Ginn and Company, New York, 1898.

A SIMPLIFIED PROOF OF THE DIVERGENCE THEOREM

DJAIRO GUEDES DE FIGUEIREDO, Instituto de Matematica
Pura e Aplicada, Rio de Janeiro

1. Introduction. Students of potential theory always wonder how to proceed in order to prove the divergence theorem for some general class of regions in R^n. We think that our proof could supply them with one at the level of advanced calculus. It is Whitney's idea [1] to get the theorem for a general region using partitions of unity and approximations, then reducing it to simpler regions. We use this method here.

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11. G. H. Hardy, J. L. Littlewood, and G. Pólya, Inequalities, 2nd ed., Cambridge University Press, New York, 1952.

12. Peter Henrici, Two remarks on the Kantorovich inequality, this MONTHLY, 68 (1961) 904–906.

13. A. S. Householder and F. L. Bauer, On certain iterative methods for solving linear systems, Numer. Math., 2 (1960) 55–59.

14. L. V. Kantorovich, Funktsionalnii analyz i prikladnaya matematika, Uspehi Mat. Nauk, (N.S.) 3 (1948) 89–185.

15. ———, Functional analysis and applied mathematics, translated by C. D. Benster and edited by G. E. Forsythe, National Bureau of Standards, Report No. 1509, Washington, D. C., 1952, pp. 106–109.

16. J. Kurschak, Math. Phys. Lapok, 23 (1914) 378.

17. Marvin Marcus and A. H. Cayford, Further results on the Kantorovich inequality, Notices Amer. Math. Soc., 9 (1962) 300.

18. Marvin Marcus and N. A. Khan, Some generalizations of Kantorovich's inequality, Portugal. Math., 20 (1961) 33–38.

19. Morris Newman, Kantorovich's inequality, J. Res. Nat. Bur. Standards, 64B (1959) 33–34.

20. G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. 1, J. Springer Verlag, Berlin, 1925, pp. 57, 213–214.

21. A. H. Schopf, On the Kantorovich inequality, Numer. Math., 2 (1960) 344–346.

22. P. Schweitzer, Egy egyenlötlensig az aritmetikai középértékröl (An inequality concerning the arithmetic mean), Mat. Fyz. Lapok, 23 (1914) 257–261.

23. W. G. Strang, On the Kantorovich inequality, Proc. Amer. Math. Soc., 11 (1960) 468.

24. G. A. Wentworth, New school algebra, Ginn and Company, New York, 1898.

A SIMPLIFIED PROOF OF THE DIVERGENCE THEOREM

DJAIRO GUEDES DE FIGUEIREDO, Instituto de Matematica Pura e Aplicada, Rio de Janeiro

1. Introduction. Students of potential theory always wonder how to proceed in order to prove the divergence theorem for some general class of regions in R^n. We think that our proof could supply them with one at the level of advanced calculus. It is Whitney's idea [1] to get the theorem for a general region using partitions of unity and approximations, then reducing it to simpler regions. We use this method here.

First we characterize the class of regions for which the theorem will be proved. A set A is said to have *p-content zero* if for each $\delta>0$ there exist k_δ spheres of radius δ that cover A and such that $k_\delta\delta^p\rightarrow 0$ as $\delta\rightarrow 0$. A *Gaussian region* is an open connected bounded set V in R^n, whose boundary S is made up of two parts, S_0 and S_1, such that: 1) S_0 is a closed set of zero $(n-1)$-content; 2) for every point x of S_1 there exists a neighborhood $N(x)$ such that $N(x)\cap S_1$ is a

regular element of surface; moreover, if we change variables so that x_1 is in the direction of the exterior normal $\nu(x)$ to S_1 at x, then $N(x)\cap S$ is represented by an equation $x_1=h(x_2, \cdots, x_n)$, where h has continuous first order derivatives and if $(x_1, x_2, \cdots, x_n)\in N(x)\cap V$ then $x_1<h(x_2, \cdots, x_n)$.

THEOREM. *Let V be a Gaussian region as defined above. Let $F=(F_1, \cdots, F_n)$ $=F(x)$ be a vector function continuous and bounded in $V\cup S_1$ and with continuous and bounded first order derivatives in V. Then*

$$\int_V \operatorname{div} F dx = \int_{S_1} F\cdot \nu d\sigma, \tag{1}$$

where $\nu=\nu(x)$ is the unit exterior normal to S_1 at x.

2. Proof of the theorem. Let us first establish (1) for some special regions.

LEMMA 1. *Formula* (1) *holds for parallelepipeds V, and F as is Theorem* 1.

This lemma can be proved easily by using iterated integrals.

A point $(x_2, \cdots, x_n)$ will henceforth be denoted by x'.

LEMMA 2. *Let V be the set of points x such that $h(x')<x_1<1$ and $-1<x'<1$, where h has continuous first order derivatives in $-1\leqq x'\leqq 1$. Let A be a curved part of S, i.e., $x_1=h(x')$. The function F is supposed to be continuous in $V\cup S$, with continuous and bounded first order derivatives in V and $F=0$ in some neighborhood of $S-A$. Then* (1) *holds.*

Proof. By the change of variables

$$y_1 = x_1 - h(x'), \qquad y' = x',$$

V goes into $V'=\{y: 0<y_1<1-h(y'),\ -1<y'<1\}$. Let K denote the parallelepiped $0<y_1<1,\ -1<y'<1$. Then the function $G(y)=F(y_1+h(y'),\ y')$ can be extended to the whole of K by defining $G(y)=0$ in $K-V'$, so that $G(y)$ is continuously differentiable in K. Applying Lemma 1 to G in K we get

$$\int_K \operatorname{div} G dy = -\int_{A'} G_1(y) dy', \tag{2}$$

where A' is the image of A. By examining the effect of the change of variables on the two integrals in (1) we see that the left-hand sides (and right-hand sides) of (1) and (2) coincide. This finishes the proof of Lemma 2.

LEMMA 3. *Suppose that all the conditions of the theorem are fulfilled and, moreover, that $F=0$ in some neighborhood N of S_0. Then* (1) *holds.*

Proof. For every point x in $V-N$ we can find a cube U centered at this point and contained in V. On the other hand, for every point x in $S-N$ we can find a cube U with one side parallel to $\nu(x)$ and such that $U\cap V$ is a region of the type

of that in Lemma 2. Since $(V\cup S)-N$ is compact we can find a finite number of such cubes U_j $(j=1,\cdots,p)$ whose union covers this set.

Now we determine p cubes U_j', $(j=1,\cdots,p)$ such that $U_j'\subset U_j$, U_j' has sides parallel to U_j and these cubes also constitute an open covering of $(V\cup S)-N$. A partition of unity for the covering (U_j') is constructed as follows. Let α_j be a C_∞-function such that $\alpha_j=0$ outside U_j', and $\alpha_j>0$ in U_j'. It is clear that the function $\alpha=\sum\alpha_j$ is different from zero on $(V\cup S)-N$. Defining $\beta_j=\alpha_j/\alpha$ we conclude that 1) β_j are C_∞-functions; 2) β_j are equal to zero outside U_j and in a neighborhood of the boundary of U_j; 3) $\sum\beta_j=1$. Using this partition of unity (β_j) we see that (1) in this general case reduces to (1) for the particular cases of Lemma 1 and 2.

LEMMA 4. *Let A and B be two open sets in R^n such that dist $(A, B)\geqq d$. Then there exists an infinitely differentiable function $\phi(x)$ such that*

$$\phi(x) = 0 \text{ in } A, \qquad \phi(x) = 1 \text{ in } B$$

and $|\text{grad}\,\phi(x)|\leqq k/d$, where k is a constant.

Proof. Let $\psi(x)$ be an infinitely differentiable function such that

$$\psi(x) = 0 \quad \text{for} \quad |x| \geqq 1$$

and

$$\int \psi(x)dx = 1.$$

Now we define C as the set of all points x such that $\text{dist}(x, B)<d/2$. It is easily verified that the function ϕ defined by

$$\phi(x) = \left(\frac{d}{2}\right)^{-n}\int \psi\left(\frac{y-x}{d/2}\right)dy,$$

satisfies all the requirements of Lemma 4.

In order to conclude the proof of the theorem we define two sequences of sets

$$A_j = \left\{x\colon \text{dist}\,(x, S_0) < \frac{1}{2^j}\right\} \qquad j = 1, 2, \cdots,$$

$$B_j = \left\{x\colon \text{dist}\,(x, S_0) > \frac{3}{2^j}\right\} \qquad j = 1, 2, \cdots.$$

Using Lemma 4 we find functions $\phi_j\in C_\infty$ that are 0 in A_j, 1 in B_j and such that

$$|\text{grad}\,\phi_j(x)| \leqq k2^{j-1}.$$

Now ϕ_jF is a function like F of Lemma 3. By Lemma 3, therefore,

$$\int_V \text{div}\,(\phi_jF)dx = \int_{S_j} \phi_jF\cdot\nu d\sigma$$

or

$$\int_V \phi_j \operatorname{div} F dx + \int_V \operatorname{grad} \phi_j \cdot F dx = \int_{S_1} \phi_j F \cdot \nu d\sigma.$$

It is easily seen that as j goes to infinity the first integral converges to $\int_V \operatorname{div} F dx$ and the last one to $\int_{S_1} F \cdot \nu d\sigma$, which can be taken as an improper integral.

If we show that

$$\lim_{j \to \infty} \int_V \operatorname{grad} \phi_j dx = 0,$$

the proof will be finished. Using the hypothesis on S_0 we conclude that for each $\delta > 0$ the set A_δ (of the points that are at a distance less than δ from S_0) is covered by k_δ spheres of radius 2δ. So the complement of B_j can be covered by k_j spheres of radius $1/2^{j-2}$, say, and k_j is so related to j that

$$k_j \left(\frac{1}{2^{j-1}} \right)^{n-1} \to 0 \quad \text{as} \quad j \to \infty.$$

Using this we obtain

$$\left| \int_V \operatorname{grad} \phi_j F dx \right| \leqq k 2^{j-1} \cdot k_j c \left(\frac{1}{2^{j-2}} \right)^{n-1} = 2^n c k k_j \left(\frac{1}{2^{j-1}} \right)^{n-1},$$

which goes to zero as j tends to infinity. Here c is the volume of the unit sphere.

3. Two remarks about Gaussian regions. First we observe that the property of having p-content zero is invariant under Lipschitz mappings.

Secondly, we may prove that a set A in R^n has zero n-content if and only if it has zero Jordan content.

As a consequence of the separability of R^n, we see that S_1 is made up of a denumerable number of regular surfaces $S_{1,j}$, each one an image by a mapping F_j of an open set A_j in $(n-1)$-dimensional space. In order to take surface integrals in $S_{1,j}$ we have to assume that A_j has Jordan content. This implies that the boundary ∂A_j of A_j has zero Jordan content. If we assume that the mapping F_j can be extended up to the boundary ∂A_j in such a way that F_j is a Lipschitz mapping there, we conclude by the above observations that the boundary of $S_{1,j}$ has zero $(n-1)$-content.

From these remarks we conclude that any region in R^n bounded by a finite number of regular surfaces is a Gaussian region. This contains all regions usually occurring in applications of the divergence theorem.

The author would like to thank Jaak Peetre for mentioning that the simple proof of Lemma 4 is due to Hörmander.

Reference

1. H. Whitney, Geometric Integration Theory, Princeton University Press, 1957.

ON TRANSFORMATIONS IN R^n AND A THEOREM OF SARD

T. M. FLETT, University of Liverpool

1. An elegant proof of the formula for change of variable in a multiple integral has been given by J. Schwartz [4] (see also Zaanen [5], p. 162) in which the theorem is reduced to the following inequality:

THEOREM A. *Let D be an open set in $\mathbf{R}^n$, let f be a continuously differentiable (1-1) mapping of D into $\mathbf{R}^n$, and let $J(x)$, the Jacobian determinant of f at x, be nonzero on D. Then for any measurable subset E of D the set $f(E)$ is measurable and*

$$m(f(E)) \leqq \int_E |J(x)|\, dx, \tag{1.1}$$

where m denotes n-dimensional Lebesgue measure.

It is, of course, true that under the hypotheses of Theorem A we have equality in (1.1), but the weaker result stated in Theorem A is sufficient for the proof of the formula for change of variable.

A complement to Theorem A is provided by a theorem of Sard (see, for example, de Rham [3], p. 9) which states

THEOREM B. *Let D be an open set in $\mathbf{R}^n$, let f be a continuously differentiable mapping of D into $\mathbf{R}^n$, let $J(x)$ be the Jacobian determinant of f at x, and let E_0 be the set of points x of D for which $J(x)=0$. Then $f(E_0)$ is of measure zero.*

Extensions of Theorem B under less restrictive conditions have been given by Rado and Reichelderfer, and in particular it has been shown that Theorem B holds if f is merely differentiable on D ([2], pp. 339, 343). Here, however, we restrict ourselves to the case stated above, which is the case most often used in the theory of differentiable manifolds.

The result of Theorem B is most naturally viewed as an extension of the inequality of Theorem A to the case in which $J(x)$ vanishes at points of D, and indeed Theorem B shows that (1.1) continues to hold in this case. It is also clear that we can remove the hypothesis in Theorem A that f is (1-1), for if f is not (1-1) the integral $\int_E |J(x)|dx$ is equal to the measure of $f(E)$ with multiply-covered volumes being counted multiply. We are therefore led to the following theorem, which contains both Theorems A and B.

THEOREM C. *Let D be an open set in $\mathbf{R}^n$, let f be a continuously differentiable mapping of D into $\mathbf{R}^n$, and let $J(x)$ be the Jacobian determinant of f at x. Then for any measurable subset E of D the set $f(E)$ is measurable and*

$$m(f(E)) \leqq \int_E |J(x)|\, dx. \tag{1.2}$$

Theorem C is a simple consequence of theorems of Rado and Reichelderfer [2, p. 363], but these theorems themselves use difficult ideas involved in the

algebraic topology of $\mathbf{R}^n$. We can also use Schwartz's elementary proof of Theorem A to deal with the subset of D in which $J(x)\neq 0$ and then appeal to Theorem B to complete the argument. However, since Sard's Theorem B is itself an immediate consequence of Theorem C, it seems worth while to give a direct and elementary proof of Theorem C, and this is the purpose of this note. It will be seen that the proof depends on a simple geometrical inequality in which the form of the result is the same whether $J(x)$ is zero or nonzero, and that this inequality leads easily to a form of (1.2) with the measure m replaced by outer measure (Lemma 5). It is only in the proof of the measurability of $f(E)$ (Lemma 6) that we reduce to the case in which $J(x)\neq 0$, and even here this reduction could be avoided by the use of more difficult ideas.

2. We begin by recalling a few definitions. For any point v_0 of $\mathbf{R}^n$ and any set of n linearly independent vectors $a_1, \cdots, a_n$ in $\mathbf{R}^n$, the *parallelotope P with initial vertex v_0 and edge-vectors $a_1, \cdots, a_n$* is the set of all points of $\mathbf{R}^n$ of the form

$$x = v_0 + \sum_{i=1}^{n} \lambda_i a_i,$$

where $\lambda_1, \cdots, \lambda_n$ are real numbers such that $0\leqq\lambda_i\leqq 1$, $i=1, \cdots, n$. The point $v_0+\frac{1}{2}\sum_{i=1}^{n} a_i$ is called the *centre* of the parallelotope.

For fixed k the set of those points of P for which λ_k has a fixed value equal to either 0 or 1 is called an *$(n-1)$-dimensional face* (or, briefly, *face*) of P, so that the number of faces of P is $2n$. The point

$$v_0 + \lambda_k a_k + \frac{1}{2}\sum_{\substack{i=1\\ i\neq k}}^{n} a_i$$

is called the centre of the face.

It is immediate that the parallelotope P with initial vertex at the origin and edge-vectors $a_1, \cdots, a_n$ is the image of the unit cube

$$C = \{x = (x^1, \cdots, x^n)\colon 0 \leqq x^i \leqq 1,\ i = 1, \cdots, n\}$$

under the nonsingular linear transformation $h\colon \mathbf{R}^n \to \mathbf{R}^n$ given by

$$h(x) = h(x^1, \cdots, x^n) = \sum_{i=1}^{n} x^i a_i,$$

and equally the image of the unit cube by any nonsingular linear transformation of $\mathbf{R}^n$ onto itself is a parallelotope of this form. It follows that P is compact, and that the frontier of P is the union of the $2n$ faces of P; also (see, for example, Zaanen [5], p. 160) the n-dimensional measure of P is equal to $|\det(h)| = |\det(a_i^j)|$, where a_i^j is the jth coordinate of a_i; and obviously these last results extend to a parallelotope with any initial vertex.

Throughout the following discussion we use the ordinary Euclidean norm

for points of $\mathbf{R}^n$ and the corresponding norm for a linear transformation of $\mathbf{R}^n$ into itself, and we denote the inner product of x and y by $x\cdot y$. We use $A(n)$ to denote a positive constant depending only on n, not necessarily the same on any two occurrences.

3. For our proof of Theorem C we require two simple geometrical inequalities.

LEMMA 1. *Let F be a set in $\mathbf{R}^n$ contained in a hyperplane H, let x_0 be a fixed point of F, and let $\|x-x_0\|\leqq d$ whenever $x\in F$. Let also G be the set of points of $\mathbf{R}^n$ whose distance from F is less than δ. Then G is measurable (since it is open) and*

$$m(G) \leqq 2^n(d+\delta)^{n-1}\delta. \tag{3.1}$$

It is evident that G lies between the two hyperplanes parallel to H and distant δ from it, and to prove (3.1) we construct a parallelotope containing G with two of its faces in these hyperplanes.

By a suitable translation we may suppose that H contains the origin, so that H is an $(n-1)$-dimensional vector subspace of $\mathbf{R}^n$. We can therefore find a unit vector a_1 such that $x\cdot a_1=0$ for all $x\in H$ (i.e. such that a_1 is orthogonal to H), and then we can find vectors $a_2, \cdots, a_n$ such that $\{a_1, a_2, \cdots, a_n\}$ is a complete orthonormal set in $\mathbf{R}^n$. Let now $y\in G$. Since every vector in $\mathbf{R}^n$ can be expressed as a linear combination of the a_i, there exist real numbers $\lambda_1, \cdots, \lambda_n$ such that

$$y - x_0 = \sum_{i=1}^{n} \lambda_i a_i.$$

Further, since the distance of y from F is less than δ, there exists $x\in F$ (possibly identical with y) such that $\|y-x\|<\delta$, and then writing

$$y - x = (y - x_0) - (x - x_0),$$

we obtain $\lambda_1=(y-x_0)\cdot a_1=(y-x)\cdot a_1-(x-x_0)\cdot a_1=(y-x)\cdot a_1$, whence $|\lambda_1|\leqq \|y-x\|\,\|a_1\|=\|y-x\|<\delta$. Also

$$\|y - x_0\| \leqq \|y - x\| + \|x - x_0\| < \delta + d,$$

so that for $i=2, \cdots, n$,

$$|\lambda_i| = |(y - x_0)\cdot a_i| \leqq \|y - x_0\|\,\|a_i\| = \|y - x_0\| \leqq d + \delta.$$

It follows that G is contained in the (fixed) parallelotope with centre x_0 and edge-vectors $2\delta a_1$, $2(d+\delta)a_i$, $i=2, \cdots, n$, and since the measure of this parallelotope is $2^n(d+\delta)^{n-1}\delta|\det(a_i^j)|=2^n(d+\delta)^{n-1}\delta$, the result follows.

LEMMA 2. *Let h be a linear transformation of $\mathbf{R}^n$ into itself, let P be the image by h of the unit cube $C=\{x=(x^1, \cdots, x^n): 0\leqq x^i\leqq 1,\ i=1, \cdots, n\}$, and let Q be the set of points of $\mathbf{R}^n$ whose distance from P is less than δ. Then Q is measurable (since it is open) and $m(Q)\leqq|\det(h)|+A(n)(\|h\|+\delta)^{n-1}\delta$.*

Suppose first that det $(h)=0$, so that h is singular. In this case P is contained in a hyperplane, and we apply Lemma 1 to $F=P$, taking x_0 to be the image of the centre w_0 of C. Since

$$\|h(w) - h(w_0)\| = \|h(w - w_0)\| \leqq \|h\| \, \|w - w_0\| \leqq \tfrac{1}{2}\sqrt{n}\|h\|$$

whenever $w \in C$, we have $\|x-x_0\| \leqq \frac{1}{2}\sqrt{n}\|h\|$ whenever $x \in P$, whence Lemma 1 gives

$$m(Q) \leqq 2^n(\tfrac{1}{2}\sqrt{n}\|h\| + \delta)^{n-1}\delta \leqq A(n)(\|h\| + \delta)^{n-1}\delta,$$

as required.

Suppose next that det $(h) \neq 0$. In this case P is a parallelotope with measure $m(P)=|\det(h)|$, and it is therefore enough to prove that the open set $Q\backslash P$ has measure not exceeding $A(n)(\|h\|+\delta)^{n-1}\delta$. Since P is compact, for each $y \in Q\backslash P$ there exists $x \in P$ such that $\|y-x\|$ is equal to the distance of y from P, and evidently x is a frontier point of P, so that x lies on one or more $(n-1)$-dimensional faces of P. Since P has $2n$ such faces and each face is the image by h of a face of C, it is now enough to prove that if B is a face of C and E is the set of points of $\mathbf{R}^n$ whose distance from $h(B)$ is less than δ, then

$$m(E) \leqq A(n)(\|h\| + \delta)^{n-1}\delta. \tag{3.2}$$

To prove this last result we observe that $h(B)$ is contained in a hyperplane, so that we can apply Lemma 1 to $F=h(B)$. We choose x_0 to be the centre of the face $h(B)$ of P, so that $\|x-x_0\| \leqq \frac{1}{2}\sqrt{(n-1)}\|h\|$ whenever $x \in h(B)$, and then Lemma 1 gives

$$m(E) \leqq 2^n(\tfrac{1}{2}\sqrt{(n-1)}\|h\| + \delta)^{n-1}\delta \leqq A(n)(\|h\| + \delta)^{n-1}\delta.$$

This proves (3.2), and completes the proof of Lemma 2.

In the case in which det $(h) \neq 0$ it is tempting to estimate $m(Q)$ by using the inequality $m(Q) \leqq m(P')$, where P' is the smallest parallelotope containing Q with sides parallel to those of P, but unfortunately the measure $m(P')$ tends to infinity as we approach the singular case, i.e. as det (h) tends to 0 (this is easily seen from a diagram illustrating the plane case). Most proofs of the change of variable formula in which the estimate of the measure of a parallelotope appears, do in fact use an estimate of the form $m(Q) \leqq m(P')$, and it is for this reason that the hypothesis inf $|J(x)| > 0$ is essential to such proofs.

From Lemma 2 we deduce immediately:

LEMMA 3. *Let C be a closed cube in $\mathbf{R}^n$ with sides parallel to the axes and of length α, let h be a linear transformation of $\mathbf{R}^n$ into itself, and let Q be the set of points of $\mathbf{R}^n$ whose distance from the set $h(C)$ is less than $\alpha\delta$. Then Q is measurable* (*since it is open*) *and*

$$m(Q) \leqq m(C)\{ |\det(h)| + A(n)(\|h\| + \delta)^{n-1}\delta \}.$$

By applying Lemma 3 to the derivative of a differentiable mapping, we ob-

tain the following result; in this we use the definition of derivative given by Dieudonné ([1], Chapter 8).

LEMMA 4. *Let C be a closed cube in $\mathbf{R}^n$ with centre x_0 and with sides parallel to the axes, let f be a differentiable mapping of C into $\mathbf{R}^n$, and let $J(x)$ be the Jacobian determinant of f at x. Then*

$$m^*(f(C)) \leqq m(C)\{ | J(x_0) | + A(n)(\|f'(x_0)\| + \eta)^{n-1}\eta\}, \tag{3.3}$$

where $\eta = \sup_{x \in C} \|f'(x) - f'(x_0)\|$ and m^* denotes outer Lebesgue measure.

To prove (3.3) let α be the length of the sides of C, and let P be the image of C by the linear transformation $f'(x_0): \mathbf{R}^n \to \mathbf{R}^n$. By the mean value theorem applied to the function $f - f'(x_0)$ (cf. [1], (8.6.2)), we have for each x of C

$$\|f(x) - f(x_0) - f'(x_0)(x - x_0)\| \leqq \eta\|x - x_0\| < \eta\alpha\sqrt{n},$$

and this inequality expresses the fact that the point $f(x) - f(x_0) + f'(x_0)(x_0)$ of the translate $f(C) - f(x_0) + f'(x_0)(x_0)$ of $f(C)$ is at a distance less than $\eta\alpha\sqrt{n}$ from the point $f'(x_0)(x)$ of P. It follows that this translate of $f(C)$ is contained in the set of points of $\mathbf{R}^n$ whose distance from P is less than $\eta\alpha\sqrt{n}$, and applying Lemma 3 (and noting that det $(f'(x_0)) = J(x_0)$) we obtain immediately the inequality (3.3).

4. The remainder of the proof of Theorem C is similar to Schwartz's proof of Theorem A (cf. [5], p. 165), but we give it here for the sake of completeness. We divide the proof into two further lemmas. It should be noted that Lemma 5 contains Sard's Theorem B.

LEMMA 5. *Let D be an open set in $\mathbf{R}^n$, let f be a continuously differentiable mapping of D into $\mathbf{R}^n$, and let $J(x)$ be the Jacobian determinant of f at x. Then for any measurable subset E of D*

$$m^*(f(E)) \leqq \int_E | J(x) | \, dx, \tag{4.1}$$

where m^* denotes outer Lebesgue measure.

Suppose first that E is a closed cube C with sides parallel to the axes. Since f' is continuous on C, we can divide C into a finite number of nonoverlapping closed cubes $C_1, \cdots, C_N$ with centers $x_1, \cdots, x_N$ and with sides parallel to the axes such that $\|f'(x) - f'(x_k)\| \leqq \epsilon$ whenever $x \in C_k (k = 1, \cdots, N)$. By Lemma 4, for each cube C_k we have

$$m^*(f(C_k)) \leqq m(C_k)\{ | J(x_k) | + A\epsilon\},$$

where A is independent of k, so that also

$$m^*(f(C)) \leqq \sum m^*(f(C_k)) \leqq \sum | J(x_k) | m(C_k) + A\epsilon m(C),$$

the summations being extended over all cubes C_k. When the maximum diameter

If E is closed, so are $E\cap C$ and $f(E\cap C)$, and hence if E is a countable union of closed sets, then $f(E\cap C)$ is measurable. Since any measurable set is the union of a set of measure zero and a set which is a countable union of closed sets, it is now enough to prove that $f(E\cap C)$ is measurable when E is of measure zero, and this follows immediately from Lemma 5. This completes the proof of Lemma 6, and so also of Theorem C.

References

1. J. Dieudonné, Foundations of modern analysis, Academic Press, New York, 1960.
2. T. Rado and P. V. Reichelderfer, Continuous transformations in analysis, Springer-Verlag, Berlin, 1955.
3. G. de Rham, Variétés différentiables, Hermann, Paris, 1955.
4. J. Schwartz, The formula for change in variables in a multiple integral, this MONTHLY, 61 (1954) 81–85.
5. A. C. Zaanen, An introduction to the theory of integration, North Holland Publ. Co., Amsterdam, 1958.

THE COMPANION MATRIX AND ITS PROPERTIES

LOUIS BRAND, University of Houston

1. Companion matrix. The companion matrix of the polynomial

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n \tag{1}$$

is defined as

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{pmatrix} \tag{2}$$

in which the first superdiagonal consists entirely of ones and all other elements above the last row are zeros. The companion matrix of $\lambda+a_1$ is $[-a_1]$. The characteristic equation of A is $\det(A-\lambda I)=0$ or

$$\begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -\lambda & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1-\lambda \end{vmatrix} = 0.$$

If E is closed, so are $E\cap C$ and $f(E\cap C)$, and hence if E is a countable union of closed sets, then $f(E\cap C)$ is measurable. Since any measurable set is the union of a set of measure zero and a set which is a countable union of closed sets, it is now enough to prove that $f(E\cap C)$ is measurable when E is of measure zero, and this follows immediately from Lemma 5. This completes the proof of Lemma 6, and so also of Theorem C.

References

1. J. Dieudonné, Foundations of modern analysis, Academic Press, New York, 1960.
2. T. Rado and P. V. Reichelderfer, Continuous transformations in analysis, Springer-Verlag, Berlin, 1955.
3. G. de Rham, Variétés différentiables, Hermann, Paris, 1955.
4. J. Schwartz, The formula for change in variables in a multiple integral, this MONTHLY, 61 (1954) 81–85.
5. A. C. Zaanen, An introduction to the theory of integration, North Holland Publ. Co., Amsterdam, 1958.

THE COMPANION MATRIX AND ITS PROPERTIES

LOUIS BRAND, University of Houston

1. Companion matrix. The companion matrix of the polynomial

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n \tag{1}$$

is defined as

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \tag{2}$$

in which the first superdiagonal consists entirely of ones and all other elements above the last row are zeros. The companion matrix of $\lambda+a_1$ is $[-a_1]$. The characteristic equation of A is $\det(A-\lambda I)=0$ or

$$\begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -\lambda & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1-\lambda \end{vmatrix} = 0.$$

If we multiply columns 2, 3, $\cdots$, n of this determinant by $\lambda, \lambda^2, \cdots, \lambda^{n-1}$ and add them to the first column, all elements of this column become zero except the last which is now $-f(\lambda)$. Since the cofactor of this element is $(-1)^{n+1}$, the characteristic equation of A is

$$\det(A - \lambda I) = (-1)^n f(\lambda) = 0; \tag{3}$$

and since the highest common divisor of all cofactors in $\det(A - \lambda I)$ is clearly 1, (3) is also the minimum equation of A.

THEOREM 1. *The companion matrix of the polynomial $f(\lambda)$ has $f(\lambda) = 0$ for its characteristic and minimum equations.*

The companion matrix is singular when and only when $a_n = 0$; for $\det A = (-1)^n a_n$.

The genesis of the companion matrix is evident when one replaces the linear differential equation

$$f(D)x = 0 \qquad (D = d/dt) \tag{4}$$

or the linear difference equation

$$f(E)x_n = 0 \qquad (E = 1 + \Delta) \tag{5}$$

by a system of n linear equations of the first order. In both cases the matrix of the system is the companion of the polynomial $f(\lambda)$.

For example, the differential equation

$$x''' + ax'' + bx' + cx = 0$$

is replaced by the system

$$\begin{aligned} x' &= \phantom{- cx -{}} y \\ y' &= \phantom{- cx - by -{}} z \\ z' &= - cx - by - az \end{aligned}$$

whose matrix is precisely the companion of the polynomial $\lambda^3 + a\lambda^2 + b\lambda + c$. Similarly the difference equation

$$x_{n+3} + ax_{n+2} + bx_{n+1} + cx_n = 0$$

may be replaced by the system

$$\begin{aligned} x_{n+1} &= \phantom{- cx_n -{}} y_n \\ y_{n+1} &= \phantom{- cx_n - by_n -{}} z_n \\ z_{n+1} &= - cx_n - by_n - az_n \end{aligned}$$

whose matrix is the companion of the same polynomial.

2. Eigenvectors. The equation $f(\lambda) = 0$ may be written in the matrix form

$$Ae(\lambda) = \lambda e(\lambda), \tag{6}$$

where the column vector

$$e(\lambda) = (1, \lambda, \lambda^2, \cdots, \lambda^{n-1});$$

for the first $n-1$ equations are the identities $\lambda^i = \lambda^i$ $(i = 1, 2, \cdots, n-1)$ and the last is

$$-a_n - a_{n-1}\lambda - \cdots - a_1\lambda^{n-1} = \lambda^n.$$

Thus if λ_i is an eigenvalue of A, equation (6) is valid for $\lambda = \lambda_i$. Moreover, the rank of the matrix $A - \lambda_i I$ is always $n-1$ even when λ_i is a multiple zero of $f(\lambda)$; for the minor of the element $(n1)$ has a determinant of value 1. The eigenvalue λ_i is therefore associated with just *one* eigenvector $e_i = e(\lambda_i)$; two eigenvectors are called *equal* if one is a scalar multiple of the other. We state this result as

THEOREM 2. *If the polynomial $f(\lambda)$ of degree n has m $(\leqq n)$ distinct zeros λ_i then its companion matrix has exactly m independent eigenvectors*

$$e_i = (1, \lambda_i, \lambda_i^2, \cdots, \lambda_i^{n-1}), \qquad i = 1, 2, \cdots, m, \tag{7}$$

associated with the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$.

These eigenvectors are linearly independent since the rank of the $m \times n$ Vandermonde matrix formed from their components is exactly m.

3. Generalized eigenvectors. When the companion matrix has an eigenvalue λ_1 of multiplicity k, λ_1 satisfies the equations

$$f(\lambda) = 0, f'(\lambda) = 0, \cdots, f^{(k-1)}(\lambda) = 0.$$

The first of these equations is equivalent to the matrix equation (6); the others are equivalent to matrix equations obtained from (6) by $k-1$ successive differentiations with respect to λ:

$$Ae^{(j)}(\lambda) = \lambda e^{(j)}(\lambda) + je^{(j-1)}(\lambda), \qquad j = 1, 2, \cdots, k-1,$$

where $e^{(0)}(\lambda)$ means $e(\lambda)$. These are equivalent to the system

$$A\,\frac{e^{(j)}(\lambda)}{j!} = \lambda\,\frac{e^{(j)}(\lambda)}{j!} + \frac{e^{(j-1)}(\lambda)}{(j-1)!}, \qquad j = 1, 2, \cdots, k-1. \tag{8}$$

Thus when $\lambda = \lambda_1$ is a k-tuple zero of $f(\lambda)$ we have the k equations

$$\begin{aligned} A\,e_1 &= \lambda_1 e_1 \\ A\,e_2 &= \lambda_1 e_2 + e_1, \\ A\,e_3 &= \lambda_1 e_3 + e_2, \\ &\cdots\cdots \\ A\,e_k &= \lambda_1 e_k + e_{k-1}, \end{aligned} \tag{9}$$

where $e_1 = e(\lambda_1)$ is *the* eigenvector and

$$e_{j+1} = \frac{e^{(j)}(\lambda_1)}{j!}, \qquad j = 1, 2, \cdots, k-1, \tag{10}$$

are, by definition, $k-1$ *generalized eigenvectors* associated with λ_1. Thus with every k-tuple eigenvalue we associate k eigenvectors of which $k-1$ are generalized. We note that the way in which equations (8) are derived guarantees their consistency. Moreover the k vectors

$$\begin{aligned}
e_1 &= (1, \lambda_1, \lambda_1^2, \lambda_1^3, \quad \cdots, \lambda_1^{n-1}) \\
e_2 &= (0, 1, 2\lambda_1, 3\lambda_1^2, \quad \cdots, (n-1)\lambda_1^{n-2}) \\
e_3 &= (0, 0, 1, \binom{3}{2}\lambda_1, \quad \cdots, \binom{n-1}{2}\lambda_1^{n-3}) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
e_k &= (0, 0, 0, 0, \quad \cdots, \binom{n-1}{k}\lambda_1^{n-k})
\end{aligned}$$

are linearly independent since the rank of the $k\times n$ matrix formed from their components is k; for the $k\times k$ determinant on the left has the value 1.

The entire set of n eigenvalues of A, simple and multiple, are now associated with n eigenvectors; m of these, given by (7), are the eigenvectors associated with the m distinct eigenvalues, whereas the $n-m$ remaining are generalized eigenvectors of the type (10). The entire set is linearly independent and may be used to reduce A to its Jordan normal form.

4. Reduction to the Jordan Normal Form. Let the $n\times n$ companion matrix A have the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ written in an order of increasing multiplicity (such as 3, 1, 1, 2, 2, 2) and denote the associated eigenvectors, actual or generalized, by $e_1, e_2, \cdots, e_n$. Since this set is linearly independent they admit a reciprocal set $e^1, e^2, \cdots, e^n$ defined by $e_i\cdot e^j = \delta_i^j$. Form the two $n\times n$ matrices

$$B = (e_1\ e_2 \cdots e_n) \quad \text{from the } n \text{ columns } e_i,$$

$$B^{-1} = \begin{pmatrix} e^1 \\ e^2 \\ \cdot \\ \cdot \\ \cdot \\ e^n \end{pmatrix} \qquad \text{from the } n \text{ rows } e^i.$$

Then $B^{-1}AB = J$ will be in the Jordan normal form. The proof is immediate; for

$$B^{-1}AB = B^{-1}(Ae_1 \mid Ae_2 \mid \cdots \mid Ae_n) = (e^iAe_j)$$

has elements e^iAe_j which are precisely the elements of a Jordan matrix by virtue of Equations (9). An example will make this clear.

which may be obtained from I by the succession of row operations

$$A = (4 - 3a)(4 - 2b)(4 - 1c)(-4d)(1432)I.$$

Here (1432) denotes a permutation of rows, $(-4d)$ means row 4 times $-d$, and $(4-1c)$ means row 4 minus c times row 1. Hence, if $d\neq 0$, and we take the inverse operations in reverse order,

$$A^{-1} = (1234)(-4/d)(4 + 1c)(4 + 2b)(4 + 3a)I$$

or

$$A^{-1} = \begin{bmatrix} -c/d & -b/d & -a/d & -1/d \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus the inverse of any companion matrix can be written down at once. It is related to the companion matrix of the polynomial

$$\lambda^4 + \frac{c}{d}\lambda^3 + \frac{b}{d}\lambda^2 + \frac{a}{d}\lambda + \frac{1}{d} = 0 \tag{12}$$

whose roots are the reciprocals of the roots of (11): The inverse of A is the companion of the polynomial (12) revolved counterclockwise 180° about its center.

This paper was presented at the joint meeting of the Texas Academy of Science and the Mathematical Association at Galveston, December 8, 1961.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the twenty-fourth William Lowell Putnam Mathematical Competition held on December 7, 1963, have been determined in accordance with the constitution of the Competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of Michigan State University, East Lansing, Michigan. The members of the team were S. E. Crick, Jr., R. E. Greene and W. A. Webb; to each of these a prize of fifty dollars is awarded.

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The second prize, four hundred dollars, is awarded to the Department of Mathematics of Brooklyn College, Brooklyn, New York. The members of the

team were William Kantor, Steven Sperber and Robert Zarrow; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of the University of Pennsylvania, Philadelphia, Pennsylvania. The members of the team were Larry Goldstein, Ralph Greenberg and E. Y. Miller; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of California Institute of Technology, Pasadena, California. The members of the team were A. C. Hindmarsh, Kenneth Kunen and V. S. Poythress; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were J. H. Spencer, Gordon Wassermann and M. H. Weinless; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are Lawrence Corwin, Harvard University; S. E. Crick, Jr., Michigan State University; R. E. Greene, Michigan State University; J. H. Spencer, Massachusetts Institute of Technology; and Lawrence Zalcman, Dartmouth College. To each of these a prize of seventy-five dollars is awarded. The William Lowell Putnam Prize Scholarship to Harvard has been awarded to Mr. Crick, who will begin his graduate work in the fall of 1965. The value of this scholarship has been increased to $2500.00 plus tuition ($1520.00), making a total monetary value of $4020.00.

The six persons ranking second highest in the examination, named in alphabetical order, are R. W. Herrick, Oberlin College; Kenneth Kunen, California Institute of Technology; Gilbert Labelle, University of Montreal; Robert Lee, Reed College; E. Y. Miller, University of Pennsylvania; and Josef Sukonick, University of Pennsylvania. To each of these a prize of thirty-five dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: Cornell University, Ithaca, New York, the members of the team being A. D. Jette, D. J. Kilbridge and J. T. Litman; Harvard University, Cambridge, Massachusetts, the members of the team being Jeffrey Cheeger, Melvin Hochster and John Mather; University of British Columbia, Vancouver, British Columbia, the members of the team being S. A. Glass, Joanne McWhirter and Bent Petersen; University of Colorado, Boulder, Colorado, the members of the team being J. M. Cushing, D. E. Maurer and R. C. Misare; and the University of Montreal, Montreal, Quebec, the members of the team being Luc Demers, Gaston Giroux and Cecile Mayrand.

Honorable mention is given to the following twenty-five individuals, named in alphabetical order: Bruce Appleby, Massachusetts Institute of Technology; L. G. Brown, Harvard University; N. H. Camien, California Institute of Technology; M. J. Cohen, California Institute of Technology; David Ebin, Harvard University; P. J. Erdelsky, Case Institute of Technology; Daniel Fendel, Harvard University; Gaston Giroux, University of Montreal; W. E. Heierman, Georgia Institute of Technology; R. B. Hodges, Rice University; A. A. Iarrobino, Jr., Massachusetts Institute of Technology; William Kantor, Brooklyn College; Frank Kaplan, City College; William Kennersley, Rensselaer Polytechnic Institute; Gary Luxton, McGill University; Cecile Mayrand, University of Montreal; V. S. Poythress, California Institute of Technology; S. W. Reyner, South Dakota School of Mines; Michael Rolle, Massachusetts Institute of Technology; Michael Schulz, Michigan

State University; R. P. Stanley, California Institute of Technology; J. J. Weinkam, Xavier University, Cincinnati; J. R. Whitney, Michigan State University; Robert Wilson, American University; and Thomas Zaslavsky, City College.

A total of seventeen hundred five contestants from two hundred five colleges and universities entered the competition. Twelve hundred sixty contestants from one hundred ninety-nine colleges and universities (one hundred seventy having teams) participated in the examination on December 7, 1963.

The individual rankings of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

Those participating in the competition were given the following problems to solve:

Part I

1. (a) Show that a regular hexagon, six squares, and six equilateral triangles can be assembled without overlapping to form a regular dodecagon.
   (b) Let $P_1, P_2, \cdots, P_{12}$ be the successive vertices of a regular dodecagon. Explain how the three diagonals P_1P_9, P_2P_{11}, and P_4P_{12} intersect.
2. Let $\{f(n)\}$ be a strictly increasing sequence of positive integers such that $f(2)=2$ and $f(mn)=f(m)f(n)$ for every relatively prime pair of positive integers m and n (the greatest common divisor of m and n is equal to 1). Prove that $f(n)=n$ for every positive integer n.
3. Find an integral formula for the solution of the differential equation

$$\delta(\delta-1)(\delta-2)\cdots(\delta-n+1)y=f(x), \qquad x\geqq 1,$$

for y as a function of x satisfying the initial conditions $y(1)=y'(1)=\cdots=y^{(n-1)}(1)=0$, where f is continuous and

$$\delta\equiv x\frac{d}{dx}\cdot$$

4. Let $\{a_n\}$ be a sequence of positive real numbers. Show that

$$\limsup_{n\to\infty} n\left(\frac{1+a_{n+1}}{a_n}-1\right)\geqq 1.$$

Show that the number 1 on the right-hand side of this inequality cannot be replaced by any larger number. (The symbol lim sup is sometimes written $\overline{\lim}$.)

5. (a) Prove that if a function f is continuous on the closed interval $[0, \pi]$ and if

$$\int_0^\pi f(\theta)\cos\theta\, d\theta=\int_0^\pi f(\theta)\sin\theta\, d\theta=0$$

then there exist points α and β such that

$$0<\alpha<\beta<\pi \quad\text{and}\quad f(\alpha)=f(\beta)=0.$$

   (b) Let R be any bounded convex open region in the Euclidean plane (that is, R is a connected open set contained in some circular disk, and the line segment joining any two points of R lies entirely in R). Prove with the help of part (a) that the centroid (center of gravity) of R bisects at least three distinct chords of the boundary of R.
6. Let U and V be any two distinct points on an ellipse, let M be the midpoint of the chord UV, and let AB and CD be any two other chords through M. If the line UV meets the line AC in the point P and the line BD in the point Q, prove that M is the midpoint of the segment PQ.

Part II

1. For what integer a does x^2-x+a divide $x^{13}+x+90$?
2. Let S be the set of all numbers of the form 2^m3^n, where m and n are integers, and let P be the set of all positive real numbers. Is S dense in P?
3. Find every twice-differentiable real-valued function f with domain the set of all real numbers and satisfying the functional equation

$$(f(x))^2 - (f(y))^2 = f(x+y)f(x-y)$$

for all real numbers x and y.
4. Let C be a closed plane curve that has a continuously turning tangent and bounds a convex region. If T is a triangle inscribed in C with maximum perimeter, show that the normal to C at each vertex of T bisects the angle of T at that vertex. If a triangle T has the property just described, does it necessarily have maximum perimeter? What is the situation if C is a circle? (A convex region is a connected open set such that the line segment joining any two points of the set lies entirely in the set.)
5. Let $\{a_n\}$ be a sequence of real numbers satisfying the inequalities

$$0 \leqq a_k \leqq 100a_n \text{ for } n \leqq k \leqq 2n \text{ and } n = 1, 2, \cdots,$$

and such that the series

$$\sum_{n=0}^{\infty} a_n$$

converges. Prove that

$$\lim_{n\to\infty} na_n = 0.$$

6. Let E be a Euclidean space of at most three dimensions. If A is a nonempty subset of E, define $S(A)$ to be the set of all points that lie on closed segments joining pairs of points of A. For a given nonempty set A_0, define $A_n \equiv S(A_{n-1})$ for $n=1, 2, \cdots$. Prove that $A_2=A_3= \cdots$. (A one-point set should be considered to be a special case of a closed segment.)

Solutions. Part I

1. (a) Place the squares externally on the sides of the hexagon. Since the angles between adjacent sides of adjacent squares are all equal to 60°, the gaps can be filled with the six equilateral triangles. Since $60°+90°=150°$ the resulting dodecagon is regular.

(b) The three diagonals are concurrent. Let the dodecagon be composed as described in part (a) in such a fashion that P_1P_{12} is the side of a square. The lines P_1P_{12}, P_2P_{12}, $\cdots$, $P_{11}P_{12}$ divide the angle 150° at P_{12} into ten equal angles of 15°. Therefore the angle $P_1P_{12}P_4$ is equal to 45°, P_4P_{12} is a diagonal of the square on P_1P_{12}, and P_1P_9 is the other diagonal. The three lines P_1P_9, P_2P_{11}, P_4P_{12} all pass through the center of this square.

2. Assume that $f(3)=3+p$, where $p\geqq 0$. Then $f(6)=6+2p$, $f(5)\leqq 5+2p$, $f(10)\leqq 10+4p$, $f(9)\leqq 9+4p$, and $f(18)\leqq 18+8p$. Also, $f(5)\geqq 5+p$, $f(15)\geqq 15+8p+p^2$, and $f(18)\geqq 18+8p+p^2$. Consequently, $18+8p+p^2\leqq 18+8p$, and hence $p=0$ and $f(3)=3$. Since $f(6)=6$, $f(n)=n$ for $n\leqq 6$. In general, if $f(n)=n$ for $n\leqq 2k$, where k is an integer >1, $f(2k-1)=2k-1$, and hence $f(4k-2)=4k-2$, and $f(n)=n$ for $n\leqq 4k-2$. Since $4k-2>2k$, induction shows that $f(n)=n$ for all positive integers n.

3. The first step is to show that

$$\delta(\delta-1)(\delta-2)\cdots(\delta-n+1) = x^n D^n.$$

Proof by induction reduces to showing that

$$D^n(\delta - n) = xD^{n+1},$$

which can itself be proved by induction. Alternatively, the initial identity can be proved by showing that it is valid when applied to every nonnegative integral power x^k, and hence valid for every polynomial, and this verification for x^k reduces to showing that

$$\delta(\delta-1)\cdots(\delta-n+1)x^k = k(k-1)\cdots(k-n+1)x^k.$$

The given differential equation becomes $x^n D^n y(x) = f(x)$, and the solution is provided by Liouville's formula for iterated integrals:

$$y(x) = \frac{1}{(n-1)!}\int_1^x (x-t)^{n-1}\frac{f(t)}{t^n}\,dt.$$

4. Assume the conclusion is false. Then there is a positive integer N such that for $n \geqq N$,

$$n\left(\frac{1+a_{n+1}}{a_n} - 1\right) < 1.$$

This inequality is equivalent to

$$\frac{1}{n+1} < \frac{a_n}{n} - \frac{a_{n+1}}{n+1}.$$

Replacing n by N, $N+1, \cdots, N+k-1$, in turn, and adding the results, we obtain

$$\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{N+k} < \frac{a_N}{N} - \frac{a_{N+k}}{N+k} < \frac{a_N}{N},$$

in contradiction to the divergence of the harmonic series.

To show that 1 cannot be replaced by a larger number, let $a_n = kn$, $n = 1, 2, \cdots$. Then

$$n\left(\frac{1+a_{n+1}}{a_n} - 1\right) = \frac{1+k}{k} \to \frac{1+k}{k},$$

which is arbitrarily near 1 for large k. Alternatively, if $a_n = n \log_e n$

$$\lim_{n\to\infty} n\left(\frac{1+a_{n+1}}{a_n} - 1\right) = 1.$$

5. (a) Since $\sin\theta > 0$ for $0 < \theta < \pi$, the second of the two assumed equations implies that $f(\alpha) = 0$ for at least one α between 0 and π. Assume now that this α is the *only* zero of f between 0 and π. Then $f(\theta)$ must change sign at α and nowhere else between 0 and π. Hence $f(\theta)\sin(\theta-\alpha)$ is of constant sign and

$$\int_0^\pi f(\theta)\sin(\theta - \alpha)d\theta \neq 0.$$

But this is inconsistent with the assumed vanishing of two integrals.

(b) Choose the center of gravity of the bounded convex domain D as the origin of a system of polar coordinates r, θ. Let $r = r(\theta)$ be the equation of the boundary curve. Obviously, there is at least one direction θ with $r(\theta) = r(\theta+\pi)$. Choose it as the positive x-axis. Put $f(\theta) \equiv r^3(\theta) - r^3(\theta+\pi)$. Since O is the center of gravity, both of the integrals given in part (a) vanish. Hence $f(\theta)$ has at least two zeros, i.e. $r(\theta) = r(\theta+\pi)$ for at least two distinct values of θ with $0 < \theta < \pi$.

6. *First solution.* By Steiner's theorem,

$$MPUV \overset{A}{\barwedge} BCUV \overset{D}{\barwedge} QMUV \barwedge MQVU.$$

Hence PQ is a pair of the involution $(MM)(UV)$. Since M is the midpoint of UV, the other invariant point of this involution is the point at infinity, and the involution relates pairs of points equidistant from M.

Second solution. Choose an oblique coordinate system so that the y-axis contains the points U and V, and the x-axis contains the midpoints of chords parallel to U and V. Let the equation of the ellipse be $y^2 = ax^2+bx+c$, and those of the lines containing the chords AB and CD be $y = mx$ and $y = nx$, respectively. Then denote:

A: (x_1, y_1), where x_1 is either root of $m^2x^2 = ax^2+bx+c$,

C: (x_2, y_2), where x_2 is either root of $n^2x^2 = ax^2+bx+c$.

The y-intercept of AC is

$$mx_1 - \frac{y_2 - y_1}{x_2 - x_1}x_1 = \frac{(m-n)x_1x_2}{x_2 - x_1}.$$

With a similar notation, the y-intercept of BD is $(m-n)\bar{x}_1\bar{x}_2/(\bar{x}_2 - \bar{x}_1)$. The problem is to show that the sum of these y-intercepts is zero, and this quickly reduces to showing:

$$\frac{x_1 + \bar{x}_1}{x_1\bar{x}_1} = \frac{x_2 + \bar{x}_2}{x_2\bar{x}_2}.$$

Finally, this follows immediately from the formulas for the sum and product of the roots of a quadratic equation.

Third solution. Replace the ellipse by a circle. Drop perpendiculars a and b from P and Q to AB, and perpendiculars c and d from P and Q to CD. Write $l = UM = MV$, $p = PM$, and $q = MQ$. We wish to prove $p = q$.

We have the following pairs of similar right triangles:

$$\triangle Ma \sim \triangle Mb, \quad \triangle Mc \sim \triangle Md, \quad \triangle Cc \sim \triangle Bb, \quad \triangle Aa \sim \triangle Dd.$$

These yield respectively

$$\frac{p}{q} = \frac{a}{b}, \quad \frac{p}{q} = \frac{c}{d}, \quad \frac{CP}{BQ} = \frac{c}{b}, \quad \frac{PA}{QD} = \frac{a}{d},$$

whence

$$\begin{aligned} \frac{p^2}{q^2} &= \frac{p}{q}\,\frac{p}{q} = \frac{a}{b}\,\frac{c}{d} = \frac{c}{b}\,\frac{a}{d} = \frac{CP}{BQ}\,\frac{PA}{QD} \\ &= \frac{CP \times PA}{BQ \times QD} = \frac{UP \times PV}{UQ \times QV} = \frac{(l-p)(l+p)}{(l+q)(l-q)} = \frac{l^2 - p^2}{l^2 - q^2} \\ &= \frac{l^2}{l^2} = 1. \end{aligned}$$

Thus $p = q$, as desired.

Finally, since this is an affine theorem that has been proved for a circle, it holds also for any ellipse.

Solutions Part II

1. $a = 2$. The cases $x = 0$ and $x = 1$ show that a divides 2. The case $x = -1$ shows that a cannot be 1 or -2. The case $x = -2$ shows that a cannot be -1. Finally, $a = 2$ can be checked by actual division.

2. Yes. This density is equivalent to the density of the numbers $m \log 2 + n \log 3$, which in turn is equivalent to the density of the numbers $m + n(\log 3)/(\log 2)$. Now, $\log 3/\log 2$ is irrational (the proof is easy), and hence the set of all $n \log 3/\log 2$ modulo 1 is dense in the unit interval.

3. Putting $y = x$ shows that $f(0) = 0$. Differentiating successively, first with respect to x and then with respect to y, we obtain

$$\begin{aligned} 2f(x)f'(x) &= f'(x+y)f(x-y) + f(x+y)f'(x-y), \\ 0 &= f''(x+y)f(x-y) - f(x+y)f''(x-y), \end{aligned}$$

and hence, for all u and v:

$$f''(u)f(v) = f(u)f''(v).$$

There are two main cases: (i) $f''(u) = 0$ identically and (ii) there exists a nonempty open interval I in which $f''(u) \neq 0$. Case (i) gives f linear and, since $f(0) = 0$, $f(x) = cx$ for some constant c. For case (ii), let v_0 be a point where $f(v_0)f''(v_0) \neq 0$, and let $c = f''(v_0)/f(v_0)$. We now have a nonzero constant c such that $f''(u) = cf(u)$ for all real u. There are two subcases: (iia): $c < 0$, (iib): $c > 0$. For case (iia), let $c = -a^2$, so that $f''(u) + a^2 f(u) = 0$, and $f(u) = A \sin au + B \cos au$. Since $f(0) = 0$, $B = 0$ and $f(u) = A \sin au$. For case (iib), let $c = b^2$, so that $f''(u) - b^2 f(u)$

$=0$, and $f(u)=C\sinh bu+D\cosh bu$. As before, $D=0$ and $f(u)=C\sinh bu$. In all cases these solutions check.

4. If the tangent line is permitted to approximate the curve in a neighborhood of a vertex where the normal to the curve does not bisect the angle, the principle of reflection shows easily that the perimeter of the triangle can be increased by a small displacement of the vertex. If an equilateral triangle is "blown up" slightly to give a smooth curve, an inscribed equilateral triangle whose vertices are near the midpoints of the sides of the original triangle has the property described but is certainly not of maximal perimeter. For a circle this property implies that the inscribed triangle is equilateral, and hence of maximal perimeter.

5. By assumption, for any positive integer n, a_{2n} is less than or equal to each of the n numbers $100\,a_n$, $100\,a_{n+1}$, $\cdots$, $100\,a_{2n-1}$, and consequently, as the result of addition and doubling,

$$2na_{2n} \leqq 200(a_n + a_{n+1} + \cdots + a_{2n-1}) \to 0.$$

Similarly, a_{2n-1} is less than or equal to each of the n numbers $100\,a_n$, $100\,a_{n+1}$, $\cdots$, $100\,a_{2n-1}$, and consequently,

$$(2n-1)a_{2n-1} \leqq 2na_{2n-1} \leqq 200(a_n + a_{n+1} + \cdots + a_{2n-1}) \to 0.$$

6. If A_0 is a subset of a line, then A_1 is the smallest interval I containing A_0, and therefore so are A_2, A_3, $\cdots$. If A_0 is a subset of a plane, but not a line, and if u, v, and w are any three points of this plane, define $T(u, v, w)$ to be the smallest convex set containing u, v, and w. If p lies on a segment joining points of the segments $[a, b]$ and $[c, d]$, and if q lies on a segment joining points of the segments $[e, f]$ and $[g, h]$, and if r is a point of the segment $[p, q]$, then r belongs to the smallest convex set containing the points $a, b, \cdots, g, h$, and therefore r belongs to $T(x, y, z)$ for a certain triplet x, y, and z of these points. But $T\subset S(S(\{x, y, z\}))$ and hence $r\in T\subset S(S(A_0))=A_2$. Therefore A_2 is convex, and hence equal to A_3, A_4, $\cdots$. If A_0 is a noncoplanar set, define $T(t, u, v, w)$ to be the smallest convex set containing t, u, v, and w. The procedure is the same as in the plane case, except that $r\in T(s, x, y, z)$ for some four points s, x, y, and z. The inclusion $T\subset S(S(\{s, x, y, z\}))$ follows from the fact that if L_1 and L_2 are two nonadjacent edges of a solid tetrahedron then every point of this solid tetrahedron lies on a segment joining a point of L_1 and a point of L_2.

Mathematical Swifties

"I'm dividing one integer by another," Tom said rationally.

"Why isn't π equal to 22/7?" Tom asked irrationally.

"The ratio of the circumference of a circle to its diameter is not 22/7," said Tom piously.

"The first derivative shows that the function is increasing," Tom stated positively.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146

SOME CONTOUR INTEGRAL SOLUTIONS TO BESSEL'S EQUATION

JAMES M. HORNER, University of Alabama

It is well known that

$$y(z) = z^n \int_C e^{izt}(1 - t^2)^{n-1/2} dt \tag{1}$$

is a solution to Bessel's equation provided that $e^{izt}(1-t^2)^{n+1/2}$ vanishes at the termini of the contour C. This result can be generalized.

THEOREM. *Let* $P(t)=at^2+bt+c$, *where* $b^2-4ac\neq 0$, *and let*

$$f(z, t) = A \exp\,[iuzP'(t)] + B \exp\,[-iuzP'(t)], \tag{2}$$

where A *and* B *are constants and* $u=\pm(b^2-4ac)^{-1/2}$. *Then*

$$y(z) = z^n \int_C f(z, t) P^{n-1/2} dt \tag{3}$$

is a solution of Bessel's equation for appropriate contours C, *assuming differentiation of* (3) *under the sign of integration.*

Proof. It is sufficient to show that

$$w_1(z) = \int_C f(z, t) P^{n-1/2} dt \tag{4}$$

satisfies

$$L[w] = zw'' + (2n + 1)w' + zw = 0. \tag{5}$$

For simplicity, let

$$g(z, t) = A \exp\,[iuzP'(t)] - B \exp\,[-iuzP'(t)]. \tag{6}$$

If (4) is substituted into (5) the result is

$$\begin{aligned} L[w_1] &= z\int_C f(z, t)P^{n-1/2}\{1 + u^2[2P''P - (P')^2]\}\,dt \\ &\quad + u\int_C P^{n-1/2}\{i(2n + 1)P'g(z, t) - 2zuP''Pf(z, t)\}\,dt. \end{aligned} \tag{7}$$

But $2P''P-(P')^2=-(b^2-4ac)=-u^{-2}$, so the first integral in (7) vanishes and (7) becomes

$$L[w_1] = 2iu\int_C \frac{\partial}{\partial t}\{g(z,t)P^{n+1/2}\}dt. \tag{8}$$

The conclusion then follows for all contours C for which $[g(z,t)P^{n+1/2}]_C=0$.

In particular if r_1 and r_2 are zeros of $P(t)$ then

$$y_n(z) = z^n\int_{r_1}^{r_2} f(z,t)P^{n-1/2}dt \tag{9}$$

is a solution to the Bessel equation when $\mathrm{Re}(n+1/2)>0$. If $A+B=0$, $y_n(z)$ is the trivial solution. When $A+B\neq 0$, $y_n(z)/z^n$ is an integral function for $\mathrm{Re}(n+1/2)>0$, so

$$y_n(z) = K_nJ_n(z). \tag{10}$$

If both members of (10) are divided by z^n and then evaluated at $z=0$, we find that

$$K_n = 2^n(A+B)\Gamma(n+1/2)\int_{r_1}^{r_2} P^{n-1/2}dt. \tag{11}$$

If $P(t)$ is written $P(t)=a(t-r_1)(t-r_2)$ the integral in (11) becomes a Beta function integral, with the change of variable $t=r_1+s(r_2-r_1)$, and

$$K_n = (-a)^{n-1/2}(r_2-r_1)^{2n}(A+B)2^{-n}\sqrt{\pi}\,\Gamma(n+1/2). \tag{12}$$

So we have the following

COROLLARY. *If* $P(t)=a(t-r_1)(t-r_2)$, $r_1\neq r_2$, $a\neq 0$, *then for* $\mathrm{Re}(n+1/2)>0$

$$J_n(z) = \frac{(A+B)^{-1}(2z)^n}{(r_2-r_1)^{2n}\sqrt{\pi}\,\Gamma(n+1/2)}\int_{r_1}^{r_2} f(z,t)[(t-r_1)(r_2-t)]^{n-1/2}dt, \tag{13}$$

where $f(z,t)$ *is given by* (2), *with* $u=\pm a(r_2-r_1)^{-1}$ *and* $A+B\neq 0$.

Reference

1. G. N. Watson, A treatise on the theory of Bessel functions, 2nd ed., Cambridge University Press, 1944.

SOME REMARKS ON ORBITS IN INVERTIBLE SPACES

ANDREW J. UMEN, State University of New York at Buffalo

In a recent note [4] Norman Levine indicated some additional local properties which are necessarily global in invertible spaces. It is the purpose of this note to exhibit some of the properties of orbits in invertible spaces and to relate the study of invertible spaces to an early paper [1] by Richard Arens.

N-TH POWERS IN THE FIBONACCI SERIES

FLOYD BUCHANAN, Buffalo, New York

The two theorems below are concerned with the Fibonacci series or Pisano series, as it is sometimes called. It is the series defined by $U_n = U_{n-1} + U_{n-2}$, U_n being the nth term and $U_1 = U_2 = 1$. The limiting ratio of the terms is equal to the positive root of the quadratic equation $x^2 - x - 1 = 0$ and has an intimate connection with the Golden Ratio of ancient Greek architecture and design. The arrangement of leaf stems on the stalks of plants is another illustration of this series. There are many interesting relations and many striking resemblances to the natural number series.

THEOREM 1. *If p is a prime, U_{p^n} is a prime or product of primes unless $p = 5$, in which case $U_{5^n} = 5^n x$, where x is a prime or product of primes.*

Proof. Let us suppose $U_{p^n} (p \neq 5)$ is divisible by the square of some odd prime q. Then $U_{p^n} = rq^2$, where r is any integer. The only terms in the series which are divisible by any divisor d are those of the form $U_{sj(d)}$, where s is any integer and $j(d)$ is the rank of the first term divisible by d. The notation $j(d)$ is my own, which I adopt for convenience.

By Lucas' theorem on divisibility of terms of this series by odd primes (see [1], p. 396-V), the rank of the first term divisible by q^k, but by no higher power than k of the prime q, is equal to $q^{k-1}j(q) = j(q^k)$. Hence with $k = 2$ we would have $p^n = sj(q^2) = sqj(q)$. It follows that $j(q)$ would be a power of p and that q itself must equal p or simply that $j(p)$ is a power of p.

But if p is a quadratic residue of 5, then $U_{p-1} \equiv 0(p)$ or $p - 1 \equiv 0\ (j(p))$ and if p is a quadratic nonresidue of 5, then $U_{p+1} \equiv 0(p)$ or $p + 1 \equiv 0(j(p))$, (see [1], p. 396-VIII with $\delta = \sqrt{5}$). Also p is a residue of 5 if 5 is a residue of p and vice-versa for nonresidue character. In either case $j(p)$ is prime to p, so $j(p)$ cannot be a power of p and U_{p^n} is not divisible by the square of any odd prime.

If $q = 2$, then $U_{p^n} = 4r$. As $j(4) = 6$, $p^n \equiv 0\ (6)$, which is not possible. Then, for all primes other than 5, the theorem is true.

When $p = 5$, since $j(5) = 5 = U_5$, we have $5^n = 5^{n-1}j(5)$, formally. Evidently $5^n = j(5^n)$ and $U_{5^n} = 5^n x$ where x is prime to 5. If $x = kq^2$ with q any odd prime and k any integer, then $5^n \equiv 0(j(q^2))$ or $5^n \equiv 0(qj(q))$ with the result $q = 5$, but then $x \equiv 0\ (5)$, which is a contradiction.

Our last step is to set $q = 2$ in the preceding paragraph. Then $x = 4k$ and, as $j(4) = 6$, we get $5^n \equiv 0\ (6)$, which is obviously incorrect. Hence, as all possible cases have been rejected, the theorem follows.

THEOREM 2. *U_{12}, U_6 and the trivial $U_1 = U_2 = 1$ are the only terms in the series which are powers of integers other than the first degree.*

Proof. Suppose that $U_n = a^m (m > 1)$, where a is any integer and n is odd with at least two prime factors. With p as any one of the prime factors of n, we can write $U_n = (U_n/U_p)U_p$, and U_n/U_p is an integer since $U_n \equiv 0\ (U_p)$ if $n \equiv 0\ (p)$,

If n is divisible by 9, then as $U_9=34$, then n would have to be divisible by 17. So 3 does not occur in more than the first power in n.

If $n\equiv 0$ (8), then as $U_8=21$, $U_n\equiv 0$ (7) so 2 does not occur in n to more than the second power.

Hence, if U_n is to be a power of an integer, we see that n must be a divisor of 12. As $U_6=8$, $U_{12}=144$ and $U_1=U_2=1$, the theorem follows.

Reference

1. L. E. Dickson, History of the Theory of Numbers, Vol. I, Divisibility and Primality, Chelsea, New York.

CONTINUOUS DEPENDENCE OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

AARON STRAUSS, University of Wisconsin

Consider the system of differential equations

$$x' = f(t, x), \tag{E}$$

where x and f are real column n-vectors and t is a real scalar. We follow the notation of [1]. Thus $|x| = \sum_{i=1}^{n} |x_i|$, where $x=(x_1, \cdots, x_n)$, while $f\in(C, \text{Lip})$ in D means that f is continuous in the pair (t, x) in D and that there exists a constant $k>0$ such that for every (t, x_1) and (t, x_2) in D,

$$|f(t, x_1) - f(t, x_2)| \leqq k|x_1 - x_2|. \tag{1}$$

It is known (see [1], Chapter 1) that if $f\in(C, \text{Lip})$, then any solution of (E) is a continuous function of its initial conditions. The standard proof uses successive approximations. Our purpose is to give a new proof, which is more direct and seems more natural. We note that similar methods have been used before to prove weaker theorems (cf. [2]).

THEOREM 1. *Let $f\in(C, \text{Lip})$ in a domain D of the $(n+1)$-dimensional (t, x) space, and suppose that ψ is a solution of* (E) *on some interval $a\leqq t\leqq b$. Define $U_\delta=\{(t, x)\in D: a<t<b,\ |x-\psi(t)|<\delta\}$. Then for any $\epsilon>0$, there exists a $\delta>0$ such that for any $(\tau, \xi)\in U_\delta$, there is a (unique) solution ϕ of* (E), *such that*

(i) $\phi(\tau)=\xi$,

(ii) *ϕ is defined on all of $a\leqq t\leqq b$,*

(iii) *$|\psi(t)-\phi(t)|<\epsilon$ on $a\leqq t\leqq b$.*

Proof. Let $\delta_1>0$ be such that $U_{\delta_1}\subseteq D$, let $0<\epsilon\leqq\delta_1$, and choose $\delta<\epsilon e^{-k(b-a)}$. Let (τ, ξ) be any point in U_δ and let ϕ be that (local) solution of (E) for which $\phi(\tau)=\xi$. Let $\psi(\tau)=\bar{\xi}$. Then for $\tau\leqq t\leqq b$,

$$\psi(t) = \bar{\xi} + \int_\tau^t f(s, \psi(s))ds.$$

If n is divisible by 9, then as $U_9=34$, then n would have to be divisible by 17. So 3 does not occur in more than the first power in n.

If $n\equiv 0$ (8), then as $U_8=21$, $U_n\equiv 0$ (7) so 2 does not occur in n to more than the second power.

Hence, if U_n is to be a power of an integer, we see that n must be a divisor of 12. As $U_6=8$, $U_{12}=144$ and $U_1=U_2=1$, the theorem follows.

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Consider the system of differential equations

$$x' = f(t, x), \tag{E}$$

where x and f are real column n-vectors and t is a real scalar. We follow the notation of [1]. Thus $|x| = \sum_{i=1}^{n} |x_i|$, where $x=(x_1, \cdots, x_n)$, while $f\in(C, \text{Lip})$ in D means that f is continuous in the pair (t, x) in D and that there exists a constant $k>0$ such that for every (t, x_1) and (t, x_2) in D,

$$|f(t, x_1) - f(t, x_2)| \leqq k|x_1 - x_2|. \tag{1}$$

It is known (see [1], Chapter 1) that if $f\in(C, \text{Lip})$, then any solution of (E) is a continuous function of its initial conditions. The standard proof uses successive approximations. Our purpose is to give a new proof, which is more direct and seems more natural. We note that similar methods have been used before to prove weaker theorems (cf. [2]).

THEOREM 1. *Let* $f\in(C, \text{Lip})$ *in a domain* D *of the* $(n+1)$-*dimensional* (t, x) *space, and suppose that* ψ *is a solution of* (E) *on some interval* $a\leqq t\leqq b$. *Define* $U_\delta=\{(t, x)\in D\colon a<t<b,\ |x-\psi(t)|<\delta\}$. *Then for any* $\epsilon>0$, *there exists a* $\delta>0$ *such that for any* $(\tau, \xi)\in U_\delta$, *there is a (unique) solution* ϕ *of* (E), *such that*

(i) $\phi(\tau)=\xi$,

(ii) ϕ *is defined on all of* $a\leqq t\leqq b$,

(iii) $|\psi(t)-\phi(t)|<\epsilon$ *on* $a\leqq t\leqq b$.

Proof. Let $\delta_1>0$ be such that $U_{\delta_1}\subseteq D$, let $0<\epsilon\leqq\delta_1$, and choose $\delta<\epsilon e^{-k(b-a)}$. Let (τ, ξ) be any point in U_δ and let ϕ be that (local) solution of (E) for which $\phi(\tau)=\xi$. Let $\psi(\tau)=\bar{\xi}$. Then for $\tau\leqq t\leqq b$,

$$\psi(t) = \bar{\xi} + \int_\tau^t f(s, \psi(s))ds.$$

Also for as long as $(t, \phi(t))$ remains in D,

$$\phi(t) = \xi + \int_\tau^t f(s, \phi(s))ds.$$

Therefore

$$\begin{aligned} |\psi(t) - \phi(t)| &\leqq |\bar{\xi} - \xi| + \int_\tau^t |f(s, \psi(s)) - f(s, \phi(s))| \, ds \\ &\leqq \delta + \int_\tau^t k |\psi(s) - \phi(s)| \, ds. \end{aligned}$$

Using Gronwall's inequality (see [1], Chapter 1, problem 1)

$$|\psi(t) - \phi(t)| \leqq \delta e^{k(b-a)} < \epsilon.$$

Thus $(t, \phi(t))$ cannot leave D, and ϕ can be continued to $\tau \leqq t \leqq b$, where $|\psi(t) - \phi(t)| < \epsilon$. A similar argument gives the same result for $a \leqq t \leqq \tau$, proving Theorem 1.

The following corollary of Theorem 1 gives the desired continuity result.

COROLLARY. *Let f and D be as in Theorem 1. Let $(\tau_0, \xi_0) \in D$ and let $\psi = \psi(t, \tau_0, \xi_0)$ be that solution of* (E) *on some interval $a \leqq t \leqq b$ for which $\psi(\tau_0, \tau_0, \xi_0) = \xi_0$. Then for any $t_0 \in [a, b]$, ψ is continuous at (t_0, τ_0, ξ_0).*

Proof. Fix any $t_0 \in [a, b]$. Let δ_1, ϵ, and δ be as in the proof of Theorem 1. Then there exists a $\delta_2 > 0$ such that $|\psi(t', \tau_0, \xi_0) - \psi(t'', \tau_0, \xi_0)| < \delta/4$ whenever $|t' - t''| < \delta_2$, uniformly for $t', t'' \in [a, b]$. Choose $\eta = \min(\delta_2, \delta/4)$. Let (t_1, τ_1, ξ_1) be any point such that $t_1 \in [a, b]$, $(\tau_1, \xi_1) \in D$, and

$$|t_0 - t_1| + |\tau_0 - \tau_1| + |\xi_0 - \xi_1| < \eta.$$

Let $\phi = \phi(t, \tau_1, \xi_1)$ be that solution of (E) for which $\phi(\tau_1, \tau_1, \xi_1) = \xi_1$. We shall show $|\psi(t_0, \tau_0, \xi_0) - \phi(t_1, \tau_1, \xi_1)| < \epsilon$. Now

$$\begin{aligned} |\xi_1 - \psi(\tau_1, \tau_0, \xi_0)| &\leqq |\xi_1 - \xi_0| + |\xi_0 - \psi(\tau_1, \tau_0, \xi_0)| \\ &\leqq |\xi_1 - \xi_0| + |\psi(\tau_0, \tau_0, \xi_0) - \psi(\tau_1, \tau_0, \xi_0)| \\ &< \delta/4 + \delta/4 = \delta/2. \end{aligned}$$

Thus $(\tau_1, \xi_1) \in U_{\delta/2}$ so that by Theorem 1, we actually have

$$|\psi(t, \tau_0, \xi_0) - \phi(t, \tau_1, \xi_1)| < \epsilon/2 \qquad \text{for } a \leqq t \leqq b.$$

Finally

$$\begin{aligned} &|\psi(t_0, \tau_0, \xi_0) - \phi(t_1, \tau_1, \xi_1)| \\ &\quad \leqq |\psi(t_0, \tau_0, \xi_0) - \psi(t_1, \tau_0, \xi_0)| + |\psi(t_1, \tau_0, \xi_0) - \phi(t_1, \tau_1, \xi_1)| \\ &\quad \leqq \epsilon/2 + \delta/4 < \epsilon, \end{aligned}$$

completing the proof.

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References

1. E. A. Coddington and Norman Levinson, Theory of ordinary differential equations, McGraw-Hill, New York, 1955.

2. Witold Hurewicz, Lectures on ordinary differential equations, Wiley and MIT Press, New York, 1958.

ANOTHER PROOF OF WEDDERBURN'S THEOREM

T. J. Kaczynski, Evergreen Park, Illinois

In 1905 Wedderburn proved that every finite skew field is commutative. At least seven proofs of this theorem (not counting the present one) are known. See [1], [2], [5] (Part Two, p. 206 and Exercise 4 on p. 219), [6] (two proofs), and [7]. Unlike these proofs, the proof to be given here is group-theoretic, in the sense that the only non-group-theoretic concepts employed are of an elementary nature.

Lemma. *Let q be a prime. Then the congruence $t^2+r^2\equiv -1 \pmod q$ has a solution t, r with $t\not\equiv 0 \pmod q$.*

Proof. If -1 is a quadratic residue, take $r=0$ and choose t appropriately. Assume -1 is a nonresidue. Then any nonresidue can be written in the form $-s^2 \pmod q$ with $s\not\equiv 0$. If t^2+r^2 is ever a nonresidue for some t, r, set $t^2+r^2 \equiv -s^2$, and we have $(ts^{-1})^2+(rs^{-1})^2\equiv -1$. (Throughout this note, x^{-1} denotes that integer for which $xx^{-1}\equiv 1 \pmod q$.) On the other hand, if t^2+r^2 is always a residue, then the sum of any two residues is a residue, so $-1\equiv q-1=1+1+\cdots +1$ is a residue, contradicting our assumption.

Proof of the theorem. Let F be our finite skew field, F^* its multiplicative group. Let S be any Sylow subgroup of F^*, of order, say, p^α. Choose an element g of order p in the center of S. If some $h\in S$ generates a subgroup of order p different from that generated by g, then g and h generate a commutative field containing more than p roots of the equation $x^p=1$, an impossibility. Thus S contains only one subgroup of order p and hence is either a cyclic or a generalized quaternion group ([3] p. 189).

If S is a generalized quaternion group, then S contains a quaternion subgroup generated by two elements a and b, both of order 4, where $ba=a^{-1}b$. Now a^2 generates a commutative field in which the only roots of the equation $x^2=1$ or $(x+1)(x-1)=0$ are ± 1, so since $(a^2)^2=1$, we have

$$a^2 = -1. \tag{1}$$

Hence $a^{-1}=a^3=-a$, so

$$ba = -ab. \tag{2}$$

Similarly,

$$b^2 = -1. \tag{3}$$

Taking q = characteristic of F $(q\cdot 1=0)$, choose t and r as specified in the lemma. Using relations (1), (2), (3), we have

$$(t+ra+b)(r^2+1+rta+tb) = r(t^2+r^2+1)a+(t^2+r^2+1)b = 0.$$

One of the factors on the left must be 0, so for some numbers $u, v, w, u \not\equiv 0 \pmod q$, we have $w+va+ub=0$, or $b=-u^{-1}va-u^{-1}w$. So b commutes with a, a contradiction. We conclude that S is not a generalized quaternion group, so S is cyclic.

Thus every Sylow subgroup of F^* is cyclic, and F^* is solvable ([4], pp. 181–182). Let Z be the center of F^* and assume $Z \neq F^*$. Then F^*/Z is solvable, and its Sylow subgroups are cyclic. Let A/Z (with $Z \subset A$) be a minimal normal subgroup of F^*/Z. A/Z is an elementary abelian group of order p^k (p prime), so since the Sylow subgroups of F^*/Z are cyclic, A/Z is cyclic. Any group which is cyclic modulo its center is abelian, so A is abelian. Let x be any element of F^*, y any element of A. Since A is normal, $xyx^{-1} \in A$, and $(1+x)y = z(1+x)$ for some $z \in A$. An easy manipulation shows that $y-z = zx-xy = (z-xyx^{-1})x$.

If $y-z=z-xyx^{-1}=0$, then $y=z=xyx^{-1}$, so x and y commute. Otherwise, $x=(z-xyx^{-1})^{-1}(y-z)$. But A is abelian, and z, y, $xyx^{-1} \in A$, so x commutes with y. Thus we have proven that A is contained in the center of F^*, a contradiction.

References

1. E. Artin, Über einen Satz von Herrn J. H. M. Wedderburn, Abh. Math. Sem. Hamburg, 5 (1927) 245.
2. L. E. Dickson, On finite algebras, Göttingen Nachr., 1905, p. 379.
3. M. Hall, The theory of groups, Macmillan, New York, 1961.
4. Miller, Blichfeldt and Dickson, Theory and applications of finite groups, Wiley, New York, 1916.
5. B. L. van der Waerden, Moderne Algebra, Ungar, New York, 1943.
6. J. H. M. Wedderburn, A theorem on finite algebras, Trans. Amer. Math. Soc., 6 (1905) 349.
7. E. Witt, Über die Kommutativität endlicher Schiefkörper, Abh. Math. Sem. Hamburg, 8 (1931) 413.

A NOTE ON PRODUCT SYSTEMS OF SETS OF NATURAL NUMBERS

T. G. McLaughlin, University of California at Los Angeles

In this note, we apply a slight twist to a trick exploited about twelve years ago by J. C. E. Dekker ([2]), our purpose being to expose a couple of elementary facts about nonempty, countable "product systems" of infinite sets of natural numbers which are, at the same time, "finite symmetric difference systems." We proceed in terms of the following definitions.

Definition. *By a product system of subsets of N (N the natural numbers), we mean a collection of subsets of N which contains, along with any two of its members, their intersection.*

Similarly,

$$b^2 = -1. \tag{3}$$

Taking q = characteristic of F $(q\cdot 1=0)$, choose t and r as specified in the lemma. Using relations (1), (2), (3), we have

$$(t + ra + b)(r^2 + 1 + rta + tb) = r(t^2 + r^2 + 1)a + (t^2 + r^2 + 1)b = 0.$$

One of the factors on the left must be 0, so for some numbers $u, v, w, u \not\equiv 0 \pmod q$, we have $w+va+ub=0$, or $b=-u^{-1}va-u^{-1}w$. So b commutes with a, a contradiction. We conclude that S is not a generalized quaternion group, so S is cyclic.

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If $y-z=z-xyx^{-1}=0$, then $y=z=xyx^{-1}$, so x and y commute. Otherwise, $x=(z-xyx^{-1})^{-1}(y-z)$. But A is abelian, and z, y, $xyx^{-1} \in A$, so x commutes with y. Thus we have proven that A is contained in the center of F^*, a contradiction.

References

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Definition. *By a product system of subsets of N (N the natural numbers), we mean a collection of subsets of N which contains, along with any two of its members, their intersection.*

function, from N *onto* N, which is one-to-one. A *recursive set* of numbers is a subset Δ of N such that each of Δ, $N-\Delta$ is empty or is the range of a recursive function. It is an elementary result that if Δ, Σ are two infinite, coinfinite recursive sets, there exists a recursive permutation mapping Δ onto Σ.

COROLLARY. *Let C be as in the theorem, with the additional property that C is closed under recursive permutations. Suppose that there exists an infinite subset Δ of N which is not immune (i.e., Δ has an infinite recursive subset), which adheres to C, and which is (modulo finite extensions) maximal with respect to adherence to C. Then C consists entirely of cofinite sets of numbers.*

Proof. Suppose that C contains a noncofinite set. Then, since Δ adheres to C, Δ is noncofinite. By the theorem, Δ itself belongs to C. By hypothesis, Δ has an infinite recursive subset Σ. By a result cited in the paragraph preceding the statement of the corollary, there is a recursive permutation, g, such that $g(\Sigma)=N-\Sigma$, $g(N-\Sigma)=\Sigma$. Since C is closed under recursive permutations, $g(\Delta)\in C$. Hence, $\Delta-g(\Delta)$ is finite. But, obviously, $\Delta-g(\Delta)$ is infinite; and from this contradiction the corollary follows.

Remark. Our proof of the corollary has points in common with the last half of the proof of Theorem 6.5 in [4].

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References

1. Martin Davis, Computability and unsolvability, McGraw-Hill, New York, 1958.
2. J. C. E. Dekker, The constructivity of maximal dual ideals in certain Boolean algebras, Pacific J. Math., 3 (1953) 73–101.
3. J. C. E. Dekker and J. R. Myhill, Recursive equivalence types, University of California Press, 1960.
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In the note "On Simultaneous Hermitian Congruence Transformations of Matrices," by K. N. Majindar, published in this MONTHLY, 70 (1963) page 844 the matrix A should be

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TANGENTS AND DIFFERENTIALS

HUGH A. THURSTON, University of British Columbia

1. A recent paper [1] makes the point that the usual elementary definition of *differential* is inadequate. The paper [2] gives a valid elementary definition of *tangent-line.* The two concepts are related: given a valid definition of tangent to a plane curve, $dx:dy$ can be defined as the direction-ratio of the tangent. This is a particularly good definition of differential for an elementary course; it is both easy to grasp and potentially rigorous (because the treatment of tangents can be made as rigorous as desired).

The definitions are given in Section 2, and the familiar formulae for $dx:dy$ in the explicit, parametric, and implicit cases follow as in Section 3. It turns out (Section 4) that the formula in the implicit case holds more generally than might be expected: it can be proved without using the implicit-function theorem (which requires continuous derivatives).

In a sense, the three problems of defining tangent, defining differential, and differentiating a function-of-a-function are equivalent. We have indicated that the first two are mutually equivalent; and it is well known that, once differentials are defined, differentiation of a function-of-a-function is trivial. Conversely, if the theorem about differentiating a function-of-a-function is known, it could be made the basis of a definition of differential (not quite as general as ours—in fact, bearing the same relation to ours as parametric tangent bears to geometric tangent: for these terms see [2]). This we show in Section 5.

Finally (Section 6) we point out that if we use two different forms of the *same* relation to calculate differentials, then the definition in this paper ensures automatically that we get the same result from each, whereas other definitions do not have this desirable property.

2. DEFINITION 1. *The line L through the point P of the point-set S is a tangent to S at P if P is a limit-point of S and if, given any cone with vertex P and axis L, the line PQ is inside the cone for every point Q of S near enough to P.*

It is clear that a given set has at most one tangent at a given point, and that all tangents to a plane set lie in the plane of the set.

DEFINITION 2. *Given a relation between two variables, say x and y, we let S be the set of points whose coordinates satisfy the relation. We define a binary function whose domain is a subset of S and whose values are ratios as follows: at any point at which S has a tangent, the value of the function is the direction-ratio of the tangent. The value of the function at (x, y) is traditionally denoted by $dx:dy$; and dx and dy are called the differentials of x and y with respect to the given relation.*

A statement such as "The differentials of x and y with respect to the relation $y=x^2$ satisfy the equation $dy=2x\cdot dx$" is traditionally stated as "If $y=x^2$, then $dy=2x\cdot dx$." (Of course, $dy=2x\cdot dx$ means neither more nor less than $dx:dy$

$=1:2x$. Indeed, the only statements that can meaningfully be made about dx and dy are statements about their mutual ratio: the statements "$dx=2$" and "$dx=(dy)^2$" mean nothing.)

3. We have at once the following results.

THEOREM 1. *If $F'(a)$ exists, then the value of dx:dy at $[a, F(a)]$ with respect to the relation $y=F(x)$ is $1:F'(a)$.*

THEOREM 2. *If $x=X(t)$, $y=Y(t)$, $t\in I$ traces a simple arc, if $a\in I$, and if the ratio $X'(a):Y'(a)$ exists: then the value of dx:dy at $[X(a), Y(a)]$ with respect to the relation $x=X(t)$, $y=Y(t)$, $t\in I$, is $X'(a):Y'(a)$.*

THEOREM 3. *If the ratio $\psi_1(a, b):\psi_2(a, b)$ exists and if $\psi(a, b)=0$ and if ψ is differentiable at (a, b), then the value of dx:dy at (a, b) with respect to the relation $\psi(x, y)=0$ is*

$$-\psi_2(a, b):\psi_1(a, b). \tag{i}$$

(Here ψ_1 and ψ_2 are the two partial derivatives of ψ.)

Proofs. Theorems 1 and 2 follow immediately from [2]. For Theorem 3, we note that, because ψ is differentiable at (a, b),

$$[\psi(a+h, b+k) - \psi(a, b) - h\cdot\psi_1(a, b) - k\cdot\psi_2(a, b)]/(h^2+k^2)^{1/2} \to 0$$

as $(h, k)\to(0, 0)$. Then if $(a+h, b+k)$ is on the graph of $\psi(x, y)=0$, we have

$$(h\cdot\ell + k\cdot m)/(h^2+k^2)^{1/2} \to 0 \quad \text{as} \quad (h, k) \to (0, 0),$$

where $\ell=\psi_1(a, b)$ and $m=\psi_2(a, b)$.

Now if u is the inclination of the line joining (a, b) to $(a+h, b+k)$ we have

$$\cos u : \sin u = h:k;$$

and if v is the inclination of a line with direction-ratio (i) we have

$$\cos v : \sin v = -m:\ell.$$

Therefore

$$|\sin(u-v)| = |\ell\cdot h + m\cdot k|/(h^2+k^2)^{1/2}\cdot(\ell^2+m^2)^{1/2} \to 0$$

as $(h, k)\to(0, 0)$. It follows easily that the line through (a, b) with direction-ratio (i) is the tangent there to the curve.

4. Note. If, in Theorem 3, we were to assume continuity of ψ_1 and ψ_2 in a neighbourhood of (a, b), instead of mere differentiability of ψ at (a, b), then the theorem would follow as an easy corollary to the implicit-function theorem. In this connection, it is interesting to notice that differentiability is not enough for the implicit-function theorem.

Specifically: if

$$\phi(x, y) = \begin{cases} x - \frac{1}{2}y - y^2 \cdot \sin y^{-1} & \text{whenever} \quad y \neq 0 \\ x & \text{whenever} \quad y = 0 \end{cases}$$

then $\phi(0, 0) = 0$, $\phi_2(0, 0) = -\frac{1}{2} \neq 0$, ϕ is differentiable at $(0, 0)$, but the equation $\phi(x, y) = 0$ is not solvable for y at $(0, 0)$.

Theorem 3 as quoted above, then, is stronger than the version obtained from the implicit-function theorem. However, we cannot remove the differentiability proviso from Theorem 3 and rely only on the existence of the two partial derivatives (not both zero). Specifically: if

$$\phi(x, y) = \begin{cases} (8x^3 - y^3)/(x^2 + y^2) & \text{whenever} \quad (x, y) \neq (0, 0) \\ 0 & \text{when} \quad (x, y) = (0, 0), \end{cases}$$

then $\phi(0, 0) = 0$, $\phi_1(0, 0) = 8$, $\phi_2(0, 0) = -1$; but the line through $(0, 0)$ with direction-ratio $1:8$ is not the tangent there to the curve $\phi(x, y) = 0$. (In fact, the tangent has direction-ratio $1:2$.)

5. Once we have a valid definition of differential, we have an immediate proof of the formula for differentiating a function of a function: if $z = G[F(x)]$ we put $F(x) = y$, whence $z = G(y)$. Then $dz = G'(y) \cdot dy$ and $dy = F'(x) \cdot dx$, whence

$$dz = G'[F(x)] \cdot F'(x) \cdot dx.$$

(At present, this proof seems to be confined to nonrigorous treatments such as [3], presumably because of the lack of rigorous definitions of tangent and differential.)

Indeed, the function-of-a-function rule and the validity of the differential are, in a sense, equivalent. If the function-of-a-function rule were proved independently (as, indeed, in most treatments it is) then it could be made the basis of an alternative (slightly less general) definition of differential, as follows. We define $dx:dy$ for a curve in parametric form $x = X(t)$, $y = Y(t)$. For the definition to be valid, this ratio must be proved independent of the choice of parameter for the given curve. The crux of the proof turns out to be the function-of-a-function rule. The details are as follows.

LEMMA. *If a simple arc S has parametrizations*

$$x = X(t), \qquad y = Y(t), \qquad t \in I$$

and

$$x = A(t), \qquad y = B(t), \qquad t \in J$$

then there is a continuous function F with inverse G such that

$$X(t) = A[F(t)], \qquad Y(t) = B[F(t)] \qquad \textit{whenever } t \in I,$$

and

$$A(t) = X[G(t)], \qquad B(t) = Y[G(t)] \qquad \textit{whenever } t \in J.$$

Moreover, if $A'[F(c)] \neq 0$, *then* $F'(c)$ *exists.*

Proof. All results, except possibly that expressed by the last sentence, are well known. To prove the last result, we let $a = F(c)$ and define a function H by

$$\begin{cases} H(t) = \dfrac{A(t) - A(a)}{t - a} & \text{whenever } t \in J \textit{ and } t \neq a \\ H(a) = A'(a). \end{cases}$$

Then $\lim_{t\to a} H(t) = A'(a)$, and H is continuous. Also

$$\frac{X(t) - X(c)}{t - c} = \frac{A[F(t)] - A[F(c)]}{t - c}$$
$$= H[F(t)] \cdot \frac{F(t) - F(c)}{t - c} \qquad \text{whenever } t \in I \text{ and } t \neq c.$$

Now

$$\lim_{t\to c} H[F(t)] = \lim_{t\to a} H(t), \text{ because } F \text{ is continuous}$$
$$= A'(a), \text{ as already proved}$$
$$\neq 0.$$

Therefore, for every t in some neighbourhood of c, $H[F(t)] \neq 0$, and so

$$\frac{F(t) - F(c)}{t - c} = \frac{X(t) - X(c)}{t - c} \cdot \frac{1}{H[F(t)]}.$$

Therefore $F'(c)$ exists (and equals $X'(c)/A'(a)$).

Note. If C is an end-point, then the various limits, neighbourhoods, etc., are one-sided, but the proof is otherwise unaltered.

THEOREM. *If the simple arc S has the parametrizations cited in the lemma, if a point has parameters c and d respectively, and if the ratios $X'(c):Y'(c)$ and $A'(d):B'(d)$ exist, then they are equal.*

Proof. The functions F and G of the lemma exist, and clearly $F(c) = d$. Hence, if $A'(d) \neq 0$, then $F'(c)$ exists. Then $X'(c) = A'(d) \cdot F'(c)$ and $Y'(c) = B'(d) \cdot F'(c)$.

Then $F'(c) \neq 0$ (for otherwise $X'(c)$ and $Y'(c)$ would both be zero and so their ratio would fail to exist) and so

$$X'(c):Y'(c) = A'(d):B'(d).$$

If, however, $A'(d) = 0$, then $B'(d) \neq 0$ and a similar proof holds.

6. Sometimes a relation can appear in various different forms. For example, the relations

(i) $y=x^{2/3}$ (ii) $x=y^{3/2}$ (iii) $x=t^3$, $y=t^2$ (iv) $x^2-y^3=0$,

are the same: every (x, y) belonging to any of them belongs to all of them.

From our definition, it follows that the differential-ratio with respect to a given relation does not depend on the form in which the relation is expressed. The traditional definition does not have this property; it can be checked in any particular case, but there is no general theorem. (The existence of a vague feeling that something of this kind is needed is shown by the inclusion in many texts of a "consistency theorem" to the effect that if dy is calculated in terms of dx from the relations $y=F(t)$, $t=G(x)$, and also from the relation $y=F[G(x)]$, then the results are the same. This covers only very special cases, and does not suffice to show consistency for any pair of the equations above. It would, however, show consistency for $y=t^2$, $t=x^{1/3}$ and (i).)

To turn for a moment to physics: Boyle's law can be written $P\cdot V=k$ or $P=k/V$ or $V=k/P$, and a physicist would unhesitatingly use any of these forms to obtain the (isothermal) differentials dP and dV, and expect (without need for checking) that they would give the same result. The physicist would be right, and any treatment of differentials which does not yield this property is inadequate for applications.

Here, then, is a point in which the present definition is superior not only to the traditional definition but to the definition in [1].

References

1. H. A. Thurston, The definition of dy:dx, this MONTHLY, 70 (1963) 539–541.
2. ———, On the definition of a tangent-line, to appear in this MONTHLY.
3. Sylvanus P. Thompson, Calculus made easy, London, 1910.

THE DENSITY OF PYTHAGOREAN RATIONALS

L. H. LANGE AND D. E. THORO, San Jose State College

If a, b, c are positive integers which satisfy $a^2+b^2=c^2$, we call the number a/b a Pythagorean rational. We give here two proofs of the following

THEOREM. *The set of all Pythagorean rationals is dense in the set of all nonnegative real numbers.*

We use the fact that if x and y are positive integers, then

$$(x^2-y^2)^2+(2xy)^2=(x^2+y^2)^2,$$

and hence, with $x>y$, $(x^2-y^2)/2xy$ is a Pythagorean rational.

Proof 1. Let α, β be any prescribed real numbers which satisfy $0\leqq\alpha<\beta<\infty$. We seek positive integers x and y, $x>y$, such that $\alpha<(x^2-y^2)/2xy<\beta$. Letting $t=x/y$, this is equivalent to the search for a rational t which satisfies $\alpha<\frac{1}{2}(t-t^{-1})<\beta$. If we let $g(t)=\frac{1}{2}(t-t^{-1})$ for all positive t, we have $g(1)=0$,

$$n_1 = \text{the smallest integer in } A,$$

$$n_{k+1} = \text{the smallest integer in } A \text{ such that } n_{k+1} > n_k, \text{ and}$$

$$a_{n_k} \leqq a_{n_{k+1}}, \qquad k = 1, 2, \cdots.$$

If B is infinite, we construct a monotone nonincreasing subsequence in an analogous way.

In the event that both A and B are finite, there exist for each $i \in C$ integers $j, k \in C, j > i, k > i$, such that $a_i < a_j$ and $a_i > a_k$. By employing the previous constructions, we can obtain both a monotone increasing subsequence and a monotone decreasing subsequence.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

THE MATHEMATICAL TRIPOS AND MATHEMATICAL EDUCATION IN GREAT BRITAIN

DANIEL PEDOE, Purdue University

I must begin by explaining that the term Tripos is the name given to the mathematical and other honours examinations held every year in the University of Cambridge. The Mathematical Tripos was the first Honours examination instituted by that University. This was in the 18th century. The term Tripos originated in the three-legged stool, or tripod, which candidates sat on when they had to prove their merit by disputation, or wrangling, before the advent of written examinations. The term Wrangler is still preserved for those who obtain honours in the Mathematical Tripos. Although examinations in other subjects are also called Triposes, nobody but a mathematician is ever called a wrangler. This makes one wonder how the old examinations in mathematics were conducted! The term Senior Wrangler was reserved for the candidate who came first in the Mathematical Tripos. Until 1910, when Wranglers were no longer listed in order, the title of Senior Wrangler was much coveted, and the list of Senior Wranglers includes many who subsequently did great work in mathematics, such as Stokes, Cayley and the astronomer John Couch Adams, if we restrict ourselves to the 19th century. Of course, many who subsequently became great did not attain to the Senior Wranglership, but came lower down the list. I need only mention James Clerk Maxwell, who was the second Wrangler in 1854.

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If B is infinite, we construct a monotone nonincreasing subsequence in an analogous way.

In the event that both A and B are finite, there exist for each $i \in C$ integers j, $k \in C$, $j > i$, $k > i$, such that $a_i < a_j$ and $a_i > a_k$. By employing the previous constructions, we can obtain both a monotone increasing subsequence and a monotone decreasing subsequence.

MATHEMATICAL EDUCATION NOTES

Edited by John R. Mayor, AAAS and University of Maryland
Collaborating Editor: John A. Brown, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

THE MATHEMATICAL TRIPOS AND MATHEMATICAL EDUCATION IN GREAT BRITAIN

Daniel Pedoe, Purdue University

I must begin by explaining that the term Tripos is the name given to the mathematical and other honours examinations held every year in the University of Cambridge. The Mathematical Tripos was the first Honours examination instituted by that University. This was in the 18th century. The term Tripos originated in the three-legged stool, or tripod, which candidates sat on when they had to prove their merit by disputation, or wrangling, before the advent of written examinations. The term Wrangler is still preserved for those who obtain honours in the Mathematical Tripos. Although examinations in other subjects are also called Triposes, nobody but a mathematician is ever called a wrangler. This makes one wonder how the old examinations in mathematics were conducted! The term Senior Wrangler was reserved for the candidate who came first in the Mathematical Tripos. Until 1910, when Wranglers were no longer listed in order, the title of Senior Wrangler was much coveted, and the list of Senior Wranglers includes many who subsequently did great work in mathematics, such as Stokes, Cayley and the astronomer John Couch Adams, if we restrict ourselves to the 19th century. Of course, many who subsequently became great did not attain to the Senior Wranglership, but came lower down the list. I need only mention James Clerk Maxwell, who was the second Wrangler in 1854.

A Senior Wrangler was assured of security, in some form or other, for the rest of his life. He could become a Fellow of his college, share in the revenues, and sit on the governing body, but since, in those days, this entailed taking Holy Orders, subscribing to the 39 Articles of the Church of England, and not getting married, a number of Senior Wranglers chose freedom, and left Cambridge for the outside world, becoming professors of mathematics elsewhere, or taking up the law, by which they earned a living, doing mathematics in their spare time, as Cayley did at first, or they became judges, actuaries, and so on. The competition for the Senior Wranglership was fierce, of course, and candidates were coached by dons who spent all their lives in making up questions for the Tripos, and in coaching for the Tripos. With Newmarket Heath, famous for horse-racing, so near to Cambridge, there have always been gentlemen of the turf resident in Cambridge, and it was natural that they should lay bets on the candidates they favoured, so that the atmosphere of the competition was very like that of a horse-race. One can imagine the anxious listening outside the closed door of a favoured candidate to hear whether he was coughing just before the examination. Then as now, however, Cambridge was a gentlemanly place, and I have never heard of a candidate being doped just before the Tripos.

The examination was in two parts. In the first part, which lasted four days, there were seven papers, each containing about ten questions on both pure and applied mathematics. I must emphasize at this point that students of mathematics in England have always had to study both pure and applied mathematics, which makes the situation very different from that existing in the States. What was both curious and significant about these seven papers, (*the first four days*, as it was called), was that no use could be made of the differential or integral calculus in solving the examination questions. (In the second set of papers, taken a week later, any methods could be used, and potential Senior Wranglers concentrated their efforts on this set.) The emphasis was on geometrical methods, as exhibited in mathematical treatises from the time of Archimedes to the time of Isaac Newton. I shall explain in a moment why I think this restriction was imposed, but in the meantime let me stress that every year for many years mathematicians at Cambridge spent a lot of their time inventing questions, some very difficult, which could be solved by geometrical methods, and that every year a large number of such questions was added to the stockpile.

If it is asked how a University examination could influence mathematics teaching in Britain, the answer is evident, once it is known that schools would train their best mathematical students to compete for entrance to the University of Cambridge where entrance then, as now, was competitive. The entrance papers would contain many questions which were best practised if the books used at the school contained similar questions. There were such books, written by former Cambridge Wranglers. I shall talk about these books in a moment.

Then, as now, most potential mathematicians wished to study at Cambridge, the University of Isaac Newton. It was Newton who was inadvertently responsi-

ble for the first seven papers of the Mathematical Tripos being restricted to geometrical solutions. The importance of Newton, as we all know, lies in the fact that he initiated the present era of science with his publication, in 1686, of the *Principia*, the Mathematical Principles of Natural Philosophy and the System of the World. In this amazing book, the motion of the planets was explained for the first time mathematically as a consequence of the universal law of gravitation, and many other natural phenomena are also investigated. The first thing which strikes one about *Principia* is that everything is done geometrically. Newton was the inventor (or discoverer, whichever you prefer) of the differential and integral calculus, and it is evident that he first obtained many of the results in *Principia* by the use of the calculus. But he refers to these methods as being somewhat "harsh," and therefore to be reckoned less geometrical, and he proves all his results by geometry.

It was the style of Newton's masterpiece which was responsible, I believe, for the emphasis on geometrical methods in the teaching of mathematics in Britain during the 18th and 19th centuries. I shall give a few examples of the methods used for solving problems in a moment, but to show how long the tradition continued, let us look at the mathematical education of a great mathematician, J. E. Littlewood, as described in his delightful book: *A Mathematician's Miscellany.*

Littlewood was educated at St. Paul's School, London. He sat for a Cambridge Scholarship in 1902, at the age of 17, and was Senior Wrangler at the age of 19. These are the books he used at school. I think they are worth listing:

Algebra and *Analytical Conics*, both by C. Smith; *Trigonometry*, Parts I and II, Loney; *Geometrical Conics*, Macaulay; *Statics and Dynamics*, Loney (no calculus used); *Differential Calculus*, Edwards; *Integral Calculus*, Williamson; *Hydrodynamics*, Besant.

These were the basic books used. In addition Littlewood read:

Sequel to Euclid, Casey; *Algebra*, Vol. II, Chrystal; *Conic Sections*, Salmon; *Trigonometry*, Hobson; *Theory of Equations*, Burnside and Panton; *Dynamics of a Particle and Rigid Bodies*, Routh; *Statics, including Theory of Attractions*, Minchin.

May I say that when I sat for a Cambridge Scholarship in 1930, I used exactly the same books which Littlewood used in 1902! These books were written by Cambridge, Oxford, or Dublin mathematicians, and, because of the remarkable exercises they contain, we shall not see their like again. Many have been preserved for the admiration of posterity in Chelsea or Dover reprints.

In all these books, even those on the calculus, there is an emphasis on geometrical methods. I shall now give some examples of these methods.

We cannot do better than to start with Newton's first Proposition in *Principia*, where he proves, geometrically, using the theorem that triangles on the same base and between the same parallels have equal area, that if a particle moves under the action of a central force, towards a fixed point, its motion is

in a plane, and the area described by the radius vector from the point to the particle describes equal areas in equal times. Next, we consider motion under uniform gravity, parabolic motion. This can be done by solving the differential equations $d^2x/dt^2=0$, $d^2y/dt^2=-g$, but a more illuminating approach is to observe that the speed of the particle at any point of the trajectory is that of a particle which falls to that point from the directrix of the trajectory. The way now lies open to constructions using the focus-directrix property of a parabola. It is easily and intuitively proved, for example, that if a particle be projected with minimum speed from a point P to pass through a point Q, the focus of the trajectory lies on the line PQ. We then obtain the beautiful properties of the enveloping parabola, much used in the old days in the design of fountains, and also of use in ballistics. Finally, the consideration of simple harmonic motion on a line is illuminated by the consideration that if a particle describes a circle with uniform angular velocity, the orthogonal projection onto any fixed diameter describes a simple harmonic motion. Difficult problems, normally involving boundary values for the solution of a second order linear differential equation, can be solved by the use of the circular motion. Naturally, by "difficult" I mean difficult for High School pupils, who have little calculus at the stage at which they may need to know about simple harmonic motion.

Littlewood mentions that in his mathematical education at school the emphasis was on a thorough training in geometrical methods in the first instance. Calculus was to be used when maturity set in. Most British schoolboys still have a fairly thorough training in geometry before going on to university. As a fair approximation to what a candidate for a Cambridge scholarship would need to know nowadays, we can mention the ordinary properties of the triangle, excluding special points like the Brocard point; a fair knowledge of circle geometry, including coaxal circles and the theory of inversion; the theory of conic sections treated both by pure geometry and coordinate geometry; and some three-dimensional analytical geometry. Is this too much? It could be, if this knowledge is gained at the expense of other important subjects. Remember that the British schoolboy has to do applied mathematics, statics, dynamics and attractions, at school, and is therefore under a heavier burden than his trans-Atlantic cousin. But some progress has been made in mathematical teaching in Britain, and perhaps less geometry will be taught in the future, with more concentration on axiomatic mathematics.

It is natural at this stage to enquire how much geometry the average American student knows when he first goes to university, or even after a few years at university? I usually test my students at Purdue with one simple question. I draw a circle on the blackboard, with two chords PQ and $P'Q'$ of the circle intersecting at a point V, and ask whether anyone can state any theorem connecting the segments VP, VP', VQ and VQ'. With one class of 27 students, I obtained 3 answers. All the three were Chinese, from Hong Kong, which is still under British influence! More recently, with a class of 40 students, all very

keen, there were no answers at all! I am told that some students are unable to learn elementary trigonometry because they have never been exposed to the properties of similar triangles while at school!

There is a struggle going on in the United States between those who want to get rid of all ordinary geometry, and to replace it with "baby" topology, theory of functions, etc., and axiomatic mathematics in general, and those who have become alarmed at the lack of geometrical teaching (of the old sort) in the ordinary High School course. The change in the type of question set in the Putnam examinations, and discussed recently by Professors Mordell and Kelly, shows that the party of the first part is winning, so far. But then the question arises, what are they doing in Communist countries, because they take mathematics very seriously there. Are they giving up geometry in their High Schools?

The recent competition for secondary school students from European communist countries, published in the MONTHLY of May, 1963, revealed to me, at any rate, that the students in these communist countries are being regaled on Mathematical Tripos examples with a strong geometrical content. Consider this example, which comes from Rumania:

Given a plane E and 3 noncollinear points A, B, C on the same side of E, the plane ABC not being parallel to E, the points A', B' and C' are arbitrary points in E. The points L, M and N are the respective midpoints of AA', BB' and CC', and S is the center of mass of the triangle LMN. Find the locus of S as A', B' and C' move in the plane E independently of each other.

A bright British schoolboy would solve this problem by considering unit particles at the points A, B, C, A', B' and C'. The point S is the center of mass of these particles, and S is also the midpoint of the join of the center of mass of the particles at A, B, and C, and this is a fixed point, to the center of mass of the particles at A', B' and C', and this is a point which varies in the plane E. The locus of S is therefore a plane parallel to the plane E.

When I suggested this solution to some eminent mathematical friends of mine, they looked dazed, as if something had hit them, so I gathered that this kind of geometrical reasoning is not common here.

I have an open mind about the educational effects of the teaching of geometry. I should be sorry to see ordinary geometry disappear from the schools. One cannot, after all, pay too much attention to the following quotation from a Socratic dialogue on mathematics, in which one of the wranglers says:

"Experience proves that anyone who has studied geometry is infinitely quicker of apprehension than one who has not."

This has not been proved to our satisfaction, perhaps, but it has not been disproved either.

Shortened version of a talk given to the Indiana Section of the Association in October, 1963.

THE BASIS OF MATHEMATICAL MISEDUCATION IN THE INDIAN UNIVERSITIES

P. MASANI, Indiana University

1. Curriculum, Instruction, Examination. Generally speaking, the mathematical training given in Indian universities today is an emasculation of what was offered by British schools and colleges three or four decades ago.

For most students college begins at the age of 16 or 17 and lasts for four or five years culminating in a Bachelor's or Master's degree. During the first two years one learns college algebra, trigonometry, coordinate geometry and calculus. For the mathematics major the next two or three years are spent on learning more algebra, geometry and calculus, as well as the elements of differential equations, analysis, mechanics and lately modern algebra, along with an optional subject such as statistics, astronomy, electricity or complex variables. The postgraduate course is of one or two years duration after the M.A. or B.A. Till recently both the undergraduate and postgraduate syllabi included some obsolete subjects such as higher trigonometry. But they are now being revised. The unrevised syllabi are substandard in comparison to those prevailing in the West and in Japan. Some of the revised courses look impressive, but it remains to be seen how well they can be handled under Indian conditions. English remains the medium of instruction in the post-sophomore courses at least, despite attempts to replace it by one or other Indian language.

In most courses the logical harmony of the subject is lost in the maze of technicalities. During my undergraduate studies at Bombay the only officially recommended books which portrayed any conceptual development were in analysis, notably Hardy's *Pure Mathematics*. This book did not, however, enjoy as much popularity as some others on analysis, which were conceptually weak but more useful for examination purposes. A conceptual development, without rigour, was also discernible in the mechanics and electricity courses. But those in algebra, geometry, higher trigonometry, differential equations and hydrostatics consisted of unrelated results derived by more or less specious reasoning.

Under the existing rules it is hard to design an examination in which uncritical cramming does not pay off. There are regulations, for instance, which say that about 50% of the questions set in university examinations shall be optional, and that the exams shall follow the same general pattern from year to year. This situation together with the absence of a system of weekly assignments and periodic tests makes for a good deal of learning by rote. During interviews one finds that students with good examination grades are unable to answer simple questions. I have come across students with A's in analysis who could not define a limit, and with A's in astronomy who had no idea how the radius of the earth is found. As performance in the final degree examination is the sole criterion of academic achievement during a two or three year period, the bulk of the students are interested only in being coached for this examination.

In this educational set-up the intellectually and creatively inclined students get a raw deal. The system allows no short-cut or other amelioration to the

gifted, in dire contrast to the policy pursued at Harvard and other enlightened institutions. They are made to go through a grinding mill in which the cramming of barren technicalities and trivialities is emphasized, clear and imaginative thought is discounted, and intellectual initiative stifled. The deprivation is colossal. I could cite evidence for this as a college teacher in India for 9 years. But more eloquent testimony is provided by the tragedy of the great mathematician S. Ramanujan. Failing in his freshman exam in 1904 when he was 16, he became so depressed that he ran away. He wandered for a year or two, fell ill, returned to appear for the same examination in 1907 but failed again. (How he then went begging for a research stipend, became a clerk in the Madras Port Trust in 1912, wrote to Professor G. H. Hardy of Cambridge, etc., are narrated in [**3**, p. xiii].) Between 1907 and 1964 India has gone through many changes. We now have a Ramanujan Professorship in Mathematics and a postage stamp in his honor. But collegiate rules are as rigid as ever in their insistence on a dreary and deadening routine, and a second Ramanujan could today meet with the fate of his predecessor.

To excel in the Indian educational set-up one must be prepared to spout and belabour the spurious and the shoddy. One must be intelligent and persevering but also uncritical and unconscientious, temporarily at least. Many Indian mathematics and science students excel in this manoeuvre. A few of strong character are able to withstand such acquiescence to unprincipled learning, and once free from examination pressure are able to pursue scientific work. The others, more or less damaged, usually end up in the administrative, legal, or teaching professions.

2. The background of miseducation. University education came to India in the middle of the 19th century. The advent culminated the efforts of a few unusually enlightened British civilians, who fought for a liberal policy towards India as early as 1830, thereby ushering in a phase of British rule that was to demarcate it from the earlier imperialisms. Indeed, the new universities soon produced a native crop of strong liberal thinkers and leaders. But, despite their auspicious origins, these institutions simply did not mature in the way they have in the West. One must understand why, in order to find out what holds back mathematical training today. There were, I feel, two major factors responsible for the stunted growth and premature decay of the universities:

(1) The educational system was geared more to the production of an officialdom loyal to the British rulers than to the creation of an elite of practical-critical thinkers and men of action, or to the dissemination of knowledge among the masses. The system stressed the doctrinaire liberal arts at the expense of the crafts and technology, and discouraged originality, innovation and a questioning attitude. It was based on an alien tongue understood by very few. Its major effect was the consolidation of an authoritarian bureaucracy with a disdain for manual work, and the creation of vested interests amongst the intelligentsia. It thus preserved the reactionary cleavage rooted in Indian tradition between

intellectual and worker. One could not expect from such a system a prototype of the emancipated American farmer, who could slaughter hogs by day and read Tom Paine at night.

(2) The educational movement, unlike that in the West, was not backed by one towards industrialization and improved technology. On the contrary, it was launched in a period of economic impoverishment resulting from destruction of the handicraft economy. It left untouched 90% of the population, but its cost fell on their shoulders. It thus became part of a system of exploitation. No wonder that a national leader such as Gandhi rejected it entirely [1, p. 7].

With the advent of modern education several educational societies sprang up and started schools and colleges. Along with the religious missions they rendered yeoman service in spreading education over India. But with the ever mounting (and often misconceived) demand for more and more education, such societies began to spawn. Today there are an enormous number, most of which survive only by maintaining huge enrollments in their colleges and cutting down costs without regard for standards. The situation in their colleges today is comparable to what prevailed in certain American schools some decades ago. The faculty is without tenure appointment, underpaid and overworked. Promotions and privileges are bestowed often on sectarian or other extraneous criteria. The administration is often high-handed; e.g., a society in Bombay serves a notice of dismissal on any faculty member who seeks an outside post! Research and scholarship are not encouraged, nor is there any real academic freedom. The enormous expansion has also led to a decline in the quality of the college teachers. To get out of this rut the colleges would need State subsidies far in excess of what an underdeveloped nation can afford.

Most Indian universities are regional federations of colleges run either by such societies or by the State. The university lays down conditions for affiliation, sets entrance requirements, prescribes syllabi, conducts examinations and awards degrees, but leaves enrollment and undergraduate instruction to the colleges themselves. (Postgraduate instruction is undertaken by the university itself, often with the help of college teachers and in a few subjects with its own specially appointed staff.) Under the existing statutes, however, the Principals of affiliated colleges and some of their appointees are ex-officio members of important university committees. Moreover, graduates of the university are eligible to contest for such membership by election. The result is that most federal universities are controlled by vested interests—educators turned administrators, or politicians acting as educationists—and the real scholars and teachers have little say. Proposals to raise standards of admission, to enhance pay scales and to improve the quality of instruction meet with opposition. Opposition to syllabus-reform also emanates from many teachers themselves. Apart from sheer inertia they fear the loss of university examinerships, which form an important source of income for many.

India also has a few centralized universities, which though comprising several units, control instruction directly. Where the number of units is large the

situation is little better than in the federal universities. But in the new, single-unit universities it is sometimes possible to affect reform by concerted action. For instance, a relatively modern syllabus in mathematics was so introduced at Baroda University in 1953. For a description see [2].

Another deterrent to educational progress is the rapid vitiation of youthful idealism in India. Apart from devigorating climate, malnutrition, poverty and economic insecurity, there is the negative impact of social injustice, misconduct in public life and bad art. There is a good deal of student unrest, but it is unintellectual and manifests itself in ploy agitation over trifling issues. There is, as stated in Section 1, a small minority of intellectually motivated students, but theirs is a voice in the wilderness.

Often my friends and I found it vivifying to contemplate an ideal State of Injustice governed by "Boyle's Laws": $pw = \text{constant}$. Here are two samples. Let $p = \text{pay}$ and $w = \text{work}$ (i.e. useful work); Boyle's law asserts that *your pay varies inversely as the amount of useful work you do.* Next, let $p = \text{penalty}$ and $w = \text{wage}$; the law states that *the penalty you receive for the commission of a crime varies inversely as your wage.*

3. Pseudo solution. During the revolutionary and Napoleonic regimes new life was infused into French education by the consolidation or founding of first-rate institutes related to the State services but independent of the universities, e.g. the École Polytechnique and the École Normale Supérieure. In India a similar development has occurred. The Central Government has either sponsored or supported autonomous or semi-autonomous institutes such as the Indian Statistical Institute, the Tata Institute of Fundamental Research, the Indian Institutes of Technology, the National Laboratories, etc. Most of these institutions perform some of the functions of American universities. For instance, mathematical research done at the Tata Institute may be submitted as theses to Bombay University for the Ph.D. degree.

These institutes are well equipped, and many offer air-conditioned offices, open-shelf libraries, cafeterias and other trappings of modernity. But they differ widely in seriousness of purpose, strength of staff and quality of accomplishment. Many, such as the Tata Institute, have fine cores of research workers. The housing of these workers is unsatisfactory, however. For many years the Tata Institute used the servant quarters of the old Royal Bombay Yacht Club for this purpose! But even this was better than the accommodation provided by some of the other institutes. Another weakness lies in dissipative organization and adherence to archaic procedures of recruitment. The imposition of arbitrary age restrictions and requirements of "previous experience" disqualify many of the best young people from attaining senior and responsible positions. These posts are filled by drawing on senior personnel from universities or government or other institutions. The result is a bureaucratic structure in which ostentatious but small-thinking paper-shufflers at the top stifle the aspirations of dedicated, promising and occasionally spiky researchers in the lower echelons.

their stay here. The scheme could thus be made nearly self-supporting. We should select those who have given evidence of pedagogical initiative, e.g., conducted seminars or put out lecture-notes conveying sound ideas, and should shun those who have written potboilers or otherwise catered to miseducation. Appearance in the Graduate Record Examination may again be made obligatory.

Lastly, let us remember that a strengthening of mathematical activity in this country will have a salutary effect, in the long run, throughout the world and particularly in the underdeveloped nations. The AMS, MAA, NSF and other foundations have contributed much to the movement for improved mathematical curricula, sound research and better teacher training. By rededicating ourselves to this movement we would also further the cause of Indian mathematics.

This article is adapted from a lecture delivered at the 1963 Fall meeting of the Indiana Section of the MAA at Indianapolis.

References

1. M. K. Gandhi, To the Students, Ahmedabad, India, 1949.
2. P. Masani, What constitutes a good mathematics syllabus? Science and Culture, 20 (1955), 487–489, 529–533.
3. S. Ramanujan, Collected Works, Cambridge, 1927.
4. P. A. Schilpp, Albert Einstein, New York, 1949.

COMMENTS ON CODDINGTON'S PAPER ON SCHOLASTIC APTITUDE TESTS

JOHN W. WILKINSON, Research and Development Ctr., Westinghouse Electric Corp., Pittsburgh, Pennsylvania and VIRGIL L. ANDERSON, Purdue University

1. In predicting college grade point average (GPA) from such sources as SAT (Math and verbal) and high school GPA, there are many sources of variation which tend to obscure their effect. Some of the sources that contribute significantly to the variability that have been emphasized in previous studies come from the differences among sections, courses and instructors. If these sources are not to be included among the independent variables then perhaps one should consider other criteria of success, such as rank within sections, so that the variable would be kept so narrow that the extraneous independent variables do not cover up the effects of the independent variables of interest.

The SAT is not likely to be a very good tool for the purpose that initially interested Professor Coddington in it, namely, the selection of candidates for undergraduate scholarships (see [1]). The SAT was not designed for this purpose and will not be powerful in discriminating among individuals scoring at the upper end of its scale, the region where most scholarship candidates will be located. Hence, unless it is not desired to discriminate among these individuals, the SAT scores will provide primarily concomitant information. Also, for the above-mentioned reason, it is not really surprising that, in examining the 77

their stay here. The scheme could thus be made nearly self-supporting. We should select those who have given evidence of pedagogical initiative, e.g., conducted seminars or put out lecture-notes conveying sound ideas, and should shun those who have written potboilers or otherwise catered to miseducation. Appearance in the Graduate Record Examination may again be made obligatory.

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scholarship students, the association between SAT and UCLA GPA was not very pronounced.

2. If one retains grade point average of the freshman year as the criterion, how should one investigate the effect of SAT scores? One must remember that the R^2 (the square of the multiple correlation coefficient) will be quite small due to the reasons given above. It is very important to include all variables that are pertinent and possible, such as instructors, courses and rank in high school, besides the ones given in the Coddington study if the R^2 is to be reasonably high.

Let us say we have all the pertinent independent variables and we wish to evaluate the effect of SAT scores on predicting success. The interest now is to show the amount of variation in the dependent variable (grade point average in the freshman class) caused by the SAT variable. To do this, one must find the R^2 for all variables and subtract from this the R^2 for all variables except the SAT variable of interest. This effect can be tested and, of even more importance, can be evaluated relative to the amount of explained variation for all variables. Hence, if the latter is small, as it usually is, the effect of SAT relative to all explained variation may, in fact, be large.

In addition, most of the students know their SAT scores. If they believe these scores to be indicative of how they are likely to perform in various subject-matter areas, then these scores will tend to influence their selection of courses in college. It is conceivable (even likely) that students having a low SAT (Math) score will avoid mathematics in college to take courses for which they have greater "aptitude" with the probable result of raising their college GPA. Such behavior will tend to obscure any relationship between SAT (Math) and college GPA. This seems to be supported by the comparison of those freshman taking calculus with a general group.

Professor Coddington indicates that HS GPA contains practically all the information necessary in determining admission and that other measures add little of significance. His example on Junior College students (page 752), however, shows the correlation of UCLA GPA with HS GPA when JC GPA's are used also, is much lower than the corresponding correlation with SAT.

3. Almost never does the regression coefficient in a multiple regression problem mean anything when the independent variables are not orthogonal. To check the latter, examine the simple correlation matrix for all combinations of independent variables. If these simple correlations are even reasonably large, the partial regression coefficients are of little value, *per se*. They are good for prediction purposes *as a group* in the model chosen or evolved, but each regression coefficient has relatively little value in explaining its role with the dependent variable.

In addition to examining the relative change in R^2 in the absence and presence of the SAT variable, it would also be desirable to examine the residuals. This might help evolve a more valuable prediction model. Also the examination

mission on the Teaching of Science, of which he is president). The symposium discussed "The Coordination of the Teaching of Mathematics and the Teaching of Science" and was attended by about 30 scientists from all fields and all parts of the world.

In October–November he was visiting professor in the Department of Mathematics, Middle East Technical University, Ankara, Turkey. In December he held the Madras appointment. In February and early March he functioned as visiting lecturer to the University of Pakistan, being sponsored by the International Mathematical Union and the Pakistan Academy of Sciences. He gave a general lecture also to the Pakistan Science Congress in Lyallpur in March.

News Release from the University of Chicago

COLLEGE ATTENDANCE BY MU ALPHA THETA MEMBERS

In January 600 questionnaires were mailed to all fifty states in a spot-check of 1963 graduates who were members of MU ALPHA THETA. These students maintained a B average in high school and showed high ability in mathematics.

The national office of MU ALPHA THETA reports that the 528 replies received show that all but 4 are now enrolled in some college or university. Most students replied that they were enjoying college; only 6 said they were not particularly enjoying their college work. The 4 students who were not in college and had no plans to enter college were all girls. One is married; the other three are employed.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

COLLABORATING EDITORS: J. BARLAZ, Rutgers—The State University; A. E. LIVINGSTON, University of Alberta; L. CARLITZ, Duke University; H. S. M. COXETER, University of Toronto; H. EVES, University of Maine; and A. WILANSKY, Lehigh University.

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solution (other than proposers') should be sent to Professor Starke.

ELEMENTARY PROBLEMS

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E 1701. *Proposed by R. F. Jackson, University of Toledo*

Prove that for any three points on a parabola with vertical axis,

$$m_1 = m_{12} + m_{13} - m_{23},$$

mission on the Teaching of Science, of which he is president). The symposium discussed "The Coordination of the Teaching of Mathematics and the Teaching of Science" and was attended by about 30 scientists from all fields and all parts of the world.

In October–November he was visiting professor in the Department of Mathematics, Middle East Technical University, Ankara, Turkey. In December he held the Madras appointment. In February and early March he functioned as visiting lecturer to the University of Pakistan, being sponsored by the International Mathematical Union and the Pakistan Academy of Sciences. He gave a general lecture also to the Pakistan Science Congress in Lyallpur in March.

News Release from the University of Chicago

COLLEGE ATTENDANCE BY MU ALPHA THETA MEMBERS

In January 600 questionnaires were mailed to all fifty states in a spot-check of 1963 graduates who were members of MU ALPHA THETA. These students maintained a B average in high school and showed high ability in mathematics.

The national office of MU ALPHA THETA reports that the 528 replies received show that all but 4 are now enrolled in some college or university. Most students replied that they were enjoying college; only 6 said they were not particularly enjoying their college work. The 4 students who were not in college and had no plans to enter college were all girls. One is married; the other three are employed.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

COLLABORATING EDITORS: J. BARLAZ, Rutgers—The State University; A. E. LIVINGSTON, University of Alberta; L. CARLITZ, Duke University; H. S. M. COXETER, University of Toronto; H. EVES, University of Maine; and A. WILANSKY, Lehigh University.

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E 1710. *Proposed by D. I. A. Cohen, Princeton University, and Ralph Greenberg, University of Pennsylvania*

If a and b are relatively prime integers, prove that there are infinitely many perfect powers of the form $an+b$.

SOLUTIONS OF ELEMENTARY PROBLEMS

An Adieu to 1963

E 1621 [1963, 890]. *Proposed by Arthur Engel, Stuttgart, Germany*

What is the smallest value of a for which 82^n+a69^n is divisible by 1963 for all odd positive integers n?

I. *Solution by R. J. Herbert, D. T. Kexel, and P. J. Welsh, John Carroll University.* Note that $1963=(151)(13)$. Since $82^n+69^n\equiv 0 \bmod 151$ and $82^n-69^n \equiv 0 \bmod 13$ for all odd n, we must have $a\equiv 1 \bmod 151$ and $a\equiv -1 \bmod 13$. The smallest positive value for a is 454.

II. *Solution by Nyles Barnert, Arcon Corporation, Lexington, Mass.* We employ the lemma: *If x^2-y^2 divides $x+ay$, then it divides $x^{2n+1}+ay^{2n+1}$ for all n.* For the given problem, we set $x=82$, $y=69$, then $x^2-y^2=1963$. We need only find the smallest a such that 1963 divides $82+69a$. This yields $a=454$ as the solution.

Also solved by Gyárfás András, Joseph Arkin, J. W. Baldwin, Merrill Barnebey, Walter Bluger, Adelaide J. Brooks, Brother R. F. Schnepp, Sarvadaman Chowla, B. G. Clark, D. I. A. Cohen, M. J. Cohen, Hüseyin Demir, C. L. Dotton, F. J. Duarte, Philip Franklin, Michael Fried, Anton Glaser, Michael Goldberg, Myron Goldstone, Jerry Goodman, Ralph Greenberg, S. H. Greene, Emil Grosswald, J. H. Halton, R. F. Jackson, J. E. Jean, Jr., Erwin Just and Norman Schaumberger (jointly), Frank Kocher, Sidney Kravitz, A. I. Lieberman, N. F. Lindquist, J. J. Malone, Jr., D. C. B. Marsh, Michael Merritt, P. N. Muller, K. A. K. Murthy, Walter Penney, Stanton Philipp, M. Raghavachari, T. S. Ravisankar, Robert Spitz, J. K. Stewart, G. C. Thompson, A. M. Vaidya, Simon Vatriquant, Gary Venter, W. C. Waterhouse, Charles Wexler, Oswald Wyler, Aleksandras Zujus, and the proposer.

Barnert and Venter showed that if nonintegral a are permitted, then $a=1881/69$; Lieberman, Merritt, and Muller showed that if n is even, then $a=1962$. To anticipate similar problems for the next two years, Franklin pointed out that $248^n+(492)243^n$ is divisible by 1964 for all odd n, and $73^n+(1049)58^n$ is divisible by 1965 for all odd n.

Convergence of Two Series

E 1622 [1963, 890]. *Proposed by Michael Gemignani, University of Notre Dame*

Determine for what values of x the following series converge:

$$(1) \qquad \sum_{n=1}^{\infty} (\sin 1/n)^x, \qquad\qquad (2) \qquad \sum_{n=1}^{\infty} (1-\cos 1/n)^x.$$

Solution by Stanton Philipp, Seal Beach, Calif. One can see from the Taylor series of $\sin 1/n$ and $(1-\cos 1/n)$ in powers of $1/n$ that $1/2n<\sin 1/n<1/n$ and

$1/4n^2<1-\cos 1/n<1/2n^2$. It follows immediately that (1) converges for $\mathrm{Re}(x)>1$ and (2) converges for $\mathrm{Re}(x)>1/2$.

Also solved by E. R. Barnes, H. L. Chow, D. I. A. Cohen, M. J. Cohen, Frank Dapkus, J. A. Faucher, Michael Fried, Ralph Greenberg, Cornelius Groenewoud, Emil Grosswald, Eldon Hansen, H. E. Heatherly, Erwin Just and Norman Schaumberger (jointly), Joel Kugelmass, E. S. Langford, R. D. Leitch, A. I. Lieberman, E. L. Magnuson, D. C. B. Marsh, Morris Morduchow, C. B. A. Peck, L. J. Pratte, George Purdy, Perry Scheinok, C. P. Seguin, D. L. Silverman, R. A. Smith and A. M. Vaidya (jointly), O. E. Stanaitis, Rory Thompson, Andy Vince, Charles Wexler, Raymond Whitney, and Oswald Wyler. Solved partially by Merrill Barnebey, Michael Goldberg, D. E. Myers, W. C. Waterhouse, and the proposer.

The Richness of Mathematical Attack

E 1623 [1963, 891]. *Proposed by R. C. Thompson, University of British Columbia*

Let $f(x)$ be a monic polynomial of degree n with distinct zeros $x_1, x_2, \cdots, x_n$. Let $g(x)$ be any monic polynomial of degree $n-1$. Show that

$$\sum_{j=1}^{n} g(x_j)/f'(x_j) = 1.$$

I. *Solution by F. R. Olson, State University of New York at Buffalo.* Let

$$f_j(x) = \prod_{i\neq j} (x - x_i) = f(x)/(x - x_j).$$

Then $f'(x_j)=f_j(x_j)$. In terms of Lagrange's interpolation formula

$$g(x) = \sum_{j=1}^{n} g(x_j)f_j(x)/f_j(x_j).$$

Division of the $(n-1)$-th derivative of each side by $(n-1)!$ yields the desired result.

II. *Solution by W. C. Waterhouse, Harvard University.* Expanding in partial fractions we have

$$g(x)/f(x) = \sum_{j=1}^{n} [g(x_j)/f'(x_j)](x - x_j)^{-1}.$$

Now multiply by x and let $x\to\infty$.

III. *Solution by A. E. Danese, State University of New York at Buffalo.* Let $r(z)=g(z)/f(z)$. Then $z=x_1, x_2, \cdots, x_n$ are the only singular points of r and they are simple poles. Hence the sum of the residues of r at these poles is

$$A = \sum_{j=1}^{n} g(x_j)/f'(x_j).$$

The residue of r at $z=\infty$ equals the residue of $-r(1/z)/z^2$ at $z=0$, which is

readily determined as -1. Since the sum of the residues at all the singular points and at the point of infinity is zero, we have that $A=1$.

Also solved by Martin Billik and Eldon Hansen (jointly), J. L. Brown, Jr., Leonard Carlitz, A. J. Chandy, D. I. A. Cohen, M. J. Cohen and Nicholas Derzko (jointly), J. B. Deeds, Hüseyin Demir, Michael Fried, Myron Goldstein, S. H. Greene, W. J. Hartman, J. C. Hickman, V. E. Hoggatt, Jr., R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), A. M. Kriegsman, D. C. B. Marsh, Jim Morrow, M. G. Murdeshwar, C. B. A. Peck, Stanton Philipp, Henry Ricardo, S. M. Robinson, Perry Scheinok, C. P. Seguin, R. F. Shanny, O. E. Stanaitis, E. C. Stopher, V. Vitek, J. E. Wilkins, Jr., A. B. Wilcox, K. S. Williams, Oswald Wyler, David Zeitlin, and the proposer.

This problem was solved by many ingenious attacks. For example, in addition to the above (which were the most commonly employed methods): Hartman, Hickman, and Robinson employed the formula for the $(n-1)$-st divided difference for an arbitrary polynomial of degree $k<n-1$; Murdeshwar used a result in the theory of equations given as Problem 4, p. 172 of vol. 1 of Burnside and Panton, *The Theory of Equations* (5th ed.); Fried, Goldstein, Scheinok, and Wyler used some theory of Vandermonde determinants; Demir employed two geometrical relations of Chasles and Euler involving n distinct fixed points and one arbitrary point of a line.

A Correct and an Incorrect Inequality

E 1624 [1963, 891]. *Proposed by C. M. Frye, San Mateo, California*

Prove, for all integers $n>2$, that $(2n-1)^n+(2n)^n<(2n+1)^n$ and that $(2n)^n+(2n+1)^n>(2n+2)^n$.

Solution by Erwin Just and Norman Schaumberger, Bronx Community College. The first inequality is equivalent to $(2+1/n)^n-(2-1/n)^n>2^n$, which is readily verified for $n>2$ by expanding the left side of the inequality. The second assertion is false; for $\lim_{n\to\infty}[(1+1/n)^n-(1+1/2n)^n]=e-\sqrt{e}>1$, and it follows that, for n sufficiently large, $(1+1/n)^n-(1+1/2n)^n>1$. Multiplying both sides of the latter inequality by $(2n)^n$ we obtain $(2n+2)^n-(2n+1)^n>(2n)^n$, which contradicts the second inequality.

Also solved by A. N. Aheart, Joseph Arkin, J. W. Baldwin, Adelaide J. Brooks, Leonard Carlitz, Allan Chuck, B. G. Clark, M. J. Cohen, Hüseyin Demir, G. C. Dodds, J. A. Faucher, C. E. Franti, Michael Fried, Myron Goldstein, R. B. Grayless, S. H. Greene, Emil Grosswald, J. R. Hanna, Eldon Hansen, Mark Hayamizu, Stephen Hoffman, R. F. Jackson, A. M. Kriegsman, N. F. Lindquist, D. C. B. Marsh and W. H. Laubach (jointly), Stanton Philipp, Arthur Porges, George Purdy, Marlow Sholander, O. E. Stanaitis, G. C. Thompson, Simon Vatriquant, Charles Wexler, Aleksandras Zujus, and the proposer. A number of these solutions were only partially correct.

It can be shown that the second inequality is true if $1\leqq n\leqq 15$, but is false if $n\geqq 16$. In connection with the first inequality, Sholander established the more general result: "Given integer $n>2$ and real numbers x, y, z such that $0<x\leqq y-1\leqq z-2$, then $x^n+y^n\geqq z^n$ implies $x>2n-1$, $y>2n$, $z>2n+1$." Hansen showed that the second inequality should be replaced by $(2n+2)^n-(2n)^n<2(2n+1)^n \sinh(1/2)$.

An Application of the Arithmetic-Geometric Inequality

E 1625 [1963, 891]. *Proposed by J. L. Brown, Jr., Pennsylvania State University*

Let n be a positive integer, $\sigma(n)$ the sum of the positive divisors of n, and

$t(n)$ the number of these positive divisors. Show that $\sigma(n)/t(n) \geqq \sqrt{n}$.

Solution by E. L. Magnuson, HRB-Singer, Inc., State College, Pa. Consider the divisors in pairs x, y, where $xy=n$. For each pair, $(x+y)/2 \geqq \sqrt{xy}=\sqrt{n}$. Summing corresponding sides of this inequality over all pairs gives $\sigma(n)/2 \geqq [t(n)/2]\sqrt{n}$, or $\sigma(n)/t(n) \geqq \sqrt{n}$.

Also solved by A. N. Aheart, Jeanne A. Baird, J. W. Baldwin, E. R. Barnes, William Becker, D. A. Breault, Leonard Carlitz, Allan Chuck, B. G. Clark, D. I. A. Cohen, D. M. Cohen, Martin Cohen, D. M. Danvers, J. B. Deeds, Hüseyin Demir, Michael Fried, Anton Glaser, David Golber, Jerry Goodman, Ralph Greenberg, Cornelius Groenewoud, Emil Grosswald, R. F. Jackson, Erwin Just and Norman Schaumberger (jointly), J. C. Lazzara, A. E. Livingston and M. G. Murdeshwar (jointly), C. R. MacCluer, Andrzej Makowski, D. C. B. Marsh, Robert Marsh, Michael Merritt, P. N. Muller, W. I. Nissen, Jr., J. H. Oppenheim, Stanton Philipp, M. Perisastri, A. M. Vaidya, Andy Vince, W. C. Waterhouse, Charles Wexler, Raymond Whitney, K. S. Williams, K. L. Yocom, and the proposer.

An Extension of the Steiner-Lehmus Theorem

E 1626 [1963, 891]. *Proposed by Cornelius Mack, Bradford Institute of Technology, Bradford, England*

Given that X, Y are points on the sides BC, AC of a triangle ABC such that $\measuredangle XAB: \measuredangle CAB = \measuredangle YBA: \measuredangle CBA = \lambda: 1$, where $0<\lambda<1$, show that

(a) $AX>BY$ implies $AC>BC$, and conversely,

(b) $CY>CX$ implies $AC>BC$, and conversely,

(c) $AY>BX$ implies $AC>BC$, and conversely, provided that $0<\lambda\leqq 0.5$, but that there exist triangles for which this is not true if $0.5<\lambda<1$.

Solution by the proposer. (a) Now $\measuredangle AXC=B+\lambda A$, $\measuredangle BYC=A+\lambda B$. Hence $AX \sin(B+\lambda A)=AC\sin C$. Similarly, $BY\sin(A+\lambda B)=CB\sin C$. Hence

$$AX/BY = \sin B \sin(A+\lambda B)/\sin A \sin(B+\lambda A).$$

Consider

$$\alpha = 2\sin B\sin(A+\lambda B) - 2\sin A\sin(B+\lambda A).$$

If we set $1-\lambda=\mu$, then

$$\alpha = \cos(\mu B - A) - \cos(A+B+\lambda B) - \cos(B-\mu A) + \cos(A+B+\lambda A).$$

Collecting the first and third, and the second and fourth terms we get

$$\alpha = 2\sin\lambda\phi\sin(2-\lambda)\theta + 2\sin\lambda\theta\sin(2+\lambda)\phi, \tag{1}$$

where $2\theta=B-A$, $2\phi=B+A$. If $B>A$, then, since $0<B+A<\pi$ and $0<\lambda<1$, we have $0<\lambda\theta<\lambda\phi<\pi/2$; $(2-\lambda)\theta<B-A<\pi$. If, further,

$$(2+\lambda)\phi \equiv (1+\lambda/2)(B+A) < \pi,$$

every term in (1) is positive, and therefore so is α. If $(1+\lambda/2)(B+A)>\pi$, nevertheless $(1+\lambda/2)(B+A)-\pi<\lambda\phi$. Hence $-\sin(2+\lambda)\phi<\sin\lambda\phi$. But

$t(n)$ the number of these positive divisors. Show that $\sigma(n)/t(n) \geqq \sqrt{n}$.

Solution by E. L. Magnuson, HRB-Singer, Inc., State College, Pa. Consider the divisors in pairs x, y, where $xy=n$. For each pair, $(x+y)/2 \geqq \sqrt{xy}=\sqrt{n}$. Summing corresponding sides of this inequality over all pairs gives $\sigma(n)/2 \geqq [t(n)/2]\sqrt{n}$, or $\sigma(n)/t(n) \geqq \sqrt{n}$.

Also solved by A. N. Aheart, Jeanne A. Baird, J. W. Baldwin, E. R. Barnes, William Becker, D. A. Breault, Leonard Carlitz, Allan Chuck, B. G. Clark, D. I. A. Cohen, D. M. Cohen, Martin Cohen, D. M. Danvers, J. B. Deeds, Hüseyin Demir, Michael Fried, Anton Glaser, David Golber, Jerry Goodman, Ralph Greenberg, Cornelius Groenewoud, Emil Grosswald, R. F. Jackson, Erwin Just and Norman Schaumberger (jointly), J. C. Lazzara, A. E. Livingston and M. G. Murdeshwar (jointly), C. R. MacCluer, Andrzej Makowski, D. C. B. Marsh, Robert Marsh, Michael Merritt, P. N. Muller, W. I. Nissen, Jr., J. H. Oppenheim, Stanton Philipp, M. Perisastri, A. M. Vaidya, Andy Vince, W. C. Waterhouse, Charles Wexler, Raymond Whitney, K. S. Williams, K. L. Yocom, and the proposer.

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Consider

$$\alpha = 2\sin B\sin(A+\lambda B) - 2\sin A\sin(B+\lambda A).$$

If we set $1-\lambda=\mu$, then

$$\alpha = \cos(\mu B - A) - \cos(A+B+\lambda B) - \cos(B-\mu A) + \cos(A+B+\lambda A).$$

Collecting the first and third, and the second and fourth terms we get

$$\alpha = 2\sin\lambda\phi\sin(2-\lambda)\theta + 2\sin\lambda\theta\sin(2+\lambda)\phi, \tag{1}$$

where $2\theta=B-A$, $2\phi=B+A$. If $B>A$, then, since $0<B+A<\pi$ and $0<\lambda<1$, we have $0<\lambda\theta<\lambda\phi<\pi/2$; $(2-\lambda)\theta<B-A<\pi$. If, further,

$$(2+\lambda)\phi \equiv (1+\lambda/2)(B+A) < \pi,$$

every term in (1) is positive, and therefore so is α. If $(1+\lambda/2)(B+A)>\pi$, nevertheless $(1+\lambda/2)(B+A)-\pi<\lambda\phi$. Hence $-\sin(2+\lambda)\phi<\sin\lambda\phi$. But

$\gamma/(2\cos\phi)>0$, and therefore $\gamma>0$. Hence $CY>CX$. If $\mu>0.5$, then, since

$$2\sin\mu A\sin\mu B = \cos(\rho+1)\theta - \cos(\rho+1)\phi,$$

we see from above that

$$\begin{aligned}\gamma/\cos\phi &= (2\sin^2\phi - \sin^2\theta)\sin\rho\theta - 2\sin\theta\sin\phi\sin\rho\phi + \sin\theta\\ &\quad + \sin\theta\{\cos\rho\theta\cos\theta - \cos\rho\phi\cos\phi\}\\ &= 2(\sin^2\phi - \sin^2\theta)\sin\rho\theta + \sin\theta\{\cos(1-\rho)\theta - \cos(1-\rho)\phi\}.\end{aligned}$$

But $0<(1-\rho)\theta<(1-\rho)\phi<\pi/2$. Hence every term in our expression for $\gamma/\cos\phi$ is positive. Hence $\gamma>0$, and therefore $CY>CX$; and so $CY>CX$ implies $AC>BC$ and conversely, while $CY=CX$ implies $AC=BC$.

Editorial Note. Simpler solutions to this problem, particularly to part (b), are invited.

Square-Free Integers

E 1627 [1963, 891]. *Proposed by Ralph Greenberg, University of Pennsylvania*

Prove that every positive integer except 1 is the sum of two square-free integers.

Solution by Sarvadaman Chowla, R. A. Smith, and A. M. Vaidya, Pennsylvania State University. If for a real number x, $Q(x)$ denotes the number of square-free positive integers less than or equal to x, then it is enough to show that $Q(x)>(x+1)/2$; for then, if $x_1, \cdots, x_k$ be the square-free positive integers $\leqq n$, consider the two sets

$$M_1\colon \{x_1, \cdots, x_k\}, \qquad M_2\colon \{n-x_1, \cdots, n-x_k\}.$$

M_1 contains k distinct positive integers $\leqq n$ and M_2 contains k distinct non-negative integers strictly less than n, that is, M_2 contains at least $k-1$ distinct positive integers strictly less than n. Since $2k-1>n$, we have an $x_i\neq n$ and an x_j such that $x_i=n-x_j$. Then n is the sum of two square-free positive integers x_i and x_j. We shall show that $Q(n)>(n+1)/2$ for $n\geqq 385$. The assertion of the problem can be verified directly for all smaller values of n.

It can easily be shown (see, e.g., Landau's *Primzahlen*, p. 581) that

$$Q(x) = \sum_{n\leqq\sqrt{x}} \mu(n)[x/n^2],$$

where $\mu(n)$ is the Möbius function and $[u]$ denotes as usual the greatest integer $\leqq u$. Therefore

$$\begin{aligned}Q(x) &= x\sum_{n=1}^{\infty}\mu(n)/n^2 - x\left\{\sum_{n>\sqrt{x}}\mu(n)/n^2 + \sum_{n\leqq\sqrt{x}}\mu(n)(x/n^2 - [x/n^2])\right\}\\ &= (6/\pi^2)x - \{S_1+S_2\}, \text{ say.}\end{aligned}$$

Now it can be proved by elementary means that

$$|S_1| \leqq 1+\sqrt{x} \quad\text{and}\quad |S_2| < \sqrt{x}.$$

Hence

$$|Q(x) - (6/\pi^2)x| < 1 + 2\sqrt{x}.$$

Now if $x \geqq 385$, then

$$1 + 2\sqrt{x} \leqq 7x/66 - 1/2.$$

So finally, for $n \geqq 385$,

$$Q(n) > (6/\pi^2 - 7/66)n + 1/2 > (20/33 - 7/66)n + 1/2 = (n + 1)/2.$$

Also solved by Gyárfás András, J. W. Baldwin, W. R. Becker, David Bienenfeld, M. J. Cohen, Frank Dapkus, George Diderrich, Michael Fried, S. H. Greene, Emil Grosswald, Ned Harrell, R. A. Jacobson, D. C. B. Marsh, Michael Merritt, Stanton Philipp, G. C. Thompson, Jack Winter, and the proposer.

A number of these solutions were open to criticism.

Vaidya called attention to T. Estermann's paper, "On the representation of a number as the sum of two numbers not divisible by kth powers," in the *J. London Math. Soc.*, 6 (1931) 37–40. It may be of interest to know that Estermann has proved (*ibid*, 219–221) that every large number is the sum of a prime and a square-free integer.

Some Triangle Inequalities Involving the Angle Bisectors

E 1628 [1963, 891]. *Proposed by Leonard Carlitz, Duke University*

Let t_a, t_b, t_c denote the angle bisectors of a triangle, r the inradius, R the circumradius, and s the semiperimeter. Show that

$$t_a^2 + t_b^2 + t_c^2 \leqq s^2, \tag{1}$$

$$t_b^2t_c^2 + t_c^2t_a^2 + t_a^2t_b^2 \leqq rs^2(4R + r), \tag{2}$$

$$t_at_bt_c \leqq rs^2. \tag{3}$$

In each case there is equality if and only if the triangle is equilateral.

Solution by Stanton Philipp, Seal Beach, Calif. It is easy to prove that $t_a = [2\sqrt{bc}/(b+c)]\sqrt{s(s-a)}$. Then $t_a \leqq \sqrt{s(s-a)}$, with equality if and only if $b=c$. Similar statements hold, of course, for t_b and t_c. Now the assertions to be proved follow immediately, since $bc+ca+ab=s^2+4rR+r^2$, $2s=a+b+c$, $rs^2=\sqrt{s^3(s-a)(s-b)(s-c)}$.

Also solved by A. N. Aheart, W. J. Blundon, H. W. Guggenheimer, J. S. Leon, Franz Leuenberger, Andrzej Makowski, D. C. B. Marsh, and the proposer.

A Condition for a Semigroup to be an Abelian Group

E 1629 [1963, 891]. *Proposed by F. M. Sioson, University of Hawaii*

Show that any associative system S satisfying the identity $x^2y=y=yx^2$ is a commutative group.

I. *Solution by Roy Dubisch, University of Washington, and B. E. Rhoades,*

Berkeley, Calif. The given identity implies $x^2=e$ and each x is its own inverse. $(xy)^2=e$ yields $xy=y^{-1}x^{-1}=yx$.

II. *Solution by C. M. Geschke, John Carroll University.* By a well-known theorem any associative system is a group with respect to a binary composition if it contains an identity from the right (r) and for every element y a right inverse (y_r) with respect to r. Now $yx^2=y\rightarrow x^2=r$ and $yy=y^2=r\rightarrow y=y_r$. Hence S is a group. Also, $xx=r=x(yy)x=(xy)(yx)\rightarrow yx=(xy)_r$. But $(xy)_r=xy$. Hence S is commutative. The proof shows that the hypothesis can be weakened by requiring only the identity $yx^2=y$ to be satisfied.

Also solved by A. N. Aheart, Joseph Altinger, W. H. Bailey, K. F. Bailie, Nyles Barnert, Ralph Bennett, D. A. Breault, Brother T. C. Wesselkamper, R. J. Bumcrot, F. B. Cannonito, Leonard Carlitz, A. J. Chandy, D. I. A. Cohen, M. J. Cohen, R. J. Cormier, D. M. Danvers, J. B. Deeds, Hüseyin Demir, George Diderrich, Roy Feinman, M. S. Fineman, T. S. Frank, Michael Fried, J. A. Glasenapp and T. C. Upson (jointly), Anton Glaser, Jack Goebel, Michael Goldberg, Myron Goldstein, D. J. Hansen, Dunstan Hayden, H. E. Heatherly, Stephen Hoffman, J. E. Homer, Jr., W. D. Jackson, R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), P. L. Kingston, Max Klicker, Joel Kugelmass, J. Kuzmanovich, E. S. Langford, J. F. Leetch, Joel Levy and P. Meyers (jointly), Jiang Luh, R. J. Lundgren, C. R. MacCluer, J. J. Malone, Jr., D. C. B. Marsh, Stephen Montague, Jim Morrow, M. G. Murdeshwar, D. E. Myers, John Nichols, W. I. Nissen, Jr., C. B. A. Peck, M. Perisastri, Stanton Philipp, D. T. Price, George Purdy, T. S. Ravisankar, P. N. Rheinstein, James Riddell, Azriel Rosenfeld, Perry Scheinok, Marlow Sholander, D. L. Silverman, John Stout, Rory Thompson, A. M. Vaidya, W. C. Waterhouse, Ron Wilder, J. E. Wilkins, Jr., A. B. Wilcox, Oswald Wyler, K. L. Yocom, and the proposer.

A Bounded Solution of a Differential Equation

E 1630 [1963, 891]. *Proposed by Reuben Hersh, Stanford University*

If the polynomial $P(x)$ has no purely imaginary zeros, and if the function f satisfies $|f(x)|<1$ for all real x, then the ordinary differential equation $P(D)u=f$ has exactly one solution $u(x)$ which is bounded for all x, and that bound can be chosen as the product of the reciprocals of the real parts of the zeros of P.

Solution by Oswald Wyler, University of New Mexico. Since no solution of $P(D)u=0$ is bounded for all real x, there is at most one bounded solution of $P(D)u=f$ for bounded f. Denote it (if it exists) by P^*f. If $P=QR$, then $P^*f=Q^*(R^*f)=R^*(Q^*f)$ if Q^* and R^* are defined. Thus it is enough to produce P^*f for $P(x)=x-c$. We put

$$(P^*f)(x)=\int_{-\infty}^{x}e^{c(x-t)}f(t)dt, \qquad \text{if Re } c<0;$$

$$(P^*f)(x)=-\int_{x}^{\infty}e^{c(x-t)}f(t)dt, \qquad \text{if Re } c>0.$$

One checks easily that $|(P^*f)(x)|\leqq K/|\text{Re } c|$ for all real x if $|f(x)|\leqq K$ for all real x, and that $D(P^*f)-c(P^*f)=f$, if either Re $c<0$ or Re $c>0$.

Also solved by the proposer.

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Also solved by the proposer.

a) homeomorphisms between subsets of S preserve the property of having nonempty interiors.

b) if p and q are any (not necessarily distinct) points of S and if U_p and U_q are any neighborhoods respectively of p and q, then there exists a neighborhood V_q of q such that $V_q \subseteq U_q$ and V_q is homeomorphic to U_p with a homeomorphism $h: U_p \to V_q$ such that $h(p)=q$.

5215. *Proposed by A. Wilansky, Lehigh University*

Prove that a topological group (with more than one element) has the discrete topology if and only if it has a compact open subset which includes no right translate of itself.

5216. *Proposed by Oswald Wyler, University of New Mexico*

Let p be a prime and let F_n be the nth Fibonacci number ($F_1=F_2=1$, $F_{n+1}=F_n+F_{n-1}$). Show that:

(a) $F_{p-1}\equiv 0 \pmod p$, $F_p\equiv 1 \pmod p$, if $p\equiv \pm 1 \pmod 5$.

(b) $F_p\equiv -1 \pmod p$, $F_{p+1}\equiv 0 \pmod p$, if $p\equiv \pm 2 \pmod 5$.

5217. *Proposed by Otomar Hajek, Prague, Czechoslovakia*

Given a real-valued function f on a compact interval $J\subset E^1$ of class Lip_A^1 [i.e., $|f(x)-f(y)|\leq A|x-y|$ for $x, y\in J$], prove that there exist polynomials p_n with $p_n\to f$ uniformly on J, p_n in the same class Lip_A^1 on J.

Using this one may show that for Lip_A^1 maps from a compact parallelepiped of E^p to E^q, there exist uniform polynomial approximations in $\mathrm{Lip}_{A\sqrt{q}}^1$ (in the Euclidean norm). Can this be sharpened to Lip_A^1?

SOLUTIONS OF ADVANCED PROBLEMS

Convergents of a Continued Fraction

5111 [1963, 672]. *Proposed by W. A. Schneider, Milwaukee, Wisconsin*

If P_n/Q_n is the nth convergent of the continued fraction for $\sqrt{(x^2+1)}$, then $\operatorname{arccot} P_{2n-1}=2\operatorname{arccot} Q_{2n}-\operatorname{arccot} P_{2n+1}$.

Solution by D. Suryanarayana, Andhra University, Waltair, India. The proposed continued fraction is

$$x+\frac{1}{2x+}\,\frac{1}{2x+}\cdots.$$

We have the following relations:

$$(1)\qquad P_{2n+1}=2xP_{2n}+P_{2n-1},\qquad\qquad (2)\qquad P_{2n}^2-P_{2n-1}P_{2n+1}=x^2+1,$$

$$(3)\qquad P_{2n}^2-(x^2+1)Q_{2n}^2=1,\qquad\qquad (4)\qquad (x^2+1)Q_{2n}=xP_{2n}+P_{2n-1}.$$

[(1) and (2) can be proved by induction on n, and for (3) and (4) see Barnard

and Child, *Higher Algebra*, pp. 534–535.] By virtue of the above four relations it is not hard to show that

$$\frac{P_{2n-1}P_{2n+1}-1}{P_{2n-1}+P_{2n+1}}=\frac{Q_{2n}^2-1}{2Q_{2n}}.$$

That is, cot $\{\text{arccot } P_{2n-1}+\text{arccot } P_{2n+1}\}=\cot\{\text{arccot } Q_{2n}\}$ which implies the required result.

Also solved by A. N. Aheart, L. Carlitz, Walter Penney, and J. M. Quoniam.

Non-Archimedean Field

5112 [1963, 672]. *Proposed by N. R. Riesenberg, University of Wisconsin*

In Dieudonné, *Foundations of Modern Analysis* (Academic Press, N. Y., 1960), a real number system is defined as a field which (1) is Archimedean ordered, and (2) possesses the nested interval property. It is well known that neither (1) nor (2) alone suffices to give a real number system and many examples of Archimedean ordered fields which are not real number systems are in the literature. Give an example of a field which is non-Archimedean ordered but which possesses the nested interval property.

Solution by David W. Dean, Duke University. Let F be the field of formal power series over the reals. Say that $\sum_0^\infty a_n z^n \geqq 0$ if $a_n \geqq 0$ for all n. An interval in F is a set of the form $[A, B]=\{C\in F \mid A\leqq C\leqq B\}$.

Suppose $[A_j, B_j]$ is a decreasing sequence of intervals, and that $A_j=\sum_{n=0}^\infty a_n^{(j)}z^n$, $B_j=\sum_{n=0}^\infty b_n^{(j)}z^n$. Then $[a_n^{(j)}, b_n^{(j)}]$ is a decreasing sequence of closed real intervals for each n, and so there exists c_n such that

$$c_n \in \bigcap_{j=1}^{\infty}[a_n^{(j)}, b_n^{(j)}].$$

The series $\sum_{n=0}^\infty c_n z^n$ is then in each $[A_j, B_j]$ and so is in $\bigcap_{j=1}^\infty [A_j, B_j]$.

Finally F is not Archimedean as 1 and z are not related at all.

Also solved by R. O. Davies.

Sum of an Infinite Series

5113 [1963, 672]. *Proposed by J. S. Frame, Michigan State University*

Sum the series

$$S=\sum_{k=0}^{\infty}\binom{2k}{k}(-16)^{-k}(2k+1)^{-2}.$$

Solution by A. Weinmann, The University, Leicester, England. The required sum S can be transformed into a definite integral by using an integral representation for $(2k+1)^{-2}$. Thus

and Child, *Higher Algebra*, pp. 534–535.] By virtue of the above four relations it is not hard to show that

$$\frac{P_{2n-1}P_{2n+1}-1}{P_{2n-1}+P_{2n+1}}=\frac{Q_{2n}^2-1}{2Q_{2n}}.$$

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Function with Uncountable Number of Horizontal Tangents

5114 [1963, 672]. *Proposed by W. E. Johnson and C. M. Petty, Lockheed Aircraft Corp., Sunnyvale, California*

Let the function $F(t)$ have a continuous derivative on $[0, 1]$ and set $S_1 = \{t: F'(t) = 0\}$, $S_2 = \{F(t): t \in S_1\}$. Show by an example that the set S_2 may be uncountable.

Solution by W. C. Waterhouse, Harvard University. Let $g(x) = 0$ on the Cantor set and $g(x) = (x-a)(b-x)$ in each interval (a, b) forming the complement of the Cantor set. Let $F(x) = \int_0^x g$; then $F' = g$ is continuous, and S_1 is the uncountable Cantor set. Since $\{x: g(x) > 0\}$ is everywhere dense, F is strictly increasing, and hence one-to-one; therefore S_2 is also uncountable.

Also solved by I. N. Baker, Robert Bowen, R. O. Davies, J. L. Denny, R. A. Jacobson, K. F. Kinneberg, K. O. Leland, Solomon Marcus, Ron Rietz, J. M. Shaw and J. F. Standish, D. E. Varberg, Oswald Wyler, Larry Zalcman, J. A. Zilber, and the proposers.

Linear Dimension of Composite Field

5115 [1963, 672]. *Proposed by Harley Flanders, Purdue University*

Let $k \leqq K$, $F \leqq \Omega$, all commutative fields. We may form the composite KF and it is known that $[KF:F] \leqq [K:k]$ if $[K:k]$ is finite. Prove that this inequality is true when $[K:k]$ is infinite, provided that $[F:k]$, the linear dimension of F over k, is countable.

Solution by Oswald Wyler, University of New Mexico. If $[KF:F]$ is finite and $[K:k]$ infinite, then $[KF:F] < [K:k]$ trivially. We assume now that $[KF:F]$ and $[K:k]$ both are infinite, and that $[F:k]$ is countable. In this case, $[KF:F] = [KF:F][F:k] = [KF:k] = [KF:K][K:k]$, and card $K = [K:k]$ card k. If k is countable, then F also is countable, and $[KF:F] = [KF:F]$ card $F =$ card $KF =$ card $K[F] =$ card $K = [K:k]$ card $k = [K:k]$. If k is uncountable and $x \in \Omega$ transcendental over k, then the uncountably many elements $(x-a)^{-1}$ of Ω, $a \in k$, are linearly independent over k. It follows that F is algebraic over k if $[F:k]$ is countable and k uncountable. But then $KF = K[F]$, and since a linear basis of F over k generates the vector space $K[F]$ over K, we have $[KF:K] \leqq [F:k]$. It follows that $[KF:K][K:k] = [K:k]$, so that, again, $[KF:F] = [K:k]$.

We note that our result is somewhat stronger than that proposed in the problem.

Also solved by the proposer.

A Double Summation

5116 [1963, 673]. *Proposed by David Greenstein, Northwestern University*

Let

$$S(A) = \sum_{j=1}^{\infty}\sum_{k=0}^{j-1} \frac{A^{j+k}}{j!k!}, \qquad (A \text{ real}).$$

An engineer needs asymptotic information about $S(A)$ as $A \to \infty$. He conjectures that $e^{-2A}S(A) \to 1$. Prove or disprove his conjecture.

Solution by R. G. Buschman, State University of New York at Buffalo. Consider

$$\sum_{n=0}^{N}\sum_{k=0}^{N} \frac{A^{n+k}}{n!k!} = \sum_{n=1}^{N}\sum_{k=0}^{n-1} \frac{A^{n+k}}{n!k!} + \sum_{k=1}^{N}\sum_{n=0}^{k-1} \frac{A^{n+k}}{n!k!} + \sum_{n=0}^{N} \frac{A^{2n}}{n!n!}.$$

If we pass to the limit on N, then we have

$$e^{2A} = 2S(A) + I_0(2A),$$

where I_0 is the modified Bessel function of the first kind. This yields the explicit formula for $S(A)$,

$$S(A) = \tfrac{1}{2}\{e^{2A} - I_0(2A)\},$$

to which the known asymptotic expansion for I_0 can be applied, giving

$$e^{-2A}S(A) = \tfrac{1}{2} + O(A^{-1/2}).$$

Also solved by C. R. Berndtson and C. G. Fain, M. S. Demos, G. Di Antonio, D. Ž. Djoković, Ralph Greenberg, Emil Grosswald, Eldon Hansen, G. W. Hedstrom, J. Koekoek, E. L. Magnuson, Stanton Philipp, D. Ramakotaiah, J. J. Schäffer, Arnold Singer, Franklin C. Smith, R. P. Tapscott, Rory Thompson and Henry Gray, W. F. Trench, J. H. van Lint, W. C. Waterhouse, A. Weinmann, J. Ernest Wilkins, Jr., Oswald Wyler, M. Wyman, and the proposer.

Roots of Unity

5117 [1963, 673]. *Proposed by L. Carlitz, Duke University*

Let η, ζ be roots of unity such that

$$a\eta + b\zeta + c = 0 \qquad (\eta^2 \neq 1, \zeta^2 \neq 1),$$

where a, b, c are nonzero integers. Show that the only possibilities are given by $a=b=c$, $\eta=\omega$, $\zeta=\omega^2$, $\omega^2+\omega+1=0$.

Solution by Harley Flanders, Purdue University. We have $-a\eta=b\zeta+c$ and $-a\bar{\eta}=b\bar{\zeta}+c$. We multiply these expressions, noting that $\bar{\eta}=\eta^{-1}$, $\bar{\zeta}=\zeta^{-1}$ since these are roots of unity:

$$a^2 = b^2 + c^2 + bc(\zeta + \zeta^{-1}).$$

Since $bc\neq 0$ we conclude that $\zeta+\zeta^{-1}=u=$ rational, and $\zeta^2-u\zeta+1=0$. Thus ζ is quadratic over the rationals so that if ζ is a primitive nth root of unity, then $n=1, 2, 3, 4, 6$ are the only possibilities; by hypothesis $n\neq 1$, $n\neq 2$. We rule out $n=4$ because if $\zeta=i$, $i^2=-1$, then η is a unity root in the field $Q(i)$ so that $\eta=\pm i$. But $a(\pm i)+bi+c\neq 0$. This leaves two cases:

$$S(A) = \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \frac{A^{j+k}}{j!k!}, \qquad (A \text{ real}).$$

An engineer needs asymptotic information about $S(A)$ as $A \to \infty$. He conjectures that $e^{-2A}S(A) \to 1$. Prove or disprove his conjecture.

Solution by R. G. Buschman, State University of New York at Buffalo. Consider

$$\sum_{n=0}^{N} \sum_{k=0}^{N} \frac{A^{n+k}}{n!k!} = \sum_{n=1}^{N} \sum_{k=0}^{n-1} \frac{A^{n+k}}{n!k!} + \sum_{k=1}^{N} \sum_{n=0}^{k-1} \frac{A^{n+k}}{n!k!} + \sum_{n=0}^{N} \frac{A^{2n}}{n!n!}.$$

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Solution by Harley Flanders, Purdue University. We have $-a\eta=b\zeta+c$ and $-a\bar{\eta}=b\bar{\zeta}+c$. We multiply these expressions, noting that $\bar{\eta}=\eta^{-1}$, $\bar{\zeta}=\zeta^{-1}$ since these are roots of unity:

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Finally, (i) and (ii) show that $\phi(t)+\psi(t)>0$ for $\pi/2n<t\leqq\pi/2$, also (i) and (iii) imply the same inequality for $0<t<\pi/2n$.

Also solved by L. Carlitz.

Convergent Sequence

5120 [1963, 673]. *Proposed by D. C. Olivier, Carleton College, Northfield, Minn.*

Define a sequence $\{v_n\}=\{v_n(x)\}$ recursively by $v_1=x$, $v_{n+1}=(2+1/n)v_n-1$, $n\geqq 1$. It is not hard to show that $\{v_n\}$ converges for at most one real value of x. Find x such that $\{v_n\}$ converges.

Solution by Roy O. Davies, The University, Leicester, England. By induction we have

$$v_n = 1\cdot 3 \cdots (2n-1)(x-s_n)/(n-1)!, \qquad n = 1, 2, \cdots,$$

where $s_1=0$ and $s_{n+1}=s_n+[n!/1\cdot 3 \cdots (2n+1)]$. If $\{v_n\}$ converges then $x-s_n\to 0$, whence

$$x = \lim s_n = \sum_{n=1}^{\infty} \frac{n!}{1\cdot 3 \cdots (2n+1)} = \frac{1}{2}\pi - 1.$$

(See, e.g., Bromwich, *Infinite Series*, 1st ed., p. 169.) For this x we find that

$$v_n = [n/(2n+1)] + [n(n+1)/(2n+1)(2n+3)] + \cdots,$$

and since the rth term here is increasing and tends to 2^{-r}, it follows that v_n indeed tends to $\frac{1}{2}+\frac{1}{4}+ \cdots =1$.

Also solved by I. N. Baker, L. Carlitz, A. J. Casson, J. H. E. Cohn, H. D. Friedman, D. R. Hayes, Fulton Koehler, J. Koekoek, R. H. C. Newton, Ron Rietz, A. A. Sastry, H. Schwerdtfeger, D. W. Showalter, Arnold Singer, Robert Singleton, J. H. van Lint, K. H. van Weerden, J. Ernest Wilkins, Jr., Oswald Wyler, Max Wyman, and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, Carleton College and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. Programmed Materials: K. O. May, Carleton College, Northfield, Minn. Films: E. P. Vance, Oberlin College, Oberlin, Ohio.

Lectures on Tensor Calculus and Differential Geometry. By Johan C. H. Gerretsen, P. Noordhoff N. V., Groningen, 1962. xii+204 pp. Dfl.25.

Here is an unusual introduction to the methods and principal results of the differential geometry of general manifolds. Frequently the development of this

subject begins abstractly with the transformation laws for tensors and the definition of a Riemann space and proceeds to the detailed formalism of tensor calculus. The present text has attempted to cultivate geometric thinking by taking a less formal approach.

The author does not study the manifolds as autonomous spaces but always as subspaces imbedded in a linear vector space with a Euclidean metric. Thus the metric of the manifolds does not appear as an intrinsic property of the manifold but is induced by the metric of the enveloping space. Despite these limitations (or rather, because of them), the book offers the advantages of always remaining close to the images of classical differential geometry and developing the tensor formalism in a natural way from vector algebra.

The book begins with a presentation of the elements of linear algebra. Tensor algebra is then based upon multilinear vector functions. The notions of tensor calculus such as covariant differentiation, the Christoffel symbols and the Riemann curvature tensor are all introduced by means of elementary operations on vectors in Euclidean n-space. Using these tools, the book develops the differential geometry of curves and hypersurfaces as well as a number of special topics such as geodesic and conformal mapping. The treatment is essentially algebraic, indicating clearly the dependence on the underlying Euclidean space. The concluding chapter deals with integrability conditions and their application to existence theorems.

The approach used by the author gives rise to the major shortcoming of the book. The metric of a manifold and the covariant differentiation process are introduced as properties derived from the enveloping space, not as intrinsic attributes of the manifold. While the isometrically invariant character of covariant differentiation is indicated, it would be well to supplement the book with an independent account of the tensor formalism as needed in the study of general spaces, without reference to an embedding space. Other disadvantages for classroom use are the complete absence of illustrative examples or of exercises for the student.

These adverse features are negligible, however, in comparison with the book's positive advantages. The text is distinguished by clarity of exposition and logical development and provides a quite complete introduction to the subject.

AARON FIALKOW, Polytechnic Institute of Brooklyn

Theory and Application of Liapunov's Direct Method. By Wolfgang Hahn, translated by Lehnigh and Hosenthien. Prentice-Hall, New York, 1963. 182 pp. $6.74.

This book is an excellent and accurate translation of "Theorie und Anwendung der direkten Methode von Ljapunov" which was published in Germany in 1959 and which was the first detailed and advanced account of Liapunov's method to appear in a western language. In addition to corrections which have been made in this translation, a new section on "Differential Equations with

Bounded Solutions" has been added, and the excellent bibliography of the German edition has been brought up-to-date (to early 1962).

Fundamental concepts and basic theorems are presented in the first two chapters. These are then amply illustrated in Chapter 3. Chapter 4 contains the converses of the main theorems and Zubov's method of constructing Liapunov functions. Stability problems for linear systems are presented in Chapter 5. Stability under perturbations, and in particular total stability (stability under constantly acting perturbations) are the subject of Chapter 6. The last two chapters of the book are brief discussions of important topics. Special investigations of critical cases are considered in Chapter 7. The important problem of stability for partial differential equations, difference-differential equations, and difference equations are briefly mentioned in Chapter 8.

This book, a thorough résumé, is a valuable addition to the growing literature in English on Liapunov's direct method.

J. P. LASALLE, RIAS

Einführung in die Verbandstheorie. By G. Szász. Akadémiai Kiadó, Budapest, 1962. 255 pp. $8.85.

This introductory text in lattice theory presents the most important concepts and methods of the subject and indicates some of the relations between lattice theory and other branches of mathematics. The chapter on complete lattices includes material on topological spaces, closure operations, Galois connections, and Dedekind cuts. Other chapters include discussions of the Kurosch-Ore theorem, complemented modular lattices and projective geometries, Boolean rings, measure theory in Boolean algebras, and the representation of distributive lattices as rings of sets.

A most valuable feature of the book is its clear presentation of the concepts of general algebra as they apply to lattices of various types and its equally clear demonstration of the usefulness of lattice theory in the study of general algebra. The final chapter studies the congruence relations of a general algebraic structure and arrives at the Schreier refinement theorem. It then specializes to consider congruence relations in lattices themselves.

Of special interest is the chapter on semi-modular lattices, which investigates relations between weak forms of complementation and conditions weaker than modularity. This chapter reflects some of the author's own research interests.

Most of the presentations and proofs are models of good organization and exposition. An exception worth mentioning is in the definition of relatively atomic lattice, which needs to be changed if Theorem 51 and its proof are to be correct.

Exercises are provided at the end of each chapter. These and the many references to the literature interspersed throughout the text help to make this an interesting introduction to lattice theory.

GEORGE N. RANEY, University of Connecticut

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GEORGE N. RANEY, University of Connecticut

It strikes this reviewer that this book would be ideal for a student, even an honors student, who wanted a good review in preparation for a comprehensive examination.

DAVID ROSEN, Swarthmore College

Stochastic Service Systems. By John Riordan. Wiley, New York, 1962. 139 pp. $6.75.

Anyone seeking an introduction to queueing theory will not suffer from a lack of material to guide him. In a remarkably short time, a large number of texts have been published, ranging from those which regard the subject as a branch of pure mathematics to others which view it entirely through the eyes of the businessman longing for greater efficiency. Riordan covers the middle ground, presenting the basic mathematical tools in the course of analyzing the most important queueing situations, while leaving aside any detailed applications. Because it largely succeeds in combining clarity and thoroughness with elegance and brevity, this book should rank among the best in its field.

After a general introduction and a short look at traffic input and service distributions considered separately, the case of the infinite server queue is taken up. (The objection of the author to the term "queueing theory" as not being inclusive enough—whence the title—might be noted here. This reviewer feels that this objection has lost any validity that it may have had, since "queue" is now used in the broadest possible sense.) This is followed by fairly detailed accounts of single server and many server systems in which many of the countless variations on the queueing theme are treated; e.g., loss systems, delay systems with Poisson input and general service time, Lindley's work on waiting time distributions, priority and random service, and the beautifully simple approach of S. O. Rice. The brief last chapter on traffic measurements is somewhat unusual. Discussions of the determination of the basic parameters of any queueing situation, e.g., arrival rates, service rates—are surprisingly rare in the statistical literature, and Riordan did well by bringing this area of potential trouble to the attention of the reader.

The author has obviously made an attempt to present the results in the most accessible form. This is no mean task in a field in which empty manipulation of virtually unusable formulae is an all-too-pervasive fault. Still, some graphs, and numerical examples would have helped to give a more intuitive feeling for the subject.

The prerequisite for this book is a good basic grounding in probability. Some acquaintance with the Wiener-Hopf method will help since the discussion in the text is rather cursory.

In sum, the reader will find here an excellent entrance into queueing theory; the superb little book by Khinchin should also be read by those who wish to penetrate further into the foundations. The editors of the new SIAM series in Applied Mathematics may be congratulated on their inaugural volume.

MARTIN A. LEIBOWITZ, Bellcomm, Inc.

Introduction to General Topology. By Maynard J. Mansfield. Van Nostrand, Princeton, 1963. 116 pp. $4.50.

This one-semester introduction to general topology, addressed to juniors and seniors who have completed an elementary calculus course, covers in a generally lucid and formal manner connectedness, compactness, the T_i-separation axioms and metric spaces.

Its scope is limited by the omission of any discussion of the composition of functions or of the product of two topological spaces. This last omission appears to restrict most of the underlying sets in examples to be either R, Z or some finite set, and furthermore it limits the book's application of function theory to real-variable theory (for example the Heine-Borel-Lebesgue theorem is proved only for R).

Except for a few missing subscripts there are no misprints; the proof of the completion theorem is false since k_ϵ depends on p and q; and the author confuses set theory with topology when he states that a function is a topological concept. It would be preferable to keep all the set theory in chapter one and to say more about the real numbers. Otherwise the book is well-organized and free from errors.

JOHN C. TAYLOR, Columbia University

Stability of Motion. Applications of Lyapunov's second method to differential systems and equations with delay. By N. N. Krasovskii. Translated by J. L. Brenner. Stanford University Press, California, 1963. 188 pp. $6.00.

This book is a translation, with alterations and additions, of N. N. Krasovskii, Nekotorye zadači teorii ustoičivosti dviženiya (Moscow, 1959).

Lyapunov functions are a generalization of the concept of potential energy of a system, and the use of these functions gives the only general method available for investigating the stability of solutions of ordinary differential equations. The existence of a Lyapunov function whose derivative along solutions of the equations possesses certain properties implies stability properties of the solutions. The converse is also true. Both aspects of this problem are treated in detail in the first five chapters of this book, and the importance of each is made clear by many specific examples.

The author indicates in the last two chapters that the proper setting for a treatment of differential equations with delayed arguments is in function space and then proceeds to a detailed discussion of Lyapunov functionals. This treatment has stimulated much research in this area, and will certainly stimulate more. The proofs are sometimes cumbersome, probably due to the fact that this approach is still in its infancy.

The translation is very good, the bibliography has been brought up to date, and appropriate notes concerning more recent developments have been inserted in the translation.

JACK K. HALE, RIAS, Baltimore, Md.

Topology of 3-Manifolds and Related Topics. Edited by M. K. Fort, Jr., Prentice-Hall, Englewood Cliffs, N. J., 1963. viii+256 pp. $7.50.

This book consists primarily of summaries or full length reports of talks given at a topology institute at the University of Georgia in 1960. The articles are grouped under the headings: 1. Decompositions and subsets of E^3; 2. n-Manifolds; 3. Knot theory; 4. The Poincaré conjecture; 5. Periodic maps and isotopies; 6. Applications.

The articles vary greatly in all respects. Some are mere statements that a certain talk was given; many are research papers; others are extensive reports on fields of current interest. The most remarkable paper, Zeeman's *The Topology of the Brain and Visual Perception*, has nothing to do with manifolds. (This article is the only "application.") Since there are over 40 articles, we shall mention only a few that are perhaps of more general interest, in spite of the fact that many of the research papers are of high quality.

R. H. Bing discusses *Decomposition of* E^3 in his usual engaging style, including useful advice about titles of mathematical articles. S. S. Cairns surveys the relations between *Differentiable and Polyhedral Manifolds.* R. H. Fox takes us on *A Quick Trip Through Knot Theory*, and also discusses *Some Problems in Knot Theory.* E. C. Zeeman presents a proof of the *Poincaré Conjecture for* $n \geqq 5$. (With these and other articles, Zeeman and Fox account for more than 100 pages.) D. B. A. Epstein gives a brief and lucid account of *Ends.*

MORRIS W. HIRSCH, University of California

Flows on Homogeneous Spaces. By L. Auslander, L. Green, and F. Hahn with the assistance of L. Markus and W. Massey, and an Appendix by L. Greenberg. Annals of Mathematics Studies, Princeton University Press, Princeton, N. J., 1963. 107 pp. $2.75.

This is a collection of research papers intended primarily for the expert in the field. The homogeneous spaces considered are compact Hausdorff spaces of the form G/D where G is a noncompact connected Lie group and D is a discrete subgroup of G; and the flows studied are the ones induced by one parameter subgroups of G. Results from the theory of Lie groups, ergodic theory, group representations and topological dynamics are used to study the dynamical properties of these flows. Thus, for example, all three dimensional simply connected noncompact Lie groups G which possess discrete subgroups D with G/D compact are determined. Then the flows induced on G/D by one parameter subgroups of G are exhaustively classified with respect to such dynamical properties as minimality, metric transitivity, ergodicity, etc. In higher dimensions the discussion is restricted to solvable and nilpotent Lie groups.

This book is a valuable contribution to the subject not only because of the many results obtained but also for the many examples against which one may test various conjectures as to what happens in higher dimensions and for more general flows.

ROBERT ELLIS, Wesleyan University, University of Pennsylvania

An Elementary Introduction to the Theory of Probability. By B. V. Gnedenko and A. Ya. Khinchin. Translated from the fifth Russian edition by Leo F. Boron. Dover, New York, 1962. xii+130 pp. $1.45.

The high reputations of the authors and the fact that this is the fifth edition since 1945 both vouch for the quality of this small book. Part I, *Probabilities*, is a 54-page treatment of the standard rules for probability in finite sample spaces: addition, multiplication, conditional probabilities, independent events, Bayes' formula, the binomial (Bernoulli) distribution, and Bernoulli's theorem with Chebyshev's proof. (A minor correction: p. 51, line 7, read "distance more than" in place of "distance not more than.") The hypergeometric distribution is not included. Part II, *Random Variables*, is a 59-page treatment of random variables and their distribution laws, mean values, mean values of sums and products (of independent r.v.), mean deviation, standard deviation, probable deviation, Chebyshev's inequality and laws of large numbers, and normal distributions. The expected value operator is not used: instead, the authors use a bar to denote mean value. The *Conclusion*, 5 pages, sketches the development of probability theory from Fermat, Pascal, and Huygens to the present, with special attention to the contributions of Russians, but with credit also to the United States, France, Great Britain, Sweden, Japan, and Hungary. The discussion of the central limit theorem gives an excellent idea of why the normal distribution arises naturally in scientific applications.

The exposition is notably clear. New concepts are well motivated. Set language and symbolism are not used. The example-rule-example pattern is followed in almost every section. In general, no mathematics beyond high school algebra is required, though a few calculations use summation sigmas. There are no exercises, but the book would be an excellent supplement for elementary courses in probability in high school or college. The many examples illustrate how probability applies to a broad variety of practical and theoretical situations. It is unavoidable that, in such a brief treatment, some fine points will be ignored (for example, in connection with the probability distribution of the square or absolute value of a random variable which may assume negative and positive values), but the authors have succeeded admirably in presenting significant parts of probability at this level.

GEORGE B. THOMAS, JR., Massachusetts Institute of Technology

Linear Algebra and Matrix Theory. By E. D. Nering. Wiley, New York, 1963. xi+289 pp. $6.95.

This book should prove to be satisfactory as a text for a one semester course at an advanced undergraduate level. A rigorous treatment of most of the topics that normally make up a linear algebra course is given in the first five (of the six) chapters. These chapter headings are: I, Vector Spaces; II, Linear Transformations and Matrices; III, Determinants, Eigenvalues, and Similarity Transformations; IV, Linear Functionals, Bilinear Forms, Quadratic Forms; V, Orthogonal and Unitary Transformations, Normal Matrices. In the last chap-

ter, Selected Applications of Linear Algebra, good but brief treatments of applications in fields such as linear programming and communication theory are given.

The author covers a good deal of material in a few pages. One omission that might bear mentioning is a discussion of the rational canonical forms for matrices under similarity. There is no development of field theory in this book. And no theory of polynomial forms is given. Knowledge of some of the elements of this theory is needed at various places, such as in the proof of the Hamilton-Cayley Theorem. In order to gain much insight into most of the applications discussed in the last chapter, a student would have to do a considerable amount of background reading. (A bibliography is given at the end of each section of this chapter.)

A good and ample supply of problems of various degrees of difficulty is found at the ends of the various sections of the chapters. Solutions or hints to solutions to many of these exercises are given at the end of the book.

P. W. CARRUTH, Swarthmore College

Solved and Unsolved Problems in Number Theory. Vol. I. By Daniel Shanks. Spartan Books, Baltimore, 1963. ix+229 pp. $7.50.

The title of this book is somewhat misleading. It is not a collection of problems, but a highly individualistic introductory textbook in number theory in which "problem—solution" is given preference over "theorem—proof." This is not to say that there are no theorems, but that the theorems are regarded not as end-products, but rather as stepping stones to the solutions of problems on which the author has already focused the reader's attention.

In the first chapter, for which the substratum is the problem of perfect numbers and Mersenne primes, one finds the unique factorization theorem, the theorems of Fermat and Euler, Euler's and Gauss's criteria and the law of quadratic reciprocity, all developed without mention of congruences, and interspersed with historical remarks, classical conjectures and much information on the results obtained by modern computers. In the second chapter congruences are introduced, and the group theoretic structure of the residue class groups is studied in much greater detail than is customary. The third chapter, built upon the Pythagorean theorem, ranges over Fermat's equation $x^n+y^n=z^n$ and its various elementary special cases, Pell's equation and continued fractions, and Lucas's criterion for primality of Mersenne numbers, with digressions on such matters as quantum physics, Pythagorean philosophy, and a purely arithmetic derivation of the Leibniz identity $\pi/4=1-1/3+1/5-\cdots$.

This is clearly not the book for a student who likes the orderly, polished and general (if not abstract) exposition to be found in most textbooks. On the other hand, the author has something to say, both philosophically and mathematically, which should be stimulating to students and enlightening even to professionals. No description of the contents can impart the flavor of the book; the interested reader is advised to examine a copy.

W. J. LEVEQUE, University of Michigan

The Language of Computers. By Bernard A. Galler. McGraw-Hill, New York, 1962. 244 pp. $8.95.

This book shows how an algebraic computer language is developed out of the requirements of various types of problems. The problems selected are graded in complexity, starting with the trivial problem of making change with the fewest coins, then the problem of collecting social security from a pay check, followed by coding and decoding secret messages, then the use of various random number generators in integration, problems of sorting, the correlation coefficient, generation of programs for simple networks of switches, and finally, the problem of systems of simultaneous equations, including a zero pivot. Each problem requires a further development of the language and there are adequate exercises at the end of each chapter. The programs are written in the MAD language developed at the University of Michigan. The last few chapters are devoted to a comparison of MAD with FORTRAN and ALGOL in terms of the criteria of completeness, efficiency of the translator, and time required for execution of the hardware program.

The book is written at a level that college freshmen or sophomore mathematics students can understand. There is a careful step by step development of the algorithms for the solution of each problem. In some cases several algorithms are offered for the solution of one problem and these are compared in terms of time and memory requirements. Unfortunately, many computation centers are not using MAD and may never use it since they are geared to FORTRAN. Also, the $8.95 price for a 244 page book may limit its use somewhat as a student text. For centers using other algebraic languages, the chief usefulness of the book would seem to lie in the logical analyses underlying the algorithms included.

HEATH K. RIGGS, University of Vermont

Local Rings. By Masayoshi Nagata. Wiley, New York, 1963. xiii+234 pp. $11.00.

A local ring is so named because it arises in a natural way in the study of local properties of an algebraic variety. Abstractly defined, a local ring is a commutative Noetherian ring with unity having a unique proper maximal ideal.

We quote from the introduction, "To illustrate this geometric aspect of local rings, let us consider an affine n-dimensional space A_n over the complex field C. Let $x_1, \cdots, x_n$ be a set of coordinates for A_n and P a point in A_n. If R_P is the set of rational functions in $x_1, \cdots, x_n$ which are regular at P, then R_P is a Noetherian ring in which $m_P = \{f \in R_P | f(P) = 0\}$ is the only maximal ideal. Thus R_P is a local ring associated with a point P of A_n. An irreducible variety V going through P defines a prime ideal of R_P, $p = \{f \in R_P | f(V) = 0\}$ and vice versa a prime ideal of R_P defines a variety through P. Further, the ring R_P/p is again a local ring which we call the local ring of P on V."

This book is a comprehensive but condensed account of the algebraic, topological, and geometric aspects of local ring theory. While it is essentially self

contained, it is not a book for the mathematical novice. However, for one well versed in the elements of commutative ring theory and topology (to say nothing of algebraic geometry) it should prove to be a gold mine of stimulating information. Each chapter contains several exercises, most of which are extensions of the theory in the book.

In view of the wide popularity of homological algebra in commutative ring theory at present, the following quotation from the preface is interesting, "We give in Chapter IV a new theory of syzygies, which takes the place of homological methods employed in the theory of local rings. Thus we shall never use homological algebra in the present book. Furthermore, it should be emphasized here that our theory is simpler than the one given by homological algebra even for readers who know the subject."

R. E. JOHNSON, University of Rochester

Integral Operators in the Theory of Linear Partial Differential Equations. By Stefan Bergman. Ergebnisse der Mathematik und ihrer Grenzgebiete, New Series, Vol. 23, Springer-Verlag, Berlin, 1961. viii+145 pp. DM 39, 80.

This book summarizes the applications of integral operators to linear partial differential equations. A major part of the development of these applications is due to the author.

As is well known, the real part of an analytic function of one complex variable is a harmonic function of two real variables. "Take the real part" applied to analytic functions is an operation yielding harmonic functions while preserving many properties of the associated analytic functions. There also exist inverse operators which yield analytic functions from harmonic functions.

The object of the integral operator approach as surveyed in this book is to generalize the above mentioned relation to relations between analytic functions of complex variables and solutions of more general linear partial differential equations. By means of the operators certain theorems about analytic functions of complex variables can be translated into corresponding theorems about solutions of certain partial differential equations.

The first chapter considers operators $P(f)$ which generate solutions u of partial differential equations in two real variables. Here $f(z)$ is an analytic function which is associated with the solution u. Different operators show that various properties of the associated analytic functions are transformed into analogous properties of the class of solutions generated. Among others, the Bergman operator of the first kind is defined and represented in various ways. This operator has a simple inverse and is useful for studying coefficient problems. Information about the behavior of particular solutions of the partial differential equations can be obtained by considering subsequences of coefficients of certain series developments of the solutions.

The second chapter considers harmonic functions in three real variables. Here harmonic functions are generated by the Whittaker-Bergman operator which provides a mapping from the set of analytic functions of two complex

variables into the set of harmonic functions. By considering various classes of analytic functions, classes of harmonic functions with interesting properties and singularities are constructed.

The third chapter describes operators transforming harmonic functions into solutions of more general differential equations in three real independent variables.

Chapter IV deals with systems of partial differential equations.

Chapter V considers equations of mixed type and elliptic equations with singular and nonanalytic coefficients.

The treatise includes a comprehensive bibliography of books and original papers. Some of the proofs have been omitted in cases where the transition to the original paper is not difficult.

Although no physical applications are considered in this book, integral operators can be applied to problems in hydrodynamics and other fields. The reader interested in such applications of integral operators might refer to the book, *Bergman's Linear Integral Operator Method in the Theory of Compressible Fluid Flow*, by M. Z. v. Krzywoblocki, Springer-Verlag, Wien, 1960.

A. M. WHITE, Harvey Mudd College

The Nature of Scientific Thought. By Marshall Walker. Prentice-Hall, Englewood Cliffs, N. J., 1963. 179 pp. $2.25.

This book aims to explain the scientific method to "the general, educated reader." Its main themes are the construction of conceptual models, the derivations of the predictions from the model, and the verification of the model by comparison between observations and predictions. The book illustrates this process by discussions of the historical development of a number of basic scientific ideas. For example, the development of Newton's laws of motion, some implications of the postulates of special relativity, and the development of quantum mechanics are very ably set forth. The operation of the senses and the nervous system in receiving, transmitting, and interpreting information is discussed. Discussions of comparative linguistics and of the nature of mathematics, and an introduction to a number of concepts used in physics and mathematics are given.

The book also has a number of other objectives. It surveys the present state of scientific research and the present methods of supporting this research, it interprets ethical and moral behavior as conduct which maximizes the probability of survival of the individual and the species, and it advocates a political program—the avoidance of war, the control of population, and a reorganization of public education to speed the education of able students and to use the time of teachers and students more effectively.

In this reviewer's opinion, the book attempts too large a task. The very strong merit of the book is the discussion of specific scientific ideas. These serve well to acquaint the reader with the scope of modern science—what results it has obtained and how it attempts to account for and predict experience. The

arguments for a position are sometimes questionable in detail, e.g., using the early creativity of a few exceptional scientists to justify changing the structure of the entire public educational system, and sometimes wrong in basic concept, e.g., claiming that "mathematics is concerned with operations on abstract symbols," ignoring the fact that the operations act on mathematical objects. The author takes a bold stand in insisting that all ethical and moral imperative statements are predictions that the probability for survival of the individual and the species is decreased if a prohibited act is performed. This seems at once too nonempirical (often one can't compute the relevant possibilities, so one can't check the predictions) and too restrictive a definition to render the meaning of many such statements. As a general criticism, this reviewer thinks that the author, having found the single cause of all human behavior—survival the goal, and the scientific method the survival technique—is not sufficiently careful to check that his explanation accounts for all the facts. Finally, the book would serve its prime function better if it provided a bibliography for those readers who want to learn more about some of the sciences but aren't sure how to go about it.

R. C. MJOLSNESS, Los Alamos

Algebraic Logic. By P. R. Halmos. Chelsea, New York, 1962. 271 pp. $3.75.

By algebraic logic the author means that branch of general algebra which deals with algebraic structures mirroring in some sense certain formal logics. Examples of such structures, in historical order of investigation, are Boolean, Brouwerian, relation, projective, and cylindric algebras. Subsequent to the study of these algebras, Halmos introduced and investigated monadic and polyadic algebras, and the present monograph is the collection of all the articles Halmos has written on these two kinds of algebras. Polyadic algebras have algebraic operations mirroring sentential connectives, first-order quantifiers, and changes of variables, while monadic algebras are just a very special kind of polyadic algebras. The class of all polyadic algebras of a given degree is an equational class, and hence by any standard is a fitting object of study; but the polyadic algebras exactly corresponding to first-order logic are the locally finite ones of infinite degree which, unfortunately, do not form an equational class. The general polyadic algebras do mirror closely certain quite general logics with infinitely long expressions which are now being intensively studied.

As Halmos indicates, one of the main problems in algebraic logic is to state and prove algebraically various important theorems of logic. This was done by Tarski and later by Halmos for Gödel's completeness theorem. Halmos also treats algebraically, e.g., the description operator. The program has been carried through by Daigneault for Feferman-Vaught generalized products and for Craig's interpolation theorem. Notably still lacking is an algebraic treatment of Gödel's incompleteness theorem. It should also be mentioned that Daigneault and Keisler have generalized Halmos' treatment of the completeness theorem by showing that any simple polyadic algebra of infinite degree is isomorphic to

an O-valued functional algebra. An open problem is to characterize those polyadic algebras of finite degree which are isomorphic to O-valued functional algebras; the reviewer has shown that not every simple algebra is of this kind.

Halmos' book is highly recommended as an introduction for those who wish to study logic from a purely algebraic point of view.

DONALD MONK, University of Colorado

An Introduction to the Calculus of Variations. By L. A. Pars. Wiley, New York, 1962. 350 pp. $8.50.

This book is somewhat similar to Bliss' Carus Monograph on the calculus of variations but is wider in scope. It was the author's intent to "give to the non-specialist a good insight into the fundamental ideas of the subject, a good working knowledge of the relevant techniques, and an adequate starting point for further study and research . . . ." It is also somewhat similar to Akhiezer's *Calculus of Variations* (English translation, Blaisdell Publishing Co., New York, 1962). There has been a need for books suitable for an introductory course in the calculus of variations. It is good to have another book as a possible choice for a text book in the subject.

The first five chapters are concerned with the ordinary problem in the plane. One chapter is devoted to concrete problems illustrating the theory. The multiplier rule for an isoperimetric problem in the plane is derived in the sixth chapter, and there is a brief discussion of the relation of the isoperimetric problem to Sturm-Liouville systems. There are many diagrams illustrating the geometric details of particular problems.

In chapter seven necessary conditions and the fundamental sufficiency theorem are given for the ordinary problem in three dimensions with an indication how to extend the theory to n-dimensions. It seems unnecessary to develop the theory first for the plane and then repeat the process for higher dimensions. It would have been simpler to treat the problem for n-dimensions from the start. Any objection a reader might have to this would be taken care of by the many concrete examples.

In chapter eight we find the multiplier rule for the Lagrange problem with n dependent and one independent variable. The next chapter deals with the parametric problem in the plane. The last chapter is entitled "Multiple Integrals." A more appropriate title might be "Dirichlet's Principle." Here some properties of harmonic functions are established. Dirichlet's principle is proved for a circular area and then, by following a method due to Poincaré, the principle is established for the general case. Since, as the author points out, a "higher standard of sophistication" is needed to follow a proof of Dirichlet's principle, it might have been preferable to use this sophistication on some theorems of Tonelli about direct methods in general.

There are thirty-five exercises at the end of the book that would keep a student very busy.

ALINE H. FRINK, Pennsylvania State University

Electronic Computers: Fundamentals, Systems, and Applications. Edited by Paul von Handel. Springer-Verlag, Vienna, and Prentice-Hall, Englewood Cliffs, N. J., 1961. 235 pp. $13.50.

The stated objective of this book is "to present an over-all view of various types of modern electronic computers" for "people . . . who have no special knowledge of computers." The main body of the work is by H. W. Gschwind, M. G. Jaenke and R. G. Tantzen of Holloman Air Force Base. There are chapters on digital computers, analog computers, digital differential analyzers and computing control systems.

In the chapter on digital computers such topics as over-all system organization, storage types, number systems, programming and applications are discussed. In the analog chapter one finds a discussion of presently available components, analog computer set-up and scaling. In these chapters there is a qualitative discussion of the error but no discussion of stability. In the case of the digital differential analyzer, an effort at error analysis is made but it does not seem to the reviewer to be adequate.

F. J. MURRAY, Duke University

Diophantine Approximations. By Ivan Niven. Wiley, New York, 1963. 68 pp. $5.00.

This self-contained monograph is an extension of the Hedrick lectures delivered by the author at the 1960 summer meeting of the MAA. It does not offer a complete survey of the field but is confined to the following topics: basic results on homogeneous and nonhomogeneous approximations of real numbers and analogous results for complex numbers; asymmetric approximation of irrational numbers; fundamental properties of the multiples of an irrational number, for both fractional and integral parts. A unique feature of this monograph is that continued fractions are not used. The inclusion of basic results for complex numbers is noteworthy, as well as the presence of many new proofs offered here for the first time. An attractive feature is the inclusion of a section entitled "Further results" at the end of each chapter to provide a bibliographic account of closely related work. The author's exposition is concise and lucid and the monograph will be extremely helpful and informative to specialist and non-specialist alike.

W. E. BRIGGS, University of Colorado

BRIEF MENTION

Self-Organizing Systems. Edited by M. C. Yovits, G. T. Jacobi, and G. D. Goldstein. Spartan Books, Washington D. C., 1962. 563 pp. $12.00.

Proceedings of a 1962 conference of workers in several of the disciplines concerned with Self-Organizing Systems—Mathematics, Physics, Psychology, Biology, Embryology, Neurophysiology, etc.

Vector Analysis Including the Dynamics of a Rigid Body. By G. D. Smith. Oxford, New York, 1962. viii+192 pp. $4.00.

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Mathematical Optimization Techniques. By Richard Bellman, editor. University of California Press, 1963. xii+346 pp. $8.50.

Papers presented at the Symposium on Mathematical Optimization Techniques, Santa Monica, October 1960. The book is divided into four parts: I. Aircraft, Rockets, and Guidance; II. Communication, Prediction, and Decision; III. Programming, Combinatorics, and Design; IV. Models, Automation, and Control.

Angles and In- and Ex-Elements of Triangles and Tetrahedra. By Kesiraju Satyanarayana. Bangalore Press, Bangalore City, 1962. xiii+135 pp. 5 Rupees.

Journal of Research in Science Teaching, vol. 1, Issue 1. J. Stanley Marshall, editor. Wiley, New York, 1963. 98 pp.

Introductory Statistical Mechanics for Physicists. By D. K. C. MacDonald. Wiley, New York, 1963. ix+176 pp. $6.75.

Elementary "applied" statistical mechanics with emphasis on solid state phenomena.

Quantum Mechanics for Mathematicians and Physicists. By Ernest Ikenberry. Oxford University Press, New York, 1962. 269 pp. $8.00.

A fresh concise introduction to the elements of the theory emphasizing the mathematical aspects of its development. A useful section-by-section bibliography is included.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor H. L. Alder, University of California, Davis, represented the Association at the Convocation held as part of the Dedication of the California State College at Hayward on May 2, 1964.

Professor R. R. Stoll, Oberlin College, represented the Association at the dedication of the Charles F. Kettering Science Center at Ashland College on March 14, 1964.

Brigham Young University: Assistant Professors K. L. Hillam and L. J. Olpin have been promoted to Associate Professors; Mr. H. E. Wickes has been promoted to Assistant Professor; Assistant Professor H. G. Moore has been granted a leave of absence and awarded an NSF Science Faculty Fellowship for study at the University of California at Santa Barbara.

Assistant Professor L. A. Fiedler, Black Hawk College, has been promoted to Associate Professor and appointed Acting Head of the Mathematics Department.

Professor Karl Menger, Illinois Institute of Technology, has been appointed Visiting Professor for the spring semester at the University of Arizona.

Associate Professor Gloria Olive, Anderson College, has been promoted to Professor and Chairman of the Mathematics Department.

Professor Emeritus L. K. Adkins, Wisconsin State College, LaCrosse, died on November 11, 1963. He was a charter member of the Association.

Professor Emeritus J. E. Dotterer, Manchester College, died on January 21, 1964. He was a charter member of the Association.

Assistant Professor Corinne R. Hattan, University of Illinois, died on February 7, 1964. She was a member of the Association for 26 years.

Professor S. S. Wilks, Princeton University, died on March 8, 1964. He was a member of the Association for 23 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Spring meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at the United States Naval Academy, Annapolis, Maryland, on April 27, 1963. Professor Herta T. Freitag, Chairman of the section, presided. The invited address on "Men and Mathematics: a Plea for the Historical Sense in Mathematics," was delivered by Dr. Philip J. Davis, Chief Numerical Analysis Division, National Bureau of Standards, Washington, D. C., the recipient of the Chauvenet Prize, 1963.

At the business meeting the following officers were elected: Chairman, Dr. John W. Wrench, Jr., Applied Mathematics Laboratory, David Taylor Model Basin, Washington, D. C.; Vice Chairmen, Professor George H. Butcher, Howard University, Washington, D. C. and Professor Raymond W. Moller, Catholic University of America, Washington, D. C.; Secretary, Professor Samuel S. Saslaw, U. S. Naval Academy, Annapolis, Maryland; Treasurer, Professor Stanley B. Jackson, University of Maryland, College Park, Maryland.

The following program was presented:

1. *Tailgater, a simultaneous compiler*, by Professor H. Kaplan, U. S. Naval Academy.

2. *A theory of primes and Cramer's conjecture*, by Commander F. B. Correia, USN, U. S. Nava Academy.

3. *Convex metrics*, by Dr. Christoph Witzgall, National Bureau of Standards, Washington, D. C.

4. *Error analysis of the magnetic attitude prediction program for the Tiros satellites*, by W. H. Land, Jr., I.B.M. Corporation, Bethesda, Maryland.

5. *A least squares unit vector perpendicular to a given set of vectors*, by H. E. Castro, U. S. Naval Weapons Laboratory, Dahlgren, Virginia.

6. *Ship location by means of an acoustic range*, by Professor R. P. Bailey, U. S. Naval Academy.

7. *A theorem on convex programming*, by Dr. A. J. Goldman, National Bureau of Standards, Washington, D. C.

S. S. SASLAW, *Secretary*

DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Fall meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at American University, Washington, D. C. on Saturday, December 14, 1963. Dr. John W. Wrench, Jr., Chairman of the section, presided. Dr. F. Joachim Weyl, Deputy Chief and Chief Scientist, gave the invited address on "Elementary

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S. S. SASLAW, *Secretary*

FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The forty-first annual meeting of the Louisiana-Mississippi Section of the MAA was held in Biloxi, Mississippi, on February 14–15, 1964, with Northeast Louisiana College the host institution. There were 239 persons registered including 119 members of the Association.

The general session on Friday afternoon included the welcome and the response, the appointment of committees and the report of the Governor of the Section, Professor R. D. Anderson.

The technical papers were presented in two concurrent sessions with Professor W. E. Timon, Jr., Vice Chairman and Professor Wilson Davis, Vice Chairman, presiding. Professor Roy Sheffield presided at the three general sessions.

The following officers were elected for the coming year: Chairman, P. K. Rees, Louisiana State University; Vice Chairman for Mississippi, Russell Stokes, University of Mississippi; Vice Chairman for Louisiana, D. E. Dupree, Northeast Louisiana College; the term of Secretary-Treasurer, Z. L. Loflin, University of Southwestern Louisiana, extends to 1966.

The invited speakers for the meeting were Professor Alfred Willcox of CUPM and Professor Gail Young of Tulane University. Dr. Willcox spoke on the CUPM Pregraduate program and discussed better counselling for high school students. Professor Young's talk on Friday evening gave a comprehensive survey of CUPM Level I progress with an apprehensive look ten years in the future. His Saturday morning address was entitled "The Concrete and the Abstract" an abstract of which appears below.

The following papers were presented:

1. *On maps with nonnegative Jacobians*, by M. L. Marx of Tulane University.

THEOREM. *Let M and N be differentiable orientable N-Manifolds. Suppose that $F:M\to N$ is differentiable with nonnegative Jacobian. Define A to be the set of limit points of sequences $\{F(X_n)\}$ where $\{X_n\}$ has no limit point in M. Let P be the set in M where the Jacobian is nonzero. Then either P is empty or $F(P)$ is contained in A or $F(M)$ contains a component of $N-A$.*

2. *Some results concerning the (f, d_n) method of summability*, by Gaston Smith of University of Southern Mississippi.

The (f, d_n) method of summability was defined and two results concerning the regularity of the method were discussed. Then a special case of the method which provides an effective means for obtaining the analytic continuation of a power series was considered.

3. *Generation of multivariable orthogonal polynomials*, by D. E. Dupree of Northeast Louisiana State College.

4. *A computation method—numerical solution of partial differential equations*, by E. E. Moyers of University of Mississippi.

A method is given for treating a problem in potential theory which arises in the study of fluid flow through porous media. The iterative method permits computation of two-dimensional flow fields for use in the determination of invasion patterns when multiple sources and sinks are present.

5. *A numerical integration procedure*, by E. B. Anders of Northeast Louisiana State College.

6. *Analysis of trade between hostile systems*, by D. L. Brito of U.S.A.A., Fort Sill, Oklahoma.

7. *A note on the spectral theorem for matrices*, by J. D. Gilbert of Louisiana Polytechnic Institute.

8. *Commuting functions problem*, by Haskell Cohen of Louisiana State University.

The question of whether two continuous commuting functions on the unit interval to itself have a common fixed point or not is considered (two functions f and g commute if $f[g(x)]=g[f(x)]$ for all x). Some of the elementary properties of such functions are exhibited. It is shown that if there is a pair of commuting functions without a common fixed point, then there is a pair of onto functions with the same property. Finally it is shown that if the additional restriction that f and g be open maps is added, then f and g must have a common fixed point.

9. *The application of the Laplace transform to the problem of the tautochrone*, by W. H. Herbert of Louisiana Polytechnic Institute.

This problem is that of determining a curve through the origin in a vertical xy plane such that the time required for a particle to slide down the curve to the origin is independent of the starting position. The required curve is called the tautochrone. Equating gain in kinetic energy to loss of potential energy, we obtain a differential equation and solve it by the method of the Laplace transform. The curve is proved to be that of a cycloid.

10. *A programmed learning experience at Louisiana State University*, by J. E. Keisler.

LSU's two years of experience with the use of programmed learning materials for the teaching of remedial mathematics was described. In initial phases of the experience, with a great deal of faculty time and effort involved, the results were slightly better than those with the traditional text-book-lecture method. The more closely the desired goal of having the students use these materials on their own to become qualified for collegiate mathematics was approximated, the larger the percentages of students with unsatisfactory grades became. The text-book-lecture method is again in effect for remedial mathematics.

11. *The concrete and the abstract*, by Gail Young of Tulane University.

Primarily pedagogical, the paper is concerned with the necessity of introducing the concepts of abstract mathematics early in the undergraduate curriculum, so that the concepts become familiar and "concrete" to the student. Several places in the curriculum where such ideas can be introduced are given.

Z. L. LOFLIN, *Secretary*

FEBRUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The annual meeting of the Northern California Section of the MAA was held at Stanford University on February 1, 1964. Professor Lester H. Lange, San Jose State College, presided at the general sessions and the Vice-Chairman, Professor Daniel F. Coulter, Jr., Hartnell College, conducted the business meeting. There were 205 people registered for the meeting of whom 162 were members of the Association.

The following officers were elected: Professor Daniel F. Coulter, Jr., Chairman; Professor E. Maurice Beesley, University of Nevada, Vice-Chairman; Dr. Joel L. Brenner, Stanford Research Institute, Secretary-Treasurer; Professor Max Kramer, San Jose State College, Program-Chairman.

The invited address entitled "The Independence of the Axiom of Choice" was delivered by Professor Paul Cohen of Stanford University.

A presentation on "Future Goals for School Mathematics" was given by Professor E. G. Begle, Stanford University, followed by a panel discussion by Professor G. Polya, Stanford University; Professor C. M. Larsen, San Jose State College; Mrs. Sarah Herriot, Cubberley Senior High School; and Martin J. Dreyfuss, San Jose City College.

The following papers were presented:

1. *A Diophantine algorithm*, by Professor Dmitri Thoro, San Jose State College.

The Diophantine equation $x^2-xy-y^2=A$ has a solution in relatively prime integers x and y

if and only if (i) $A=5^eA'\neq 0$, where $e=0$ or 1 and (ii) if p is a prime factor of A', then $p\equiv 1$ or -1 (mod 10). For $|A|\leqq 4^k$, no more than $k+2$ unimodular transformations are required to obtain a solution. This algorithm was programmed in FORTRAN and run on an IBM 1620.

2. *A composite generator*, by R. C. Orr, Humboldt State College.

The function $f(k, n, r, s,)=$

$$(6n+k)\left[\frac{k(k+2)(k-2)(k-3)(6r+s)}{9k-3}+1\right]$$

in which n and r are natural numbers, s is 0 or -2, and k is 0, ± 1, ± 2, or 3, covers the composite natural numbers and never assumes a noncomposite value.

3. *Some new Fibonacci identities*, by Professor V. E. Hoggatt, Jr., San Jose State College.

It is well known that the Fibonacci numbers raised to powers satisfy linear recursion formulas. From a certain matrix of order three there can be derived Fibonacci Squares identities like the following:

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}F_{j+i}^2 = 5^nF_{2(n+j)+1} \qquad (n\geqq 0\ j \text{ any integer}).$$

Similar identities may be found by matrix methods for Fibonacci Fourth Powers. The author desires similar formulas for Fibonacci Cubes and higher powers.

4. *An algorithm for finding certain partial fraction expansions*, by Dr. Martin Blumberg, Stanford Linear Accelerator Center.

An algorithm, and some formulas derived therefrom, is presented for finding partial fraction expansions of algebraic expressions with multiple poles of finite order and with unit numerator. The coefficients of the linear factors turn out to be functions of the differences of the roots of the various terms multiplied by binomial coefficients (taken vertically rather than horizontally). If some of the factors have complex conjugate roots the computation is somewhat simplified; the form of their inverse Laplace transforms is easily identified. A method is given for expanding expressions with a polynomial numerator. The formulas are illustrated with numerical examples.

5. *A technique for solving a type of integral equation*, by Dr. Richard Bellman, RAND Corporation and Dr. Paul Brock, Hughes Aircraft Corporation.

The application of mathematical models to physiological processes frequently gives rise to Volterra equations of the form $u(t)=f(t)+\int_{T_0}^t k(t-s)u(s)ds$. These renewal equations arise in many other connections as well. The solution of this equation is expressed in terms of the solutions of linear systems of differential equations, derived from similar integral equations whose kernels are exponential forms. These auxiliary equations are obtained by means of differential approximation and quasilinearization.

B. J. LOCKHART, *Secretary*

ANNOUNCEMENT OF CHANGES IN THE 1964–65 COMBINED MEMBERSHIP LIST

The format used in listing members of AMS, MAA, and SIAM will differ in the next issue of the *Combined Membership List*, which will go to press in October, 1964. Not all the listings will be changed before the next issue, but new addresses and changes of address will be processed in the new format as they are received. The changes are:

1. The entry for a member will list his name, title, and place of employment, and only one address, the mailing address. *Both* the home and the business addresses will no longer be included.

2. Certain standard abbreviations will be introduced; e.g., the names of states in all state colleges and universities will be abbreviated.

3. ZIP code numbers will be added to the addresses.

These changes are being undertaken to reduce the size and production cost of the *Combined Membership List.*

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CALENDAR OF FUTURE MEETINGS

Forty-fifth Summer Meeting, University of Massachusetts, Amherst, August 24–26, 1964.

Forty-eighth Annual Meeting, Denver-Hilton Hotel, Denver, Colorado, January 28–30, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Southern Illinois University, Carbondale, May 14–15, 1965.
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 19–20, 1965.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN, University of Michigan, Ann Arbor, Michigan, March, 1965.
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY, Rutgers, The State University, New Brunswick, November 7, 1964.
NORTHEASTERN, Worcester Polytechnic Institute, Worcester, Massachusetts, November 28, 1964.
NORTHERN CALIFORNIA, College of San Mateo, California, February 6, 1965.
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Drexel Institute of Technology, Philadelphia, Pennsylvania, November 21, 1964.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont Men's College, California, March 13, 1965.
SOUTHWESTERN, Arizona State University, Tempe, Arizona, Spring, 1965.
TEXAS
UPPER NEW YORK STATE
WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN MATHEMATICAL SOCIETY, Amherst, Massachusetts, August 25–28, 1964.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Illinois Institute of Technology, Chicago, June 21–25, 1965.
ASSOCIATION FOR COMPUTING MACHINERY, Philadelphia, August 25–28, 1964
CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Detroit, November 26–28, 1964.
INSTITUTE OF MATHEMATICAL STATISTICS, Berne, Switzerland, September 14–16, 1964.
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Leamington Hotel, Minneapolis, August 20–21, 1964.
OPERATIONS RESEARCH SOCIETY OF AMERICA, Hotel Leamington, Minneapolis, October 7–9, 1964.
PI MU EPSILON, Amherst, Mass., August 25–26, 1964.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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THE AMERICAN MATHEMATICAL MONTHLY

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AUGUST-SEPTEMBER 1964

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10053.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

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A DIVISION ALGEBRA FOR SEQUENCES AND ITS ASSOCIATED OPERATIONAL CALCULUS

LOUIS BRAND, University of Houston

1. Introduction. Mikusiński [1] has developed an operational calculus which is essentially a division algebra in which the elements are the set of continuous, real or complex valued functions over the interval $0 \leqq t < \infty$. This set forms a commutative ring [2] with the operations

$$\textit{Addition}: \quad (a + b)(t) = a(t) + b(t).$$

$$\textit{Convolution}: \quad (ab)(t) = \int_0^t a(\tau)b(t - \tau)d\tau.$$

Titchmarsh's theorem [3] shows that there are no divisors of zero in this algebra. The ring may therefore be extended into a division algebra whose elements a/b are called *operators* [4]. Mikusiński then applies this algebra to the solution of both ordinary and partial differential equations. The calculations are formally very similar to those in which the Laplace transform is used; but the method is more general since it is free from convergence considerations and applies to equations such as $x' - x = 2(t-1)e^{t^2}$ [5] in which the right member is not transformable.

In applying operational methods to the solution of difference equations [6], however, Mikusiński uses a method which, in his words, "is not connected in any essential way with the operational calculus" [7]. The purpose of this paper is to set up a commutative ring in which the elements are sequences and the operations are the addition and convolution (Cauchy product) of sequences $\{f(t)\}$, over the integers 0, 1, 2, $\cdots$.

$$\textit{Addition}: \quad (a + b)(t) = a(t) + b(t)$$

$$\textit{Convolution}: \quad (ab)(t) = \sum_{\tau=0}^{t} a(\tau)b(t - \tau).$$

This ring proves to have no divisors of zero for the analog of Titchmarsh's theorem is readily proved. The ring may therefore be extended into a division algebra with unique identity elements. This algebra is then applied to the solution of difference equations in much the same way as Mikusiński's algebra is applied to differential equations.

This was first done by Josef Eliáš [8] who defined the convolution of sequences and constructed an operational calculus for sequences that closely parallels that of Mikusiński. His definition of convolution differs from that above in that the initial term is zero; his formulas are given in terms of the operator $1/\{1\}$ (the inverse of the sum operator) and are applied to solve difference equations in the Δ-form.

T. A. Newton [9] used the convolution of sequences in the form $(ab)(t)$ given above to obtain recurrence relations for the power series solution of linear

difference equations. Finally D. H. Moore [10] showed that this could be done by purely algebraic operations by defining the algebraic derivative of a sequence as $D\{f(t)\} = \{(t+1)f(t+1)\}$ and expressing some basic sequences in terms of the shift operator $s = \{0, 1, 0, 0, \cdots\}$.

The transition from differential equation to recurrence relation is facilitated when the equation is expressed in terms of the operator $\vartheta = xD$. Since $\vartheta(\vartheta-1) = x^2D^2$, the equation

$$(1 + x^2)y'' + 2xy' - 2y = 0,$$

considered by both Newton and Moore, becomes

$$\vartheta(\vartheta - 1)y + x^2(\vartheta + 2)(\vartheta - 1)y = 0.$$

This discloses the indices $\lambda_1 = 0$, $\lambda_2 = 1$ and the solutions

$$y_1 = \sum_{n=0}^{\infty} c_n x^{2n},$$

where $(2n-1)c_n + (2n-3)c_{n-1} = 0$ and $y_2 = x$.

We shall also express sequences in terms of the shift operator s. In a formula such as $\{a(t)\} = A(s)$, $A(s)$ is precisely the generating function of the sequence in the sense of Laplace,

$$A(s) = \sum_{t=0}^{\infty} a(t)s^t,$$

but in which convergence no longer poses a germane question. Thus this operational calculus gives the well-known formulas for generating functions a new interpretation just as the calculus of Mikusiński reinterprets Laplace transforms —in both cases without reference to convergence.

2. Commutative ring. Let $\mathcal{S}$ be the set of real or complex valued sequences defined over the integers $t = 0, 1, 2, \cdots$. With operations of addition and convolution the elements form a commutative ring satisfying the six postulates. (We parallel Mikusiński's notation in using single letters to denote the elements (sequences) of our algebra):

A_1: $a + b = b + a$;

A_2: $(a + b) + c = a + (b + c)$;

A_3: *Given* a, b, *there exists an element* x *such that* $a + x = b$;

M_1: $ab = ba$;

M_2: $(ab)c = a(bc)$;

D: $a(b + c) = ab + bc$.

M_1 and D are obvious and M_2 is easily proved. From A_1, A_2, A_3 the unicity of x and the existence and unicity of the identity element for addition may be de-

duced. For the set $\mathcal{S}$ it is the zero sequence $\{0, 0, 0, \cdots\}$.

We now show that this ring has no divisors of zero.

THEOREM. *The convolution of two sequences is zero when and only when one of the sequences is zero.*

Proof. Let $c=ab=0$. If $a(t)\equiv 0$ there is nothing to prove. Therefore let $a(n)\neq 0$ be the first nonzero element of $\{a(t)\}$. Then

$$c_n = \sum_{\tau=0}^{n} a(\tau)b(n-\tau) = a(n)b(0) = 0 \qquad b(0) = 0,$$

$$c_{n+1} = \sum_{\tau=0}^{n+1} a(\tau)b(n+1-\tau) = a(n)b(1) = 0 \qquad b(1) = 0,$$

and in general, by induction, $b(t)\equiv 0$.

3. Quotient field. From the commutative ring we now construct a division algebra whose elements are ordered pairs (a, b) with $b\neq 0$ and having the equivalence relation

(1) $$(a, b) = (c, d) \quad \text{iff} \quad ad = bc.$$

In particular for any nonzero sequence f,

(2) $$(a, b) = (af, bf), \qquad f \neq 0.$$

Addition and multiplication are defined as for fractions $(a, b)=a/b$:

(3) $$(a, b) + (c, d) = (ad + bc, bd),$$

(4) $$(a, b) \cdot (c, d) = (ac, bd).$$

Since b, $d\neq 0$, $bd\neq 0$ by the theorem above, and hence, the sum and product of the ordered pairs are elements of the field. The zero element is now $(0, f)$, $f\neq 0$; for from (3) and (2)

$$(a, b) + (0, f) = (af, bf) = (a, b).$$

These ordered pairs satisfy all six postulates of a commutative ring. Moreover they satisfy the postulate

M_3. *Given (a, b), (c, d) with $a\neq 0$ (as well as b, d, $\neq 0$), there exists a unique pair (x, y) such that $(a, b)\cdot(x, y)=(c, d)$.*

Proof. By hypothesis $ab\neq 0$, $ad\neq 0$. Hence

$$(a, b)\cdot(bc, ad) = (abc, bad) = (c, d)$$

by (2); thus we may take $(x, y)=(bc, ad)$. Moreover (x, y) is unique; for if $(a, b)\cdot(x, y)=(a, b)\cdot(u, v)$ or $(ax, by)=(au, bv)$ then $ax\,bv=by\,au$ from (1). Since $ab\neq 0$, $xv=yu$ or $(x, y)=(u, v)$.

If f is any nonzero sequence, (f, f) is the multiplicative identity.

4. Numerical operators. The sequences $\mathbf{k} = \{k, 0, 0, \cdots\}$ are isomorphic with the numbers k; for

$$\mathbf{k}_1 + \mathbf{k}_2 = \{k_1 + k_2, 0, 0, \cdots\},$$
$$\mathbf{k}_1\mathbf{k}_2 = \{k_1k_2, 0, 0, \cdots\}.$$

On occasion we may use boldface $\mathbf{k}$ to denote the above sequence; but ordinarily italic k will serve. Thus k times a sequence in the usual sense is the same as $\mathbf{k}$ times the sequence in the sense of our algebra.

We shall also write

$$k\{a\} \quad \text{for} \quad \{k, 0, 0, \cdots\}\{a(0), a(1), a(2), \cdots\}$$
$$k\{1\} \quad \text{for} \quad \{k, 0, 0, \cdots\}\{1, 1, 1, \cdots\}.$$

All powers of the sequence $\mathbf{1} = \{1, 0, 0, \cdots\}$ are $\mathbf{1}$ and $1\{a\} = \{a\}$. From (3) and (4)

$$(a, 1) + (b, 1) = (a + b, 1), \qquad (a, 1)\cdot(b, 1) = (ab, 1);$$

thus the pair $(f, 1)$ may be regarded as another notation for the sequence f; and we shall write $(f, 1) = f$. In particular the multiplicative identity $(f, f) = (1, 1) = \mathbf{1}$.

5. Fraction notation. If we write $(a, b) = a/b$ $(b \neq 0)$ equations (1) to (4) are the familiar rules for numerical fractions when multiplication is replaced by convolution. Any sequence $f(t)$ may be written as $f/1$; and the identity elements for addition and convolution are simply $0/1 = 0$ and $1/1 = \mathbf{1}$.

If $a = bc$ $(b \neq 0)$ we have $a/b = c/1 = c$, a sequence. But if a and b are arbitrary, a/b may not be a sequence; for example, if $a(0) \neq 0$, $b(0) = 0$, a/b is not a sequence c for $b(0)\ c(0) = 0$ cannot equal $a(0)$. We call the elements of our division algebra *operators*; they include, in particular, numbers and sequences.

6. Shift operator. The sequence

$$s = \{0, 1, 0, 0, \cdots\} \tag{5}$$

is called the *shift operator* on account of the convolution

$$s\{f(0), f(1), \cdots\} = \{0, f(0), f(1), \cdots\}$$

in which each element of f is shifted one step to the right. We have $s^2 = \{0, 0, 1, 0, 0, \cdots\}$, and in general

$$s^n = \{\underbrace{0, 0, \cdots, 0}_{n}, 1, 0, 0, 0, \cdots\}. \tag{6}$$

When $n = 0$ we define $s^0 = \{1, 0, 0, \cdots\}$ in agreement with (6). Thus any sequence $\{f(t)\}$ may be written as an infinite series

$$\{f(t)\} = \sum_{t=0}^{\infty} f(t)s^t = F(s). \tag{7}$$

The question of convergence is not germane to this notation, for (7) merely states that the term $f(n)$ of the sequence occupies the same place as the 1 in s^n.

From (7) we have

$$\{r^t f(t)\} = \sum_{t=0}^{\infty} f(t)(rs)^t = F(rs). \tag{8}$$

Moreover $\{tf(t)\} = \sum_{t=0}^{\infty} tf(t)\, s^t = s\sum_{t=0}^{\infty} f(t)\, ts^{t-1}$, or since the last sum is the formal derivative of the series $F(s)$ in (7),

$$\{tf(t)\} = sF'(s). \tag{9}$$

Formulas (8) and (9) have been proved when $F(s)$ is a simple series as in (7). These formulas also hold when $F(s) = A(s)/B(s)$, the quotient of two simple series. For if

$$\{f(t)\} = \frac{\{a(t)\}}{\{b(t)\}} = \frac{A(s)}{B(s)} = F(s)$$

we have the convolution $\sum_{\tau=0}^{t} f(\tau)b(t-\tau) = a(t)$. The identity

$$\sum_{\tau=0}^{t} r^\tau f(\tau) r^{t-\tau} b(t-\tau) = r^t a(t) \quad \text{or} \quad \{r^t f(t)\}\{r^t b(t)\} = \{r^t a(t)\}$$

now implies that

$$\{r^t f(t)\} = \frac{\{r^t a(t)\}}{\{r^t b(t)\}} = \frac{A(rs)}{B(rs)} = F(rs). \tag{8'}$$

Moreover the identity

$$\sum_{\tau=0}^{t} \tau f(\tau)b(t-\tau) + \sum_{\tau=0}^{t} f(\tau)(t-\tau)b(t-\tau) = ta(t)$$

implies that $\{tf(t)\}\{b(t)\} + \{f(t)\}\{tb(t)\} = \{ta(t)\}$. Therefore

$$\{tf(t)\}B(s) + \frac{A(s)}{B(s)} sB'(s) = sA'(s),$$

$$\{tf(t)\} = s\,\frac{BA' - AB'}{B^2} = s\left(\frac{A}{B}\right)' = sF'(s). \tag{9'}$$

7. Sum and difference operators. The sequence

$$\sigma = \{1, 1, 1, \cdots\} \tag{10}$$

is called the *sum operator* since the convolution

$$\sigma f = \left\{ \sum_{\tau=0}^{t} f(\tau) \right\}. \tag{11}$$

The reciprocal of σ, namely

$$\delta = \frac{1}{\sigma} = \{1, -1, 0, 0, \cdots\} = 1 - s \tag{12}$$

is called the *difference operator* from the property

$$\delta f = \{f(0), \Delta f(0), \Delta f(1), \cdots\}.$$

Writing this $(1-s)\{f(t)\} = f(0) + s\{\Delta f(t)\}$, and putting $\Delta f(t) = f(t+1) - f(t)$, we get the important result

$$s\{f(t+1)\} = \{f(t)\} - f(0) \tag{13}$$

in which $f(0)$ is a *number*. In particular when $f(t) = 1$, $s\{1\} = \{1\} - 1$, or

$$\{1\} = \frac{1}{1-s}; \tag{14}$$

since $\{1\} = \sigma$, this also follows from (12). Again with $f(t) = r^t$, we have $sr\{r^t\} = \{r^t\} - 1$, or

$$\{r^t\} = \frac{1}{1-rs} \tag{15}$$

which also follows from (8) and (14).

From (13), $s\{f(t+2)\} = \{f(t+1)\} - f(1)$; hence

$$s^2\{f(t+2)\} = \{f(t)\} - f(0) - sf(1), \tag{16}$$

and in general

$$s^n\{f(t+n)\} = \{f(t)\} - f(0) - sf(1) - \cdots - s^{n-1}f(n-1). \tag{17}$$

The powers of σ are readily computed:

$$\sigma^2 = \sigma\{1\} = \sum_{\tau=0}^{t} 1 = \tau \Big|_{\tau=0}^{t+1} = \{t+1\},$$

$$\sigma^3 = \sigma\{t+1\} = \sum_{\tau=0}^{t} (\tau+1) = \frac{(\tau+1)^{(2)}}{2}\Big|_{\tau=0}^{t+1} = \frac{\{(t+2)^{(2)}\}}{2!},$$

and in general, by induction (cf. [11] for notation), $\sigma^{n+1} = \{(t+n)^{(n)}\}/n!$; or since, $\sigma = (1-s)^{-1}$

$$\{(t+n)^{(n)}\} = \frac{n!}{(1-s)^{n+1}}. \tag{18}$$

If we put $f(t)=t^{(n)}$ in (17) and note that $f(0)=f(1)=\cdots=f(n-1)=0$, we have

$$\{t^{(n)}\} = s^n\{(t+n)^{(n)}\} = \frac{n!s^n}{(1-s)^{n+1}}. \tag{19}$$

From (18) and (8) we have the useful result:

$$\{r^t(t+n)^{(n)}\} = \frac{n!}{(1-rs)^{n+1}}. \tag{20}$$

8. Operational formulas. Starting with $\{1\}=(1-s)^{-1}$ and using (9) we find successively

$$\{t\} = \frac{s}{(1-s)^2}, \qquad \{t^2\} = \frac{s(s+1)}{(1-s)^3}, \qquad \{t^3\} = \frac{s(s^2+4s+1)}{(1-s)^4}. \tag{21}$$

Again from these formulas and (8),

$$\{r^t\} = \frac{1}{1-rs}, \qquad \{tr^t\} = \frac{rs}{(1-rs)^2}, \cdots. \tag{22}$$

From (22)

$$\{e^{i\alpha t}\} = \frac{1}{1-se^{i\alpha}} = \frac{1-se^{-i\alpha}}{1-2s\cos\alpha+s^2},$$

and on taking real and imaginary parts,

$$\{\cos\alpha t\} = \frac{1-s\cos\alpha}{1-2s\cos\alpha+s^2}, \tag{23}$$

$$\{\sin\alpha t\} = \frac{s\sin\alpha}{1-2s\cos\alpha+s^2}. \tag{24}$$

When $\alpha=\pi/2$ these become

$$\left\{\cos\frac{\pi}{2}t\right\} = \frac{1}{1+s^2}, \tag{25}$$

$$\left\{\sin\frac{\pi}{2}t\right\} = \frac{s}{1+s^2}. \tag{26}$$

Also from (9)

$$\left\{t\cos\frac{\pi}{2}t\right\} = \frac{-2s^2}{(1+s^2)^2}, \tag{27}$$

$$\left\{t\sin\frac{\pi}{2}t\right\} = \frac{s(1-s^2)}{(1+s^2)^2}, \tag{28}$$

and hence

$$\left\{\frac{t}{2}\cos\frac{\pi}{2}t+\cos\frac{\pi}{2}t\right\}=\frac{1}{(1+s^2)^2},\tag{29}$$

$$\left\{\frac{t}{2}\sin\frac{\pi}{2}t+\frac{1}{2}\sin\frac{\pi}{2}t\right\}=\frac{s}{(1+s^2)^2}.\tag{30}$$

When the left members of equations (23) to (30) are multiplied by r^t, we must replace s by rs in the right members.

If s were a real or complex variable in equation (7), $F(s)$ would be the generating function of the sequence $\{f(t)\}$. Apart from considerations of convergence, formulas (8) and (9) are formal consequences of (7) and show that $F(rs)$ and $sF'(s)$ are generating functions of the sequences $\{r^tf(t)\}$ and $\{tf(t)\}$ respectively. With the aid of these results all the operational formulas were deduced from the generating function $(1-s)^{-1}$ of the sequence $\{1\}$. Consequently the formulas above have a double meaning: 1°, when s is the shift operator they express a sequence $\{f(t)\}$ as a rational function $F(s)$ in a division algebra; 2°, when s is a numerical variable, $F(s)$ is the generating function of $\{f(t)\}$.

9. Difference equations. To solve a difference equation of order n with constant coefficients

$$a_0y(t+n)+a_1y(t+n-1)+\cdots+a_ny(t)=f(t),$$

multiply by s^n and use (17) to put the left member in the form

$$(a_0+a_1s+\cdots+a_ns^n)y+G(s),$$

where $G(s)$ is a polynomial of degree $n-1$ at most and whose coefficients depend upon the first n terms of the sequence y given as initial conditions. Express $\{f(t)\}$ as $F(s)$ by means of the appropriate operational formulas, solve the equation for

$$y=\frac{s^nF(s)-G(s)}{a_0+a_1s+\cdots+a_ns^n},$$

decompose into partial fractions and interpret them as sequences in t. The method is illustrated in the following examples in which $sy(t+1)=y-y_0$, $s^2y(t+2)=y-y_0-sy_1$, are used to transform the left members and $y_n=y(n)$.

Example 1. $y(t+2)+y(t)=\sin(\pi/2)t$, $y_0=1$, $y_1=0$. Multiply by s^2; then

$$y-1+s^2y=\frac{s^3}{1+s^2},$$

$$\begin{aligned}y&=\frac{1+s}{1+s^2}-\frac{s}{(1+s^2)^2}\\&=\left\{\cos(\pi/2)\,t+\sin(\pi/2)t-\tfrac{1}{2}(1+t)\sin(\pi/2)t\right\}\end{aligned}$$

on using (25), (26) and (30).

therefore

$$y = \tfrac{1}{5}\{\sin(\pi/2)t\} + \tfrac{2}{5}\{\cos(\pi/2)t\} + \tfrac{3}{5}\{2^{t/2}\cos(\pi/4)t\} - \tfrac{4}{5}\{2^{t/2}\sin(\pi/4)t\},$$

$$y(t) = \tfrac{1}{5}(\sin(\pi/2)t + 2\cos(\pi/2)t) + (\tfrac{1}{5}2^{t/2})(3\cos(\pi/4)t - 4\sin(\pi/4)t).$$

References

1. Jan Mikusiński, Operational Calculus, Pergamon Press, London and New York, 1959.
2. ———; *ibid.* p. 342.
3. ———, *ibid.* part 1, chapter 2.
4. ———, *ibid.* part 1, chapter 3.
5. ———, *ibid.* p. 42, ex. 1. The equation should read $x'-x=(2t-1)e^{t^2}$.
6. ———, *ibid.* part 2, chapter 3.
7. ———, *ibid.* p. 164, 12.
8. Josef Eliáš, O Operátorovej Metóde Riesenia Diferenčyyeh Rovnik, Mat.-Fyz. Časopis, Sav. VIII, 4(1958) 203–227.
9. T. A. Newton, An application of the generating function to differential equations, this Monthly, 68(1961) 242–244.
10. D. H. Moore, Convolution products and quotients and algebraic derivatives of sequences, this Monthly, 69(1962) 132–138.
11. L. Brand, Binomial expansions in factorial powers, this Monthly, 67(1960) 953–957.

LAPLACE TRANSFORMS AND CANONICAL MATRICES

HARRY HOCHSTADT, Polytechnic Institute of Brooklyn

I. The purpose of this article is to demonstrate the intimate connection between systems of differential equations and canonical matrices. In particular it will be shown, using only some basic concepts of differential equations and linear algebra, that for every matrix a similarity transformation can be found which will put the given matrix into the Jordan canonical form. Subsequently some special cases will be treated, in particular normal matrices. The method leads to a canonical form naturally and is also a constructive method. Although the basic results are not new, it is believed that the treatment is original.

II. This section will summarize some of the basic facts of differential equations, Laplace transforms and linear algebra which will be used. The class of matrices to be discussed will be square $n\times n$ matrices, whose entries are complex numbers; such a matrix will be denoted by A. Its hermitian transpose or adjoint will be denoted by A^*, and is defined as the complex conjugate of the transpose of A, that is

$$A^* = \bar{A}^T.$$

The inner product of two vectors X and Z, with entries x_i and z_i will be given by

$$(Z, X) = \sum_{i=1}^{n} \bar{z}_i x_i.$$

therefore

$$y = \tfrac{1}{5}\{\sin(\pi/2)t\} + \tfrac{2}{5}\{\cos(\pi/2)t\} + \tfrac{3}{5}\{2^{t/2}\cos(\pi/4)t\} - \tfrac{4}{5}\{2^{t/2}\sin(\pi/4)t\},$$

$$y(t) = \tfrac{1}{5}(\sin(\pi/2)t + 2\cos(\pi/2)t) + (\tfrac{1}{5}2^{t/2})(3\cos(\pi/4)t - 4\sin(\pi/4)t).$$

References

1. Jan Mikusiński, Operational Calculus, Pergamon Press, London and New York, 1959.
2. ———; *ibid.* p. 342.
3. ———, *ibid.* part 1, chapter 2.
4. ———, *ibid.* part 1, chapter 3.
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6. ———, *ibid.* part 2, chapter 3.
7. ———, *ibid.* p. 164, 12.
8. Josef Eliáš, O Operátorovej Metóde Riesenia Diferenčyyeh Rovnik, Mat.-Fyz. Časopis, Sav. VIII, 4(1958) 203–227.
9. T. A. Newton, An application of the generating function to differential equations, this Monthly, 68(1961) 242–244.
10. D. H. Moore, Convolution products and quotients and algebraic derivatives of sequences, this Monthly, 69(1962) 132–138.
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$$A^* = \bar{A}^T.$$

The inner product of two vectors X and Z, with entries x_i and z_i will be given by

$$(Z, X) = \sum_{i=1}^{n} \bar{z}_i x_i.$$

We also know that $(Z, AX) = (A^*Z, X)$. The characteristic polynomial $\Delta(s)$ will be defined as the determinant of $sI - A$, that is

$$\Delta(s) = |sI - A|,$$

where I is the identity matrix. For an $n \times n$ matrix $\Delta(s)$ is a polynomial of degree n. For every zero of that polynomial, say s_i, it must be possible to find a row vector R_i such that

$$R_i A = s_i R_i.$$

The s_i and R_i are known as eigenvalues and eigenvectors respectively. We shall not make a distinction between row and column vectors in our notation, but a product like RA will make sense only if R is interpreted as a row vector, whereas AC, where C is a vector, will make sense only if C is a column vector.

If T is a nonsingular matrix then TAT^{-1} has the same characteristic polynomial $\Delta(s)$ as A, and the eigenvalues of A and TAT^{-1} must coincide.

Linear, homogeneous systems of differential equations with constant coefficients can be written in the form

$$x_i'(t) = \frac{dx_i(t)}{dt} = \sum_{j=1}^{n} a_{ij}x_j(t) \qquad (i = 1, 2, \cdots, n).$$

It is convenient to write such a system in the form

$$X'(t) = AX(t),$$

where A is a matrix with constant entries a_{ij}, and X a column vector with entries $x_i(t)$. $X'(t)$ has the entries $x_i'(t)$. It is known that such a system has precisely n linearly independent solutions, and that for every initial vector $X(0)$ a unique solution $X(t)$ exists for all t.

The Laplace transform $y(s)$ of a function $x(t)$ is defined by

$$\int_0^\infty x(t)e^{-st}dt = y(s).$$

The above integral exists for all $x(t)$ of exponential growth, if the real part of s is sufficiently large. Only such functions will be considered here. By integration by parts one can prove the following

$$\int_0^\infty x'(t)e^{-st}dt = sy(s) - x(0),$$

$$\int_0^\infty t^k e^{-s_a t}e^{-st}dt = \frac{k!}{(s + s_a)^{k+1}},$$

where s_a is fixed. It is also known that if

$$\int_0^\infty x_1(t)e^{-st}dt = \int_0^\infty x_2(t)e^{-st}dt$$

and $x_1(t)-x_2(t)$ is continuous then $x_1(t)=x_2(t)$. In other words the Laplace transform has a unique inverse in the space of continuous functions.

III. We now turn our attention to the system

$$X'(t) = AX(t) \tag{1}$$

with the initial condition $X(o)=X_0$. By multiplying the above by e^{-st} and integrating over the interval $0\leqq t<\infty$, and defining $Y(s)=\int_0^\infty X(t)e^{-st}dt$ we obtain

$$sY(s) - X_0 = AY(s). \tag{2}$$

This can be rewritten as an algebraic system

$$(sI - A)Y(s) = X_0. \tag{3}$$

The matrix $(sI-A)$ must have an inverse, $B(s)$, unless s is an eigenvalue, that is a zero of $\Delta(s)=|sI-A|$. Since $B(s)$ must satisfy

$$B(s)(sI - A) = I, \tag{4}$$

it follows that each entry in $B(s)$ must be a rational function of s, so that

$$B(s) = \left(\frac{p_{ij}(s)}{r_{ij}(s)}\right),$$

where $p_{ij}(s)$ and $r_{ij}(s)$ are suitable polynomials. If for some s at least one of the $r_{ij}(s)$ vanishes, $B(s)$ will fail to exist for that s. But $B(s)$ must exist for all s with the exception of the zeros of $\Delta(s)$. Therefore we can, without loss of generality, assume that

$$B(s) = \left(\frac{p_{ij}(s)}{\Delta(s)}\right). \tag{5}$$

From (4) we see that

$$\operatorname*{Lim}_{s\to\infty} B(s)\left(I - \frac{1}{s}A\right) = \operatorname*{Lim}_{s\to\infty} \frac{1}{s} I = 0$$

and it follows that all entries in $B(s)$ in (5) must be proper rational functions. If s_a is a zero of $\Delta(s)$ of multiplicity m_a, and there are l distinct zeros we can write

$$\Delta(s) = \prod_{a=1}^{l} (s - s_a)^{m_a}.$$

Accordingly, we can decompose $B(s)$ into partial fractions so that

$$B(s) = \sum_{a=1}^{l}\sum_{b=1}^{m_a} \frac{B_{a,b}}{(s - s_a)^b}, \tag{6}$$

where the $B_{a,b}$ are constant matrices.

Applying the inverse $B(s)$ in the form (6) to (3) shows that

$$Y(s) = \sum_{a=1}^{l} \sum_{b=1}^{m_a} \frac{B_{a,b}X_0}{(s - s_a)^b}. \tag{7}$$

One can now invert this Laplace transform to obtain the solution to the differential equation

$$X(t) = \sum_{a=1}^{l} \sum_{b=1}^{m_a} \frac{t^{b-1}e^{s_a t}}{(b-1)!} B_{a,b}X_0. \tag{8}$$

From (4) and (6) we obtain

$$\begin{aligned} &\sum_{a=1}^{l} \sum_{b=1}^{m_a} \frac{B_{a,b}[(s - s_a)I + (s_aI - A)]}{(s - s_a)^b} \\ &\quad = \sum_{a=1}^{l} \left\{ \frac{B_{a,m_a}(s_aI - A)}{(s - s_a)^{m_a}} + \sum_{b=1}^{m_a-1} \frac{B_{a,b}(s_aI - A) + B_{a,b+1}}{(s - s_a)^b} + B_{a,1} \right\} = I. \end{aligned} \tag{9}$$

For $m_a = 1$ the second inner summation vanishes. Since the right side of (9) has no singularities all singular terms on the left must vanish. Then

$$\begin{aligned} B_{a,m_a}(s_aI - A) &= 0 \\ B_{a,b}(s_aI - A) &= -B_{a,b+1} \qquad \begin{pmatrix} b = 1, 2, \cdots, m_a - 1 \\ a = 1, 2, \cdots, l \end{pmatrix} \end{aligned} \tag{10}$$

and incidentally $\sum_{a=1}^{l} B_{a,1} = I$. The latter will be of little consequence in the sequel, except as a possible check on computational accuracy.

We now examine (10) in more detail. From the first equation we deduce that every row of B_{a,m_a} must either vanish or be an eigenvector of A corresponding to s_a. If such an eigenvector can be found, say R_{a,m_a}, then B_{a,m_a-1} must contain a row R_{a,m_a-1} such that

$$R_{a,m_a-1}(s_aI - A) = -R_{a,m_a}.$$

This follows from equation (10) for $b = m_a - 1$. Similarly, we find rows in the remaining equations such that

$$\begin{aligned} R_{a,m_a}(s_aI - A) &= 0 \\ R_{a,b}(s_aI - A) &= -R_{a,b+1} \qquad \begin{pmatrix} b = 1, 2, \cdots, m_a - 1 \\ a = 1, 2, \cdots, l \end{pmatrix}. \end{aligned} \tag{11}$$

Thus we have found a system of m_a vectors satisfying (11).

It could happen however that B_{a,m_a} vanishes identically. Then

$$B_{a,m_a-1}(s_aI - A) = 0.$$

Here, as before, B_{a,m_a-1} either vanishes identically or contains an eigenvector. In that case we obtain a system similar to (11), but involving only $m_a - 1$ such equations.

In any case we claim that we can extract one or more such equations from (10) such that

$$
\text{(12)}\qquad
\begin{aligned}
&R^{(\tau)}_{a,n_\tau}(s_aI - A) = 0\\
&R^{(\tau)}_{a,b}(s_aI - A) = -R^{(\tau)}_{a,b+1}
\end{aligned}
\qquad
\begin{pmatrix} b = 1, 2, \cdots, n_\tau - 1\\ a = 1, 2, \cdots, l \end{pmatrix}
$$

and τ is an index running over the number of such systems contained in (10).

Next we shall show that the number of linearly independent row vectors in (10) which satisfy (12) is precisely m_a, that is, $\sum n_\tau = m_a$. That there cannot be more than m_a will be shown somewhat later. It will be shown that if there were more, s_a would have to have multiplicity higher than m_a. Suppose there are fewer than m_a. Recall that $\sum m_a = n$, the order of the original system of differential equations. It would follow that all rows of all $B_{a,b}$ are linearly dependent on fewer than n linearly independent vectors. In this case we could select an $X_0 \neq 0$ such that $B_{a,b}X_0 = 0$ for all a and b. Then from (8) we would find that $X(t) = 0$. But corresponding to a nonzero initial condition we cannot obtain a vanishing solution. Therefore the vectors satisfying (12) for all a, b, and τ are n in number and linearly independent.

We will now arrange these systems of vectors (12) first in order of a then in τ and then in b. Then we will form a matrix T with these as rows. T is nonsingular since its n rows are linearly independent. We will also define the row vectors δ_k to have 1 in the kth column and zeros elsewhere.

Next we observe that

$$
\text{(13)}\qquad
\begin{aligned}
\delta_1 TAT^{-1} &= R^{(1)}_{1,1}AT^{-1} = (s_1R^{(1)}_{1,1} + R^{(1)}_{1,2})T^{-1} = s_1\delta_1 + \delta_2\\
\delta_2 TAT^{-1} &= R^{(1)}_{1,2}AT^{-1} = (s_1R^{(1)}_{1,2} + R^{(1)}_{1,3})T^{-1} = s_1\delta_2 + \delta_3\\
&\cdot\\
&\cdot\\
\delta_k TAT^{-1} &= R^{(1)}_{1,k}AT^{-1} = (s_1R^{(1)}_{1,k} + R^{(1)}_{1,k+1})T^{-1} = s_1\delta_k + \delta_{k+1}\\
&\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \leqq k \leqq n_1 - 1)\\
&\cdot\\
\delta_{n_1} TAT^{-1} &= R^{(1)}_{1,n_1}AT^{-1} = s_1R^{(1)}_{1,n_1}T^{-1} = s_1\delta_{n_1},
\end{aligned}
$$

and so on for all rows. The matrix

$$
J = \left(\begin{array}{cccccc|ccc}
s_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots\\
0 & s_1 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots\\
\cdot & & & & & & & & \\
\cdot & & & & & 1 & & & \\
\cdot & & & & & & & & \\
0 & 0 & 0 & 0 & \cdots & s_1 & 0 & 0 & \cdots\\
\hline
0 & 0 & 0 & & \cdots & 0 & & & \\
0 & 0 & 0 & & & 0 & & & \\
\cdot & \cdot & \cdot & & \cdots & \cdot & & & \\
\cdot & \cdot & \cdot & & & \cdot & & & \\
\cdot & \cdot & \cdot & & & \cdot & & &
\end{array}\right)
$$

will yield (13) when applied to the vectors δ_k. It follows that

$$J = TAT^{-1},$$

where J has the general structure

$$J = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & M_p \end{bmatrix}, \tag{14}$$

in which each M is of the form

$$M_1 = \begin{bmatrix} s_1 & 1 & 0 & \cdots & 0 \\ 0 & s_1 & 1 & \cdots & 0 \\ 0 & 0 & s_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdot 1 \\ 0 & 0 & 0 & & s_1 \end{bmatrix}. \tag{15}$$

The form (14) is called the Jordan canonical form. For each system in (12) there will be one such block. Since several such systems may be contained in (10), there will be a number of blocks with s_a in a main-diagonal. But the total number of s_a in the main-diagonal cannot exceed m_a. Otherwise $\Delta(s) = |sI - J|$ would have s_a as a zero to a multiplicity higher than m_a. This fact was used previously.

Example:

$$A = \begin{bmatrix} 5 & -1 & 1 & 1 & 0 & 0 \\ 1 & 3 & -1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Delta(s) = |sI - A| = (s-4)^5(s-2)$$

$$B(s) = \frac{1}{\Delta(s)} \begin{bmatrix} (s-4)^3(s-3)(s-2) & -(s-4)^3(s-2) & (s-4)^2(s-2)^2 & (s-4)^2(s-2)^2 & 0 & 0 \\ (s-4)^3(s-2) & (s-5)(s-4)^3(s-2) & -(s-6)(s-4)^2(s-2) & -(s-6)(s-4)^2(s-2) & 0 & 0 \\ 0 & 0 & (s-4)^4(s-2) & 0 & (s-4)^3(s-2) & (s-4)^3(s-2) \\ 0 & 0 & 0 & (s-4)^4(s-2) & -(s-4)^3(s-2) & -(s-4)^3(s-2) \\ 0 & 0 & 0 & 0 & (s-4)^4(s-3) & (s-4)^4 \\ 0 & 0 & 0 & 0 & (s-4)^4 & (s-4)^4(s-3) \end{bmatrix}$$

$$B(s) = \frac{B_{1,5}}{(s-4)^5} + \frac{B_{1,4}}{(s-4)^4} + \frac{B_{1,3}}{(s-4)^3} + \frac{B_{1,2}}{(s-4)^2} + \frac{B_{1,1}}{s-4} + \frac{B_{2,1}}{s-2}$$

$B_{1,5} = B_{1,4} = 0$

$$B_{1,3} = \begin{pmatrix} 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_{1,2} = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_{1,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

$$B_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & -1/2 & 1/2 \end{pmatrix}$$

To construct T we select as its first three rows the first rows of $B_{1,1}$, $B_{1,2}$ and $B_{1,3}$ in that order. The next two rows are the fourth rows of $B_{1,1}$ and $B_{1,2}$ and the last row of T is the fifth row of $B_{2,1}$. A calculation now shows that

$$TAT^{-1} = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix};$$

$$0 = ((A - s_aI)X_{m_a-1}, (A - s_aI)X_{m_a-1}) + (m_a - 1)^2(X_{m_a}, X_{ma}).$$

From this we can conclude that for $m_a > 1$

$$(A - s_aI)X_{m_a-1} = 0.$$

By repeating this argument successively in (17) we find that $X_b = 0$ for $b > 1$. This shows that the solution in (16) contains only one nonvanishing element. This implies that all $B_{a,b}$ in (8) vanish for $b > 1$. Therefore the Jordan canonical form of A is diagonal.

This last result can be strengthened considerably. We require the following lemma; its proof is trivial.

LEMMA. *If X_a and X_b are eigenvectors of a normal matrix corresponding to distinct eigenvalues s_a and s_b then $(X_a, X_b) = 0$; that is X_a and X_b are orthogonal.*

The eigenvectors of A corresponding to a multiple eigenvalue can always be orthogonalized by the Gram-Schmidt process. This means that the transformation matrix T, which is composed of the eigenvectors of A can be so chosen as to be unitary. That is $T^*T = I$. We summarize these results in the following:

THEOREM. *If A is normal we can find a unitary matrix T such that TAT^* is diagonal.*

This work was supported by the NSF, under grant GP-165.

A PROOF OF THE PRIME NUMBER THEOREM

E. GROSSWALD, University of Pennsylvania

1. Introduction. Let $\pi(x)$ denote the number of primes not exceeding some real number x and define the symbol of asymptotic equivalence by stipulating that $f(x) \sim g(x)$ shall mean the same as $\lim_{x\to\infty} f(x)/g(x) = 1$. It is our purpose to give a proof of the following statement, known as

THE PRIME NUMBER THEOREM: $\pi(x) \sim x/\log x$.

The proof, while neither as short as Landau's [2], nor as elementary as the proofs of Selberg [3], Erdös [1], or Wright [5], seems to have the advantage of great clarity. Like Landau's proof, it uses only some easily established properties of the Riemann zeta function in the half plane Re $s \geqq 1$.

Let p stand for primes, n for natural integers and define, as usual, $\zeta(s)$ for $s = \sigma + it$ by

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_n n^{-s} \qquad \sigma > 1 \tag{1}$$

and by analytic continuation otherwise. We take for granted the following properties:

$$0 = ((A - s_aI)X_{m_a-1}, (A - s_aI)X_{m_a-1}) + (m_a - 1)^2(X_{m_a}, X_{ma}).$$

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$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_n n^{-s} \qquad \sigma > 1 \tag{1}$$

and by analytic continuation otherwise. We take for granted the following properties:

(2) $\zeta(s) - (s-1)^{-1}$ and $\dfrac{\zeta'}{\zeta}(s) + (s-1)^{-1}$ are analytic for $\sigma \geqq 1$;

(3) for $s = 1 + it$, $\zeta(s) \neq 0$ and $\zeta(s)$, $\dfrac{\zeta'}{\zeta}(s)$, $\dfrac{\zeta''}{\zeta}(s)$ are all $O(\log^\alpha t)$

$(0 < \alpha < \infty)$ as $t \to \infty$.

Here and in what follows, all logarithms, including $\log \zeta(s)$, are obtained by direct analytic continuation from the branch that is real for real $s > 1$. The first result of (2) (and much more) may be obtained from the Euler-Maclaurin sum formula. The first assertion of (3) follows (after de la Vallée-Poussin) from the observation that (1) leads, for $\sigma > 1$, to $|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geqq 1$, which is inconsistent, on account of (2), with the assumption that $\zeta(1+it) = 0$; the other assertions of (3) may be obtained essentially by the Euler-Maclaurin sum formula. The second assertion of (2) follows from the first, because of (1) and the first of (3). We shall also have to use the following two well-known lemmas:

LEMMA 1 (RIEMANN-LEBESGUE). *Let the function $f(t)$ be differentiable and absolutely integrable on* $(0, \infty)$, *then the improper integral* $J(y) = \int_0^\infty f(t)e^{ity}dt$ *converges for every real y and* $J(y) = o(1)$ *as* $y \to \infty$.

LEMMA 2 (TAUBERIAN). *Let $f(x)$ be positive and nondecreasing; if* $\int_1^x u^{-1}f(u)du \sim x(\log x)^{-1}$, *then* $f(x) \sim x(\log x)^{-1}$.

2. Sketch of the proof. From (1) with $\sigma > 1$ we obtain

$$\begin{aligned}
\log \zeta(s) &= -\sum_p \log(1 - p^{-s}) = -\sum_{n=2}^\infty \{\pi(n) - \pi(n-1)\} \log(1 - n^{-s}) \\
&= \sum_{n=2}^\infty \pi(n)\{\log(1 - (n+1)^{-s}) - \log(1 - n^{-s})\} \\
&= \sum_{n=2}^\infty \pi(n) \int_n^{n+1} \frac{d}{dx}(\log(1 - x^{-s}))dx \\
&= \sum_{n=2}^\infty \pi(n) s \int_n^{n+1} x^{-1}(x^s - 1)^{-1}dx = s\int_2^\infty x^{-1}(x^s - 1)^{-1}\pi(x)dx.
\end{aligned}$$

These formal operations are easily justified if $\sigma > 1$. After division by s, the right hand side is almost exactly the Mellin transform of $\pi(x)$. It actually *is* a Mellin transform, not quite of $\pi(x)$, but, as we shall show, of the closely related function $f(x) = \sum_{m=1}^\infty m^{-1}\pi(x^{1/m})$. The difference between $f(x)$ and $\pi(x)$ is comparatively small. If q is the greatest integer not exceeding $\log x/\log 2$, then, for $m > q$, $x^{1/m} < 2$ so that $\pi(x^{1/m}) = 0$; hence, $f(x) = \sum_{m=1}^q m^{-1}\pi(x^{1/m}) = \pi(x) + \sum_{m=2}^q m^{-1}\pi(x^{1/m})$. But

$$0 \leqq \sum_{m=2}^q m^{-1}\pi(x^{1/m}) \leqq \sum_{m=2}^q m^{-1}x^{1/m} \leqq \sum_{m=2}^q \tfrac{1}{2}x^{1/2} = \tfrac{1}{2}(q-1)x^{1/2} < (2\log 2)^{-1}x^{1/2}\log x,$$

so that

$$\pi(x) = f(x) + O(x^{1/2}\log x); \tag{4}$$

clearly, $f(x)=0$ for $x<2$. For later use we also note that $f(x)\leqq \pi(x)+x^{\frac{1}{2}}\log x$, so that, a fortiori,

$$f(x) < 2x. \tag{5}$$

In order to prove the Prime Number Theorem it is sufficient, therefore, to prove that $f(x)=x\ (\log x)^{-1}\cdot(1+o(1))$, because then the same is true of $\pi(x)$, the error term in (4) being of lower order than $o(x(\log x)^{-1})$.

Assuming for a moment that $\int_2^\infty x^{-1}(x^s-1)^{-1}\pi(x)dx=\int_1^\infty f(x)x^{-s-1}dx$, we have obtained so far that, for $\sigma>1$,

$$s^{-1}\log \zeta(s) = \int_1^\infty f(x)x^{-s-1}dx. \tag{6}$$

Equation (6) can be "solved" for $f(x)$, by the classical theorem on the inversion of Mellin transforms, which yields

$$f(x) = (2\pi i)^{-1}\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-1}x^s\log\zeta(s)ds, \tag{7}$$

valid for any $c>1$. And once we have found $f(x)$, our problem is completely solved by (4), which gives $\pi(x)$.

One may indeed attempt to evaluate the integral in (7) directly. By (2), $\log\zeta(s)=-\log\ (s-1)+\log\ h(s)$, with $\log\ h(s)=\log((s-1)\zeta(s))$ analytic in $\sigma\geqq 1$, and, by routine computations (almost identical to those that we shall perform here)

$$-(2\pi i)^{-1}\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-1}x^s\log\,(s-1)ds$$

is found to be equal to $x(\log x)^{-1}+o(x(\log x)^{-1})$. The proof that the "error term"

$$\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-1}x^s\log h(s)ds$$

is sufficiently small is not trivial. While one may now take even $c=1$ (because $\log h(s)$ stays analytic for $s=1+it$) Lemma 1 is not directly applicable; indeed, for $s=1+it$, (3) only shows that $s^{-1}\log h(s)=O(t^{-1}\log t)$. This is not sufficient to insure the absolute convergence of the integral, which would be the simplest way to show that the convergence is uniform with respect to x.

In order to avoid these difficulties, it is preferable to return for a moment to (6) for a very slight change which will insure the absolute convergence of the integrals involved. For that purpose, set $g(x)=\int_1^x u^{-1}f(u)du$; then $g(x)=0$ for $x<2$ (because $f(x)=0$ for $x<2$) and, by (5), $g(x)<\int_1^x u^{-1}(2u)du<2x$. Also, $g'(x)=x^{-1}f(x)$ and, after an integration by parts, the second member of (6) may be

rewritten as

$$\int_1^\infty g'(x)x^{-s}dx = g(x)x^{-s}\Big|_1^\infty + s\int_1^\infty g(x)x^{-s-1}dx.$$

For $\sigma > 1$, the integrated term vanishes and instead of (6) we obtain

$$s^{-2}\log\zeta(s) = \int_1^\infty g(x)x^{-s-1}dx \qquad (\sigma > 1); \tag{6'}$$

hence instead of (7) we obtain

$$g(x) = (2\pi i)^{-1}\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-2}x^s\log\zeta(s)ds \qquad (c > 1). \tag{7'}$$

If we replace $\log\zeta(s)$, as before, by $-\log(s-1)+\log h(s)$, then (7′) becomes

$$g(x) = I_1(x) + I_2(x), \tag{8}$$

where

$$I_1(x) = -(2\pi i)^{-1}\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-2}x^s\log(s-1)ds,$$

$$I_2(x) = (2\pi i)^{-1}\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-2}x^s\log h(s)ds.$$

We shall show first that $I_2(x) = o(x(\log x)^{-1})$; next, that $I_1(x) = x(\log x)^{-1} + o(x(\log x)^{-1})$. Then it follows from (8) that $g(x) = x(\log x)^{-1} + o(x(\log x)^{-1})$. Lemma 2 leads then to $f(x) = x(\log x)^{-1} + o(x(\log x)^{-1})$ and, on account of (4), the theorem will be proven.

3. Proof of the lemmas. For completeness, we indicate here the proof of the lemmas used.

Proof of Lemma 1. The function $f(t)$ being absolutely integrable, we can determine T_1 and ϵ_1 so that, for $T_1 \leqq T$, $\epsilon \leqq \epsilon_1$

$$\left|\int_T^\infty f(t)e^{ity}dt\right| \leqq \int_T^\infty |f(t)|\,dt < \tfrac{1}{3}\eta, \quad \left|\int_0^\epsilon f(t)e^{ity}dt\right| \leqq \int_0^\epsilon |f(t)|\,dt < \frac{\eta}{3}$$

for arbitrarily small $\eta > 0$. Next, keeping T and ϵ fixed and integrating by parts, we obtain

$$\left|\int_\epsilon^T f(t)e^{ity}dt\right| = \left|(iy)^{-1}\left\{e^{ity}f(t)\Big|_\epsilon^T - \int_\epsilon^T f'(t)e^{ity}dt\right\}\right|$$

$$\leqq y^{-1}\left\{|f(T)| + |f(\epsilon)| + \int_\epsilon^T |f'(t)|\,dt\right\} < \tfrac{1}{3}\eta$$

for sufficiently large y and the lemma follows.

Proof of Lemma 2. Given any $\epsilon > 0$, set $y = x(1+\epsilon)$ and, for sufficiently large x, consider the quantity $\phi = \int_x^y u^{-1} f(u) du$. On the one hand, by hypothesis,

$$(1 - \epsilon^2)x(\log x)^{-1} < \int_1^x u^{-1} f(u) du < (1 + \epsilon^2)x(\log x)^{-1},$$

so that

$$\phi = \int_1^y u^{-1} f(u) du - \int_1^x u^{-1} f(u) du < (1 + \epsilon^2)y(\log y)^{-1} - (1 - \epsilon^2)x(\log x)^{-1}$$
$$< (\log x)^{-1}\{(1 + \epsilon^2)y - (1 - \epsilon^2)x\} = x(\log x)^{-1}\{(1 + \epsilon^2)(1 + \epsilon) - (1 - \epsilon^2)\}$$
$$= x(\log x)^{-1}\epsilon(1 + \epsilon)^2;$$

on the other hand, by the monotonicity of $f(x)$, $\phi = \int_x^y u^{-1} f(u) du \geqq f(x) \int_x^y u^{-1} du = f(x) \log (y/x) = f(x) \log (1+\epsilon)$. Hence,

$$f(x) \leqq \phi/\log(1 + \epsilon) < x(\log x)^{-1}\{\epsilon(1 + \epsilon)^2/\log (1 + \epsilon)\} \leqq x(\log x)^{-1}(1 + \epsilon)^3$$

and $(x^{-1}\log x)f(x) < (1+\epsilon)^3$ for arbitrarily small $\epsilon > 0$; similarly, one shows that $(x^{-1}\log x)f(x) > (1-\epsilon)^3$ for arbitrarily small $\epsilon > 0$, if only x is large enough and this finishes the proof of the Lemma.

4. Proof of the theorem. It only remains to fill in the details of the different steps sketched without proofs in Section 2.

(a) *Proof of* (6). From $f(x) = \sum_{m=1}^{\infty} m^{-1}\pi(x^{1/m}) = \sum_{m=1}^{q} m^{-1}\pi(x^{1/m})$ it follows that

$$\int_1^\infty f(x)x^{-s-1}dx = \int_1^\infty \sum_{m=1}^{q} m^{-1}\pi(x^{1/m}) \cdot x^{-s-1}dx = \sum_{m=1}^{\infty} \int_1^\infty m^{-1}\pi(x^{1/m})x^{-s-1}dx,$$

because of the uniform convergence of the series. In each integral we make the change of variable $x = y^m$, obtaining

$$\int_1^\infty f(x)x^{-s-1}dx = \sum_{m=1}^{\infty} \int_1^\infty \pi(y)y^{-ms-1}dy = \int_1^\infty \left(\sum_{m=1}^{\infty} \pi(x)x^{-ms-1} \right) dx$$
$$= \int_1^\infty x^{-1}(x^s - 1)^{-1}\pi(x)dx,$$

the termwise integration being again justified by the uniform convergence of the series (for $1 \leqq x < \infty$, and constant $\sigma > 1$).

(b) *Estimation of* $I_2(x)$. By Cauchy's theorem on residues,

$$I_2(x) = (2\pi)^{-1} \lim_{T\to\infty} \int_{-T}^{+T} (1 + it)^{-2}x^{1+it}g_1(t)dt,$$

where $g_1(t) = \log h(1+it)$. Hence, setting

$$I_3(y) = \lim_{T\to\infty} \int_{-T}^{+T} (1+it)^{-2} g_1(t) e^{ity} dt, \qquad I_2(x) = (2\pi)^{-1} x I_3(\log x).$$

Integrating by parts, we obtain

$$I_3(y) = \lim_{T\to\infty} \left\{ (iy)^{-1} e^{ity}(1+it)^{-2} g_1(t) \Big|_{-T}^{+T} - (iy)^{-1} \int_{-T}^{+T} (1+it)^{-3} g_2(t) e^{ity} dt \right\}$$

with

$$\begin{aligned} g_2(t) &= (1+it) g_1'(t) - 2i g_1(t) \\ &= i\left\{ s\left((s-1)^{-1} + \frac{\zeta'}{\zeta}(s) \right) - 2 \log ((s-1)\zeta(s)) \right\}_{s=1+it}. \end{aligned}$$

Using (3), we see that $g_1(t)$ and $g_2(t)$ are both differentiable for real t,

$$g_1(t) = \log \{(s-1)\zeta(s)\}_{s=1+it} = \log t + O(\log \log t),$$

and $g_2(t) = o\ (t \log^\alpha t)$ for $t \to \infty$. Hence, the integrated term of $I_3(y) \to 0$ as $T \to \infty$ and Lemma 2 is applicable to the last integral. It follows, as claimed, that

$$I_3(y) = o(y^{-1}) \quad \text{and} \quad I_2(x) = (2\pi)^{-1} x I_3(\log x) = o(x(\log x)^{-1}).$$

(c) *Computation of* $I_1(x)$. In $I_1(x)$ we move the line of integration to $c=1$, with a small semi-circular indentation Γ around the singularity $s=1$. This is permitted by Cauchy's theorem on residues, because the integrand goes to zero as $t \to \infty$ and has no singularities for $\sigma \geqq 1$, $s \neq 1$. Hence,

$$-2\pi i I_1(x) = \lim_{T\to\infty} \left\{ \int_{1-iT}^{1-i\eta} + \int_\Gamma + \int_{1+i\eta}^{1+iT} s^{-2} x^s \log (s-1) ds \right\}.$$

The contribution of the integral along Γ can be made arbitrarily small, by taking η sufficiently small. Indeed,

$$\begin{aligned} \left| \int_\Gamma s^{-2} x^s \log (s-1) ds \right| &= \left| \int_{-\pi/2}^{\pi/2} (1+\eta e^{i\theta})^{-2} x^{1+\eta e^{i\theta}} \log (\eta e^{i\theta}) \cdot \eta i e^{i\theta} d\theta \right| \\ &\leqq (1-\eta)^{-2} x^{1+\eta} \cdot \eta \int_{-\pi/2}^{\pi/2} \left| \log (\eta e^{i\theta}) \right| d\theta \\ &\leqq (1-\eta)^{-2} x^{1+\eta} \eta \int_{-\pi/2}^{\pi/2} (\log \eta^{-1} + |\theta|) d\theta \\ &= (1-\eta)^2 x^{1+\eta} \eta \{ \pi \log \eta^{-1} + \tfrac{1}{4}\pi^2 \} \to 0 \end{aligned}$$

as $\eta \to 0$. Hence,

$$\lim_{T\to\infty} \int_{c-iT}^{c+iT} s^{-2} x^s \log (s-1) ds = ix \lim_{\eta\to 0} \lim_{T\to\infty} \left\{ \int_{-T}^{-\eta} + \int_\eta^T (1+it)^{-2} x^{it} \log (it) dt \right\},$$

and $I_1(x)$ becomes $-(2\pi)^{-1} x I(\log x)$, with

lows from Lemma 2 that $f(x)=x(\log x)^{-1}+o(x(\log x)^{-1})$ and, on account of (4) the proof is complete.

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References

1. P. Erdös, Proc. Nat. Acad. Sci., 35 (1949) 374–384.
2. E. Landau, Sitzungsberichte, Preuss. Akad. Wiss., (1932) 514–521.
3. A. Selberg, Ann. of Math., 50 (1949) 305–313.
4. E. C. Titchmarsh, The Theory of the Riemann Zeta Function, Clarendon Press, Oxford, 1951.
5. E. M. Wright, Proc. Roy. Soc. Edinburgh, 6 (1952) 257–267.

ON GENERATORS AND DEFINING RELATIONS FOR THE UNIMODULAR GROUP $\mathfrak{M}_2$

GEORGE K. WHITE, University of Toronto

1. Introduction. In this paper, $\mathfrak{M}_n$ ($n=2, 3, \cdots$) shall denote the unimodular group of all real $n\times n$ matrices with integral entries and determinant ± 1.

Burrowes Hunt [5; 6] has for some time interested himself and his student Beldin in the unimodular groups, and the latter [1] has given in his thesis (unpublished) an expression which yields for any given n two elements $A, B\in\mathfrak{M}_n$ which suffice to generate $\mathfrak{M}_n$. Trott [7] has since deduced independently another pair of generators U_2, U for $\mathfrak{M}_n$. Two other recent papers of interest, kindly brought to my attention by Professor Hunt, are one by Brenner [2] (who finds yet another pair of generators S, T' of $\mathfrak{M}_n$), and one by Sze-Chien Yien [8] who gives a set $\{B_{ij}, U\}$ of more than two generators, and goes on to find complete defining relations for all $\mathfrak{M}_n$, using the sets $\{B_{ij}, U\}$.

This brief bibliography by no means exhausts the literature: there are, for instance, further references in [3], chapter 7. In this paper we restrict our attention to the particular group $\mathfrak{M}_2$, and the Beldin generators $\{A, B\}$. Our goal is to supply a simple abstract 2-generator definition of $\mathfrak{M}_2$ by providing suitable defining relations for A, B. We achieve this in sections 2 and 3 by first imposing relations (3) on the symbols A, B to define an abstract group $\mathfrak{H}$, and then proving that $\mathfrak{H}\simeq\mathfrak{M}_2$ (Lemma 1) utilizing results set forth in [3]. In section 4, we make use of the Fibonacci numbers to deduce and state some further simple relations between A, B, and to find concise expressions for $Z=-E$ (where E denotes the identity), in terms of A, B.

I should like to thank Professor Coxeter for his interest and help in connection with this paper. I am also grateful to Professor Hunt for, among other things, providing us with material from Beldin's thesis.

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References

1. P. Erdös, Proc. Nat. Acad. Sci., 35 (1949) 374–384.
2. E. Landau, Sitzungsberichte, Preuss. Akad. Wiss., (1932) 514–521.
3. A. Selberg, Ann. of Math., 50 (1949) 305–313.
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2. Two lemmas. Consider the abstract group $\mathfrak{G}$ defined by

$$\mathfrak{G} = \{R_1, R_2, R_3\},$$

$$\begin{cases} R_1^2 = R_2^2 = R_3^2 = E, \\ (R_1R_2)^3 = (R_1R_3)^2 = Z, \\ Z^2 = E. \end{cases} \tag{1}$$

It can be shown [**3**, pp. 85–87] that

$$\mathfrak{G} \simeq \mathfrak{M}_2,$$

under the correspondence

$$R_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad R_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It then follows that

$$Z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -E,$$

so that

$$g \in G \quad \text{implies} \quad gZ = Zg. \tag{2}$$

LEMMA 1. *Suppose that the abstract group $\mathfrak{H}$ is generated by two elements A, B with defining relations*

$$(A^{-1}B)^2 = (A^{-2}B^3)^2 = (A^{-4}B^2)^3 = E. \tag{3}$$

Then $\mathfrak{H} \simeq \mathfrak{M}_2$, under the correspondence

$$A \leftrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B \leftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

We remark that A and B, so identified, are the matrices shown by Beldin [**1**] to generate $\mathfrak{M}_n$, in the particular case $n=2$. We shall prove our lemma by utilizing the result quoted in [**3**, p. 2, ll. 1–6]. First, however, we alter slightly our definition (3) of $\mathfrak{H}$, and calculate the relations between R_1, R_2, R_3 and A, B.

LEMMA 2. *In our abstract definition for $\mathfrak{H}$*, (3) *may be replaced by*

$$(A^{-1}B)^2 = (A^{-2}B^3)^2 = (A^{-2}B^2)^6 = E. \tag{4}$$

Proof. Assume $(A^{-1}B)^2 = (A^{-2}B^3)^2 = E$ only. Then $A^{-1}B = B^{-1}A$, and

$$A^{-2}BA^{-2}B^2 = A^{-1}A^{-1}BA^{-1}A^{-1}BB = A^{-1}(B^{-1}AA^{-1}B^{-1}A)B = A^{-1}B(B^{-3}A^2)A^{-1}B.$$

Thus $(A^{-2}BA^{-2}B^2)^2 = E$, $A^2B^{-2}A^2 = BA^{-2}B^2A^{-2}B$. So

$$(A^{-2}B^2)^2 = B^{-1}BA^{-2}B^2A^{-2}BB = B^{-1}A^2B^{-2}A^2B,$$

and the following five relations are equivalent:

$$(A^{-2}B^2)^6 = E, \quad (B^{-1}A^2B^{-2}A^2B)^3 = E, \quad (A^2B^{-2}A^2)^3 = E,$$
$$A^2(B^{-2}A^4)^2B^{-2}A^2 = E, \quad (B^{-2}A^4)^3 = E.$$

The following expressions for R_1, R_2, R_3, originally derived with the help of some results of Beldin [1], may be verified by direct matric multiplication:

$$\text{(5)} \qquad \begin{cases} R_1 = A^{-1}B, \\ R_2 = A^{-1}B^{-1}A^2, \\ R_3 = A^{-1}B^2A^{-2}BA. \end{cases}$$

Using only (5), and $R_1^2=E$, we may next solve for A, B in terms of R_1, R_2, R_3:

$$\text{(6)} \qquad \begin{aligned} A &= BR_1^{-1}, \\ R_2 &= A^{-1}(B^{-1}A)A = A^{-1}R_1^{-1}A, \\ R_3 &= (A^{-1}B)(BA^{-1})(A^{-1}B)A = R_1BA^{-1}R_1A = R_1BR_2^{-1}. \end{aligned}$$

Therefore,

$$\text{(7)} \qquad B = R_1R_3R_2, \qquad A = R_1R_3R_2R_1,$$

and

$$\text{(8)} \qquad \text{(5) implies (7) when } R_1^2 = E.$$

3. Proof of Lemma 1. Since (3) is equivalent to (4) by Lemma 2, it is sufficient to show that (4) and (5) both hold if and only if both (1) and (7) hold [**3**, p. 2, ll. 1–6]. First, assume (1) and (7). Then,

$$\begin{aligned} A^{-1}B &= R_1R_2R_3R_1R_1R_3R_2 = R_1, \\ A^{-1}B^{-1}A^2 &= R_1R_2R_3(R_1R_2R_3R_1R_1R_3R_2R_1)(R_1R_3)R_2R_1 \\ &= R_1R_2R_3(E)(ZR_3R_1)R_2R_1 \\ &= ZR_1R_2R_1R_2R_1(R_2R_2) = Z^2R_2 = R_2, \\ A^{-1}B^2A^{-2}BA &= (A^{-1}B)B(R_2^{-1}) = R_1(R_1R_3R_2)R_2 = R_3. \end{aligned}$$

So (5) holds. Also, by (5) and (1),

$$\begin{aligned} (A^{-1}B)^2 &= R_1^2 = E. \\ A^{-2}B^3 &= A^{-1}(A^{-1}B)B^2 = A^{-1}R_1B^2 \\ &= R_1R_2(R_3R_1)R_1R_1R_3R_2(R_1R_3)R_2 \\ &= R_1R_2ZR_1R_3R_3R_2ZR_3R_1R_2 \\ &= (R_1R_2)^2R_3R_1R_2. \end{aligned}$$

So

$$\begin{aligned}(A^{-2}B^3)^2 &= (R_1R_2)^2R_3(R_1R_2)^3R_3R_1R_2\\ &= Z(R_1R_2)^2R_1R_2 = Z^2 = E.\\ A^{-2}B^2 &= A^{-1}(A^{-1}B)B = A^{-1}R_1B = R_1R_2(R_3R_1)R_1R_1R_3R_2\\ &= ZR_1R_2R_1R_3R_3R_2 = Z(R_1R_2)^2,\end{aligned}$$

$(A^{-2}B^2)^6 = Z^6(R_1R_2)^{12} = E$, and thus (4) is satisfied. Next, assume (4) and (5). We have

$$R_1^2 = (A^{-1}B)^2 = E,$$

and so by (8), (7) holds. Also, by (6),

$$\begin{aligned}R_2^2 &= (A^{-1}R_1^{-1}A)^2 = A^{-1}R_1^2A = E.\\ R_3^2 &= A^{-1}B^2A^{-2}B^3A^{-2}BA = A^{-1}B^{-1}(B^3A^{-2}B^3A^{-2})BA = E.\\ R_1R_2 &= A^{-1}BA^{-1}(B^{-1}A)A = A^{-1}BA^{-2}BA.\end{aligned}$$

$$\begin{aligned}(9)\qquad (R_1R_2)^3 &= A^{-1}B(A^{-2}B^2A^{-2})B^2A^{-2}BA = A^{-1}B(A^{-2}B^2)^3B^{-1}A\\ R_1R_3 &= A^{-1}BA^{-1}B^2A^{-2}BA\\ (R_1R_3)^2 &= A^{-1}B(A^{-1}B^2A^{-2}B^2A^{-1})B^2A^{-2}BA.\end{aligned}$$

But

$$A^{-1}B^2A^{-1} = (B^{-1}A)(AB^{-1}) = B^2(B^{-3}A^2)B^{-1} = B^2(A^{-2}B^3)B^{-1} = B^2A^{-2}B^2.$$

Hence $(R_1R_2)^3 = (R_1R_3)^2 = Z$, say. By (9),

$$Z^2 = A^{-1}B(A^{-2}B^2)^6B^{-1}A = A^{-1}BB^{-1}A = E.$$

Thus (1) is satisfied, and our proof of Lemma 1 is complete.

4. Further relations between the Beldin generators A, B; some expressions for Z. One might ask if there are simple relations between A, B other than those in (3), (4). It is well known that A and B are of infinite period and that in fact,

$$(10)\qquad A^m = B^n \quad\text{implies}\quad m = n = 0,$$

$$(11)\qquad A^m = \begin{bmatrix}1 & m\\ 0 & 1\end{bmatrix},\qquad B^n = \begin{bmatrix}f_{n+1} & f_n\\ f_n & f_{n-1}\end{bmatrix}\qquad (m, n = 0, \pm 1, \pm 2, \cdots),$$

where $\{f_n\}$ is the Fibonacci sequence defined by

$$(12)\qquad \begin{cases}f_1 = f_2 = 1\\ f_{n+2} = f_{n+1} + f_n\end{cases}\qquad (n = 0, \pm 1, \pm 2, \cdots).$$

The following remark provides additional simple relations for A, B, as well as giving some simple expressions for Z.

REMARK 1. In M_2, the only elements A^mB^n ($m\neq 0$, $n>0$) of finite period are the six given by

$$(13)\qquad \begin{cases}(A^{-1}B)^2 = (A^{-2}B^3)^2 = (A^{-4}B^2)^3 = E,\\ (A^{-3}B^2)^2 = (A^{-2}B^2)^3 = (A^{-2}B^4)^3 = Z,\ Z^2 = E.\end{cases}$$

We remark that the following relations are equivalent:

$$(A^mB^n)^r = E, \qquad (A^{-m}B^{-n})^r = E, \qquad (B^nA^m)^r = E, \qquad (B^{-n}A^{-m})^r = E,$$

and thus provide alternative formulations of the relations found in (13). We shall show also that no relations of the type $A^mB^nA^sB^t=E$ (and thus none of the type $B^nA^sB^tA^m=E$) are to be found, other than those given in (13).

REMARK 2. *If* $A^mB^nA^sB^t=E$ ($m\neq 0$, $n\neq 0$), *then* $s=m$, $t=n$.

Proof of Remarks. From (11),

$$(14)\qquad A^mB^n = \begin{bmatrix} f_{n+1}+mf_n & f_n+mf_{n-1}\\ f_n & f_{n-1}\end{bmatrix},$$

$$(15)\qquad B^lA^k = \begin{bmatrix} f_{l-1} & kf_{l+1}+f_l\\ f_l & kf_l+f_{l-1}\end{bmatrix}.$$

Suppose $A^mB^nA^sB^t=E$, and set $s=-k$, $t=-l$. Then $A^mB^n=B^lA^k$ and so by (14), (15), $f_l=f_n$, $f_{l+1}=f_{n+1}+mf_n$, $kf_l+f_{l-1}=f_{n-1}$. Thus $l=\pm n$, and $l=-n$, since otherwise $m=0$. Hence by (12),

$$(16)\qquad k = (f_{n-1}-f_{-n-1})/f_{-n} = -(f_{-n+1}-f_{n+1})/f_n = -m,$$

since $n\neq 0$. Thus our second remark is proved.

Next, let $A^mB^n=M$ ($m\neq 0$, $n>0$), and suppose M has period r. Denote by χ, Δ, the trace and determinant of M. Then $M\neq E$, $M\neq Z$ by (14), and a classical argument shows that either

$$(17)\qquad \Delta = -1 \quad \textit{and} \quad \chi = 0, \quad r = 2$$

or

$$(18)\qquad \Delta = 1 \quad \textit{and} \quad (\chi, r) = (1, 6), \quad (-1, 3), \quad \textit{or} \quad (0, 4).$$

For let ω_1, ω_2 be the characteristic roots of M. Then, by hypothesis, $\omega_1\omega_2=\Delta=\pm 1$, $\omega_1+\omega_2=\chi$ is real, and $\omega_1^r=\omega_2^r=1$ since $M^r=E$. Since $M\neq Z$ or E, (17) and (18) follow. Furthermore, if $r=6$, $\omega_1^3=\omega_2^3=-1$, and hence the matrix $(A^mB^n)^3$ of period 2 must actually be Z. Reasoning similarly for $r=4$ we obtain

$$(19)\qquad r = 4 \quad \text{or} \quad r = 6 \qquad \text{implies that } (A^mB^n)^{r/2} = Z.$$

Clearly

$$(20)\qquad \Delta = (-1)^n$$

and, by (14),

$$\chi = f_{n+1} + f_{n-1} + mf_n \qquad (m = \pm 1, \pm 2, \cdots; n = 1, 2, \cdots). \tag{21}$$

Thus any element A^mB^n ($m \neq 0$, $n > 0$) of period r is one satisfying (17)–(21); we note in particular that $|\chi| \leqq 1$ always by (17), (18).

We now show that (21) has no solution when $|\chi| \leqq 1$, unless $n \leqq 5$. If we define $u_n = f_{n-1}/f_n$ then (21) becomes, on rearranging,

$$u_n = \tfrac{1}{2}\chi f_n - \tfrac{1}{2}(m + 1) \qquad (m = \pm 1, \pm 2, \cdots; n = 1, 2, \cdots). \tag{22}$$

It is well known that the sequences $\{u_{2t}\}$ and $\{u_{2t+1}\}$ ($t = 1, 2, \cdots$) are monotonely decreasing and increasing respectively, with a common limit point as $t \to \infty$. Thus if $n \geqq 6$, $|\chi| \leqq 1$, $.615 < u_7 \leqq u_n \leqq u_6 = .625$ and $\frac{1}{2}|\chi| f_{n-1} \leqq \frac{1}{2} f_5 < .063$; when we substitute this information into (22) we get a contradiction. Thus the solution of (17)–(21) reduces to a simple enumeration of case (χ, n) with $\chi = 0$ and $n = 1, 3, 5$ or $\chi = \pm 1$, 0 and $n = 2, 4$. This enumeration establishes (13), thus completing the proof of our first remark.

5. A further remark. The subgroup of $\mathfrak{M}_2$ consisting of matrices of determinant $+1$ is denoted by $\mathfrak{M}_2^+$. Let A, B be the Beldin generators of $\mathfrak{M}_2$. Then it is well known [4] that $\mathfrak{M}_2^+$ is generated by A and

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = AB^{-2}A^2,$$

with defining relations $S^2 = -E$, $(SA)^3 = E$. We note that these are equivalent to the relations $(A^{-3}B^2)^2 = Z$, $(A^{-4}B^2)^3 = E$ found in (13).

6. Another pair of generators for $\mathfrak{M}_2$. Closely related to the Beldin generators A, B for $\mathfrak{M}_2$ are the pair R, U mentioned in [3, p. 88]. In fact,

$$A = \overset{-1}{R_1} U R_1, \qquad B = \overset{-1}{R_1}\overset{-1}{R} R_1.$$

Hence Lemma 1 is immediate from Lemma 2, if we make use of the defining relations (7.29) given in [3]. We have retained our own proof of Lemma 1, however, to make the subject more self-contained.

References

1. D. Beldin, Thesis, Reed College, Oregon (1957, unpublished).

2. J. L. Brenner, Generators of the unimodular group and the restricted unimodular group, Rev. Ci. (Lima), 62 (1960) 20–23.

3. H. S. M. Coxeter, and W. O. J. Moser, Generators and Relations for Discrete Groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Heft 14, Springer Verlag, Berlin, 1957.

4. L. K. Hua, and I. Reiner, Automorphisms of the unimodular group, Trans. Amer. Math. Soc., 71 (1951) 331–348.

5. Burrowes Hunt, A Farey-dissection of the plane and the corresponding simultaneous-approximation algorithm, Bull. Amer. Math. Soc., 60 (1954) 476.

6. ———, Continued fractions and indefinite forms, *ibid.*, 62 (1956) 553.

7. S. M. Trott, A pair of generators for the unimodular group, Canad. Math. Bull. 5, (1962) 245–252.

8. Sze-Chien Yien, Defining relations of n-dimensional modular groups, Science Record (Peking) (N. S.), 4 (1960) 313–316.

THE POLYGONAL INEQUALITIES

D. M. SMILEY AND M. F. SMILEY, Riverside, California and
the University of California, Riverside

Introduction. For a unitary space X, the triangle inequality

$$T(x, y) = |x| + |y| - |x+y| \geqq 0, \tag{1}$$

and the quadrilateral inequality

$$|x| + |y| + |z| + |x+y+z| \geqq |x+y| + |y+z| + |z+x| \tag{2}$$

are known. (A unitary space is a vector space X over the complex field equipped with a positive definite sesqui-linear inner product (x, y), i.e., $(y, x) = \overline{(x, y)}$, $(x+y, z) = (x, z) + (y, z)$, $(\alpha x, y) = \alpha(x, y)$, $(x, x) = |x|^2$ is positive for x nonzero.) Let us define, for n vectors $x_1, \cdots, x_n$ in X,

$$S_k = \sum (|x_{i_1} + \cdots + x_{i_k}| : 1 \leqq i_1 < i_2 < \cdots < i_k \leqq n).$$

One might guess that since (2) is $S_1 + S_3 \geqq S_2$ that $\sum((-1)^k S_k; k = 1, \cdots, n) \leqq 0$. (Cf. T. Popovici, Colloq. Math., 3 (1955) 172.) For $n \geqq 4$, however, this guess is wrong. If we take $x_i = x_1 \neq 0$ $(i = 1, \cdots, n-1)$ and $x_n = -x_1$ then the left-hand member of this inequality becomes $|x_1|$ times

$$\begin{aligned}
&-n + 2C(n-1, 2) - C(n-1, 2) - 3C(n-1, 3) + 2C(n-1, 3) \\
&+ 4C(n-1, 4) + \cdots + (-1)^{n-1}[(n-3)C(n-1, n-2) + (n-1)] \\
&+ (-1)^n(n-2) \\
= &- n + C(n-1, 2) - C(n-1, 3) + \cdots + (-1)^{n-1}[n-1-n+2] \\
= &- n - 1 + n - 1 = -2,
\end{aligned}$$

where the C's are binomial coefficients. On the other hand, if we take $x_i = x_1 \neq 0$ $(i = 1, \cdots, n-1)$ and $x_n = -2x_1$, then the left-hand member of this inequality becomes $|x_1|$ times

$$\begin{aligned}
&-n - 1 + C(n-1, 1) + 2C(n-1, 2) - 0C(n-1, 2) - 3C(n-1, 3) \\
&+ C(n-1, 3) + 4C(n-1, 4) + \cdots + (-1)^{n-1}[(n-4)C(n-1, n-2) \\
&+ (n-1)] + (-1)^n(n-3) \\
= &- n - 1 + C(n-1, 1) + 2C(n-1, 2) - 2C(n-1, 3) + \cdots \\
&+ 2(-1)^{n-2}C(n-1, n-2) + (-1)^{n-1}[n-1-n+3] \\
= &- n - 1 + n - 1 - 2 + 2(n-1) = 2n - 6.
\end{aligned}$$

Thus, for $n \geqq 4$, neither the stated inequality nor its dual holds.

The purpose of this paper is to prove that for $k = 2, \cdots, n-1$,

$$C(n-2, k-1)S_1 + C(n-2, k-2)S_n - S_k \geqq 0. \tag{3}$$

We call (3) the *polygonal inequalities*. Summing the inequalities (3), we find that for $n \geqq 3$,

$$S_1 + S_n \geqq (S_2 + \cdots + S_{n-1})(2^{n-2} - 1)^{-1}. \tag{4}$$

To see the geometric meaning of (4), let Γ be the polygon of $n+1$ sides whose vertices P_i are the end-points of the vectors $\sum(x_j, j=1, \cdots, i)$ and the origin. The origin and the end-points of the vectors $x_{i_1} + \cdots + x_{i_k}$ for $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$ and $k=1, \cdots, n$ form the vertices of what may be called (by stretching our language a bit) the parallelepiped Δ spanned by $x_1, \cdots, x_n$. The *vectors* $x_{i_1} + \cdots + x_{i_k}$ for $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$ and $k=2, \cdots, n$ themselves may then be called the diagonals of Δ issuing from the origin. Then the perimeter of Γ is not less than the fraction $(2^{n-2}-1)^{-1}$ times the sum of the lengths of all possible diagonals of Δ issuing from the origin, except the main diagonal of $\Delta: x_1 + \cdots + x_n$.

Our method of proof permits a relatively succinct description of the conditions for equality in (3). We then turn to a discussion of the logical relations amongst (1), (2), and (3). Let X be a real (or complex) linear space in which $|x|$ is a real number for each x in X. If (2) holds in X, then so do (1) and (3). But even if X is a normed linear space (so that (1) holds), (2) may fail. There is, then, a class of normed linear spaces, which we call *quadrilateral spaces* for which (2) (and hence also (3)) is valid. Our paper concludes with a statement of the meager information we have been able to derive so far about these quadrilateral spaces.

We wish to express our appreciation to Professor Victor L. Klee for bringing our attention to this problem. We are also indebted to a referee for several helpful suggestions.

We will be amazed to learn that our simple observations about Euclidean geometry have escaped notice until this time. Even if our results are not new, we feel that our exposition is not without interest.

1. The triangle inequality. Let us begin by proving (1). We have

$$\begin{aligned} T(x,y) \geqq 0 \text{ iff } (|x| + |y|)^2 \geqq |x+y|^2 &\text{ iff } 2|x|\,|y| \geqq (y,x) + (x,y) \\ &\text{iff } 2|x|^2|y|^2 \geqq (|x|y, |y|x) + (|y|x, |x|y) \\ &\text{iff } (|x|y - |y|x, |x|y - |y|x) \geqq 0. \end{aligned}$$

By suppressing the inequality signs in these equivalences we see that $T(x, y) = 0$ iff $|y|x = |x|y$.

2. The quadrilateral inequality. Now we may easily prove (2). With $P(x, y) = |x| + |y| + |x+y|$ we note that $T(x, y)P(x, y) = 2|x|\,|y| - (x, y) - (y, x)$. One then sees that

$$S_1^2 - S_3^2 = T(x,y)P(x,y) + T(y,z)P(y,z) + T(z,x)P(z,x).$$

Since S_1+S_3, being the perimeter of the quadrilateral Γ, is at least as large as $P(x, y)$, $P(x, z)$, $P(y, z)$; it follows that

$$S_1 - S_3 \leqq T(x, y) + T(y, z) + T(z, x) = 2S_1 - S_2.$$

This proves (2).

3. The polygonal inequalities. After the preliminaries of Sections 1 and 2, we are ready to prove our result (3). We begin with the identity

$$(5) \qquad (S_1 - S_n)(S_1 + S_n) = \sum (T(x_i, x_j)P(x_i, x_j) \colon 1 \leqq i < j \leqq n).$$

Since, by the triangle inequality (1),

$$S_n + |-x_{k+1}| + \cdots + |-x_n| \geqq |x_1 + \cdots + x_k|,$$

we see that

$$(6) \qquad S_1 + S_n \geqq |x_1| + \cdots + |x_k| + |x_1 + \cdots + x_k|.$$

Note that

$$(7) \qquad \left(\sum_{r=1}^{k} |x_r| + \left|\sum_{r=1}^{k} x_r\right|\right)\left(\sum_{r=1}^{k} |x_r| - \left|\sum_{r=1}^{k} x_r\right|\right) = \sum (T(x_i, x_j)P(x_i, x_j) \colon 1 \leqq i < j \leqq k).$$

We now assume (as we may) that $S_1+S_n \neq 0$ and infer that

$$(8) \qquad S_1 - S_n \leqq \sum_{r=1}^{k} |x_r| - \left|\sum_{r=1}^{k} x_r\right| + (S_1 + S_n)^{-1} \sum (T(x_i, x_j)P(x_i, x_j) \colon 1 \leqq i < j,\ k < j \leqq n).$$

There are $C(n, k)$ ways of selecting k vectors from the original n vectors $x_1, \cdots, x_n$. For each such selection an analogue of (8) is valid. Adding these $C(n, k)$ inequalities yields

$$(9) \qquad C(n, k)(S_1 - S_n) \leqq (k/n)C(n, k)S_1 - S_k + C(S_1 - S_n),$$

where $C = [C(n, 2) - C(k, 2)]C(n, k)(C(n, 2))^{-1}$. A minor bit of computation shows that (9) is equivalent to (3).

4. Polygonal equalities. We shall now derive the conditions that our polygonal inequalities (3) become equalities. For this purpose we shall say that the vectors $y_1, \cdots, y_t \in X$ are *codirectional* in case the nonzero y_i $(i = 1, \cdots, t)$ all have the same direction. Since

$$\left(\sum_{i=1}^{t} |y_i| - \left|\sum_{i=1}^{t} y_i\right|\right)\left(\sum_{i=1}^{t} |y_i| + \left|\sum_{i=1}^{t} y_i\right|\right) = \sum (2|y_i||y_j| - (y_i, y_j) - (y_j, y_i) \colon 1 \leqq i < j \leqq t),$$

$y_1, \cdots, y_t$ are codirectional iff

$$\sum_{i=1}^{t} |y_i| = \left|\sum_{i=1}^{t} y_i\right|.$$

It is clear that equality holds in (9) iff equality holds in every analogue of (8). Equality holds in (8) iff the right hand member of (7) is equal to

$$(S_1 + S_n)\left(\sum_{r=1}^{k} |x_r| - \left|\sum_{r=1}^{k} x_r\right|\right).$$

Hence equality holds in (8) iff $x_1, \cdots, x_k$ are codirectional or

$$S_1 + S_n - \sum_{r=1}^{k} |x_r| - \left|\sum_{r=1}^{k} x_r\right| = 0,$$

which may be rewritten as

$$\sum_{r=k+1}^{n} |x_r| + \left| -\sum_{r=1}^{n} x_r\right| = \left|\sum_{r=1}^{k} - x_r\right|.$$

Hence equality holds in (8) iff $x_1, \cdots, x_k$ are codirectional or $x_{k+1}, \cdots, x_n$, $-\sum_{i=1}^{n} x_i$ are codirectional. These observations prove the following:

LEMMA. *We have equality in* (3) *iff for every permutation* $i \rightarrow i\tau$ *of* $1, \cdots, n$ *either the vectors* $x_{1\tau}, \cdots, x_{k\tau}$ *are codirectional or the vectors* $x_{(k+1)\tau}, \cdots, x_{n\tau}$, $-\sum_1^n x_i$ *are codirectional.*

It is desirable to express the conditions for equality in (3) in more geometric terms. Let us set $\sigma_n = \sum_1^n x_i$ and write ζ for the number of distinct indices i for which $x_i \neq 0$.

THEOREM. *Equality holds in* (3) *iff one of the following conditions holds*: (i) $\zeta = 2$, (ii) $\zeta = 3$ *and* $\sigma_n = 0$, (iii) $k = n-1$ *and* $\sigma_n = 0$, (CD) $x_1, \cdots, x_n$ *are codirectional*, (E) $x_1, \cdots, \hat{x}_i, \cdots, x_n$ *are codirectional for some* $i = 1, \cdots, n$, *and* x_i, σ_n *are also codirectional.* (The sign ^ placed over a vector indicates that this vector is to be deleted from the sequence.)

Proof. The lemma shows at once that each of these five conditions implies equality in (3). To prove the converse, let equality hold in (3) and assume that the first four conditions fail.

If $\sigma_n \neq 0$, we have x_i, $-\sigma_n$ not codirectional for some $i = 1, \cdots, n$ because (CD) fails. The lemma then yields that x_p, x_q are codirectional for all $p, q \neq i$. Since (CD) fails, we must have x_p, x_i not codirectional for all indices $p \neq i$ for which $x_p \neq 0$. Then the lemma shows that x_q, $-\sigma_n$ are codirectional for all $q \neq i$. But, because $x_i = \sigma_n - (x_1 + \cdots + \hat{x}_i + \cdots + x_n)$, we see that x_i, σ_n are codirectional and (E) holds.

If $\sigma_n = 0$, then $\zeta \geqq 4$ and $n - k \geqq 2$. For any set of four distinct indices i, j, p, q,

It is obvious that $X(1, n)$ and $X(2, n)$ are quadrilateral spaces. The choice of $x=(1, 1, -1, 0, \cdots, 0)$, $y=(1, -1, 1, 0, \cdots, 0)$, $z=(-1, 1, 1, 0, \cdots, 0)$ in $X(p, n)$ reduces (2) to the inequality $4(3)^{1/p}\geqq 6$, so that $X(p, n)$ is not a quadrilateral space if $n\geqq 3$ and $p>(\log 3)/(\log 1.5)\doteq 2.7$. We conjecture that $X(p, n)$ is a quadrilateral space for all n if $1\leqq p\leqq 2$, but we are unable to prove this result.

It also seems likely that every real normed linear space of dimension two is a quadrilateral space, but again we are unable to prove this result.

H. Hornich (Math. Z., 48 (1942) 268–274) used calculus to derive inequalities related to, but different from ours. His paper contains a proof of quadrilateral inequality due to Hlawka. A recent paper of R. Lučić (Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 101–106 (1963) 5–6) derives Hornich's inequalities from the quadrilateral inequality.

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POLYNOMIAL REPRESENTATIONS OF SUMS OF TWO SQUARES

ROBERT SPIRA, Duke University

If one applies Brahmagupta's formula

$$(1)\qquad (x^2+y^2)(u^2+v^2) = (xv \pm yu)^2 + (xu \mp yv)^2$$

to the product of a prime >2, $P=a^2+b^2$, with itself, one obtains

$$(2)\qquad P^2 = (a^2+b^2)^2 + 0^2 = (a^2-b^2)^2 + (2ab)^2,$$

and these indeed must be the two representations of P^2 as a sum of two squares, since neither term of the second can be zero. In this paper it will be shown that if the representation of P_r is $a_r^2+b_r^2$, then the representations of $\Pi P_r^{j_r}$ are certain polynomials in the letters a_r and b_r.

To discuss this problem further, let us recall the basic facts on the form x^2+y^2 as given by Dickson [1] (Chap. IV), [2] (Chap. V), and [3] (Vol. II, Chap. VI). The numbers represented by x^2+y^2 are of the form $2^\alpha S^2T$, where S is composed of primes $\equiv 3 \pmod 4$ and T of primes $\equiv 1 \pmod 4$. For $T=P_0^{j_0}P_1^{j_1} \cdots P_k^{j_k}$ there are 2^k primitive representations, and the total number of representations is

$$(3)\qquad \tfrac{1}{2}[\tau(T)+\chi(T)],$$

where τ is the number of divisors and $\chi(T)$ is 1 if T is a square and is 0 if T is not a square. The number of representations given by these last formulas does not take into account changes of sign or of order of x and y.

To carry out the construction of the representations, we need the following:

It is obvious that $X(1, n)$ and $X(2, n)$ are quadrilateral spaces. The choice of $x=(1, 1, -1, 0, \cdots, 0)$, $y=(1, -1, 1, 0, \cdots, 0)$, $z=(-1, 1, 1, 0, \cdots, 0)$ in $X(p, n)$ reduces (2) to the inequality $4(3)^{1/p}\geqq 6$, so that $X(p, n)$ is not a quadrilateral space if $n\geqq 3$ and $p>(\log 3)/(\log 1.5)\doteq 2.7$. We conjecture that $X(p, n)$ is a quadrilateral space for all n if $1\leqq p\leqq 2$, but we are unable to prove this result.

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If one applies Brahmagupta's formula

$$(x^2+y^2)(u^2+v^2) = (xv \pm yu)^2 + (xu \mp yv)^2 \tag{1}$$

to the product of a prime >2, $P=a^2+b^2$, with itself, one obtains

$$P^2 = (a^2+b^2)^2 + 0^2 = (a^2-b^2)^2 + (2ab)^2, \tag{2}$$

and these indeed must be the two representations of P^2 as a sum of two squares, since neither term of the second can be zero. In this paper it will be shown that if the representation of P_r is $a_r^2+b_r^2$, then the representations of $\Pi P_r^{j_r}$ are certain polynomials in the letters a_r and b_r.

To discuss this problem further, let us recall the basic facts on the form x^2+y^2 as given by Dickson [1] (Chap. IV), [2] (Chap. V), and [3] (Vol. II, Chap. VI). The numbers represented by x^2+y^2 are of the form $2^\alpha S^2 T$, where S is composed of primes $\equiv 3 \pmod 4$ and T of primes $\equiv 1 \pmod 4$. For $T=P_0^{j_0}P_1^{j_1}\cdots P_k^{j_k}$ there are 2^k primitive representations, and the total number of representations is

$$\tfrac{1}{2}[\tau(T)+\chi(T)], \tag{3}$$

where τ is the number of divisors and $\chi(T)$ is 1 if T is a square and is 0 if T is not a square. The number of representations given by these last formulas does not take into account changes of sign or of order of x and y.

To carry out the construction of the representations, we need the following:

THEOREM 1. *If m and n are sums of two squares, and e^2+f^2 a representation of the product $m\cdot n$, then e^2+f^2 can be obtained from suitable representations of m and n by Brahmagupta's formula.*

Proof. Note that G.C.D. (m, n) is not necessarily 1. There does not seem to be a short algorithm to find the representations of m and n.

Let $m\cdot n=e^2+f^2$. We seek a, b, c, d such that $a^2+b^2=m$, $c^2+d^2=n$, $ac-bd=e$, and $ad+bc=f$. We know that $m=S_1^2m_1$, $n=S_2^2n_1$, where S_1 and S_2 are made up of primes $\equiv 3 \pmod 4$, whereas m_1 and n_1 have no such primes in their factorizations. A prime q dividing $S_1^2S_2^2$ to the highest power 2α is easily seen, considering successive powers, to divide e and f to the α-th power. Thus, $(e/S_1S_2)^2+(f/S_1S_2)^2=m_1\cdot n_1$, or $e_1^2+f_1^2=m_1\cdot n_1$, and no prime $q\equiv 3 \pmod 4$ divides $m_1\cdot n_1$. If we solve the problem for m_1, n_1, e_1, and f_1, we can also solve the problem for m, n, e, and f.

Now, factor m_1 and n_1 in the Gaussian integers: $m_1=\Pi(a_j+b_ji)(a_j-b_ji)$, $n_1=\Pi(c_j+d_ji)(c_j-d_ji)$. Noting that $e_1^2+f_1^2=(e_1+f_1i)(e_1-f_1i)$, we choose the pair (a_1+b_1i), (a_1-b_1i). If a_1+b_1i divides e_1+f_1i, set e_2+f_2i equal to $(e_1+f_1i)/(a_1+b_1i)$; then $e_2-f_2i=(e_1-f_1i)/(a_1-b_1i)$. If a_1+b_1i does not divide e_1+f_1i, then a_1-b_1i must. Set e_2+f_2i equal to $(e_1+f_1i)/(a_1-b_1i)$; then $e_2-f_2i=(e_1-f_1i)/(a_1+b_1i)$. Continue this process through the successive pairs (a_j+b_ji), (a_j-b_ji), and then through the pairs (c_j+d_ji), (c_j-d_ji). Then, adjusting a unit,

$$\begin{aligned} e_1+f_1i &= \prod(a_j+\epsilon_jb_ji)\cdot\prod(c_j+\epsilon_jd_ji)\cdot\text{unit}, \quad \text{where} \quad \epsilon_j=\pm 1 \\ &= (A+Bi)\cdot(C+Di), \end{aligned} \tag{4}$$

and $m_1=A^2+B^2$, $n_1=C^2+D^2$, $e_1=AC-BD$, $f_1=AD+BC$.

LEMMA. *If G.C.D. $(m, n)=1$, then each representation $e^2+f^2=m\cdot n$ is obtained exactly once by Brahmagupta's formula.*

Proof. By Theorem 1, each representation is obtained at least once. On the other hand, we shall show that we can count the representations obtained and see that there are just enough. Setting $N(x)$ equal to the number of representations of x as a sum of two squares (without regard to order or sign of the numbers squared), and setting m equal to $2^\alpha S_1^2T_1$ and n equal to $2^\beta S_2^2T_2$, where the primes dividing S_1 and S_2 are congruent to 3 (mod 4) and the primes dividing T_1 and T_2 are congruent to 1 (mod 4), we know that:

$$N(m) = \tfrac{1}{2}[\tau(T_1)+\chi(T_1)]$$
$$N(n) = \tfrac{1}{2}[\tau(T_2)+\chi(T_2)].$$

The proof consists of the following three cases: (i) neither T_1 nor T_2 is a square; (ii) exactly one of T_1 and T_2 is a square; and (iii) both T_1 and T_2 are squares. Without loss of generality we can assume that $\alpha=\beta=0$.

We carry out the proof of case (ii). Since exactly one of T_1 and T_2 is a square, we let T_2 be a square and T_1 not a square. Then,

$$N(m) = \tfrac{1}{2}\tau(T_1)$$
$$N(n) = \tfrac{1}{2}[1 + \tau(T_2)].$$

Now the greatest possible number of representations of $m \cdot n$ obtainable is

$$2N(n)\cdot N(m) - N(m),$$

since one of the representations of n is $(\sqrt{n})^2+0^2$; this gives rise to only one representation of $m \cdot n$ on application of Brahmagupta's identity to a representation of m.

Substituting for $N(n)$ and $N(m)$ in this formula, we obtain:

$$\begin{aligned} 2N(n)\cdot N(m) - N(m) &= 2(\tfrac{1}{2}[1 + \tau(T_2)])\cdot(\tfrac{1}{2}\tau(T_1)) - \tfrac{1}{2}\tau(T_1) \\ &= \tfrac{1}{2}\tau(T_1)[1 + \tau(T_2) - 1] = \tfrac{1}{2}\tau(T_1)\tau(T_2) \\ &= \tfrac{1}{2}\tau(T_1T_2) = N(mn). \end{aligned}$$

Thus, each representation is obtained once and only once. The proofs of cases (i) and (iii) are similar.

Now we take up Theorem 2, the polynomial representation theorem.

For each prime $P_r \equiv 1 \pmod 4$, we set $P_r = a_r^2 + b_r^2$, in some definite way, say $a_r > b_r > 0$. We will find polynomials in the a_r's and b_r's giving the primitive representations of $\prod P_r^{j_r}$ as a sum of two squares. The imprimitive representations will also be found. The polynomial representations of a general sum of two squares (having the additional factor $2^\alpha S^2$ where primes dividing S are $\equiv 3 \pmod 4$) are seen to be slight modifications of the representations obtained.

For Theorem 2, we use the following general formula obtained from Brahmagupta's identity (where z^* means the conjugate of z),

$$\begin{aligned} &\{[(u + u^*)/2]^2 + [(u - u^*)/2i]^2\}\cdot\{[(v + v^*)/2]^2 + [(v - v^*)/2i]^2\} \\ (5)\qquad &= [(uv + u^*v^*)/2]^2 + [(uv - u^*v^*)/2i]^2 \\ &= [(u^*v + uv^*)/2]^2 + [(u^*v - uv^*)/2i]^2. \end{aligned}$$

In order to state Theorem 2 concisely, the following notation is used.

DEFINITION. *If t is a nonnegative integer and α_r is a Gaussian integer then*

$$(6)\qquad \chi_t(\alpha_r^{j_r}) = \begin{cases} \alpha_r^{j_r} & \textit{if } 2^r \textit{ does not appear in representation of } t \textit{ in the binary system} \\ \alpha_r^{*j_r} & \textit{otherwise.} \end{cases}$$

DEFINITION.

$$(7)\qquad \overset{*}{\chi}_t(\alpha_r^{j_r}) = \chi_t(\alpha_r^{*j_r}).$$

DEFINITION.

$$(8)\qquad A(P_0^{j_0}, P_1^{j_1}, \cdots, P_s^{j_s}, t) = \left[\prod_{r=0}^{s} \chi_t(\alpha_r^{j_r}) + \prod_{r=0}^{s} \overset{*}{\chi}_t(\alpha_r^{j_r})\right] \Big/ 2,$$

$$B(P_0^{j_0}, P_1^{j_1}, \cdots, P_s^{j_s}, t) = \left[\prod_{r=0}^{s} \chi_t(\alpha_r^{j_r}) - \prod_{r=0}^{s} \chi_t^*(\alpha_r^{j_r}) \right] \Big/ 2i. \tag{9}$$

The number t is called the *tag*.

We now prove Theorem 2, the fundamental representation theorem for representations of sums of two squares.

THEOREM 2. *Let* $P_r = a_r^2 + b_r^2$, $P_r \equiv 1 \pmod 4$, $\alpha_r = a_r + ib_r$, $(r = 0, 1, \cdots, k)$, *then*:

PART A: *The primitive representations of* $\prod_{r=0}^{k} P_r^{j_r}$, *provided all* $j_r \geqq 1$, *are given by*:

$$[A(P_0^{j_0}, \cdots, P_k^{j_k}, t)]^2 + [B(P_0^{j_0}, \cdots, P_k^{j_k}, t)]^2, \qquad 0 \leqq t \leqq 2^k - 1.$$

PART B: *All representations are given by*:

$$(P_0^{s_0} P_1^{s_1} \cdots P_k^{s_k})^2 \{ [A(P_0^{j_0 - 2s_0}, \cdots, P_k^{j_k - 2s_k}, t)]^2 + [B(P_0^{j_0 - 2s_0}, \cdots, P_k^{j_k - 2s_k}, t)]^2 \},$$

where if any $j_r - 2s_r = 0$, *the entry* $P_r^{j_r - 2s_r}$ *is to be omitted, and* $0 \leqq t \leqq 2^\theta - 1$ *(or* 0 *if* $2^\theta - 1$ *is negative) where* $\theta = -1 +$ *(number of nonzero exponents* $j_r - 2s_r$*), running over all sets* $(s_0, s_1, \cdots, s_k)$, *where* $0 \leqq s_r \leqq [j_r/2]$.

Example. For P^k, $P = a^2 + b^2$, we have the single primitive representation,

$$P^k = [((a + ib)^k + (a - ib)^k)/2]^2 + [((a + ib)^k - (a - ib)^k)/2i]^2,$$

and all representations are given by:

$$(P^0)^2\{[A(P^k, 0)]^2 + [B(P^k, 0)]^2\}$$
$$(P^1)^2\{[A(P^{k-2}, 0)]^2 + [B(P^{k-2}, 0)]^2\}$$
$$(P^2)^2\{[A(P^{k-4}, 0)]^2 + [B(P^{k-4}, 0)]^2\}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$(P^{[k/2]})^2\{[A(P^{k-2[k/2]}, 0)]^2 + [B(P^{k-2[k/2]}, 0)]^2\}.$$

Note that

$$A(P^0, 0) = [(a + ib)^0 + (a - ib)^0]/2 = 1,$$
$$B(P^0, 0) = [(a + ib)^0 - (a - ib)^0]/2i = 0.$$

Proof. The proof is by induction on k. Let $k = 0$. By Theorem 1, the single primitive representation of $P_0^{j_0}$ is a polynomial in a_0 and b_0. We are to verify that the primitive representation of $P_0^{j_0}$ is $[A(P_0^{j_0}, 0)]^2 + [B(P_0^{j_0}, 0)]^2$, where $A(P_0^{j_0}, 0) = (\alpha_0^{j_0} + \alpha_0^{*j_0})/2$ and $B(P_0^{j_0}, 0) = (\alpha_0^{j_0} - \alpha_0^{*j_0})/2i$. In the examples above, this was shown for $j_0 = 0$ and 1. Assume it now for j_0 and prove it for $j_0 + 1$. Since imprimitive representations give rise to imprimitive representations, the primitive representation of $P_0^{j_0+1}$ must arise from the primitive representations of P_0 and $P_0^{j_0}$ by Brahmagupta's formula (using Theorem 1). Applying now identity (5) with $u = \alpha_0$, $v = \alpha_0^{j_0}$, we obtain:

$$P_0\cdot P_0^{j_0} = \{[(\alpha_0 + \overset{*}{\alpha_0})/2]^2 + [(\alpha_0 - \overset{*}{\alpha_0})/2i]^2\}$$

$$\cdot\{[(\alpha_0^{j_0} + \alpha_0^{*j_0})/2]^2 + [(\alpha_0^{j_0} - \alpha_0^{*j_0})/2i]^2\}$$

$$= \begin{cases}[(\alpha_0^{j_0+1} + \alpha_0^{*j_0+1})/2]^2 + [(\alpha_0^{j_0+1} - \alpha_0^{*j_0+1})/2i]^2 \\ [(\alpha_0\alpha_0^{*j_0} + \alpha_0\alpha_0^{*j_0})/2]^2 + [(\alpha_0\alpha_0^{*j_0} - \alpha_0\alpha_0^{*j_0})/2i]^2\end{cases}$$

$$\begin{cases}[A(P_0^{j_0+1}, 0)]^2 + [B(P_0^{j_0+1}, 0)]^2 \\ [\alpha_0\overset{*}{\alpha_0}]^2\{[(\alpha_0^{*j_0-1} + \alpha_0^{j_0-1})/2]^2 + [(\alpha_0^{*j_0-1} - \alpha_0^{j_0-1})/2i]^2\}\end{cases}$$

and as $\alpha_0\alpha_0^* = (a_0+b_0i)(a_0-b_0i) = a_0^2+b_0^2 = P_0$, we obtain (changing the sign inside the square bracket):

$$\begin{cases}[A(P_0^{j_0+1}, 0)]^2 + [B(P_0^{j_0+1}, 0)]^2 \\ P_0^2\{[A(P_0^{j_0-1}, 0)]^2 + [B(P_0^{j_0-1}, 0)]^2\}.\end{cases}$$

Thus, the result of applying Brahmagupta's formula to P_0 and $P_0^{j_0}$ gives rise to two representations, one primitive of the required form and the other imprimitive.

To complete the induction on k, we assume Part A of Theorem 2 for all integers $\leqq k$ and Part B for all integers $<k$. We prove Part B for k and Part A for $k+1$. For Part B, observe that the formula given yields distinct representations (since by the induction hypothesis the primitive representations for each set $(s_0, s_1, \cdots, s_k)$ are all distinct), so that we have only to verify that the number of representations of the formula agrees with the a priori known number of representations. We think of summing the range of t over the various possibilities for the s_r's. Without loss of generality, we can rearrange the product and take the first k_1+1 j_r's as odd. Assume first that $k_1 \geqq 0$. We rewrite $j_{i_r} = 2j_r' + 1$ $(0 \leqq r < k_1+1)$, and $j_{i_r} = 2j_r'$ $(k_1 < r \leqq k)$. Now we drop the primes.

We classify the representations according to the number of nonzero exponents. Counting first the representations with no nonzero exponents, which gives 2^k as the range of t, we have

$$\prod_{i=0}^{k_1} (j_i + 1)\cdot \prod_{k_1+1\leqq i\leqq k} j_i$$

possibilities, since if $0 \leqq i \leqq k_1$ there are $j_i + 1$ numbers u such that $0 \leqq u \leqq [(2j_i+1)/2]$; and for $k_1+1 \leqq i \leqq k$ there are j_i numbers u such that $0 \leqq u < [j_i]$. (The strict inequality holds since there are *no* nonzero exponents.)

For the range of $t = 2^{k-1}$, one of the primes with an even exponent is raised to the highest power, so we obtain

$$\sum_{j=k_1+1}^{k} \left[\left(\prod_{i=0}^{k_1} (j_i + 1)\right)\left(\prod_{\substack{i\neq j \\ k_1+1\leqq i\leqq k}} j_i\right)\right]$$

possibilities.

and for each representation for $0 \leqq t \leqq 2^k - 1$, we pick up two representations conforming to the definitions of χ_t and the polynomials

$$A(P_0^{j_0}, \cdots, P_{k+1}^{j_{k+1}}, t), \quad B(P_0^{j_0}, \cdots, P_{k+1}^{j_{k+1}}, t).$$

Thus, we have constructed polynomial representations of the representations as sums of two squares of all numbers $\prod P_r^{j_r}$ where the P_r's are $\equiv 1 \pmod 4$.

In the author's thesis [4] it was shown that a similar representation theory exists for the forms x^2+2y^2 and x^2+3y^2, and a slightly restricted portion of the theory holds for the form x^2+7y^2. Since for $D>7$, x^2+Dy^2 represents primitively composite numbers, of which it does not represent the prime factors, the theory above cannot be extended to any other forms x^2+Dy^2.

For the forms $x^2+xy+Dy^2$, it appears likely that few, if any, will have a polynomial representation theory.

Finally, it should be mentioned that this theory was used to count the number of times a given representation $e^2+f^2=m \cdot n$ is obtained from representations of m and n by Brahmagupta's formula. This function of m, n, e and f is non-multiplicative, but is very often equal to τ (G.C.D. (m, n, e, f)).

Further investigation is being made on the extension of Theorem 1 to other compositions.

References

1. L. E. Dickson, Modern Elementary Theory of Numbers, University of Chicago Press, 1939.
2. ———, Introduction to the Theory of Numbers, University of Chicago Press, 1929.
3. ———, History of the Theory of Numbers, Carnegie Institute, Washington, Chelsea, 1952.
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QUERY

Daihachiro Sato, University of Saskatchewan

Could anyone supply me with information on books or papers in which the identities, natural consequences of a Stieltjes integral,

$$\text{(1)} \qquad \sum_{i=p}^{q} a_i = -\sum_{i=q+1}^{p-1} a_i \quad \text{and} \quad \text{(2)} \quad \prod_{i=p}^{q} a_i = 1 \Big/ \prod_{i=q+1}^{p-1} a_i$$

are explicitly defined, or are used intentionally in the literature?

I am collecting the cases in which the above definitions hold naturally as an extension of $\sum$ and $\prod$ when they have a negative number of terms or factors.

Address: Daihachiro Sato, University of Saskatchewan, Regina Campus, Regina, Saskatchewan, Canada.

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Address: Daihachiro Sato, University of Saskatchewan, Regina Campus, Regina, Saskatchewan, Canada.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146

ON SUBSERIES OF DIVERGENT SERIES

C. R. BANERJEE AND B. K. LAHIRI, Barackpore R.S.N. College, West Bengal and University of Calcutta

1. Let $u(1)+u(2)+u(3)+\cdots$ be a given series. Each increasing sequence $n_1, n_2, n_3, \cdots$ of positive integers determines a series

$$u(n_1) + u(n_2) + u(n_3) + \cdots \tag{1}$$

that is called a *subseries* of the given series. Kakeya [4] proved the following theorem on subseries of special convergent series of positive terms.

THEOREM 1 (KAKEYA). *If* $u(1)+u(2)+u(3)+\cdots$ *is a series of positive terms which converges to* s, *if* $u(1) \geqq u(2) \geqq u(3) \geqq \cdots$, *if*

$$u(n) \leqq u(n+1) + u(n+2) + u(n+3) + \cdots, \qquad (n = 1, 2, 3, \cdots), \tag{2}$$

and if $0 < P < s$, *then there is a subseries* (1) *which converges to* P.

It is the purpose of this note to give explicit rules for construction of a special subseries of the type required to prove the following theorem.

THEOREM 2. *Let*

$$u(1) + u(2) + u(3) + \cdots \tag{3}$$

be a divergent series of positive terms for which

$$\lim_{n\to\infty} u(n) = 0. \tag{4}$$

Let P *be a positive number. Then there is a subseries* (1) *which converges to* P.

To start the construction, let K_1 be the least integer such that $u(k) < P/2$ when $k \geqq K_1$. Let L_1 be the greatest integer for which

$$s_1 = u(K_1) + u(K_1 + 1) + \cdots + u(L_1) < P. \tag{5}$$

Then $P/2 < s_1 < P$. Let K_2 be the least integer such that $K_2 > L_1$ and $u(k) < (P - s_1)/2$ when $k \geqq K_2$. Let L_2 be the greatest integer for which

$$s_2 = s_1 + u(K_2) + u(K_2 + 1) + \cdots + u(L_2) < P. \tag{6}$$

Then $P - P/2^2 < s_2 < P$. Let K_3 be the least integer such that $K_3 > L_2$ and $u(k) < (P - s_2)/2$ when $k \geqq K_3$. Let L_3 be the greatest integer for which

$$s_3 = s_2 + u(K_3) + u(K_3 + 1) + \cdots + u(L_3) < P. \tag{7}$$

Then $P - P/2^3 < s_3 < P$. Continuation of the construction yields integers

$K_1, K_2, \cdots$ and $L_1, L_2, \cdots$ such that

$$K_1 < L_1 < K_2 < L_2 < K_3 < L_3 < \cdots, \tag{8}$$

$$s_q = s_{q-1} + u(K_q) + u(K_q + 1) + \cdots + u(L_q), \qquad (q = 2, 3, \cdots) \tag{9}$$

and $P - P/2^q < s_q < P$. It is easy to see that the required subseries converging to P is obtained by letting $n_1, n_2, n_3, \cdots$ be, in order, the integers

$$K_1, K_1 + 1, \cdots, L_1, K_2, K_2 + 1, \cdots, L_2, K_3, K_3 + 1, \cdots. \tag{10}$$

2. There are different ways of developing the basic idea that if a series fails to converge absolutely, then "most" of its subseries are divergent. Theorems of Agnew [2], Hill [3] and Tsuchikura [7], [8], bear on this matter. In some respects, investigations of subseries are quite analogous to investigations of rearrangements of series due to Agnew [1] and Sengupta [5], [6].

In [1], [3], [5], [6], and [8], the Fréchet formula

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|} \tag{11}$$

is used to obtain a metric space E in which points x and y are sequences $x_1, x_2, x_3, \cdots$ and $y_1, y_2, y_3, \cdots$ in a specified class. For investigation of rearrangements of series, E is the Fréchet space E_1 in which each point x is a permutation $x_1, x_2, x_3, \cdots$ of the sequence $1, 2, 3, \cdots$. For investigation of subseries, E is the Fréchet space E_2 in which each point x is an increasing sequence $x_1, x_2, x_3, \cdots$ of positive integers. Some of the properties of E_1 and E_2 are discovered in the papers cited above.

Let $u(1)+u(2)+u(3)+\cdots$ be a divergent series of positive terms for which $\lim_{n\to\infty} u(n)=0$. Let A be the set of points x in E_2 for which the subseries

$$u(x_1) + u(x_2) + u(x_3) + \cdots \tag{12}$$

is convergent, and let B be the complementary set of points x in E_2 for which the subseries is divergent. The authors have investigated the structures of the sets A and B. Without going into details, we remark that A and B are both dense in E_2, that A is of the first category, and that B is of the second category. Proofs of these results are quite similar to proofs appearing in papers cited above.

The authors are most grateful to the referee for his invaluable help in improving the paper.

References

1. R. P. Agnew, On rearrangements of series, Bull. Amer. Math. Soc., 46 (1940) 797–799.
2. ———, Subseries of series which are not absolutely convergent, ibid., 53 (1947) 118–120.
3. J. D. Hill, Some theorems on subseries, ibid., 48 (1942) 103–108.
4. S. Kakeya, On the partial sums of an infinite series, Science Reports of the Tôhoku Imperial University (1), 3 (1914) 159–163.
5. H. M. Sengupta, On rearrangements of series, Proc. Amer. Math. Soc., 1 (1950) 71–75.
6. ———, Rearrangements of series, ibid., 7 (1956) 347–350.
7. T. Tsuchikura, On some divergence problems, Tôhoku Math. J. (2), 2 (1950) 30–39.
8. ———, On the theory of series with function terms, ibid., 3 (1951) 104–113.

THE MIDTERM COEFFICIENT OF THE CYCLOTOMIC POLYNOMIAL $F_{pq}(x)$

SISTER MARION BEITER, Rosary Hill College, Buffalo

Introduction. The interested reader will find the background in cyclotomy in [3] and [4] sufficient for the purpose of this note, although the investigation is based on results in [1] and [2].

The monic polynomial whose roots are the primitive mth roots of unity is defined to be the cyclotomic polynomial $F_m(x)$. By Dedekind's inversion formula ([4] p. 114),

$$F_m(x) = \prod_{d|m} (x^d - 1)^{\mu(m/d)}. \tag{1}$$

In [1] it is proved that if m is a product of two distinct odd primes, p and q, then the coefficients of $F_{pq}(x)$ can equal only ± 1 or 0.

General coefficient. Let $F_{pq}(x) = \sum_{n=0}^{\phi(pq)} c_n x^n$.

THEOREM I. *In* $F_{pq}(x)$

$$c_n = \begin{cases} (-1)^\delta & \text{if } n = \alpha q + \beta p + \delta \text{ in exactly one way,} \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where α and β are nonnegative integers and $\delta = 0, 1$.

Proof. From (1) it follows that

$$\begin{aligned} F_{pq}(x) &= (x^{pq} - 1)(x - 1)/(x^p - 1)(x^q - 1) \\ &= (1 - x)(1 + x^q + \cdots + x^{(p-1)q})(1 + x^p + x^{2p} + \cdots) \\ &= \sum_{\alpha=0}^{p-1} x^{\alpha q} \sum_{\beta=0}^{\infty} x^{\beta p} - \sum_{\alpha=0}^{p-1} x^{\alpha q+1} \sum_{\beta=0}^{\infty} x^{\beta p} \\ &= \sum_{\alpha,\beta,\delta} (-1)^\delta x^{\alpha q+\beta p+\delta}, \end{aligned}$$

where α runs through the integers from zero to $p-1$, β is any nonnegative integer, and $\delta = 0, 1$. Then c_n in $F_{pq}(x)$ is the sum of the coefficients of all terms on the right with exponent $\alpha q + \beta p + \delta = n$. Where no such partition exists, c_n is zero. If there is exactly one partition, c_n equals $(-1)^\delta$.

Assume that n can be partitioned in two ways:

$$\begin{aligned} n &= \alpha_1 q + \beta_1 p + \delta_1 \\ &= \alpha_2 q + \beta_2 p + \delta_2, \end{aligned}$$

with $\delta_1 = \delta_2$. Then $q(\alpha_1 - \alpha_2) = p(\beta_2 - \beta_1)$. This implies that p divides $\alpha_1 - \alpha_2$. But since $\alpha < p$, $|\alpha_1 - \alpha_2| < p$. Therefore $\alpha_1 - \alpha_2 = \beta_2 - \beta_1 = 0$, and the two partitions are identical. Hence, when two distinct partitions of n in the form (2) exist, in one of them $\delta = 1$, in the other $\delta = 0$. In this case c_n is $(-1)^1 + (-1)^0 = 0$, and the theorem is proved.

A discussion similar to this occurs in [1]

Midterm coefficient. Set $n=\phi(pq)/2$ in (2). Then

$$\begin{aligned}(p-1)(q-1)/2 &= \alpha q+\beta p+\delta,\\ p(2\beta+1) &\equiv 1-2\delta \pmod{q},\\ px &\equiv \pm 1 \pmod{q}.\end{aligned}$$

Let k be the solution of $px\equiv 1 \pmod{q}$, $1\leqq k\leqq q-1$. Then $q-k$ is a solution of $px\equiv -1 \pmod{q}$.

Consider $pk\equiv 1 \pmod{q}$. Then

$$\begin{aligned}pk &= 1+qh, \qquad & h &= (pk-1)/q,\\ \beta &= (k-1)/2 & \alpha &= (p-1)/2-h/2.\end{aligned}$$

In the case k is odd, these values of α and β are integral, $\delta=0$, and the midterm coefficient is 1.

If k is even, $q-k$ is odd, $\delta=1$, and the midterm coefficient is -1. Thus we have

THEOREM II. *In $F_{pq}(x)$, when $n=\phi(pq)/2$, $c_n=(-1)^{k-1}$, where k is the least positive solution of the congruence $px\equiv 1$* (mod q).

Remarks. In the special case $q=sp+1$, k is odd and the midterm coefficient is $+1$. Similarly, for $q=sp-1$, k is even and the midterm coefficient is -1.

In any case, the roles of p and q in the congruences may be reversed, without affecting the oddness or evenness of k.

The following table gives the value of the midterm coefficient c_n of $F_{pq}(x)$ when p is 3, 5, or 7. All values of $m=pq$ and less than 143 reduce to one of these special cases.

p	a	c_n
3	1	± 1 according as $q\equiv \pm a \pmod{p}$.
5	1, 2	
7	1, 3, 5	

The author thanks the referee for his suggestions.

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References

1. A. S. Bang, Om Ligningen $\phi_n(x)=0$, Nyt Tidsskrift for Mathematik, 6 (1895) 6–12.

2. A. Migotti, Zur Theorie der Kreisteilungsgleichung, S.-B. der Math.-Naturwiss. Classe der Kaiserlichen Akademie der Wissenschaften, Wien, (2) 87 (1883) 7–14.

3. T. Nagell, Introduction to Number Theory ,Wiley, New York, 1951.

4. B. L. van der Waerden, Modern Algebra, transl. Fred Blum, Stechert-Hafner, New York, 1949.

A SIMPLE PROOF OF MORLEY'S THEOREM

HAIM ROSE, Kiriat Shmonah, Israel

Morley's well-known theorem states that the points of intersection of the adjacent trisectors of the interior (or exterior) angles of any triangle meet at the vertices of an equilateral triangle.

We shall give a simple proof for the case of exterior angles (see [1] p. 24 and [2] pp. 345–349). We shall distinguish in the proof between three cases: acute angled triangle, right angled triangle, and obtuse angled triangle.

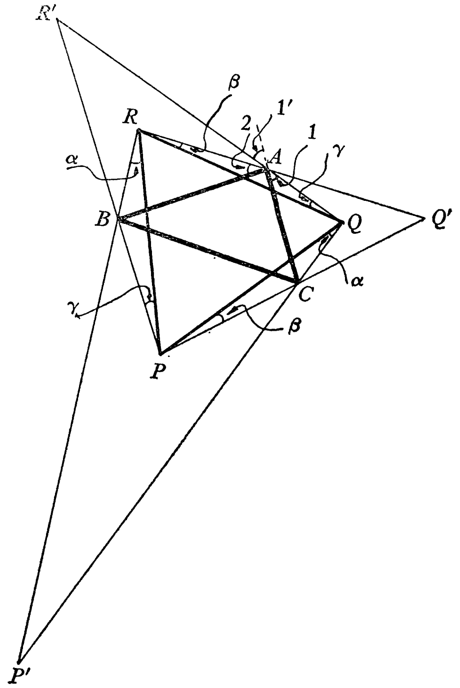

FIG. 1

Acute Triangle. In Figure 1, on the respective sides QR, RP, PQ, of a given equilateral triangle PQR, erect isosceles triangles $P'QR$, $Q'RP$, $R'PQ$, whose respective base angles, $60°+\alpha$, $60°+\beta$, and $60°+\gamma$, satisfy the equation and inequalities

$$\alpha + \beta + \gamma = 60°, \qquad \alpha < 30°, \beta < 30°, \gamma < 30°.$$

The vertices of the isosceles triangles are on the same sides as those of the equilateral triangle respective to their bases. The intersections of the sides of the isosceles triangles meet in points A, B, C. We have to prove that the sides of the isosceles triangles trisect the exterior angles of the triangle ABC.

Figure 1 shows that

$$\sphericalangle PR'Q = 180° - (120° + 2\gamma) = 60° - 2\gamma, \qquad \therefore \tfrac{1}{2}\sphericalangle PR'Q = 30° - \gamma.$$

Referring to triangle $AR'B$, we see that

$$\begin{aligned}\sphericalangle ARB &= 60° + \beta + \alpha = 60° + \beta + \alpha + \gamma - \gamma = 120° - \gamma\\ &= 90° + (30° - \gamma) = 90° + \tfrac{1}{2}\sphericalangle BR'A.\end{aligned}$$

By symmetry, RR' bisects $\sphericalangle AR'B$ and so this last result proves that point R is the incenter of the triangle $AR'B$. Likewise Q is the incenter of $AQ'C$, and P of $BP'C$. Hence $\sphericalangle 2 = \sphericalangle 1 = \sphericalangle 1'$; in other words, the external angles of the triangle ABC are trisected.

Now we look for relations between the angles α, β, γ and the triangular angles A, B, C. Again from Figure 1 we have:

$$\sphericalangle R'RQ = 180° - \tfrac{1}{2}\sphericalangle PR'Q - \gamma = 180° - (30° - \gamma) - \gamma = 150°;$$

$$\sphericalangle RAR' = 180° - (30° - \gamma) - (150° - \beta) = \gamma + \beta = 60° - \alpha.$$

If $60° - \alpha = (180° - A)/3$ then $A = 3\alpha$, or $\alpha = A/3$; similarly, $\beta = B/3$, $\gamma = C/3$. Because $\alpha < 30°$, $\beta < 30°$, $\gamma < 30°$, it is obvious that triangle ABC is acute angled.

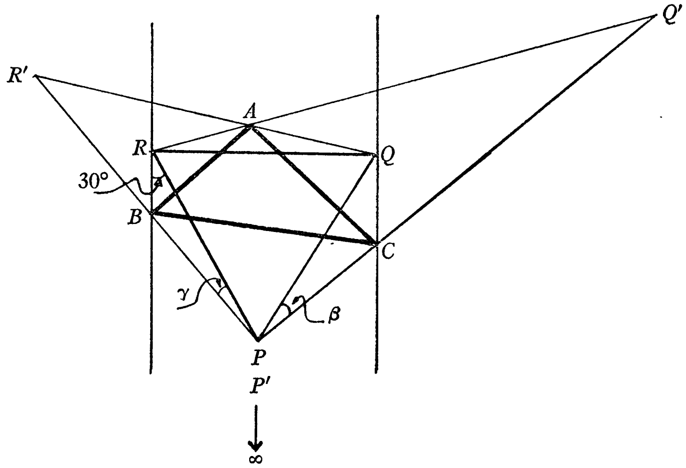

FIG. 2

Right Triangle. The base angles of the isosceles triangles in Figure 2 satisfy the equations

$$\alpha + \beta + \gamma = 60°, \qquad \alpha = 30°, \quad \beta + \gamma = 30°.$$

Considerations referring to triangles $AR'B$ and $AQ'C$, are as above. Triangle $BP'C$ has vertex P' at infinity.

$$\sphericalangle BP'C = 0°, \qquad \sphericalangle BPC = 60° + \beta + \gamma = 90° = 90° + \tfrac{1}{2}\sphericalangle BP'C.$$

The last equation proves that point P is the incenter of triangle $BP'C$. It follows that the exterior angles of ABC are trisected as in the case of an acute angled triangle. Again, $A = 3\alpha = 3 \cdot 30° = 90°$. Hence, $\triangle ABC$ is a right triangle.

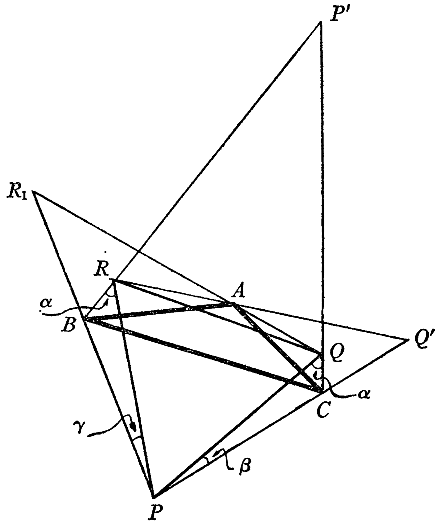

FIG. 3

Obtuse Triangle. In Figure 3 the base angles of the isosceles triangles satisfy the equation and inequalities

$$\alpha + \beta + \gamma = 60°, \qquad \alpha > 30°, \quad \beta + \gamma < 30°.$$

The isosceles triangle $RP'Q$ does not enclose the equilateral triangle PQR. From Figure 3, we see that

$$\alpha + 60° = [180° - (\alpha + 60°)] + \measuredangle RP'Q$$

$$\therefore \quad \measuredangle RP'Q = 2\alpha - 60°; \quad \text{or} \quad \tfrac{1}{2}\measuredangle RP'Q = \alpha - 30°.$$

$$\measuredangle BPC = \gamma + 60° + \beta = 120° - \alpha = 90° - (\alpha - 30°) = 90° - \tfrac{1}{2}\measuredangle BP'C.$$

The last equation proves that point P is the excenter of the triangle $BP'C$ relative to vertex P'. In this case too, therefore, the sides of the isosceles triangles trisect the external angles of ABC. By choosing the above values for the base angles of our isosceles triangles, we can ensure that the above procedure yields a triangle ABC that is similar to any given triangle. This completes the proof.

It can also be proved analytically that

$$PQ = QR = RP = 4\rho \cdot \sin\left(60° - \frac{A}{3}\right) \cdot \sin\left(60° - \frac{B}{3}\right) \cdot \sin\left(60° - \frac{C}{3}\right),$$

where 2ρ denotes the circumdiameter of ABC.

References

1. H. S. M. Coxeter, Introduction to Geometry, Wiley, New York, 1961, p. 24.
2. H. F. Baker, Introduction to Plane Geometry, Cambridge University Press, England, 1943, pp. 345–349.

A NOTE ON THE RELATION BETWEEN PERIODIC AND ORTHOGONAL FUNDAMENTAL SOLUTIONS OF LINEAR SYSTEMS, II

F. S. VAN VLECK, University of Kansas

1. Introduction. In [1] Van Vleck gave necessary and sufficient conditions for the fundamental solution matrix $X(t)$ of a linear system $x' = Ax(' = d/dt)$ to be periodic if $X(t)$ was orthogonal for all time t. There x was a real n-vector, A was a real constant n by n matrix, and $X(t)$ was the unique matrix solution which satisfied $X(0) = I$ (I is the identity matrix). The purpose of this note is to consider the same situation in case the real matrix $A = A(t)$ is a continuous function of t.

2. Consequences of orthogonality. From the theory of systems of linear differential equations we know ([2], pp. 67–70) there exists a unique matrix $X(t)$ (called the fundamental solution) which satisfies

$$X'(t) = A(t)X(t), \quad \text{and} \quad X(0) = I. \tag{1}$$

Also, we know ([2], p. 67) that

$$\begin{aligned} \det X(t) &= \det X(0) \exp\left(\int_0^t \operatorname{trace} A(s)ds\right) \\ &= \exp\left(\int_0^t \operatorname{trace} A(s)ds\right) \end{aligned} \tag{2}$$

and hence X is nonsingular for all t.

We need a characterization of matrices $A(t)$ for which $X(t)$ is orthogonal. The following theorem does this completely. A proof may be found in [3].

THEOREM 1. *If $A(t)$ is continuous, then the fundamental solution $X(t)$ which satisfies* (1) *is orthogonal for all time t if and only if $A(t)$ is skew-symmetric for all time t.*

From (2) it follows that if $X(t)$ is orthogonal, then

$$\det X(t) = 1 \qquad \text{for all } t. \tag{3}$$

If n is an odd integer, then it follows from (3) that 1 is a characteristic root of $X(t)$ for each fixed t. Thus, letting $\sigma > 0$ be a given real number, we have the following theorem since any solution $x = \phi(t)$ of $x' = A(t)x$ may be written $\phi(t) = X(t)\phi(0)$.

THEOREM 2. *If n is odd and $X(t)$ is orthogonal for all time t, then there is a periodic solution of $x' = A(t)x$ of period σ.*

3. Periodicity. Since X is nonsingular for all t (from (2)), we have

$$A(t) = X'(t)X^{-1}(t) \tag{4}$$

and hence if $X(t)$ is to be periodic, we must have $A(t)$ periodic. Therefore, we

Proof. The necessity follows from the first comment of the preceding paragraph. To show the sufficiency, suppose that every characteristic exponent of $A(t)$ is an integral multiple of $2\pi i/\sigma$. Then the characteristic roots of σS, which may be taken to be the characteristic exponents of $A(t)$, are integral multiples of $2\pi i/\sigma$. Now σS is a real skew-symmetric matrix, so its roots are simple and σS is real orthogonally similar to a diagonal block matrix ([4], p. 285). In fact, if the characteristic roots of σS are $\pm i2\pi n_j/\sigma$ and 0, then there exists a real orthogonal matrix Q such that

$$\bar{S} = Q(\sigma S)Q^{-1} = \{(2\pi n_1/\sigma)J, \cdots, (2\pi n_k/\sigma)J, 0, \cdots, 0\}$$

and hence

$$e^{t\bar{S}} = \{\bar{U}(2\pi n_1 t/\sigma), \cdots, \bar{U}(2\pi n_k t/\sigma), 1, \cdots, 1\},$$

where J and $\bar{U}(s)$ are the 2 by 2 matrices

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \bar{U}(s) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}.$$

Now $X(t) = P(t)Q^{-1}e^{tQ(\sigma S)Q^{-1}}Q = P(t)Q^{-1}e^{t\bar{S}}Q$ and hence $X(t)$ is periodic of period σ since $e^{t\bar{S}}$ (and e^{tS}) are periodic of period σ.

If $\sigma > 0$ is the least period of $A(t)$, then from what we have just proved, we have finally the following theorem which is the natural generalization of Theorem 1 of [1]. Let β_j denote the imaginary part of the characteristic exponent λ_j of $A(t)$.

THEOREM 6. *If X is an orthogonal fundamental solution of* (1), *then a necessary and sufficient condition that $X(t)$ be periodic is that there exist a positive number β and integers n_0 and m_j such that $n_0\sigma = \beta$ and $m_j\beta_j = \beta$ for each nonzero β_j.*

References

1. F. S. Van Vleck, A Note on the Relation between Periodic and Orthogonal Fundamental Solutions of Linear Systems, this MONTHLY, 71 (1964) 406–408.

2. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

3. R. L. Eisenman and D. R. Barr, Matrices Applied to Relative Motion, this MONTHLY, 70 (1963) 82–84.

4. F. R. Gantmacher, Matrix Theory, vol. 1, Chelsea, New York, 1959

A NON-HAUSDORFF TOPOLOGY SUCH THAT EACH CONVERGENT SEQUENCE HAS EXACTLY ONE LIMIT

PAUL SLEPIAN, Rensselaer Polytechnic Institute

1. Introduction. It is well known that if T is a Hausdorff topology, then each sequence to the space of T which is convergent in the classical sense has exactly one limit. If classical convergence is replaced by the more sophisticated

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THEOREM 6. *If X is an orthogonal fundamental solution of* (1), *then a necessary and sufficient condition that $X(t)$ be periodic is that there exist a positive number β and integers n_0 and m_j such that $n_0\sigma = \beta$ and $m_j\beta_j = \beta$ for each nonzero β_j.*

References

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1. Introduction. It is well known that if T is a Hausdorff topology, then each sequence to the space of T which is convergent in the classical sense has exactly one limit. If classical convergence is replaced by the more sophisticated

Moore-Smith convergence, or by the equivalent concept of filter convergence, then as is also well known, the converse of the above theorem is true.

In particular, if T is a topology such that each sequence to the space of T which is convergent in this more sophisticated sense has exactly one limit, then T is a Hausdorff topology.

It seems to be an accepted fact that the converse quoted above is false if convergence is restricted to the classical convergence. This writer, however, could find no such example in the literature. In this note we exhibit a topology T such that

(i) T is not a Hausdorff topology.

(ii) Each sequence to the space of T which is convergent in the classical sense has exactly one limit.

2. Notation.

(i) 0 is the empty set.

(ii) $\omega = \{x \mid x \text{ is a positive integer}\}$.

(iii) $R = \{x \mid x \text{ is a real number}\}$.

3. Remark. To make matters precise we shall give the definition of classical convergence. In particular, we shall say that u is T-convergent to z if and only if

(i) T is a topology.

(ii) $z \in \bigcup_{b \in T} b$.

(iii) u is on ω to $\bigcup_{b \in T} b$.

(iv) If $z \in b \in T$, there exists $n \in \omega$ such that $\{u_j \mid j \in \omega \text{ and } n \leqq j\} \subset b$.

4. Example. Let $T = \{R - A \mid A \subset R \text{ and } A \text{ is countable}\} \cup \{0\}$. Then,

(i) T is a topology.

(ii) T is not a Hausdorff topology.

(iii) If u is T-convergent to x and if u is T-convergent to y, then $x = y$.

Proof of (i): It is easy to verify that

$$(a \in T \text{ and } b \in T) \rightarrow (a \cap b \in T).$$

$$(B \subset T) \rightarrow \Big(\bigcup_{b \in B} b \in T\Big).$$

Proof of (ii): Let $x \in a \in T$ and let $y \in b \in T$. Suppose that $a \cap b = 0$. We shall show that this is impossible. There exists $A \subset R$ such that A is countable and $a = R - A$. But $a \cap b = 0$ implies that $b \subset A$, and thus, b is countable. Also, there exists $B \subset R$ such that B is countable and $b = R - B$. Since $B = R - b$, we conclude that $R - b$ is countable. Finally, $R = b \cup (R - b)$, and since b and $R - b$ are both countable, we conclude that R is countable, which is false.

Proof of (iii): Suppose that u is T-convergent to x and u is T-convergent to y. Suppose that $x \neq y$. Note that $x \in R - \{y\} \in T$. Thus, there exists $n \in \omega$ such that

$$\{u_j \mid j \in \omega \text{ and } n \leqq j\} \subset R - \{y\}.$$

But $\{u_j \mid j \in \omega \text{ and } n \leqq j\}$ is countable. Thus, $y \in R - \{u_j \mid j \in \omega \text{ and } n \leqq j\} \in T$,

implying the existence of $m\in\omega$ such that

$$\{u_j \mid j\in \omega \text{ and } m \leqq j\} \subset R - \{u_j \mid j \in \omega \text{ and } n \leqq j\}.$$

In particular, we have $u_{m+n}\notin \{u_j|j\in\omega \text{ and } n\leqq j\}$, which is impossible.

REPEATED INDEPENDENT TRIALS AND A CLASS OF DICE PROBLEMS

EDWARD O. THORP, New Mexico State University

1. Introduction. This discussion was originally motivated by a class of dice problems. They are illustrated by the following examples, which will be referred to in the sequel.

Assume that two true dice are rolled repeatedly.

Problem 1. Find the probability that both the totals 5 and 9 appear before a 7 appears.

Problem 2. Find the probability that both the totals 4 and 6 appear before a 7 appears.

Problem 3. Find the probability that 4, 5, 6, 8, 9, 10 all appear before 7 appears.

Problem 4. Find the probability that all totals different from 7 appear before a 7 appears.

Problem 1 is intuitively quite simple when we observe that on any one trial $P(5)=P(9)=4/36$ and $P(7)=6/36$, where $P(T)$ is the probability that a total of T occurs on any given trial. We might argue loosely that the probability that either a 5 or a 9 occurs before a 7 is 8/14. The probability that the other one then occurs before a 7 is 4/10. The probability that both 5 and 9 appear before 7 is thus $(8/14)(4/10)=8/35$.

Problem 2 is surprisingly difficult by comparison with problem 1. This is due to the fact that $P(4)=3/36\neq 5/36=P(6)$. We give below the solution to a general problem concerning repeated independent trials, of which problems 2, 3 and 4 are special cases which we will solve as illustrations. Finally we discuss some useful approximations to the general solution.

2. Formal solution of a class of problems. Consider a series of repeated independent trials with the outcomes of each trial being events in a given (fixed) sample space. Let $E_1, \cdots, E_m, B$, be $m+1$ events with B disjoint from each of the E_i. What is the probability that all the events E_i will occur before the event B occurs?

Let A_1^i ($A_2^i, \cdots, A_m^i, B^i$, respectively) be the event that E_1 ($E_2, \cdots, E_m, B$, respectively) does not occur in the first i trials. Let B_{i+1} be the event "B occurs on trial $i+1$" and let F_{i+1} be the event $(A_1^i \cup \cdots \cup A_m^i)B^iB_{i+1}$, where $i=1, 2, \cdots$. Let $F_1=B_1$. Thus F_i is the event that B occurs for the first time on trial i and not all the $E_1, \cdots, E_m$ have yet occurred. Hereafter we refer to

CLASSROOM NOTES

Edited by Gertrude Ehrlich, University of Maryland

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PERIODIC ENTIRE FUNCTIONS

R. P. Boas, Jr., Northwestern University

An entire function $f(z)$ is said to be of exponential type if $|f(z)| \leqq Ae^{B|z|}$ for some numbers A and B. Evidently e^{ikz} is of exponential type and so is any trigonometric polynomial $\sum_{-n}^{n} a_k e^{ikz}$. A trigonometric polynomial is, in addition, periodic with period 2π. It is natural to expect that the converse holds: an entire function of exponential type which has period 2π is necessarily a trigonometric polynomial. This theorem has been discovered and proved a number of times (cf. [1] p. 109). The following very short proof reduces the theorem to the level of a classroom exercise.

Let $|f(z)| \leqq Ae^{B|z|}$ and let $f(z)$ have period 2π. Then $g(w) = f(-i \log w)$ is uniform, and if n is an integer greater than B we have

$$w^n g(w) = \begin{cases} O(|w|^{2n}), & |w| \to \infty, \\ O(1), & |w| \to 0. \end{cases}$$

The second estimate shows that $w^n g(w)$ has a removable singularity at 0. Remove it. Then the first estimate shows that $w^n g(w)$ is a polynomial, so that $f(z) = g(e^{iz})$ is a trigonometric polynomial.

Reference

1. R. P. Boas, Jr., Entire functions, Academic Press, New York, 1954.

A THEOREM IN ELEMENTARY NUMBER THEORY

Eckford Cohen, University of Tennessee

Let $\sigma(n)$ denote the sum of the divisors of the positive integer n. The estimate,

$$\sigma(n) = O(n^{1+\delta}), \tag{1}$$

for every positive δ, is proved in Hardy and Wright ([1] Theorem 322) as a consequence of an analogous result (Theorem 327) for the Euler ϕ-function:

$$\frac{\phi(n)}{n^{1-\delta}} \to \infty \tag{2}$$

for every $\delta > 0$. The latter result is shown to follow from the

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for every $\delta > 0$. The latter result is shown to follow from the

LEMMA (Theorem 316). *If $f(n)$ is multiplicative, and $f(p^e)\to 0$ as $p^e\to\infty$ (p prime, $e\geqq 1$), then $f(n)\to 0$ as $n\to\infty$.*

The relation (Theorem 329),

$$A < \frac{\sigma(n)\phi(n)}{n^2} < 1, \qquad (n > 1) \tag{3}$$

where A is a positive constant, is the means by which (1) is deduced from (2).

The interest of the Hardy-Wright proof lies in the fact that no estimates of sums, finite or infinite, are needed. The point of this note is to make the observation that if one assumes no more than the simple estimate,

$$\sum_{k\leqq n} \frac{1}{k} = O(\log n),$$

then the refinement,

$$\sigma(n) = O(n \log n), \tag{4}$$

can be obtained quite simply without appeal to other arithmetical results, because $\sigma(n)=\sum_{d|n} d=\sum_{d\delta=n}(n/\delta)\leqq n\sum_{\delta\leqq n}1/\delta$. Furthermore, in view of (3), the estimate (2) can be replaced by

$$\frac{1}{\phi(n)} = O\left(\frac{\log n}{n}\right). \tag{5}$$

The estimate (1) can be generalized easily by the Hardy-Wright approach. Let $\sigma_s(n)$ denote the sum of the sth powers of the divisors of n; then we have

$$\sigma_s(n) = O(n^{s+\delta}), \qquad s > 0, \tag{6}$$

for all $\delta>0$. In fact, for primes p,

$$\frac{1}{1-2^{-s}} \geqq \frac{1}{1-p^{-s}} = 1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots \geqq 1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\frac{1}{p^{es}}$$
$$= \frac{p^{es}+p^{(e-1)s}+\cdots+1}{p^{es}} = \frac{\sigma_s(p^e)}{p^{es}}.$$

Hence $\sigma_s(p^e)/p^{e(s+\delta)}\to 0$ as $p^e\to\infty$, and the above lemma applies to give (6).

The result (6) also holds for $s=0$ (Theorem 315); if $s>1$, then (6) is valid with $\delta=0$ by virtue of the convergence of $\sum_{n=1}^{\infty} n^{-s}$.

Reference

1. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1938. (The numbers of the theorems referred to have been retained in the later editions.)

For $kX_i=\theta$, the character that maps A onto 0, $k(aX_i)=0$ for all a in G $\leftrightarrow k(a_iX_i)=0 \leftrightarrow k/n_i$ is an integer$\leftrightarrow k\equiv 0 \pmod{n_i}$.

THEOREM 5. *Let π be a permutation on the set S and let π partition S by cycles as $S=\bigcup S_\alpha$ where π induces the cycle π_α on S_α. Then the order of π is the l.c.m. of the lengths of the cycles π_α.*

Marshall Hall, The Theory of Groups, Macmillan, 1959, page 54, uses Lemma 2 to prove this theorem. The order of π_α is $|S_\alpha|$, the cardinality of S_α. And π^k is the identity on $S\leftrightarrow$each π_α^k is the identity on $S_\alpha\leftrightarrow k$ is a common multiple of the lengths $|S_\alpha|$.

A MIXED NON-GROUP

COLONEL JOHNSON, JR., Southern University

In a first course in abstract algebra, it is usually proved that the Right Axioms (or the Left Axioms) imply the Classical Axioms for a group. Examples are seldom available to show that a set which is closed under an associative binary operation and has, say, a left identity and right inverses for all the elements, need not be a group. Such an example is given in the following.

Let Q be the set of all 2×2 matrices of the form

$$M=\begin{pmatrix} x & y \\ x & y \end{pmatrix},$$

where x, y are real numbers and $x+y\neq 0$.

(i) It is easy to verify that Q is closed under (matrix) multiplication.

(ii) Multiplication is associative.

(iii) The matrix

$$J=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

belongs to Q and serves as a left identity since, for each matrix M in Q, $JM=M$.

(iv) If M is any matrix in Q, then the matrix

$$\begin{pmatrix} 0 & (x+y)^{-1} \\ 0 & (x+y)^{-1} \end{pmatrix}$$

belongs to Q and serves as a right inverse for M with respect to J.

Since J is not a right identity, Q does not form a group under matrix multiplication.

A COUNTEREXAMPLE RELATED TO PRODUCT TOPOLOGY

Z. Z. YEH, University of Hawaii, Honolulu

Let $(X, \mathfrak{I})$ and $(Y, \mathfrak{U})$ be topological spaces, and let $(X\times Y, \mathfrak{I}\times\mathfrak{U})$ be their product space. It is known ([1] pp. 158–159) that if $V\subset X\times Y$ is open relative to the product topology $\mathfrak{I}\times\mathfrak{U}$, then $V[x]=\{y\mid(x, y)\in V\}$ is open relative to $\mathfrak{U}$ for any $x\in X$, and $V[y]=\{x\mid(x, y)\in V\}$ is open relative to $\mathfrak{I}$ for any $y\in Y$. This leads us to consider the following converse: if A is a subset of $X\times Y$ such that $A[x]=\{y\mid(x, y)\in A\}$ is open relative to $\mathfrak{U}$ for any $x\in X$, and $A[y]=\{x\mid(x, y)\in A\}$ is open relative to $\mathfrak{I}$ for any $y\in Y$, is A necessarily open relative to $\mathfrak{I}\times\mathfrak{U}$?

The following counterexample shows that the answer is in the negative: let both $(X, \mathfrak{I})$ and $(Y, \mathfrak{U})$ be the real line R with its usual topology; then their product space is the plane $R\times R$ with its usual topology. Let A be the complement of the two diagonal lines in the plane, with the modification that the origin $(0, 0)$ be an element of A. Clearly for any $x\neq 0$, $A[x]$ is the complement of two points in R, hence is open; while $A[0]$ is R, hence is also open. Likewise, $A[y]$ is open for any $y\in Y$. But A is not open, because $(0, 0)$ is not an interior point of A.

As a simpler counterexample, we may take the complement of any line segment with one end open laid in a slanting position in the plane. For example, consider $A=R\times R-\{(x, y)\mid x=y,\ 0<x\leqq 1\}$.

Reference

1. J. L. Kelley, General Topology, Van Nostrand, Princeton, 1955.

ELECTRIC BY ONE VOLT

MARLOW SHOLANDER, Western Reserve University

Mary had a little lamp.
How farad ohm! Like snow!
Mhost every weber Mary went
It cast oerstedy glow.

When it ampered at coulomb day,
Which was abhenry rule,
Said teacher Max, "Well I'm agaussed.
Though ergsome, watt a joule!"

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1. J. L. Kelley, General Topology, Van Nostrand, Princeton, 1955.

ELECTRIC BY ONE VOLT

MARLOW SHOLANDER, Western Reserve University

Mary had a little lamp.
How farad ohm! Like snow!
Mhost every weber Mary went
It cast oerstedy glow.

When it ampered at coulomb day,
Which was abhenry rule,
Said teacher Max, "Well I'm agaussed.
Though ergsome, watt a joule!"

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington D. C. 20005

THE Ph.D. CLASS OF 1951

G. S. YOUNG, Tulane University

There were 217 Ph.D.'s in mathematics given in 1951 by universities in the United States and Canada. The purpose of this note is to say something about their publications and their present activities, and to point out some implications of these facts.

A. *Present activity.* The positions of the class members were determined first by looking in the 1963 Combined Directory of the Association, the Society and SIAM, then by looking in American Men of Science. Table I summarizes the results.

TABLE I. Location and rank around 1963 of 1951 Ph.D.'s

Group	Number		Papers		Mean
	N	%	N	%	
University Personnel					
Professors	68	31.4	584	49.9	8.6
Associate Professors	45	20.7	222	19.0	4.9
Assistant Professors	13	6.0	79	6.8	6.1
Other	18	8.3	68	5.8	3.8
All University Personnel	144	66.4	953	81.5	6.6
Industry	45	20.7	130	11.1	2.9
Unknown	28	12.9	87	7.4	3.1
Total	217	100.0	1,170	100.0	5.4

"Industry" is a classification that includes government work, or anything else that was clearly not teaching. The term "Unknown" merely means that the person was not found in the two sources; it includes several women who have married, several people who have died, and at least one prominent mathematician who simply isn't listed.

B. *Publications.* The number of papers published by each member of the class was determined by searching the indexes of the Mathematical Reviews. This means that the totals represent papers published through 1960 or 1961, probably, due to the delay in reviewing. No one published more than 29 papers in this period. Table II summarizes the data.

TABLE II. Number of papers published by class of 1951

Number of papers	Number of persons	Cumulative
$\geqq 20$	16	16
$20 > n \geqq 10$	24	40
$10 > n \geqq 5$	42	82
$n=4$	13	95
$n=3$	18	113
$n=2$	20	133
$n=1$	44	177
$n=0$	40	217

Roughly $\frac{1}{5}$ of the class have published ten or more papers; $\frac{1}{5}$ have published one paper, presumably the thesis, and $\frac{1}{5}$ have published nothing in a mathematical journal.

C. *Type of school.* I turn now to the classification of the Ph.D.'s in university and college teaching. I selected a list of what I felt were the strongest 25 departments and measured rank and publication in these; then I did the same for the other schools. The next table gives the result.

TABLE III. Schools, ranks, and publications for the class of 1951

Group	Number		Papers		Mean
	N	%	*N*	%	
Twenty-five strong schools					
Professor	29	20.1	435	45.6	15.0
Associate	17	11.8	107	11.2	6.3
Assistant	6	4.2	51	5.4	8.5
Other	9	6.3	58	6.1	6.4
Total	61	42.4	651	68.3	10.7
Other universities and colleges					
Professor	39	27.1	149	15.6	3.8
Associate	28	19.4	115	12.1	4.1
Assistant	7	4.9	28	2.9	4.0
Other	9	6.2	10	1.1	1.1
Total	83	57.6	302	31.7	3.6
All universities					
Professor	68	47.2	584	61.3	8.6
Associate	45	31.3	222	23.3	4.9
Assistant	13	9.0	79	8.3	6.1
Other	18	12.5	68	7.1	3.8
Total	144	100.0	953	100.1	

crisis. The strong schools have had no real trouble in getting some Ph.D.'s, perhaps less than they wanted, and perhaps not as able as they wanted, but Ph.D.'s. This will presumably continue. The other schools are in for trouble, with college enrollments going up sharply, and a greater percentage of students majoring in mathematics. Dr. Clarence Lindquist has kindly let me see figures before publication that indicate around 40,000 students graduating in 1970 with first degrees in mathematics, about one out of 19 such degrees, and 50,000 in 1975, about one out of 17. In 1951 there were 5753 bachelor's degrees in mathematics; in 1961, there were 11,437 [3b]. Even if there is a 25% error in Dr. Lindquist's figures, prospects are alarming.

Is it not time for the mathematical community to face up to the fact that for a long time most undergraduate teaching will be done by non-Ph.D.'s, and to begin a study of means of identifying competency among such persons?

I wish to thank Mrs. Irene Vines for her careful preparation of the data used.

References

1. Amer. Math. Soc. Notices, Special Issue, Assistantships and Fellowships in Mathematics in 1962–63, Vol. 8, No. 7, Part II, December, 1961.

2. Conference Board of Mathematical Sciences, Report of a Conference on Mathematical Manpower, 1963.

3. (a) Clarence B. Lindquist, Mathematics and Statistics Degrees during the Decade of the Fifties, this MONTHLY, 68 (1961). (b) C. B. Lindquist, Degrees in the Biological and Physical Sciences, Mathematics and Engineering; 1949–60, Office of Education, U. S. Department of Health, Education, and Welfare. (c) C. B. Lindquist, Trends of degrees, 1954–55 to 1969–70, Physical and Biological Sciences, Mathematics and Engineering, The Educational Record, Spring 1964.

GRANTS FOR STAFF IN SMALL COLLEGES

CHARLES H. SCHAUER, Research Corporation, New York

Several years ago Research Corporation initiated a program of grants intended to help strengthen research-oriented science departments at liberal arts colleges and smaller universities. During the past two years, the mathematics departments of two liberal arts colleges (Bowdoin and Knox) and of one smaller university (Idaho State College) have received departmental grants under the program, and mathematics has participated in grants of broader scope at several other colleges.

Research Corporation's all-too-finite resources have dictated the necessity of a high degree of selectivity even in the consideration of proposals for these grants. Accordingly, proposals are accepted only on invitation by a staff member of the foundation, usually after he has visited the campus and discussed the possibility with the faculty members involved and with college administrators.

A primary consideration is evidence of existing strength and research inter-

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The conference examined the increasing demand for highly skilled engineers and scientists and the concomitant decreasing supply of trained man power in practically all fields of science and technology. Ways and means were sought to identify and put to use the largest untapped source of engineering aptitudes and talents and to establish the need to encourage training of latent woman power now only in token use.

COURSE CONTENT WORK AT STANFORD

Three course content groups supported by the National Science Foundation convened for an eight-week writing session at Stanford University, June 22 to August 15. There were forty to sixty participants in each. The groups included the School Mathematics Study Group and a group sponsored by the Committee on Educational Media of the Mathematical Association of America which had panels on preservice training of elementary teachers, calculus, programmed learning, and number systems. The AAAS Commission on Science Education sponsored a group of scientists, including mathematicians, to work on the development of experimental course materials in science for grades K-5. The AAAS experimental course materials contain exercises on mathematics, introducing such topics as division, rational numbers, fractions, measurement and probability in the early grades so that these topics may be used in science exercises.

THE LAW OF INERTIA

I should rejoice to see mathematics taught with that life and animation which the presence and example of her young and buoyant sister, empirical science, could not fail to impart; short roads preferred to long ones; Euclid honorably shelved or buried "deeper than did ever plummet sound" out of the school boy's reach; morphology introduced into the elements of algebra; projection, correlation, and motion accepted as aids in geometry; the mind of the student quickened and elevated and his faith awakened by early initiation into the ruling ideas of polarity, continuity, infinity, and familiarization with the doctrine of the imaginary and inconceivable.—J. J. Sylvester, *Nature*, January 6, 1870. (Contributed by K. O. May.)

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E 1716. *Proposed by J. D. Brillhart, University of San Francisco, and R. L. Graham, Bell Telephone Laboratories*

If $(a_1, a_2, \cdots, a_n)$ denotes some arrangement of the first n positive integers, show that at least $\sqrt{n}$ distinct residue classes modulo n occur in the set $\{\sum_{j=1}^{k} a_j: 1 \leqq k \leqq n\}$.

E 1717. *Proposed by V. F. Ivanoff, San Carlos, California*

Prove the dual of Ptolemy's theorem on the spherical surface, namely: In the circumscribed spherical quadrilateral with the sides a, b, c, d touching the incircle in that order,

$$\cos \tfrac{1}{2}(a, c) \cdot \cos \tfrac{1}{2}(b, d) = \cos \tfrac{1}{2}(a, b) \cdot \cos \tfrac{1}{2}(c, d) + \cos \tfrac{1}{2}(a, d) \cdot \cos \tfrac{1}{2}(b, c).$$

(Note. This proposition holds for any plane quadrilateral whether circumscribed or not.)

E 1718. *Proposed by J. D. Cloud, Manhattan Beach, California*

Prove that, in every solution in positive integers of the Diophantine equation $x^n = y!$, either $n=1$ or $x=1$.

E 1719. *Proposed by D. I. A. Cohen, Princeton University*

Let R be the number formed by reversing the digits of the n-digit number N, $R>N$. Prove that $\sqrt{R} \leqq R-N$, and prove that if $10\sqrt{R} \geqq R-N$ then n is even.

E 1720. *Proposed by R. L. Caskey, Oklahoma State University*

Find a locus such that the tangent and the normal at each of its points intersect two given lines in four concyclic points. (Dedicated to Dr. N. A. Court.)

SOLUTIONS OF ELEMENTARY PROBLEMS

Integer Solutions of $axy+bx+cy+d=0$

E 1631 [1963, 1005]. *Proposed by Roy Feinman, Rutgers University*

Let a, b, c, d be integers, with $a \neq 0$. Can $axy+bx+cy+d=0$ have infinitely many solutions in integral x and y?

Solution by Richmond G. Albert, West Newton, Massachusetts. The equivalent equation $a^2xy+abx+acy+ad=0$ can be written as $(ax+c)(ay+b)=bc-ad$. If $bc-ad \neq 0$, $ax+c$ is a factor of $bc-ad$ and, hence, x can have at most a finite number of integral values. Similarly for $ay+b$ and y. If $bc-ad=0$, then there is a solution, and hence infinitely many, if and only if $a|c$ or $a|b$ (in which case $x=-c/a$ and y integral or x integral and $y=-b/a$, respectively, are solutions).

Also solved by Joseph Arkin, J. W. Baldwin, Joseph Bechely, E. D. Bender, W. J. Blundon, Allan W. Brunson, Leonard Carlitz, Allan Chuck and Peter Goldstein (jointly), J. D. Cloud, D. I. A. Cohen, M. J. Cohen, M. S. Demos, George Diderrich, E. S. Eby, R. L. Farrell, Gregory Forster, Michael Goldberg, Jerry Goodman, Ralph Greenberg, R. A. Jacobson, Hajna János and Horváth Sándor (jointly), Erwin Just and Norman Schaumberger (jointly), L. C. Larson, Charles Lewis,

D. C. B. Marsh, J. B. Muskat, A. E. Newman, Stanton Philipp, J. F. Polk, Jr., E. H. Primoff, George Purdy, A. S. Rosenthal, C. M. Sandwick, Robin Sibson, D. J. Silverman, Richard Sinkhorn, Eric Sturley, Simon Vatriquant, Andrew Vince, and the proposer. Not all these solutions were complete.

Polynomials Prime for Prime Argument

E 1632 [1963, 1005]. *Proposed by Michael Fried, Bell Aerosystems, Wheatfield, N. Y.*

Show that there is no polynomial which assumes prime values for all prime values of the argument.

Solution by Stanton Philipp, Seal Beach, California. The result is of course false as stated. We prove instead that if $P(x)$ is a polynomial with complex coefficients such that $P(x)$ is prime for all prime x, then either $P(x)\equiv x$ or $P(x)\equiv c$, a constant. *Proof*: Suppose that $P(x)=a_0+a_1x+\cdots+a_nx^n$ is prime for prime x, and that $P(x)\not\equiv x, P(x)\not\equiv c$. If $p_1, p_2, \cdots, p_{n+1}$ are $n+1$ distinct primes, then the a_i are rationally expressible in terms of the p_i; so the coefficients of $P(x)$ are rational. Then there is an integer K such that $f(x)\equiv KP(x)$ has integral coefficients. There exists a prime π such that $(\pi, f(\pi))=1$; for if not, $p|KP(p)$ for all primes p, and it would follow that $P(x)\equiv 0$ or $P(p)=p$ for the infinitely many primes p satisfying $(K, p)=1$, and, hence, $P(x)\equiv x$. Now, by Dirichlet's Theorem, there are infinitely many integers q_i such that $q_if(\pi)+\pi$ is prime, say $q_if(\pi)+\pi=s_i$, $i=1, 2, \cdots$. But $f(\pi)|f\{q_if(\pi)+\pi\}$ and then $KP(\pi)|KP(s_i)$, $P(\pi)|P(s_i)$, $P(\pi)=P(s_i)$, $P(x)\equiv P(\pi)$, a contradiction.

Also solved by Jack Abad, J. W. Baldwin, W. R. Becker, E. D. Bender, William Bonney, Leonard Carlitz, D. I. A. Cohen, M. J. Cohen, Robert Cohen, E. S. Eby, Michael Goldberg, Peter Goldstein, Jerry Goodman, H. S. Hahn, Erwin Just and Norman Schaumberger (jointly), Sidney Kravitz, E. S. Langford, E. L. Magnuson, D. C. B. Marsh, L. A. Ringenberg, F. G. Schmitt, Jr., E. V. Schuman, Robin Sibson, E. C. Stopher, Rory Thompson, A. M. Vaidya, Simon Vatriquant, W. C. Waterhouse, and the proposer.

Some of these solutions consisted only of the obvious counterexamples to the problem as originally stated.

A Diophantine Equation in the Calculus

E 1633 [1963, 1005]. *Proposed by Helen M. Marston, Douglass College*

A common problem in elementary calculus is: "An open box is made by cutting squares of side x from the corners of an a by b rectangle and folding up the sides. For what value of x is the volume a maximum?" When $a=b$, the answer to this problem is $a/6$. For what integers a and b, $a\neq b$, $(a, b)=1$, is the answer rational, and how many such a by b rectangles are there of length less than 50?

Solution by Loren C. Larson, St. Olaf College, Northfield, Minnesota. It is easily verified by the methods of elementary calculus that x is rational if and only if a^2+b^2-ab is a perfect square.

Suppose $0<a<b$. It follows that $a^2<a^2+b^2-ab<b^2$. Hence if a^2+b^2-ab is a perfect square, there exist integers m and n such that $a^2+b^2-ab=(a+n)^2$

and $b=a+m$. Solving these latter two equations for a and b we find that $a=(n^2-m^2)/(m-2n)$ and $b=(n(n-2m))/(m-2n)$. On the other hand, if a and b are given by these expressions then

$$a^2+b^2-ab=\left[\frac{n^2+m^2-nm}{m-2n}\right].$$

From the preceding, it is clear that for every pair of integers $0<n<m$, $a=m^2-n^2$ and $b=n(2m-n)$ will give a rectangle with sides proportional to the rectangles we desire. Since $a<b$ we may set $m=2n-j$ and write $a=(n-j)(3n-j)$ and $b=n(3n-2j)$. Without loss of generality, $(n,j)=1$. It follows that $(a,b)=1$ unless j is a multiple of 3 in which case $(a/3, b/3)=1$.

There are twelve such rectangles of length less than 50; *viz.* 5×8, 16×21, 33×40, 7×15, 3×8, 8×15, 24×35, 35×48, 11×35, 13×48, 5×21, and 7×40.

Also solved by J. W. Baldwin, Merrill Barnebey, M. S. Demos, Michael Goldberg, S. H. Greene, T. R. Hoffman and W. C. Stone (jointly), J. E. Jean, Jr., Erwin Just and Norman Schaumberger (jointly), D. C. B. Marsh, R. S. McDowell, F. D. Parker, Stanton Philipp, Eric Sturley, J. E. Tyson, Simon Vatriquant, Raymond Whitney, E. J. Ziarelli, and the proposer. Not all solutions were complete.

Several references were given for the solution of the Diophantine equation $a^2-ab+b^2=c^2$: L. E. Dickson, *Introduction to the Theory of Numbers*, Exercise 6, p. 48; this MONTHLY, 21 (1914) 98–99, and 44 (1937) 113–115.

Application of an Isoperimetric Problem

E 1634 [1963, 1005]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College*

Let a_i, $i=1, 2, \cdots, n$, be the sides of a convex polygon with area K. Prove that

$$\sum_{i=1}^{n} a_i^2 \geqq 4K\tan(\pi/n).$$

Solution by D. C. B. Marsh, Colorado School of Mines. We use the fact that of all n-gons with the same perimeter, the regular n-gon has greatest area. Thus $K\leqq na^2/[4\tan(\pi/n)]$ where $na=\sum_{i=1}^n a_i$. However, the Cauchy-Schwarz Inequality gives

$$na^2=n^{-1}\left(\sum_{i=1}^{n} a_i\right)^2 \leqq n^{-1}\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} 1\right)=\sum_{i=1}^{n} a_i^2,$$

whence the desired inequality follows.

Also solved by W. H. Bailey and Guy Torchinelli (jointly), J. W. Baldwin, Leonard Carlitz, D. I. A. Cohen, M. J. Cohen, Michael Goldberg, G. A. Kandall, W. W. Leutert, Stanton Philipp, Andy Vince, and the proposers.

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There are twelve such rectangles of length less than 50; *viz.* 5×8, 16×21, 33×40, 7×15, 3×8, 8×15, 24×35, 35×48, 11×35, 13×48, 5×21, and 7×40.

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Finally it is clear that $f(2n)=2f(2n-1)$ (since every admissible sequence of length $2n-1$ gives rise to two admissible sequences of length $2n$). Therefore

$$f(2n) = 2\binom{2n-1}{n-1} = \binom{2n}{n}.$$

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* The proposed problem may be identified with "Determine the number of different outcomes of a sequence of n games which may be played by an individual with a \$1 stake, who is not ruined during the course of play, by betting \$1 a game against an infinitely rich adversary, where each has an equal probability of winning each game." This formulation is completely analysed by J. V. Uspensky in his *Introduction to Mathematical Probability* (pp. 147–151). A simple adaptation of his equation (12) gives the desired function as

$$2^n - \sum_{k=0}^{m} \frac{(2k)!}{k!(k+1)!} 2^{n-1-2k} \quad \text{where} \quad m = [\tfrac{1}{2}(n-1)].$$

[An easy induction shows this sum to have the same value as that in Solution I. *Ed.*]

Also solved by J. R. Blum, D. I. A. Cohen, Ira Ewen, Michael Goldberg, J. D. Haggard, R. E. Jones, David Klappholz, E. S. Langford, S. G. Mohanty, Stanton Philipp, H. S. Piper, Jr., Stanley Rabinowitz, V. K. Rohatgi, Max Rosenberg, F. G. Schmitt, Jr., Robin Sibson, Robert Singleton, Rory Thompson, B. B. Winter, and the proposer.

The given problem with solution was located in various guises in W. Feller, *An Introduction to Probability Theory and its Applications*, 1st ed., pp. 249–257, and 2nd ed., pp. 70–75; P. A. MacMahon, *Combinatory Analysis*, p. 127; and J. Amer. Stat. Assoc., 57, pp. 327–337.

Number of Fibonacci Numbers Not Exceeding N

E 1636 [1963, 1005]. *Proposed by J. D. Cloud, North American Aviation, Inc.*

Given a positive integer N, how many Fibonacci numbers are there not exceeding N?

Solution by William D. Jackson, State University College, Oswego, N. Y. The Fibonacci number u_n is the nearest whole number to $[(1+\sqrt{5})/2]^n/\sqrt{5}$ (N. N. Vorob'ev, *Fibonacci Numbers*, p. 22). Therefore, given N, $u_n \leqq N$ if and only if $[(1+\sqrt{5})/2]^n/\sqrt{5} < N+\frac{1}{2}$; in other words, if and only if $n<\log[(N+\frac{1}{2})\sqrt{5}]/\log[(1+\sqrt{5})/2]$. Since $u_1=u_2=1$, the number of Fibonacci numbers not greater than N is the greatest integer less than

$$\frac{\log[(N+\frac{1}{2})\sqrt{5}]}{\log\dfrac{1+\sqrt{5}}{2}} - 1.$$

Also solved by J. L. Brown, Jr., Flor Cartuyvels, D. I. A. Cohen, Michael Goldberg, S. H. Greene, Cornelius Groenewoud, Kenneth Kramer, D. C. B. Marsh, F. D. Parker, Stanton Philipp, Robert Singleton, Rory Thompson, and the proposer.

Finally it is clear that $f(2n)=2f(2n-1)$ (since every admissible sequence of length $2n-1$ gives rise to two admissible sequences of length $2n$). Therefore

$$f(2n) = 2\binom{2n-1}{n-1} = \binom{2n}{n}.$$

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* The proposed problem may be identified with "Determine the number of different outcomes of a sequence of n games which may be played by an individual with a \$1 stake, who is not ruined during the course of play, by betting \$1 a game against an infinitely rich adversary, where each has an equal probability of winning each game." This formulation is completely analysed by J. V. Uspensky in his *Introduction to Mathematical Probability* (pp. 147–151). A simple adaptation of his equation (12) gives the desired function as

$$2^n - \sum_{k=0}^{m} \frac{(2k)!}{k!(k+1)!} 2^{n-1-2k} \quad \text{where} \quad m = [\tfrac{1}{2}(n-1)].$$

[An easy induction shows this sum to have the same value as that in Solution I. *Ed.*]

Also solved by J. R. Blum, D. I. A. Cohen, Ira Ewen, Michael Goldberg, J. D. Haggard, R. E. Jones, David Klappholz, E. S. Langford, S. G. Mohanty, Stanton Philipp, H. S. Piper, Jr., Stanley Rabinowitz, V. K. Rohatgi, Max Rosenberg, F. G. Schmitt, Jr., Robin Sibson, Robert Singleton, Rory Thompson, B. B. Winter, and the proposer.

The given problem with solution was located in various guises in W. Feller, *An Introduction to Probability Theory and its Applications*, 1st ed., pp. 249–257, and 2nd ed., pp. 70–75; P. A. MacMahon, *Combinatory Analysis*, p. 127; and J. Amer. Stat. Assoc., 57, pp. 327–337.

Number of Fibonacci Numbers Not Exceeding N

E 1636 [1963, 1005]. *Proposed by J. D. Cloud, North American Aviation, Inc.*

Given a positive integer N, how many Fibonacci numbers are there not exceeding N?

Solution by William D. Jackson, State University College, Oswego, N. Y. The Fibonacci number u_n is the nearest whole number to $[(1+\sqrt{5})/2]^n/\sqrt{5}$ (N. N. Vorob'ev, *Fibonacci Numbers*, p. 22). Therefore, given N, $u_n \leqq N$ if and only if $[(1+\sqrt{5})/2]^n/\sqrt{5} < N+\frac{1}{2}$; in other words, if and only if $n < \log[(N+\frac{1}{2})\sqrt{5}]/\log[(1+\sqrt{5})/2]$. Since $u_1 = u_2 = 1$, the number of Fibonacci numbers not greater than N is the greatest integer less than

$$\frac{\log[(N+\frac{1}{2})\sqrt{5}]}{\log\dfrac{1+\sqrt{5}}{2}} - 1.$$

Also solved by J. L. Brown, Jr., Flor Cartuyvels, D. I. A. Cohen, Michael Goldberg, S. H. Greene, Cornelius Groenewoud, Kenneth Kramer, D. C. B. Marsh, F. D. Parker, Stanton Philipp, Robert Singleton, Rory Thompson, and the proposer.

Continuous Connected Multiplicative Subgroups of the Complex Numbers

E 1639 [1963, 1006]. *Proposed by Reuben Hersh, Stanford University*

Find all subsets of the complex plane which, like the positive real axis or the unit circle, are (1) subgroups of the multiplicative group of complex numbers, and (2) continuous connected curves.

I. *Solution by Miltiades S. Demos, Drexel Institute of Technology.* If the subset contains $se^{i\alpha}(s\neq 1, \alpha\neq 0)$ then it must contain $s^n e^{in\alpha}$ by (1) and $s^t e^{it\alpha}$ by (2), where t is any real number. Therefore, setting $|z|=r$, arg $z=\theta$, we get $r=s^t$, $\theta=t\alpha$ and consequently $r=s^{\theta/\alpha}=e^{k\theta}$ or $z=e^{(k+i)\theta}$. So all equiangular spirals, passing through the point 1, (including the two degenerate ones mentioned in the problem), are the only sets satisfying the conditions of the problem.

II. *Solution by the proposer.* Evidently if w is an arbitrary fixed complex number, and θ is a real parameter running from minus infinity to plus infinity, the points $z=e^{\theta w}$ are a solution. To prove there are no others, we first remark, as is easily verified, that our set can contain a segment of a ray only if it consists solely of the positive real axis; but this case is included above, if we take w real. For all other cases, then we may use $\theta=\arg z$ as the parameter of the curve; $z=r(\theta)^{ei\theta}$. Then, if $z_1=r(\theta_1)e^{i\theta_1}$ and $z_2=r(\theta_2)^{i\theta_2}$ are two points in our set, there must be a value of $\theta=\theta_3$ such that $z_1z_2=z(\theta_3)$, or $r(\theta_1)e^{i\theta_1}r(\theta_2)e^{i\theta_2}=r(\theta_3)e^{i\theta_3}$, so $\theta_1+\theta_2=\theta_3$, $r(\theta_1)r(\theta_2)=r(\theta_3)=r(\theta_1+\theta_2)$. Let $f(\theta)=\log r(\theta)$. Then $f(\theta_1)+f(\theta_2)=f(\theta_1+\theta_2)$. But it is well-known that the only continuous functions f satisfying this equation are $f(\theta)=c\theta$. So $r=e^{c\theta}$, and $z=e^{(c+i)\theta}$.

Also solved by Flor Cartuyvels, D. I. A. Cohen, Michael Goldberg, Stephen Hoffman, D. C. B. Marsh, and Donna Seaman.

Cubes from Pyramids

E 1640 [1963, 1006]. *Proposed by C. W. Trigg, Los Angeles City College*

The edges of a triangular pyramid and a quadrilateral pyramid are all equal. Show that the two pyramids may be dissected into pieces which may be reassembled into a single cube.

Solution by Kenneth Kramer, Columbia College. Consider a cube with "top" vertices labelled P_1, P_2, P_3, P_4 (in a clockwise sense, say) and corresponding "bottom" vertices P_5, P_6, P_7, P_8. Joining P_1, P_3, P_6, and P_8 to each other yields a triangular pyramid whose edges (of length a) are diagonals of the faces of the cube. The remaining four congruent solids may then be reassembled to form a quadrilateral pyramid with edge lengths a by "matching" right-triangular faces as follows: $P_2P_3P_6\leftrightarrow P_4P_3P_8$, $P_1P_2P_6\leftrightarrow P_8P_5P_8$, $P_1P_5P_6\leftrightarrow P_3P_7P_6$. To solve the given problem, simply reverse the process.

Also solved by R. G. Albert, J. W. Baldwin, J. Basile, D. I. A. Cohen, D. E. Daykin, C. M. Frye, Michael Goldberg, J. D. Haggard, Ned Harrell, D. C. B. Marsh, Robert Singleton, and the proposer.

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Clearly, S takes $C_{[0,1]}$ into (in fact, onto) itself. Prove that there exists a unique function K on $C_{[0,1]}$ such that for all f, (1) $K(f)\in$Range (f), and (2) $K(Sf) = K(f)$.

5222. *Proposed by M. Rajagopalan and A. Wilansky, Lehigh University*

Must every convergent net in a metric space have a subnet whose range has exactly one limit point?

5223. *Proposed by C. R. MacCluer, University of Michigan*

Let L be the additive Abelian group of all ∞-tuples $(a_1, a_2, \cdots)$ where the nth entry is drawn from the integers modulo p^n, p a fixed prime, and let addition in L be coordinate-wise. Let G be the torsion subgroup and H the subgroup of all elements that have almost all zero entries. Show that H is not a direct summand of G. (This provides an example showing that purity of H does not imply that H is a direct summand even in the torsion case.)

5224. *Proposed by L. Carlitz, Duke University*

Let $\psi(a) = (a/p)$, the Legendre symbol. Show that, if $abcd \not\equiv 0 \pmod p$,

$$S = \sum_{x,y,z=0}^{p-1} \psi(ax^2 + by^2 + cz^2 - 2dxyz) = -p\{\psi(a) + \psi(b) + \psi(c) + \psi(-abc)\}.$$

5225. *Proposed by Solomon Marcus, University of Bucharest, Romania*

Let B be a given, nowhere dense, and perfect set on the real line. For any function F, with continuous derivatives on $[0, 1]$, put

$$S_1 = \{t\colon F'(t) = 0\}, \qquad S_2 = \{F(t)\colon t \in S_1\}.$$

Does there exist a function F such that $S_2 = B$? (See a closely related problem, no. 5114 [1964, 693].)

5226. *Proposed by E. O. Thorp, New Mexico State University*

Let S and T be arbitrary point sets. The Banach space $l_1(S)$ is the set of functions x on S such that $x(s) \neq 0$ for at most countably many s and such that the norm of x, defined by $|x| = \sum_{s\in S} |x(s)|$ is finite. The Banach space $l_\infty(S)$ is the set of bounded functions x on S with the norm of x defined by $|x| = \sup_{s\in S} |x(s)|$. When is $l_1(S)$ isometrically isomorphic to $l_\infty(T)$?

SOLUTIONS OF ADVANCED PROBLEMS

Weakly Continuous Mapping in a Hausdorff Space

5069 [1963, 97; 1964, 102]. *Proposed by D. R. Andrew, University of Southwestern Louisiana*

Prove or disprove the following statement. If S is a Hausdorff space and $f\colon S\to T$ is a weakly continuous one-to-one mapping of S onto the space T such that $f^{-1}\colon T\to S$ is weakly continuous, then T is necessarily a Hausdorff space. (See Norman Levine, *A Decomposition of Continuity in Topological Spaces*, this MONTHLY, 1961, pp. 44–46, for the definition of a weakly continuous function.)

2. *Solution by Kenneth E. Whipple, Auburn University.* T is not necessarily Hausdorff, as the following counterexample will show.

Denote by R and I the set of rationals and the set of irrationals respectively. Let $E_1=(0, 1]\cdot R$, $E_2=(1, 2)\cdot I$, and $E_3=(2, 3]\cdot R$, and let S be the space having as points, points of $E_1\cup E_2\cup E_3$, and having a basis $B=B_1\cup B_2\cup B_3$, where

$$B_1 = \{(x, y)\cdot R\colon 0 < x < y < 1\} \cup \{(x, 1]\cdot R\colon 0 < x < 1\},$$
$$B_2 = \{(x, y)\cdot R \cup (x + 1, y + 1)\cdot I \cup (x + 2, y + 2)\cdot R\colon 0 < x < y < 1\},$$
$$B_3 = \{(x, y)\cdot R\colon 2 < x < y < 3\} \cup \{(x, 3]\cdot R\colon 2 < x < 3\}.$$

It follows that S is Hausdorff, in particular note that if $x\in E_1$ and $y\in E_2$ then $y-x\neq 1$ since x is rational and y is irrational, which allows a separation by elements of B_1 and B_2.

Let T be the space having as points, points of S and having a basis $C=C_1 \cup C_2\cup C_3$, where

$$\begin{aligned} C_1 &= \{(x, y)\cdot R\cup (x + 1, y + 1)\cdot I\colon 0 < x < y < 1\} \\ &\quad \cup \{(x, 1]\cdot R\cup (x + 1, 2)\cdot I\colon 0 < x < 1\}, \\ C_2 &= \{(x, y)\cdot I\colon 1 < x < y < 2\}, \\ C_3 &= \{(x, y)\cdot I\cup (x + 1, y + 1)\cdot R\colon 1 < x < y < 2\} \\ &\quad \cup \{(x, 2)\cdot I\cup (x + 1, 3]\cdot R\colon 1 < x < 2\}. \end{aligned}$$

T is not Hausdorff since the point 1 cannot be separated from the point 3.

Let f be the identity transformation from S onto T. Then f is weakly continuous. Suppose that $p\in T$ and V is an element of C containing p. If $p\in E_1$ (or E_3), then there is an element U of B_1 (or B_3) containing p such that $U\subset V$. If $p\in E_2$, then C_2 contains an element V' of the form $(x, y)\cdot I$, $1<x<y<2$, such that $p\in V'\subset V$. Then $\overline{V}'=[x-1, y-1]\cdot R\cup[x, y]\cdot I\cup[x+1, y+1]\cdot R$, hence B_2 contains an element U such that $p\in U\subset\overline{V}'\subset\overline{V}$.

f^{-1} is weakly continuous. Suppose that $p\in S$ and U is an element of B containing p. If $p\in E_2$, then C_2 contains an element V such that $p\in V\subset U$. If $p\in E_1$, then B_1 contains an element U' of the form $(x, y)\cdot R$ or $(x, y]\cdot R$, $0<x<y\leq 1$, such that $p\in U'\subset U$. Then $\overline{U}'=[x, y]\cdot R\cup[x+1, y+1]\cdot I$ and C_1 contains an element V such that $p\in V\subset\overline{U}'\subset\overline{U}$. A similar argument holds if $p\in E_3$.

Editorial Note. The former solution [1964, 102] has been criticized because a topology has not been properly defined. In particular, $\{A_n\}$ is not a base at 1.

Angle Preserving Map

5084 [1963, 335; 1964, 223]. *Proposed by Robert Spira, University of California, Berkeley*

Find a one-to-one continuous function on the unit disk into the plane which is angle-preserving, yet not analytic.

Further comment by George Bergman, Harvard University. The problem has been solved: See D. Menchoff, *Sur la représentation conforme des domaines plans*, Math. Annalen, 95 (1926) 641–670. Menchoff proves: any one-to-one continuous map of a region of the plane onto a region of the plane which is angle-preserving except perhaps at countably many points, is analytic. Rademacher, (*Über streckentreue und winkeltreue Abbildungen*, Math. Zeitschrift, 4 (1919), 34–35) gets the same conclusion with the hypothesis "one-to-one" replaced by "$|(f(z')-f(z))/(z'-z)|$ bounded."

Convex Function

5088 [1963, 336; 1964, 330]. *Proposed by Joe Lipman, Queen's University, Canada*

In Meschkowski, *Unsolved and Unsolvable Problems in Geometry* (Vieweg & Son, 1960) a function defined on a convex set C of reals is called convex if $2f((x_1+x_2)/2) \leqq f(x_1)+f(x_2)$ everywhere in C.

a) Show that if f is bounded above on some subinterval of C, then this definition agrees with the usual one, viz: for any t in $(0, 1)$, $f(tx_1+(1-t)x_2) \leqq tf(x_1)+(1-t)f(x_2)$ everywhere in C.

b) Find a function on $(-\infty, -\infty)$ which is not convex (in the usual sense) but satisfies the inequality $2f((x_1+x_2)/2) < f(x_1)+f(x_2)$ for all $x_1 \neq x_2$.

Editorial Note. The proposer calls attention to an omission in the published proof [1964, 330], where the implicit assumption is made that the function f is bounded on every subinterval $[x_1, x_2]$ of C. The hypothesis, however, states only that f is bounded on some subinterval of C.

To complete the proof we show that if $2((f(x_1+x_2)/2) \leqq f(x_1)+f(x_2)$ and if f is bounded on some subinterval of C, then f is bounded on every proper subinterval of C. Let f be bounded on the closed interval I: $[a, b]$ and suppose that the interval I^+: $[b, 2b-a]$ is in C (otherwise start with a subinterval of I). If y is in I^+, then $(a+y)/2$ is in I and $f(y) \geqq 2f((a+y)/2)-f(a)$; that is, f is bounded from below on I^+. For y in the left half of I^+, $x=2y+a-2b$ is in I and $2f(y) \leqq f(x)+f(2b-a)$. Therefore $f(y)$ is bounded on the left half of I^+. Similarly we can extend the original interval of boundedness on the left, and in a finite number of steps such extensions can include any subinterval in the interior of C.

Asymptotic Distribution of Sequences

5090 [1963, 336; 1964, 332]. *Proposed by Fred Suvorov, Princeton University*

Let $\{x_n\}$ be a sequence of positive real numbers. Consider the set A of all real numbers a such that $\{x_n\}$ converges to 0 (mod a), and show that this set has measure 0. The sequence $\{x_n\}$ is said to converge to 0 (mod a) if the residue classes of the x_n on the circle R/Ra converge to 0, where R is the real numbers considered as an additive group, and Ra is the subgroup generated by a.

Comment by Paul Erdös, University of Windsor, Ontario. In his Remark 3, Professor Schoenberg has left open the question as to the enumerability of the set B for which $\{x_n\}$ is not equidistributed mod b, $b \in B$. We shall exhibit a set of positive integers $\{x_n\}$, $x_n \to \infty$ for which the set B has the cardinality of the continuum. Let $x_n = n!$ and observe first that $x_n \to 0$ mod every rational. Also (see Remark 4 in Schoenberg's solution [1964, 332]) $\{x_n\}$ is equidistributed mod a for almost all a.

LEMMA. *Let $\{x_n\}$ be a sequence of reals, $x_n \to \infty$ and let there be a dense set D such that $x_n \to 0 \bmod d$ whenever $d \in D$. Then there is a dense set B which is a G_δ such that $x_n \to 0 \bmod b$, $b \in B$; B has cardinality c.*

Proof. Given $d_1, d_2, \cdots, d_k$, we may find an $n_k(>n_{k-1})$ such that

$$x_n \pmod{a_i} < 1/2^k, \qquad i = 1, 2, \cdots, k, \quad n > n_k. \tag{1}$$

Consequently there is an open set O_k containing $d_1, d_2, \cdots, d_k$ so that $x_n \pmod b$ $<2^{-k}$, $n_k < n \leqq n_{k+1}$, for every $b \in O_k$.

Define B as the set of all elements contained in all but a finite number of the sets O_k, i.e.,

$$B = \prod_{l=1}^{\infty}\left(\sum_{l<k} O_k\right).$$

Then B is a G_δ, and for all $b \in B$, $x_n \to 0 \pmod b$.

B is now a set G_δ containing the dense set D. Therefore B has the cardinality c (see Hausdorff, *Mengenlehre*, 1935, p. 136).

The application to the sequence $\{n!\}$ is now immediate.

(Note a slight misprint in the earlier solution. The summation sign, p. 333, should carry n as upper limit, instead of ∞.)

Finitely Generated Groups

5121 [1963, 764]. *Proposed by J. B. Kruskal, Bell Telephone Laboratories, Murray Hill, N. J.*

By definition, let $G_1 > G_2$ if group G_1 contains a subgroup isomorphic to G_2, but G_2 does not contain a subgroup isomorphic to G_1. Can an infinite descending sequence $G_1 > G_2 > G_3 > \cdots$ exist where G_1 is finitely generated?

Solution by J. J. Feltmacher, Jr., University of Illinois. Such sequences exist˙ For example, let $p_1, p_2, \cdots$ be the sequence of primes, let $\sigma(p_n)$ denote the cyclic group of order p_n, and let $G_n = \sum_{i \geq n} \sigma(p_i)$. Now $G_1 > G_2 > \cdots$ for G_{n+1} contains no element of order p_n. By the theorem of Higman, Neumann and Neumann (see also Kurosch, *Theory of Groups*, v. II, p. 54), since G_1 is countable, there exists a group G_0 which can be generated by two elements and which contains G_1.

Also solved by J. L. Alperin, George Bergman, Joseph Lehner, and W. R. Scott.

Irreducible Polynomials

5122 [1963, 764]. *Proposed by Harley Flanders, Purdue University*

Let k be a field of characteristic p and let $f(x)$ be an irreducible polynomial in $k[x]$ in the variables $x = (x_1, \cdots, x_r)$. Form the polynomial $g(x)$ by $\{g(x)\}^p$ $= f(x^p)$, and the field F obtained by adjoining to k all coefficients of g. Prove that $g(x)$ is irreducible in $F[x]$.

Solution by Veselin Perić, Sarajevo, Yugoslavia. Let

$$f(x) = \sum a_i x_1^{\nu_{1i}} x_2^{\nu_{2i}} \cdots x_r^{\nu_{ri}}.$$

Then, for b_i with $b_i^p = a_i$ and for

$$g(x) = \sum b_i x_1^{\nu_{1i}} x_2^{\nu_{2i}} \cdots x_r^{\nu_{ri}}$$

we have $\{g(x)\}^p = f(x^p)$. Any element $a \in F$ is a polynomial $h(b_1, b_2, \cdots)$ in $b_1, b_2, \cdots$ with coefficients in k and, consequently, $a^p = h_1(b_1^p, b_2^p, \cdots) \in k$. Suppose $g(x) = \phi(x) \cdot \psi(x)$, where $\phi(x)$ and $\psi(x)$ are nonconstant polynomials in $F[x]$. Then

$$\{\phi(x)\}^p = \phi_1(x^p), \qquad \{\psi(x)\}^p = \psi_1(x^p),$$

where $\phi_1(x)$ and $\psi_1(x)$ are nonconstant polynomials in $k[x]$, for the coefficients of these polynomials are the pth powers of the coefficients of $\phi(x)$ and $\psi(x)$, and belong to k. From $f(x^p) = \{g(x)\}^p = \phi_1(x^p) \cdot \psi_1(x^p)$ we conclude that $f(x) = \phi_1(x) \cdot \psi_1(x)$ in $k[x]$, contrary to the hypothesis that $f(x)$ is irreducible in $k[x]$.

Also solved by George Bergman, L. Carlitz, and the proposer.

Dense Sets of Functions

5123 [1963, 765]. *Proposed by R. W. Newcomb, Stanford University*

Prove that the set of infinitely differentiable functions of support bounded on the left is dense in the set of distributions of support bounded on the left.

Solution by the proposer. We use the notation of L. Schwartz, *Théorie des distributions*, vol. I, 1957, p. 89, and vol. II, 1959, p. 28. An existence type proof for the problem follows immediately from theorems in Schwartz, vol. I. We offer the following constructive approach.

Consider a sequence $\{\phi_j\}$ of infinitely differentiable functions of bounded support chosen so that $\lim_{j\to\infty} \phi_j = \delta$; $\phi_j \in \mathfrak{D} \subset \mathcal{E}'$. For any $T \in \mathfrak{D}'_+ \subset \mathfrak{D}'$, form $\phi_j * T$. We know that $\phi_j * T \in \mathfrak{D}_+$ [Schwartz, v. II, p. 29]. Since $\phi_j \in \mathcal{E}'$ and $T \in \mathfrak{D}'$, it follows from the continuity of the convolution for such distributions [Schwartz, p. 13] that

$$\lim_{j\to\infty} \phi_j * T = \delta * T = T.$$

Consequently, $\mathfrak{D}_+$ is dense in $\mathfrak{D}'_+$.

Addition Chains of Vectors

5125 [1963, 765]. *Proposed by Richard Bellman, The RAND Corporation, Santa Monica, California*

It is easy to determine the minimum number of multiplications required to generate a^N from a. What is the minimum number required to generate $a^M b^N$ starting with a and b?

Solution by Veselin Perić, Sarajevo, Yugoslavia. Let

$$f(x) = \sum a_i x_1^{\nu_{1i}} x_2^{\nu_{2i}} \cdots x_r^{\nu_{ri}}.$$

Then, for b_i with $b_i^p = a_i$ and for

$$g(x) = \sum b_i x_1^{\nu_{1i}} x_2^{\nu_{2i}} \cdots x_r^{\nu_{ri}}$$

we have $\{g(x)\}^p = f(x^p)$. Any element $a \in F$ is a polynomial $h(b_1, b_2, \cdots)$ in $b_1, b_2, \cdots$ with coefficients in k and, consequently, $a^p = h_1(b_1^p, b_2^p, \cdots) \in k$. Suppose $g(x) = \phi(x) \cdot \psi(x)$, where $\phi(x)$ and $\psi(x)$ are nonconstant polynomials in $F[x]$. Then

$$\{\phi(x)\}^p = \phi_1(x^p), \qquad \{\psi(x)\}^p = \psi_1(x^p),$$

where $\phi_1(x)$ and $\psi_1(x)$ are nonconstant polynomials in $k[x]$, for the coefficients of these polynomials are the pth powers of the coefficients of $\phi(x)$ and $\psi(x)$, and belong to k. From $f(x^p) = \{g(x)\}^p = \phi_1(x^p) \cdot \psi_1(x^p)$ we conclude that $f(x) = \phi_1(x) \cdot \psi_1(x)$ in $k[x]$, contrary to the hypothesis that $f(x)$ is irreducible in $k[x]$.

Also solved by George Bergman, L. Carlitz, and the proposer.

Dense Sets of Functions

5123 [1963, 765]. *Proposed by R. W. Newcomb, Stanford University*

Prove that the set of infinitely differentiable functions of support bounded on the left is dense in the set of distributions of support bounded on the left.

Solution by the proposer. We use the notation of L. Schwartz, *Théorie des distributions*, vol. I, 1957, p. 89, and vol. II, 1959, p. 28. An existence type proof for the problem follows immediately from theorems in Schwartz, vol. I. We offer the following constructive approach.

Consider a sequence $\{\phi_j\}$ of infinitely differentiable functions of bounded support chosen so that $\lim_{j\to\infty} \phi_j = \delta$; $\phi_j \in \mathfrak{D} \subset \mathcal{E}'$. For any $T \in \mathfrak{D}'_+ \subset \mathfrak{D}'$, form $\phi_j * T$. We know that $\phi_j * T \in \mathfrak{D}_+$ [Schwartz, v. II, p. 29]. Since $\phi_j \in \mathcal{E}'$ and $T \in \mathfrak{D}'$, it follows from the continuity of the convolution for such distributions [Schwartz, p. 13] that

$$\lim_{j\to\infty} \phi_j * T = \delta * T = T.$$

Consequently, $\mathfrak{D}_+$ is dense in $\mathfrak{D}'_+$.

Addition Chains of Vectors

5125 [1963, 765]. *Proposed by Richard Bellman, The RAND Corporation, Santa Monica, California*

It is easy to determine the minimum number of multiplications required to generate a^N from a. What is the minimum number required to generate $a^M b^N$ starting with a and b?

Choosing $r = \log\log |n|/k \log 2 - c \log\log\log |n|$ for suitable c we obtain from (3),

$$(4) \qquad l(n) \leqq \frac{\log |n|}{\log 2}\left\{1 + O\left(\frac{1}{\log\log |n|}\right)\right\}.$$

Combining (1) and (4) we have finally

$$(5) \qquad \lim_{|n| \to \infty} l(n) \frac{\log 2}{\log |n|} = 1.$$

Also solved (partially) by H. F. Bennett.

The proposer agrees that the problems he has posed are not easy and that the minimum chain is not known, even for $k=1$. He is in the process of developing a computational algorithm.

Closed Sets of Irrationals

5126 [1963, 765]. *Proposed by Erwin Just, Bronx Community College, New York*

Prove that there exists an uncountable closed subset of the irrationals in $[0, 1]$. More generally, does every uncountable set of reals contain an uncountable closed subset?

Solution by Fred Galvin, St. Paul, Minnesota. (a) Let $r_2, r_3, r_4, \cdots$ be the rational points of $[0, 1]$. Let I_n be an open interval of length 2^{-n} about r_n. Then $[0, 1] - (I_2 \cup I_3 \cup I_4 \cup \cdots)$ is a closed set of irrational numbers which is certainly uncountable, since it has positive Lebesgue measure. More generally, any set of real numbers that has positive inner measure contains a closed set which has positive measure and is uncountable.

(b) It is not true that every uncountable set of reals contains an uncountable closed subset; if this were true, then the continuum hypothesis would follow, for every uncountable closed set has cardinality c. Moreover, assuming the continuum hypothesis, Sierpinski proves (*Hypothèse du Continue*, p. 80) that there is an uncountable set of real numbers which does not contain an uncountable measurable subset.

Editorial Note. Galvin's solution contains also the proof of the following generalization: *If S is a complete metric space with no isolated points, and if X is a subset of S which is of the first category relative to S, then $S-X$ contains an uncountable closed subset.* Therefore, if P is a perfect set of real numbers and X is of the first category relative to P, then $P-X$ contains an uncountable closed set. Thus there is an uncountable closed set of irrational points of the Cantor set.

Also solved by R. J. Gregorac, J. D. Hill, Solomon Marcus (Romania), and the proposer; also part (a) by G. A. Heuer, and R. A. Jacobson.

Ordered Chains of Sets

5127 [1963, 765]. *Proposed by Alain Etcheberry, Universidad de Chile, Santiago*

Let $S = \{a_1, a_2, \cdots\}$ be a countable, infinite set. Does there exist a strictly increasing and uncountable chain of subsets of S? (The problem is suggested by a comment of J. E. Hafstrom [1962, 249].)

Editorial Note. The problem is equivalent to problem 2 of part II of the 1962 William Lowell Putnam Competition [1963, 714; solution on p. 716]. The base example for the chain is the set of Dedekind cuts of the rationals and may be found also in the solution of problem 4871 [1961, 300]. There can, of course, be no well-ordered chain. A more general example is used for the solution of problem 4901 [1961, 578].

If S is not countable, and assuming the general continuum hypothesis, Johnston proves that there is a maximal chain C in the Boolean algebra of all the subsets of S such that $|C| = 2^{|S|}$.

Also solved by S. T. M. Ackermans (Netherlands), Bruce Blum, P. J. Erdelsky, P. Flor (Austria), Fred Galvin, Larry Gerstein, R. J. Gregorac, Melvin Henriksen, G. A. Heuer, A. E. Hoffman, L. C. House, J. B. Johnston, E. S. Langford, L. C. Larson, Elinor Lerner, Brockway McMillan, B. L. Osofsky, S. M. Robinson, Daihachiro Sato, J. A. Schatz, W. R. Scott, D. L. Silverman, George Van Zwalenberg, W. C. Waterhouse, A. Wilansky, and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, Carleton College, and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457. Programmed Materials: K. O. May, University of California, Berkeley, Calif. 94704. Films: E. P. Vance, Oberlin College, Oberlin, Ohio. 44074

MAA Studies in Mathematics, Volume 2, *Studies in Modern Algebra.* By Saunders MacLane, R. H. Bruck, Charles W. Curtis, Erwin Kleinfeld, Lowell J. Paige. Edited by A. A. Albert. Distributed by Prentice Hall, Englewood Cliffs, N. J., 1963. 190 pp. $4.00.

This little book consists of six expository articles on modern algebra, with an introduction by A. A. Albert. The first two articles are by Saunders MacLane and are entitled: "Some recent advances in algebra" and "Some additional advances in algebra." The first of these was published in the MONTHLY in 1939; the second has been written as a supplement to the original article. In this later article are presented some recent significant results on old problems as well as an indication of certain new trends in algebra. In particular, the emphasis is on new results in the theory of finite groups and on the general ideas of homological algebra. Each of these articles ends with a brief discussion of the nature of algebra, and it is interesting and instructive to compare these two accounts written some twenty-four years apart.

The other four articles are concerned primarily with nonassociative systems. They are as follows: "What is a loop?," by R. H. Bruck; "The four and eight square problem and division algebras," by Charles W. Curtis; "A characterization of the Cayley numbers," by Erwin Kleinfeld; and, "Jordan algebras," by Lowell J. Paige.

Of course, in a book of this size it would have been impossible to mention all recent important advances in algebra, and no attempt has been made to do so

either in MacLane's articles or in the choice of subject matter of the others. Nevertheless, a surprising amount has been included. The expositions are all of an unusually high quality and this book is therefore a welcome addition to the literature on modern algebra.

NEAL H. MCCOY, Smith College

Analytic Geometry and Calculus, with Vectors. By Ralph P. Agnew. McGraw-Hill, New York, 1962. vii+738 pp. $8.95.

This volume is not a treatise on mathematics, but a "teaching" textbook from which a lot about mathematics and many of its applications can be learned. The approach of the book is for the most part traditional, with the inclusion of the vector treatment a valuable feature. Probably its most attractive and distinctive feature is its wealth of problems, many of them quite extensive, in which the student is led or enabled to find his way through applications of the theory or extensions of the theory already developed in the body of the text. The author's contention, as stated in the preface, is well-founded: it is a text in which average students can make satisfactory progress, and in which superior students have ample opportunities to acquire large amounts of additional information and skill.

It is obligatory for a reviewer these days to discuss the rigor of any text in mathematics. The author defines "rigorous" as "being free from blunders," and by these lights this text can probably be said to pass muster. Those with a more rigorous definition of "rigorous" will probably not find the book satisfactory in some respects. Informal discussions frequently lead to definitions which are valid only under the rather special conditions of the discussion; flex points (p. 304), for example, could have been more generally defined, and then the usual theorems about them, with the conditions on the derivatives included, could still have been proved. For another example, conditions stated in the preliminary discussion of area (p. 236) are omitted in the formal definition which follows. The proof of Theorem 4.13 does not immediately follow the statement, but is postponed until somewhat later in the text, although an extended discussion of some of the implications of the theorem does follow the statement immediately. Some of these things are understandable in the spirit of the text, which in many parts is that of a teacher talking informally to a student, but the unevenness of this approach may be unsettling to those who do not relish seeing this informality in print.

The notion of "function" is expounded from both the "transformer" and "ordered pairs" points of view, and the decision as to which (if either) of these to emphasize is left to the teacher. Fig. 3.151, a fanciful schema illustrating the "transformer" approach, may be decisive in inclining a reader toward one point of view or the other.

The style in which Professor Agnew's books are written is, I believe, well-known. The book is replete with sage observations on matters mathematical, and

there are even some remarks that can best be described as homilies. This reviewer thinks it likely that this text will in many cases, to use the author's own words, "enable students to decide whether they have the interests and the aptitudes required for life-long careers in pure mathematics or in another science in which mathematics plays a major role," for the book is clearly the work of a teacher with wide experience who desires and is able to communicate his enthusiasm for and knowledge of mathematics to the student.

ROBERT C. STEWART, Trinity College

A Programmed Introduction to Vectors. By Robert A. Carman. Wiley, New York, 1963. xi+121 pp. $2.75.

Content: The simplest manipulations of vector algebra in two and three dimensions, including cross and dot products, but not including solid analytics, vector functions, calculus, proofs, or any depth in theory or applications. *Quality*: Satisfactory. Some minor errors and one poor choice of terms ("group" instead of "set"). *Possible uses:* As a study aid for about one week in an elementary calculus course (the material is covered more thoroughly in from 8 to 16 pages of widely used text books), or for self-study by physics students who need these manipulations before reaching them in a mathematics course. *Format*: A branching, scrambled program in which the student is usually referred to different pages depending on his response to a multiple choice question. There are some variations in this pattern, including review loops and diagnostic tests. A glossary and suggestions for further study complete the pamphlet.

Undoubtedly such programs may be helpful to students, but are they worth the cost? If the whole calculus course were programmed at this superficial level, the student would be confronted with a three foot book shelf of over 10,000 pages, costing him several hundred dollars and quite worthless for reference, quick review, or connected study of mathematical ideas and methods behind and beyond mechanical manipulations.

KENNETH O. MAY, University of California, Berkeley

Projective and Euclidean Geometry. By W. T. Fishback. Wiley, New York, 1962. 244 pp. $7.50.

This book is primarily an introduction to projective plane geometry based upon a "knowledge of elementary synthetic and analytic Euclidean geometry." A few isolated supplementary sections are concerned with the geometry of space. Both synthetic and analytic methods are employed. The development is appropriate for an advanced undergraduate course. The selection of topics is very good for an introduction of projective geometry. In view of its title the book is disappointing, at least to this reviewer, in its lack of emphasis upon Euclidean geometry.

B. E. MESERVE, University of Vermont

Tomorrow's Math. By C. Stanley Ogilvy. Oxford University Press, New York, 1962. 182 pp. $5.00.

This book contains statements of more than 150 problems which, according to the author, are "unsolved problems for the amateur." In addition, there are included some related expository material and references to the history of the problems and to additional information about them. The chapter headings indicate the subject-matter areas of the problems—geometry, arithmetic, topology, probability and combinatorics, analysis, game theory, set theory, etc. As the author points out, the title is somewhat misleading in that the book is not to be considered a listing of topics which will be of major interest to tomorrow's mathematicians.

While most of the problems are stated in nontechnical language so that the general ideas can be understood by the intelligent layman, it seems unlikely that amateurs or students can seriously attack many of them without further study and more precise definitions and formulations. For the general reader, the book provides enjoyment and an acquaintance with and appreciation of some aspects of mathematics and the nature of mathematical problems. For students from high school to graduate school the book provides an inspiration for supplementary study and a source of interesting problems, some of which are classical unsolved problems of long standing.

In general, the style is clear and effective. Such terms as *curve of constant width* (p. 53), *connectivity* (p. 107), and *function* (p. 134) may not be known to the amateur and need to be defined more precisely for the student. There are very few misprints or errors. The index is fairly complete, and numerous diagrams help to illustrate the concepts.

L. AILEEN HOSTINSKY, Connecticut College

Psychological Statistics, 3rd ed. By Quinn McNemar. Wiley, New York, 1962. 442 pp. $7.75.

As in earlier editions, McNemar's text offers unusually complete coverage of statistical techniques used in psychology and other behavioral sciences. The presentation is, for the most part, pragmatic and intuitive rather than theoretical. Sampling distributions are introduced by means of empirical demonstrations which can be duplicated by students or by enumerative methods. Underlying assumptions are clearly stated and a careful analysis of interpretation of results given. As the author states in his preface, " . . . the level, although designed for an intermediate course, is not beyond the grasp of students in elementary courses who are unafraid of mathematical reasoning."

The new edition has provided additional problems and questions of a more challenging nature in the list of exercises for each chapter given at the end of the book. Other additions include further derivations or formula developments and a chapter on trend analysis.

GRACE E. BATES, Mount Holyoke College

A Simplified Guide to Statistics for Psychology and Education, 3rd ed. By G. Milton Smith. Holt, Rinehart, and Winston, New York, 1962. 164 pp. $1.75.

A manual for applying elementary statistical methods in common use in psychology and education. Topics covered include elements of descriptive statistics, significance tests for large and small samples, chi-square tests for goodness of fit and for independence, and correlation techniques. The new edition contains directions for machine computation of means, variances, and correlation and regression coefficients, and a section on prediction by means of regression equations and the standard error of estimate.

GRACE E. BATES, Mount Holyoke College

Analysing Qualitative Data. By A. E. Maxwell. Methuen's Monographs on Applied Probability and Statistics. Wiley, New York, 1961. 163 pp. $3.00.

A short treatment of chi-square tests including some recent developments on testing for trends in contingency tables, partitioning chi-square values, and combining and comparing results from different investigations. Ranking methods are also treated and some new tests of value in the analysis of serial data. The last two chapters deal with classification procedures and item analysis.

This monograph should be useful to the research worker in the social or biological sciences. Illustrative examples are drawn mainly from studies made at the Institute of Psychiatry of which the author is a staff member. References are generously provided at the end of each chapter.

GRACE E. BATES, Mount Holyoke College

An Introduction to Vector Analysis. By F. Max Stein. Harper and Row, New York, 1963. 209 pp. $6.25.

This text, intended primarily for mathematicians, physicists, and engineers who have had a year of calculus and an introduction to elementary differential equations, is well written, clear, and intelligible. Your reviewer found one trivial typographical error. In addition to the usual material found in such a text, there have been appended three chapters on the application of vectors in theoretical physics, which will make a course from this book more meaningful as well as useful in subsequent courses. It should be added that this is a text that most students will find difficult, although stimulating.

J. LAWRENCE BOTSFORD, University of Idaho

Algebra: An Introduction to Finite Mathematics. By Israel H. Rose. Wiley, New York, 1963. 489 pp. $6.95.

This book is intended for a one semester freshman course which could accommodate students with a variety of backgrounds. The first eight chapters contain a careful review of high school algebra with emphasis on the fundamental properties of real and complex numbers while later chapters treat such

topics as matrices, groups, probability and statistics, and mathematics of finance. The level of rigor is quite high although not uniformly so. For example, the inductive definition of exponentiation for natural number exponents is not mentioned, although at a later point the determinant is discussed in complete detail, via permutation functions.

The strong points of the book include a careful discussion of the little theorem that is proved whenever a student solves a simple equation and several very interesting historical excursions. On the other side of the ledger there are the author's "definition" of the principal square root of negative numbers with resulting confusion in the laws for exponents, the nonstandard use of Σ when the only saving is typographical, and the equality axioms (which seem unnecessary in view of the assumed intuitive set theory). The reviewer must also object to the characterization of rigor in proofs (page 87) although he does not claim to have a substitute.

H. E. CHRESTENSON, Reed College

Eléments de Mathématique, VI. Première partie. Livre II: *Algèbre.* Chapitre 2: *Algèbre linéaire.* By N. Bourbaki (Troisième édition). Actualités Sci. Indust. No. 1236. Hermann, Paris, 1962. 316 pp.+inserts. 36 francs.

Both clearly written and definitive, this book should be acquired by all serious students and scholarly libraries; the cost is modest for its size, scope and worth. One need only glance at the first edition of this volume (1947) to realize the extent of the changes in the subject which have taken place in the last fifteen years. The age of the exact sequence is upon us! The numbers following refer to major subdivisions. 1. Modules over a ring with a unity: exact sequences, products and sums of families of modules, length, bases, cyclic modules. 2. Projective modules and duals. 3.4. Tensor products and homomorphisms. 5. Extending the ring of scalars. 6. Direct and inverse limits. 7. Vector spaces: bases, (co)dimension, rank, dual spaces, linear equations. 8. Restrictions to a scalar subfield. 9. Affine and projective spaces. 10. Matrices. 11. Graded modules and rings. There is a short appendix on pseudomodules, i.e., the ring of operators is not necessarily provided with a unity. There are 42 pages of exercises followed by indices of notations and terminology, a table of contents, a concordance table with the second edition, a table of principal definitions and axioms, a table of principal properties of exactness and a table of canonical homomorphisms.

F. HAIMO, Washington University

Eléments de Mathématique, XX. Livre I: *Théorie des ensembles.* Chapitre 3: *Ensembles ordonnés. Cardinaux, Nombres entiers.* By N. Bourbaki (Seconde édition). Actualités Sci. Indust. No. 1243. Hermann, Paris, 1963. 149 pp. +inserts. 36 francs.

A clear exposition of ordered sets and of elementary cardinal arithmetic is given here, for the most part independent of material in prior Bourbaki volumes.

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A clear exposition of ordered sets and of elementary cardinal arithmetic is given here, for the most part independent of material in prior Bourbaki volumes.

Generalized Functions and Partial Differential Equations. By Avner Friedman. Prentice-Hall, Englewood Cliffs, N. J., 1963. 340 pp. $10.00.

This book contains a modern treatment of the theory of generalized functions based on the work of L. Schwartz and Gelfand and Shilov. It contains much subject matter with great emphasis on concepts of function and functional spaces in the theory of generalized functions. In general the subject matter is well organized and the book is pretty much self-contained. For those unfamiliar with point set topological language a first chapter is included for survey purposes. The treatment of the subject matter is not uniformly good. Sometimes the wording is a little awkward and there is very little effort devoted to introductory or motivating remarks.

One also feels that the title of the book is a little misleading. Of eleven chapters, four are devoted to partial differential equations. Two of the chapters on differential equations are of special interest, containing original contributions of the author. Yet, the four chapters cover such a small portion of the role that distributions and generalized functions play in the theory of partial differential equations that the title "Generalized Functions with Application to Partial Differential Equations" would have been more appropriate.

To understand the subject matter a knowledge of the theories of real variables, complex variables and some functional analysis is necessary. The bibliography contains 66 references, a majority of which are Russian. Some bibliographical remarks supply useful historical background on the subject matter of each chapter. A listing of the chapters follows.

1. Linear Topological Spaces; 2. Spaces of Generalized Functions; 3. Theory of Distributions; 4. Convolutions and Fourier Transforms of Generalized Functions; 5. *W* Spaces; 6. Fourier Transforms of Entire Functions; 7. The Cauchy Problem for Systems of Partial Differential Equations; 8. The Cauchy Problem in Several Time Variables; 9. *S* Spaces; 10. Further Applications to Partial Differential Equations; 11. Differentiability of Solutions of Partial Differential Equations.

MURRAY WACHMAN, General Electric Company

Modern College Mathematics. By W. K. Smith and S. F. Dice. Allyn and Bacon, Boston, 1963. 411 pp. $7.95.

The authors state in the preface that this book grew from three sources: an institute for secondary mathematics teachers, a class for select high school students, a college course for students whose major interests were in the humanities. On the assumption that the social and behavioral sciences are included in the humanities, I would say that the authors have achieved a nice balance of simplification, exposition, and rigor.

The book is modern in spirit, starting with an axiomatic treatment of fields and going on to the number systems, sets (with a little logic and switching circuits), functions and analytic geometry, the "polynomial" calculus, two-by-two

matrix theory, and concluding with probability over finite sample spaces. There is no mention of trigonometry.

ARNOLD GRUDIN, Denison University

University Calculus. By Howard E. Taylor and Thomas L. Wade. Wiley, New York, 1962. xxi+765 pp. $9.95.

Subsets of the Plane: Plane Analytic Geometry. By Howard E. Taylor and Thomas L. Wade. Wiley, New York, 1962. vii+97 pp. $1.95.

Together these books constitute a first course in the calculus with the analytic geometry necessary to accompany it. The plane analytic geometry of the circle, line, and conic sections is contained in *Subsets of the Plane.* This booklet is paperbound and is suggested by the authors for courses in high schools or for use along with, or supplementary to, their *University Calculus.* It is carefully written and modern in its treatment, the graph of an equation being taken as the set of points whose coordinates are the ordered pairs of a relation.

University Calculus begins with a detailed discussion of sets, relations, and functions. This forms the foundation for the remainder of the book, set notation being used throughout and a careful distinction constantly made between a function F, the correspondent $F(x)$ of x under F, and the sentence or formula specifying F. Limits are presented by the ϵ, δ-definition and all necessary theorems on limits and continuity proved from this definition. Anti-derivatives are introduced by recognition from known results of differentiation. Thus the first "integral" to be considered is the definite (Riemann) integral. This also is given by the ϵ, δ-definition and is in terms of partitions and augmentations of the closed interval of the domain of the function. Several geometrical and physical problems lead to definite integrals, which are evaluated by anti-derivatives before indefinite integrals and integration by formula are considered. Additional applications are then followed by solid analytic geometry, the calculus of functions with several independent variables, and an introduction to sequences and series. An interesting variation is that all approximation procedures are considered together in a final chapter. The book confines itself to real variables and functions and the only differential equations included are the separable and exact types.

This is a substantial book, both in its size and in the mathematics presented. The careful and detailed explanations, however, make the text probably no more difficult than the average. There is an admirable consistency throughout. All results are given as theorems with the hypotheses fully stated and, in most instances, with complete proofs provided in the text or exercises. When such proofs are not included, as for instance for the existence of the definite integral, no attempt is made to justify the result intuitively, but a reference is given to a more advanced text where the proof will be found.

Illustrative examples and figures are supplied freely and the exercise sets are ample. The typography is excellent. Outlines of possible courses from the book are suggested by the authors. One interesting inclusion is an extensive develop-

ment of the curvilinear (line) integral, together with Green's Theorem and its applications. On the other hand, a number of topics usually found in a first course in the calculus are missing. For instance, volumes of revolution when the element is a hollow cylindrical shell are not included; and there are several others. The very care with which explanations and statements are given sometimes makes for considerable repetition, and the insistence on detail, while excellent for the advanced student, may not appeal to one just beginning the study of the calculus. Furthermore, with the emphasis on precision there is less reliance on intuition and fewer comments on why and how some techniques work than a student might desire. Such matters, however, can be taken care of by the instructor, if it is desired to include additional topics or to provide greater motivation for those considered. Altogether, the authors have produced a stimulating, fresh text. What the student learns here will serve as a sound foundation for any further work in analysis, and nothing will have to be learned over or learned differently later.

DONALD H. BALLOU, Middlebury College

Homology. By Saunders MacLane. Grundlehren der mathematischen Wissenschaften, Band 114. Springer, Berlin, 1963. x+422 pp. $15.50.

The process of assigning to a topological space its homology (or cohomology) groups is a two-step affair: first is produced a chain (or cochain) complex; from it, in turn, is calculated the (co)homology. MacLane's book treats, in the main, the second procedure—its tools (categories and functors, diagrams, spectral sequences), its algebraic applications (Tor, Ext, and the cohomology of monoids, groups, modules, and algebras), its consequences (dimension theory, products) and its generalization to abelian categories (derived functors, relative homological algebra, bar resolution).

The topological aspects of homology theory, though not dominant in this book, are nevertheless not totally ignored. The second chapter, in particular, where chain complexes and their homology are introduced, is a fragrant blend of topology and algebra. Elsewhere, in connection with products and spectral sequences, respectively, complete semisimplicial (c.s.s.) complexes and fiber spaces are introduced.

The book is very clearly written and, despite its length, finds space for exercises, historical comments, and remarks (without proof) on theorems and conjectures generalizing and extending the results presented. A novel and intuitively appealing feature is that Ext and Tor are initially defined directly, the first, with Yoneda, as equivalence classes of long exact sequences, the second by means of explicit generators and relations. Their usual definitions then become theorems useful for motivating, say, derived functors. For further details the interested reader is referred to the table of contents and, more especially, to the introduction; to the latter, as a model of lucidity, worthy of emulation, are also referred all prospective authors.

F. E. J. LINTON, Wesleyan University

An Introduction to Linear Programming and the Theory of Games. By Abraham M. Glicksman. Wiley, New York and London, 1963. 131 pp., paper $2.25, cloth $4.95.

The author states in the Preface of the book: "This is by far the simplest nontrivial exposition of linear programming and the theory of games which has yet appeared. It is eminently suitable for high school 'honors' programs, for college students, for teachers, and for interested laymen." Fortunately, this claim is not unjustified. The book achieves the desired goals, and, at the very elementary level, it constitutes an excellent contribution to the hypertrophic literature on the subject.

The work is divided into five parts, of which the first three contain an introductory exposition of the linear programming problem illustrated by examples, followed by a rigorous study of the elementary theory of convex functions on the euclidean plane and by a treatment of the simplex method of a predominantly practical type. The last two parts include an introduction to two person zero sum matrix games, solutions of $2\times n$ games by graphic methods and the general solution of games by linear programming techniques. At this point the simplex method is used to obtain both the duality theorem of linear programming and the minimax theorem on games. There are numerous well-motivated examples and exercises with solutions or indications as to how to solve them. The exposition is detailed and the language precise. A few misprints are of minor character.

A. G. AZPEITIA, University of Massachusetts

Banach Spaces of Analytic Functions. By Kenneth Hoffman. Prentice-Hall, Englewood Cliffs, N. J., 1962. xiii+217 pp. $9.00.

The author presents here the theory of the Hardy spaces H^p—for $1\leqq p\leqq\infty$, H^p is the Banach space of functions f which are analytic in the interior of the unit disc and for which the set of functions $f_r(\theta)=f(re^{i\theta})$ is bounded in L^p-norm, with $\|f\|$ taken to be the least upper bound of $\|f_r\|_p$—and he does so thoroughly and engagingly. The first three chapters contain a summary of prerequisites and some basic results on Fourier series and boundary values. Chapters 4, 5, and 8 contain a description of H^1, the factorization theory of H^p functions, and the theory of H^p functions on a half-plane. Although the results are drawn mainly from the analytic function and Fourier theories of 1936 and earlier, the viewpoint taken is that of functional analysis and many of the proofs are recent. The material of the remaining four chapters is solely functional analysis and covers the subject's development through 1961. Topics treated are the closed ideals of the algebra of functions continuous on and analytic inside the disc, multiplication by z as an operator on H^2, extreme points of the unit ball of H^p, projections from L^p to H^p, and the still largely uncompleted theory of the Banach algebra H^∞.

As a consequence of the care with which the contents have been organized and of generous, well-written discussion, the order of events strikes one as im-

pressively natural and easy. Open problems are discussed and at the end of each chapter there are historical notes, suggestions for supplementary reading, and exercises.

The author attempts to indicate the level of the book by prescribing as prerequisite a knowledge of the material discussed in the first chapter—roughly that of a good measure theory course and a little on Banach and Hilbert spaces. It seems to this reviewer that a student with no more than this as background and the limited experience in analysis it implies is likely to have considerable difficulty with the book. For the proofs are written with rather severe economy, and in a number of places more advanced results are used. Usually in such a case a reference to a proof is given, but there are exceptions, as in the use of group algebra properties on page 54. There are a few errors (three occur in a proof on page 90). They are minor ones, however, and neither they nor the other difficulties for the novice will seriously inconvenience the informed reader. Advanced students of functional analysis and of function theory will find the book an excellent source of results and techniques presented with imagination and freshness.

G. PHILIP JOHNSON, Wesleyan University

Topics In Mathematics. Translated and adapted from the Russian Series *Popular Lectures In Mathematics* by the Survey of Recent East European Mathematical Literature. Under the direction of Alfred L. Putnam and Izaak Wirszup. D. C. Heath and Company, Boston, 1963.

Thus far 14 booklets have been published. Translations of other booklets are in preparation. Two of the titles on the following list appear also in the Blaisdell Scientific Paperbacks series.

The names of the translators and adaptors appear in parentheses below.

1. *Configuration Theorems*. By B. I. Argunov and L. A. Skornyakov. (Edgar E. Enochs and Robert B. Brown) 41 pp. $1.40.

Elementary ideas of projective geometry, including brief mention of the algebraic meaning of configuration theorems.

2. *Equivalent and Equidecomposable Figures*. By V. G. Boltyanskii. (Alfred K. Henn and Charles E. Watts) 68 pp. $1.40.

Sophisticated material presented simply.

3. *Mistakes in Geometric Proofs*. By Ya. S. Dubnov. (Alfred Henn and Olga A. Titelbaum) 57 pp. $1.40.

Includes a chapter on mistakes in reasoning connected with the concept of limit.

4. *Proof in Geometry*. By A. I. Fetisov. (Theodore M. Switz and Luise Lange) 55 pp. $1.40.

Construction of proofs, examples of fallacious reasoning, discussion of the axioms of geometry.

5. *Induction in Geometry*. By L. I. Golovina and I. M. Yaglom. (A. W. Goodman and Olga A. Titelbaum) 106 pp. $1.40.

Interesting uses of mathematical induction to obtain familiar and unfamiliar results.

6. *Computation of Areas of Oriented Figures.* By A. M. Lopshits. (J. Massalski and Coley Mills, Jr.) 58 pp. $1.40.

Simple discussion of oriented areas and the planimeter.

7. *Areas and Logarithms.* By A. I. Markushevich. (Roland S. Toczek and Reuben Sandler) 48 pp. $1.40.

The natural logarithm defined as the area under $y=x^{-1}$.

8. *Summation of Infinitely Small Quantities.* By I. P. Natanson. (Stephen Whelan and Coley Mills, Jr.) 59 pp. $1.40.

The "limit of a sum" applied to the problems of water pressure, work, volume, and area.

9. *Hyperbolic Functions.* By V. G. Shervatov. (A. Gordon Foster and Coley Mills, Jr.) 55 pp. $1.40.

In analogy with the geometric properties of the trigonometric functions, the hyperbolic functions are presented in relationship to the hyperbola. Natural logarithms, the base e, and series expansions are also discussed.

10. *How to Construct Graphs.* By G. E. Shilov. (Jerome Kristian and Daniel A. Levine) Simplest Maxima and Minima Problems. By I. P. Natanson. (C. Clark Kissinger and Robert B. Brown) 53 pp. $1.40.

The first essay discusses plotting "by points" and "by operation" (addition, multiplication, division). The second essay solves extremum problems algebraically, by completing the square.

11. *The Method of Mathematical Induction.* By I. S. Sominskii. (Luise Lange and Edgar E. Enochs) 48 pp. $1.40.

Contains many exercises, with solutions.

12. *Algorithms and Automatic Computing Machines.* By B. A. Trakhtenbrot. (Jerome Kristian, James K. McCawley, and Samuel A. Schmitt) 101 pp. $1.40.

An unusually comprehensive treatment, beginning with the Euclidean algorithm, continuing with machine algorithms, and culminating in the demonstration of the nonexistence of an algorithm for the general word problem.

13. *An Introduction to the Theory of Games.* By E. S. Venttsel'. (Jerome Kristian and Michael B. P. Slater) 66 pp. $1.40.

Simplified discussion with almost no proofs.

14. *The Fibonacci Numbers.* By N. N. Vorobyov. (Norman D. Whaland, Jr. and Olga A. Titelbaum) 47 pp. $1.40.

Origin of the sequence, algebraic and number-theoretical properties, continued fractions, geometry.

Correction: The reviewer of "Elements of Set Theory" by Zehna and Johnson wishes to replace the fourth sentence of the second paragraph of his review (70 (1963) 686) by the following sentence:

The proofs of several tautological implications are called for in a set of exercises, yet references to them in justifying steps of later proofs are rare.

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NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

The following mathematicians have been elected to membership in the National Academy of Sciences: Professors Lipman Bers and Fritz John, New York University; Raoul Bott, Harvard University; and Hans Lewy, University of California, Berkeley.

Dean Mina S. Rees of the City University of New York has been appointed a member of the National Science Board, governing body of the National Science Foundation, for a six year term.

Case Institute of Technology has awarded the honorary degree of Doctor of Science to Professor Garrett Birkhoff of Harvard University.

Professor E. A. Cameron, University of North Carolina, represented the Association at the inauguration of Samuel P. Massie, Jr. as the Third President of the North Carolina College at Durham on April 25, 1964.

Professor Stevens Heckscher, Swarthmore College, represented the Association at the inauguration of William W. Hagerty as President of the Drexel Institute of Technology on May 12, 1964.

Professor R. E. Horton, Los Angeles City College, represented the Association at the inauguration of Franklyn A. Johnson as President of the California State College at Los Angeles on May 15, 1964.

Professor R. A. Rosenbaum, Wesleyan University, represented the Association at the National Conference on the Nuclear Scientific Era: The Child and his Education, held in New York at the Hotel Biltmore on April 24–26, 1964.

Professor H. S. Thurston, University of Alabama, represented the Association at the inauguration of Howard M. Phillips as President of Birmingham Southern College on May 14, 1964.

Professor F. E. Ulrich, Rice University, represented the Association at the inauguration of William H. Hinton as President of the Houston Baptist College on May 8, 1964.

Carleton College: Associate Professor John Dyer-Bennet has been appointed Chairman of the Department of Mathematics and Astronomy; Associate Professor K. W. Wegner has been promoted to Professor.

Case Institute of Technology: Associate Professors R. A. Clark and F. C. Leone have been promoted to Professor; Mr. F. J. Sansone has been promoted to Assistant Professor.

Lehigh University: Professors G. E. Raynor and C. A. Shook retired on July 1, 1964.

Pennsylvania State University: Professor Nathan Fine, University of Pennsylvania, has been appointed Professor; Professor Sarvadaman Chowla, University of Colorado, has been appointed Research Professor.

Associate Professor E. Z. Andalafte, Southwest Missouri State College, has been appointed Associate Professor at the University of Missouri, St. Louis.

Dr. N. M. Blachman, on leave from Sylvania Electronic Systems, Mountain View, California, will spend the academic year 1964–65 as a Fulbright Lecturer at the University of Madrid and at the Escuela Tecnica Superior de Ingenieros de Telecomunicacion.

Assistant Professor Zakkula Govindarajulu, Case Institute of Technology, has been promoted to Associate Professor.

Mr. George Grossman, Board of Education of the City of New York, has been appointed Acting Director of Mathematics.

Dr. J. H. Halton, University of Colorado, has accepted a position of Mathematician in the Applied Mathematics Department of the Brookhaven National Laboratory, Upton, Long Island, New York.

Professor G. A. Hedlund, Yale University, has been appointed to a two-year term as Chairman of the Division of Mathematics of the National Academy of Sciences-National Research Council effective July 1, 1964. He succeeds Professor E. J. McShane of the University of Virginia. Professor Mark Kac of the Rockefeller Institute has been named chairman designate.

Dr. H. N. Laden, Chesapeake and Ohio Railway Company, Cleveland, Ohio, has been named Director-Data Systems of the Chesapeake and Ohio Railway Company-Baltimore and Ohio Railroad Company.

Professor G. R. MacLane, Rice University, has been appointed Head of the Division of Mathematical Sciences of Purdue University.

Professor W. D. Maness, Howard Payne College, has been appointed Associate Professor at Austin College.

Professor K. O. May, Carleton College, is spending the academic year 1964–65 at the University of California at Berkeley doing research in the history of mathematics.

Mr. A. C. Moeller, Marquette University, has been appointed Dean of the College of Engineering.

Associate Professor C. S. Ogilvy, Hamilton College, has been promoted to Professor.

Dr. A. M. Peiser, Socony Mobil Oil Company, New York, New York, has been promoted to Engineering Consultant.

Assistant Professor Alvaro Prieto, Xavier University, has been promoted to Associate Professor.

Dr. Perry Scheinok, Radio Corporation of America, Moorestown, New Jersey, has been appointed Director of the Computing Center and Assistant Professor of Pharmacology at Hahnemann Medical College.

Assistant Professor B. J. Schweitzer, San Jose State College, has been promoted to Associate Professor.

Dr. T. I. Seidman, Boeing Scientific Research Laboratories, Seattle, Washington, has been appointed Associate Professor at Wayne State University.

Dr. Seymour Sherman, Republic Aviation Corporation, Farmingdale, Long Island, has been promoted to Chief of the Applied Mathematics Subdivision.

Assistant Professor E. T. Stapleford, Jamestown Community College, has been promoted to Associate Professor.

Mr. M. W. Stone, Rohm and Haas Company, Huntsville, Alabama, has accepted a position as Chief of the Mathematics Section, Astran Division of Space Craft, Inc., Huntsville, Alabama.

Associate Professor A. J. Terzuoli, Polytechnic Institute of Brooklyn, has been promoted to Professor.

Assistant Professor Abraham Weinstein, Nassau Community College, has been promoted to Associate Professor.

Dr. F. J. Weyl, Deputy Chief and Chief Scientist of the Office of Naval Research, Washington, D. C., has been chosen to receive one of the ten 1964 Career Service Awards.

Dr. Stanley Winkler, on leave from the International Business Machines Corporation, Bethesda, Maryland, has been appointed to the professional staff of the Institute for Defense Analyses, Washington, D. C.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for the three year term July 1, 1964 to June 30, 1967, by a mail vote of the membership of the Association in the Sections indicated:

Kansas	Calvin Foreman, Baker University
Missouri	J. J. Andrews, St. Louis University
New Jersey	L. F. McAuley, Rutgers, The State University
Northeastern	Howard Eves, University of Maine
Ohio	Wade Ellis, Oberlin College
Pacific Northwest	A. T. Lonseth, Oregon State University
Southeastern	J. R. Wesson, Vanderbilt University
Southwestern	Harvey Cohn, University of Arizona
Upper New York State	D. E. Kibbey, Syracuse University

The highest percentage of voters was 49% in the Missouri Section, followed by the Kansas Section with 43% and the Southwestern Section with 40%.

RAOUL HAILPERN, *Associate Secretary*

1965 COOPERATIVE SUMMER SEMINAR

A second Cooperative Summer Seminar will be conducted by MAA during the summer of 1965 at Bowdoin College, Brunswick, Maine, from June 21 to August 13. Grants toward the financial support of the seminar have been received from the Research Corporation, and the Alfred P. Sloan Foundation. There will be two main series of lectures, one in analysis by Professor Lynn H. Loomis of Harvard University and the other in algebra by Professor Israel N. Herstein of the University of Chicago.

Stipends of $1,800 plus travel allowances will be awarded to 30 members of mathematics departments in institutions which offer an undergraduate major but not a Ph.D. in mathematics. Further information and application blanks can be obtained by writing the Director of the Seminar, E. A. Cameron, Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27514.

THE 1964 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twenty-fifth annual William Lowell Putnam Mathematical Competition will be held on Saturday, December 5, 1964. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduates in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed to the regular mailing list about October first. If an application blank is not received by October fifteenth, one may be secured by writing the director, Professor L. E. Bush, 308 Merrill Hall, Kent State University, Kent, Ohio. Your application must be filed with the director not later than November 2, 1964. For further details of the examination and the list of prizes, including the William Lowell Putnam Prize Scholarship to Harvard University, see the announcement which will accompany the application blank. The value of the scholarship has recently been increased to $2500.00 plus tuition at Harvard.

Reports of the previous competitions and the examinations may be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August–September 1947;

December 1948; August–September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; December 1956; August–September (announcement of winners) and November (questions and solutions) 1957; August–September 1958; August–September 1959; January (questions and solutions for eighteenth, nineteenth and twentieth competitions) 1961; August–September 1961; October 1962; August–September 1963 and June–July 1964.

REMUNERATION OF AUTHORS FOR EXPOSITORY WRITING

In accordance with the Association's policy of remuneration of authors for expository writing felt to be of sufficient quality to justify remuneration (for details see page 1045 of the November 1963 issue of this MONTHLY), payments at the rate of $6.00 per printed page were made to authors of seven papers in the MONTHLY and eight papers in the MATHEMATICS MAGAZINE during 1963. The total amount expended was $876.00.

HENRY L. ALDER, *Secretary*

SIXTH EDITION OF PROFESSIONAL OPPORTUNITIES IN MATHEMATICS

A new edition of this booklet was published by the Association in April 1964. The sixth edition is a completely revised version of an article which appeared originally in the MONTHLY for January 1951. The present edition was prepared by a committee consisting of K. J. Arnold, C. R. Phelps, Mina S. Rees, W. H. Schmidt, C. E. Sealander, and J. S. Frame, Chairman.

The sixth edition of *Professional Opportunities in Mathematics* is a booklet of 32 pages. The price remains 25¢ for single copies and 20¢ each for five or more copies. Orders with payment should be sent to the Buffalo office of the Association.

DECEMBER MEETING OF THE TEXAS SECTION

The Texas Section of the Mathematical Association of America met jointly with the sixty-seventh meeting of the Texas Academy of Science at Abilene Christian College in Abilene, Texas, December 6–7, 1963. The program was arranged by James F. Gray, S.M., St. Mary's University, Vice-President for Mathematics of the TAS, who presided at the Friday afternoon session; H. C. Parrish, North Texas State University, President of the Texas Section of the MAA, chaired the Saturday morning session. An invited address on "Order Statistics" was given by Professor S. S. Wilks of Princeton University on Friday afternoon, through the cooperation of the NSF-Visiting Lecturer Program in Statistics. Professor Charles R. Sherer of Texas Christian University was elected Vice-President of Section I—Mathematics of the TAS for 1964.

The following papers were presented:

1. *A class of iterative techniques for the factorization of polynomials*, by H. A. Luther, Texas A and M University.

2. *The joint distribution of the sums of squares of blocks and treatments (unadjusted) for a symmetric balanced incomplete block design*, by J. T. Webster, Southern Methodist University.

3. *A necessary condition for a nontrivial solution in integers of* $X^p+Y^p+Z^p=0$, *p an odd prime*, by W. E. Christilles, Saint Mary's University.

4. *Order statistics*, by S. S. Wilks, Princeton University. (An invited hour addressed through the cooperation of the NSF-Visiting Lecturer Program in Statistics.)

5. *A characterization of compact operators*, by H. E. Lacey, Abilene Christian College.

6. *A linear space of functions which is a two sided ideal in C′*, by D. E. Ryan, University of Texas.

7. *A theorem on cubics*, by E. E. Moyers, Tidewater Oil Company.

8. *Primary Abelian groups: Their history and some recent developments*, by H. E. Heatherly, Texas A and M University.

9. *Some milestones in the development of group theory*, by W. S. McCulley, Texas A and M University.

10. *An arithmetic identity and its application to the Fermat problem*, by Don Edmondson, University of Texas.

11. *The place of modern algebra methods in the current public school mathematics curriculum*, by Woodard Robbins, Abilene Christian College.

Comments on and evaluation of the reactions of students and secondary teachers to curriculum changes in algebra were presented, emphasizing the positive good being accomplished and the remaining problems that must be realistically faced.

12. *On an alternate definition of the pseudo-inverse*, by Patrick Odell and James Scroggs, University of Texas.

13. *Remarks on closed convex curves of constant width*, by Brock Barton, Abilene Christian College (undergraduate).

This paper presented an expository background on convex curves of constant width and answers a question posed by B. Grunbaum (Seventh Symposium in Pure Mathematics, June 1961) with the Theorem: *Precisely three sides of a rhombus cannot be support lines for a convex closed curve if and only if the curve is a curve of constant width.*

14. *A behavioral science test of multiple standard errors of estimate*, by Ray Whiteside, University of Texas.

Bottenburg and Ward (Lackland AFB) have demonstrated some expanded uses of multiple linear regression analysis. With this technique it is possible to achieve quite efficient predictor systems with regard to many facets of the behavioral sciences. One of the values of the technique is that nonnormally distributed predictor variables may be used, even dichotomous data. It was pointed out in this paper that although we may not be too confident about estimating the distribution of error scores when predicting with nonnormal data, this does not nullify the fact that often nonnormal predictors are more efficient.

15. *Pro's and con's, problems and potential of the freshman combined analytics and calculus course*, a panel discussion by Professors J. F. Gray, S.M., St. Mary's University (moderator), Dale Maness, Howard Payne College, M. E. Mullings, Abilene Christian College, and H. C. Parrish, North Texas State University.

C. R. SHERER, *Secretary*

MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on Saturday, March 28, 1964, at Michigan State University, East Lansing, Michigan, in conjunction with the meeting of the Michigan Academy of Science, Arts and Letters. Professor Keith Moore, Albion College, presided at the morning session and Professor J. M. Calloway, Kalamazoo College, at the business meeting and afternoon session. A total of 91 persons attended the meeting including 84 members of the Association.

At the business meeting, the following officers were elected: Professor George E. Hay, The University of Michigan, Chairman; Professor Paul J. Zwier, Calvin College, Vice-Chairman; Professor James H. Powell, Western Michigan University, Secretary-Treas-

8. *Primary Abelian groups: Their history and some recent developments*, by H. E. Heatherly, Texas A and M University.

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urer. A revised version of the By-Laws of the Michigan Section of the Mathematical Association of America was approved unanimously. A copy of these is on file with the Secretary. Professor Lyle Mehlenbacher of the University of Detroit gave the report of the activities of the Board of Governors. The report of the Seventh Annual Michigan Mathematics Prize Competition was given by Professor James H. McKay, Oakland University. The competition was held in 536 Michigan high schools with 20,068 students participating. A copy of the report is on file with the Secretary.

The following papers were presented:

1. *Real algebraic geometry*, by Professor A. H. Wallace, Indiana University (by invitation).

2. *A Riemann-like method for solving hyperbolic difference equations*, by Professor C. S. Duris, Michigan State University.

This paper describes a method for determining the solutions of a certain class of partial difference equations in two discrete variables. These difference equations are termed hyperbolic because they arise by using difference quotients rather than derivatives in linear hyperbolic differential equations. The method of solution described here is patterned after Riemann's method for solving hyperbolic partial differential equations. The procedure is as follows: (1) a discrete Green's theorem is obtained, which gives rise to a discrete adjoint operator; (2) the adjoint operator defines a discrete Riemann-Green function, which is used in conjunction with Green's theorem to express the solution of the original difference equation in terms of the initial conditions.

3. *Tsuji functions and their misbehavior*, by Professor George Piranian, The University of Michigan.

A function f is a Tsuji function provided it is meromorphic in the unit disk D and maps the circles $|z|=r$ $(0<r<1)$ onto curves of bounded spherical length. Some Tsuji functions exhibit astonishingly pathological behavior. For example, a Tsuji function of the form $f(z)=\sum a_n/(z-z_n)$ can have each boundary point of D as a Julia point.

4. *Elementary characterization of the geometric and exponential distributions*, by Professor Herman Rubin, Michigan State University.

An interesting property of the geometric (exponential) distribution is that of independence of the digits. Also the excess of a geometric (exponential) random variable over a number has the same distribution, conditional on being nonnegative. As a converse, we obtain the following result: *If for a nonnegative random variable* $X=\sum A_n b^n$, A_n *integers between* 0 *and* $b-1$, *the* A*'s are independent, and the distribution of* $X-\alpha$ *given* $X-\alpha\geqq 0$ *has the same property for all* $\alpha>0$, *then the* A_n *are constant for* $n<k-1$ *and* $\sum_{j=1}^{\infty} A_{n+j}b^j$ *is geometric. If* $k=-\infty$, X *is exponential.*

5. *A pathological example related to surface area*, by Professor A. H. Copeland, Sr., The University of Michigan.

The theory of surface area is surprisingly difficult. One should expect to be able to approximate the area of a surface by the area of an inscribed polyhedron and to base a theory of area on these approximations. Certainly one shouldn't expect trouble in applying such a theory to a spherical surface. However, it is possible to inscribe in a sphere a polyhedron which has arbitrarily large area and which does not cross itself. This paper contains an unusually simple example of such a polyhedral surface.

6. *The integration by parts theorem*, by Professor T. H. Hildebrandt, The University of Michigan (by invitation).

7. *A dissection problem for sets of polygons*, by Professors B. M. Stewart and Michael Goldberg, Michigan State University.

8. *An improvement of Milne's method*, by Professor Diran Sarafyan, Michigan State University.

J. H. POWELL, *Secretary-Treasurer*

MARCH MEETING OF THE SOUTHEASTERN SECTION

The Citadel was host for the 43rd annual meeting of the Southeastern Section of the Mathematical Association of America on March 20–21, 1964. Professors Winston Massey, A. D. Wallace, Lee P. Hutchison, George E. Reeves, and Henry Sharp, Jr., presided over General and Divisional Sections. Addresses by R. P. Agnew of Cornell University and M. K. Fort, Jr., of the University of Georgia were featured. There were 242 members and guests of the Section in attendance. At the Business Meeting a motion was passed that the Section should accept future invitations for places to meet only from schools who can provide completely integrated facilities and where integrated housing is available in the area.

The following officers were elected: Chairman, Professor M. K. Fort, Jr., Vice Chairman, Professor B. F. Bryant, Vanderbilt University. Elected a year ago, Professor Henry Sharp, Jr., Emory University, started a three year term as our new Secretary. The invitation from Wake Forest College to act as host for the 1965 meeting was reaffirmed and an invitation from Emory University for the 1966 meeting was accepted. The following program was presented:

1. *Multiplicative triples in a Baer *-semigroup*, by Professor R. J. Greechie, University of Florida.

Using the terminology of D. J. Foulis, Relative Inverses in a Baer *-semigroup (Michigan Math. J., 10 (1963) 65–84), let S be a Baer *-semigroup, and let $P'(S)$ be the orthomodular lattice of closed projections in S. For $e, f \in P'(S)$, $a \in S$, define $\mathfrak{M}(e, f, a)$ to mean $(ea)'' \wedge (^f a)'' = [(e \wedge f)a]''$, and say that (e, f, a) is a *multiplicative triple*. Theorem: *For* $e, f \in P'(S)$, *T.A.E.:* (1) eCf, (2) $\mathfrak{M}(e, f, f)$, (3) $\mathfrak{M}(e, e', f)$ *and* $\mathfrak{M}(f, f', e)$. *Hence, multiplicative triples characterize the center of* $P'(S)$. Other properties of multiplicative triples are given.

2. *On embedding the halfring* $(P, +, \cdot)$ *of positive integers in the ring* $(I, +, \cdot)$ *of integers*, by Professor S. G. Bourne, University of Florida.

Definition 1. *A semiring* $(S, +, \cdot)$ *is a system consisting of a set S and two binary operations in S called addition and multiplication such that* (a) $(S, +)$ *is a semigroup;* (b) $(S, \cdot)$ *is a semigroup;* (c) $(S, +, \cdot)$ *satisfies the left and right distributive laws.*

The set P of positive integers and the compositions of ordinary addition and multiplication in P is a semiring with commutative addition. The product $P \times P$ forms a semiring with commutative addition and cancellative multiplication.

Definition 2. *A subset I of S is a 2-sided ideal in S if* $(S, +)$ *is a semigroup,* $SI \subseteq S$ *and* $IS \subseteq S$.

The diagonal $\Delta = \{(p, p) \mid p \in P\}$ is a 2-sided ideal in $P \times P$.

Definition 3. $(p_1, q_1) \equiv (p_2, q_2)$ (Δ) *iff* $(p_1, q_1) + (x, x) = (p_2, q_2) + (y, y)$ *for some* (x, x) *and* (y, y) *in* Δ.

Theorem. $P \times P/\Delta \cong I$.

3. *Semigroups of N by N matrices over semirings*, by Professor D. J. Foulis, University of Florida.

The problem is to give a intrinsic characterization of the multiplicative semigroup of N by N matrices over a semiring. This has been done for N by N matrices over a field by L. M. Gluskin [M.R. *16*, 4]. Let S be a multiplicative semigroup with 0. Two idempotents e, f in S are (*algebraically*) *equivalent* iff there exist x, y in S with $xy = e$, $yx = f$. The family $\{e_1, e_2, \cdots, e_n\}$ of idempotents in S is *homogeneous* if $e_i e_j = 0$ for $i \neq j$ and e_i is equivalent to e_j for all $i, j = 1, 2, \cdots, n$. The family $\{e_1, e_2, \cdots, e_n\}$ of idempotents in S forms a *basis* for S iff it is homogeneous and given n^2 arbitrary elements $x_{ij} \in e_i S e_j$ there exists a unique x in S with $e_i x e_j = x_{ij}$, $i, j = 1, 2, \cdots, n$. Theorem: *If S has a basis* $(n \geq 2)$, *then S is isomorphic to all N by N matrices over a semigroup.*

4. *The order of an element in a group*, by Professor R. W. Ball, Auburn University.

This article appears in this issue of the Monthly, pp. 784–785.

5. *Projective planes constructed from generalized ternary rings*, by Professor J. R. Wesson, Vanderbilt University.

The "ternary ring" used to coordinatize the general projective plane has axioms which require the existence of a "zero" and a "unit". These axioms are dropped, and the system R_1 satisfying the remaining axioms is used to construct a plane π_1. New operations on the elements of R_1 are defined so that R_1 is changed into a ternary ring R_2 with a zero and unit. The plane π_2 induced by R_2 is isomorphic to π_1. At an intermediate stage of the development, the plane dual to π_1 is constructed.

6. *Generalized totients over certain classes of n-square matrices*, by Professor M. O. LeVan, University of Florida.

7. *The status of mathematics in Shakespeare's time*, by Professor G. E. Reves, The Citadel.

This paper is presented as part of the commemoration of the quadri-centennial of the birth of Shakespeare. Shakespeare's life fell in the important transition period from ancient mathematics to modern mathematics. Developments in arithmetic were primarily of a computational nature and no significant advances were made in number theory or geometry. Trigonometry was somewhat systematized and Vieta made important contributions to algebra which set the stage for analytic geometry and the calculus.

8. *A secondary mathematics teacher training program*, by Professor J. L. Tilley, Clemson College.

This is a report on a new program for the training of mathematics secondary school teachers as developed at Clemson. A unique item is that a new general curriculum and degree had to be created by the Mathematics Department in order to fit the concepts of the desired curriculum. A complementary program under the Master of Education Degree curriculum has also been proposed.

9. *Mathematical libraries in Colombia*, by Professor W. L. Furman, Spring Hill College.

This is a survey of the mathematical books and journals contained in the principal libraries of Colombia.

10. *Inverse functions*, by Professor W. M. Perel, Charlotte College.

An easy method of finding the values of the inverse of a function, given the values of the function is presented. It is the contention of the author that this presentation generates a more nearly correct concept of function in the mind of the student than the more usual treatments.

11. *On traditional versus modern algebra*, by Professors W. S. Cannon and P. E. Campbell, Presbyterian College.

The difficulties encountered in the changing of the basic algebra course at a small liberal arts college from a traditional to a modern approach are outlined. The fact is noted that there is little difference between the modern approach system and the traditional approach in the results, in grades, and in student performance. It has been observed that the students with the modern approach do better work in future mathematics courses. A comparison of grades under the two systems confirms this.

12. *Generalized integers*, by Mrs. A. F. Horadam, Visiting Professor at the Univerity of North Carolina from University of New England, Australia.

Generalized integers are defined as follows: Suppose given a sequence $\{p\}$, finite or infinite, of positive numbers (generalized primes) such that $1<p_1<p_2<p_3< \cdots$. Form all possible products $p_1^{v_1}\cdot p_2^{v_2}\cdots p_k^{v_k}$ where $v_1, v_2, \cdots v_k$ are integers $\geqq 0$. Call these numbers generalized integers $\{1_n\}$ and assume they can be arranged as an increasing sequence: $1=l_1<l_2<l_3<\cdots<l_n<\cdots$. Define $[l_n]$ to be the number of generalized integers $\leqq l_n$ and define $l_n!=1\cdot l_2\cdot l_3\cdots l_n$. Then the highest exponent of the generalized prime p occurring in $l_n!$ is $[l_n/p_1]+[l_n/p_2]+[l_n/p_3]+\cdots$.

13. *Acceptable pairings of finite sets of integers*, by Professor G. B. Huff, University of Georgia.

Mok-Kong Shen and Tsen-Pao Shen have asked (Bull. Amer. Math. Soc., 68, p. 557) if it is possible, for each $n \geqq 3$, to group the first $2n$ natural numbers into n pairs (a_1, b_1), $(a_2, b_2), \cdots,$ (a_n, b_n) with $b_i > a_i$ and such that the $2n$ numbers $a_i + b_i$, $b_i - a_i$ are all different. More recently, M. Slater (Bull. Amer. Math. Soc., 69, p. 333) has conjectured that, for $n \neq 2$, 3 and 6, it is possible to do this so that $1 \leqq a_i \leqq n$ and $n+1 \leqq b_i \leqq 2n$. Professor Huff has two results which show the existence of several infinite collections of pairings which satisfy both conditions. Lemma I: If $a-1$, a, $a+1$ are all relatively prime to $n+1$ and $a^2+1 \equiv 0$, mod $(n+1)$, then the function of f defined by $f(x) \equiv ax$, mod $(n+1)$ and $1 \leqq f(x) \leqq n$, is a 1-1 mapping of the set $\{1, 2, \cdots, n\}$ onto itself such that the $2n$ integers $f(i)+i$ and $f(i)-i$ are all different. Lemma II: If -1 is a square modulo $(n+1)$, there is a pairing (a_1, b_1), $(a_2, b_2), \cdots, (a_n, b_n)$ of the first $2n$ natural numbers such that the $2n$ numbers $a_i + b_i$, $b_i - a_i$ are all different and $1 \leqq a_i \leqq n$ and $n+1 \leqq b_i \leqq 2n$.

14. *Taylor series for* $1/[f(z)]^a$, by Professor R. C. Meacham, Florida Presbyterian College.

Recurrence formulae are developed for the generation of Taylor series coefficients for the expansion of $F(z) = 1/[f(z)]^a$, where $a > 0$, $f(0) \neq 0$, and the Taylor series expansion of $f(z)$ is known. Specifically, if $f(z) = \sum_{p=0}^{\infty} a_p z^p$, then $F(z) = \sum_{p=0}^{\infty} A_p z^p$ where $A_0 = (1/a_0)^a$, and $A_p = (-1/pa_0) \sum_{q=0}^{p-1} [q + a(p-q)] A_q a_{p-q}$ for $p = 1, 2 \cdots$.

15. *Aspects of generalized Fibonacci numbers*, by Professor A. F. Horadam, Visiting Professor at the University of North Carolina from University of New England, Australia.

From the Fibonacci and Lucas numbers we define for arbitrary integers p, q the generalized Fibonacci sequence $\{H_{pq}\}$ for which the nth term $(n \geqq 2)$ is $H_n = H_{n-1} - H_{n-2} = pF_n + qF_{n-1}$ where F_n is the nth Fibonacci number. Among the many results obtained the most useful is the "Pythagorean property," namely, $(H_n H_{n+3})^2 + (2H_{n+1} H_{n+2})^2 = 2H_{n+1} H_{n+2} + H_n^2$ which may be used to prove the theorem that all primitive Pythagorean triples are generalized Fibonacci triples. Interesting aspects of the generalized numbers include the extension to complex numbers and quaternions, the use of matrices, and the applications to generating functions for powers of the generalized numbers and to tape sorting problems in computers.

16. *Using Grace's theorem to prove a special case of a theorem of W. Specht* (*SCTS*), by Professor Carroll Weber, East Carolina College.

Given any monic polynomial $f \in Cx[z]$, there is g, apolar to f, with all its zeros on a circle centered at an arbitrary point x, with radius $\sqrt[n]{f(x)}$, n the degree of f and g. This circle must contain a zero Z of f, and may help us make a good initial guess for numerical determination of Z. The case $x=0$ gives SCTS (Enz.der. Math. Wiss., I.1.3.II, I.1.8., p. 27, $p=1$). This method of proof may be new. There seems to be great unused potential of elementary function theory in numerical analysis.

17. *Certain properties of P-intersective sets*, by Professors A. R. Bednarek and A. D. Wallace, University of Florida.

For a given reflexive transitive relation P one can define minimal P intersective sets and show that, among other things, if M and N are minimal P-intersective sets then there is a unique one-to-one function f from M onto N such that x and $f(x)$ are P-related for each x in M. This extends considerably some results of Dubreil and Lefebvre for semigroups.

18. *Some characteristic properties of minimal P-intersective sets*, by Professors A. D. Wallace and A. R. Bednarek, University of Florida.

A minimal P-intersective set M, defined for a transitive relation P on X, has several characteristic properties. Principal amongst these are: (1) M is P-intersective and M is a maximal P-scattered set; and (2) $M \subset K$ and $M \cap P_a$ contains a single element for each $a \in K$, where K is the set of P-minimal elements and $P_a = \{x \mid x \cup Px = a \cup Pa\}$.

19. *The concept of a null set*, by Professor C. G. Phipps, Tennessee Polytechnic Institute.

Among the properties usually assigned to the null set, how many are assumed, how many are assigned by definition, and how many are derived as theorems? In most current treatments, cer-

points), and tree. For each of these spaces, a monotone relation has been found which preserves the space and which gives a metric image when the domain is metric.

26. *Cauchy's functional equation in the positive domain and the area of the rectangle*, by Professor J. Aczel, University L. Kossuth of Debrecen Hungary and University of Florida.

A modern version of Legendre's deduction of the area-formula of rectangles is given under supposition of positivity and additivity of the area. This leads to the questions of the general solutions of Cauchy's functional equation on the positive domain and of the non-existence of a Hamel-basis of the positive real numbers.

C. L. SEEBECK, *Secretary*

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The forty-fourth regular meeting of the Southern California Section of the Mathematical Association of America was held at San Fernando Valley State College, Northridge, California, on March 14, 1964. The registered attendance was 184, including 145 members of the Association. Professor Fred Marer, Chairman of the Section, presided at the morning and afternoon sessions.

At the business meeting the report of Professor R. H. Sorgenfrey, Chairman of the Nominating Committee, was read. The following officers were elected to serve for the year beginning July 1, 1964: Chairman, Professor Charles J. A. Halberg, Jr., University of California, Riverside; Vice-Chairman, Professor Tom M. Apostol, California Institute of Technology. The Secretary, R. B. Herrera, was elected for a three-year term. The following members of the Program Committee for the 1965 meeting were also elected on the ballot: Professor Charles L. Clark, Los Angeles State College, Chairman; Dr. Edward C. Posner, Professor M. F. Smiley, and Professor Paul B. Yale. Professor John A. Ferling, Claremont Men's College, was also appointed to the Program Committee by the Executive Committee of the Section, to serve as representative of the host institution for the 1965 meeting.

Professor Paul J. Kelly, Governor for the Section, reported on actions taken by the Board of Governors at the Annual Meeting of the Association. He reported in particular on the plan approved by the Board of Governors to have representatives of the MAA appointed in all universities and colleges, including junior colleges, in the United States and Canada.

The following program was presented:

1. *An extension of the inverse power method*, by Professor John Blattner, San Fernando Valley State College.

2. *In praise of Leibniz*, by Professor Abraham Robinson, University of California, Los Angeles.

Leibniz was a protagonist of the use of infinitely small and infinitely large numbers in the development of the Calculus. He regarded such entities as ideal additions to the ordinary (real) numbers and likened their introduction to the introduction of complex numbers. Leibniz' program failed at the time, perhaps because he and his successors were unable to see how to satisfy the requirement (stated explicitly by Leibniz) that the new numbers should have the same properties as the old ones, in spite of the evident failure of Archimedes' axiom. This difficulty has been resolved by the use of modern logic, and it has been shown how Leibniz' ideas can be realized effectively.

3. *Homogeneity*, by Professor R. H. Bing, University of Wisconsin, President of the Mathematical Association of America.

4. *The invisible industrial mathematician*, by Dr. Andrew Vazsonyi, Scientific Advisor, North American Aviation, El Segundo, California.

5. *Some inequalities of the Markoffs*, by Professor John Todd, California Institute of Technology.

6. Panel Discussion: *The preparation of high school mathematics teachers*, Professor Paul B. Johnson, moderator. Members of the Panel were Professor F. Lynwood Wren, San Fernando Valley State College; Professor Charles W. Seekins, Occidental College; Dr. Marian Cliffe Herrick and Mrs. Lois Whitman, Los Angeles City Schools.

Preparation of teachers must be at a truly professional level. Proper attention should be given to the various facets of teaching as a career, with emphasis on continuing professional growth on the job. Breadth of background rather than advanced specialization should be emphasized in courses for teachers. Every effort should be made to interest capable students in secondary school teaching as a career, beginning in high school mathematics courses, and continuing through college matnematics courses.

R. B. HERRERA, *Secretary*

APRIL MEETING OF THE MISSOURI SECTION

The annual meeting of the Missouri Section of the Mathematical Association of America was held at the University of Missouri, Columbia, Missouri, on April 18, 1964. Professor Nola A. Haynes, Section Chairman, presided. Seventy-six persons, of whom sixty-two were members, signed the register.

Officers elected for 1964–65 are: Chairman Raymond Freese, St. Louis University; Vice-Chairman, Russell Michel, Southeast Missouri State College; Secretary-Treasurer, Leo J. Lange, University of Missouri.

Professor R. H. Bing, President of the MAA was present at the meeting and gave the afternoon lecture, titled "Homogeneity"

The following papers were presented at the morning session:

1. *A result in number theory*, by Frank Gillespie, University of Missouri.

2. *On the estimation of a parameter*, by Gerald Haas, School of Mines and Metallurgy.

3. *On some problems in theory and automatic control*, by David Gorman, Washington University.

4. *The Madison Project: A supplementary mathematics program for elementary school teachers*, by Miss Katherine Kharas, Webster College.

MARY L. CUMMINGS, *Secretary*

APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at the Texas Technological College at Lubbock, Texas, on April 10–11, 1964. Professor Herbert C. Parrish, Chairman of the Section, and Professor E. A. Hazlewood, presided. There were 200 persons in attendance, including 160 official registrants. Dr. R. H. Bing, President of the Mathematical Association, was a distinguished visitor and gave an hour lecture on Homogeneity.

At the business meeting the following officers were elected: Chairman, Professor E. R. Heineman, Texas Technological College; Vice-Chairman, Professor Charles R. Deeter, Texas Christian University; Secretary-Treasurer, Professor Charles R. Sherer, Texas Christian University.

The program was as follows:

1. *Near rings with identities defined on simple groups*, by J. J. Malone Jr., University of Houston.

It is immediate that, if the order of the group is a prime p, the only near ring which may be defined on the group is the field of order p. In the case in which the group has even composite order it is shown, by considering the maximal sub-C-ring and the maximal sub-Z-ring, that no near ring can be defined on the group.

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2. *Structural determination of a class of primary abelian groups*, by Henry E. Heatherly, Texas A and M University.

Let G be a p-primary abelian group and k a positive integer such that $h(x+y) \leq k[h(x)+h(y)]$ for each x and y in G, where $x+y \neq 0$. (Note: $h(a)$ is the height of a in G.) The structure (direct sum decomposition) of such groups is determined by the following Theorem: *If G is a group as described above, then G is (exclusively) either (1) the direct sum of groups each isomorphic to the additive group of rationals or to a quasicyclic group or (2) the direct sum of cyclic groups each of order p.*

3. *On subgroups of primary abelian groups*, by Charles Megibben, Texas Technological College.

L. Fuchs (see "Recent results and problems on abelian groups" in *Topics in abelian groups*, Chicago, 1963) has raised the following question: If G is an abelian group, which subgroups H of G are the intersection of a family of pure subgroups of G? The solution of this problem for p-groups (and hence for torsion groups) is the content of the following Theorem. *If G is a p-group and H is a subgroup of G, then H is the intersection of a family of pure subgroups of G if and only if for each nonnegative integer, $(p^nG)[p] \subseteq H$ implies $p^nG \subseteq H$.*

4. *Nonparametric statistical classification*, by Carl F. Kossack, Graduate Research Center, Dallas, Texas.

One of the important applications of population discrimination theory is that of statistical classification sometimes referred to as the "diagnosis problem." In particular, the two-population classification problem is that of classifying a single p-variate observation into one of two p-variate populations using samples of individuals from each population. The author considers the nonparametric discrimination theory of Hodges and Fix for the development of a nonparametric classification technique. The weakness in their theory that revolves around the need for a metrix, is resolved by introducing the concept of an average ranking for each individual in the samples. This ranking is then used to evolve a classification technique for new individuals requiring classification with attention being given to an empirical method for measuring the operational effectiveness of the method when applied to a given situation.

5. *On the problem of choosing the largest mean*, by R. P. Bland, Southern Methodist University, Department of Mathematical and Experimental Statistics.

The problem of choosing, from three or more normal populations with equal variances, the population with the largest mean is considered as a multiple decision problem. The decision is to be based on independent, random samples of equal size drawn from each population. The loss associated with a decision is the sum of losses for each of the means involved, i.e., the loss is additive. A Bayes (minimum average risk) procedure is found for the case of three populations where the population means have independent identical normal prior densities.

6. *Optimum provisioning policies for reparable spares*, by C. H. Boll, Department of Mathematical and Experimental Statistics, Southern Methodist University.

A birth-and-death process is sometimes applicable in real-life systems design problems concerning redundancy and in inventory problems of reparable spares. Using this model, inventory policies for stand-by spares which minimize total expected cost are obtained in this paper which take into account the following parameters: interchangeability, repair rate, failure rate, unit cost, interest rate, length of program, and shortage penalties.

These optimal policies can be described as follows. The cost indifference curves between having N and $(N+1)$ spares is a straight line (in the region of interest) when plotted using as coordinates the logarithm of unit cost and logarithm of ratio of repair and failure rates. The asymptotic slopes of the indifference lines are given, and an iteration is suggested for obtaining a point on the line in the region of interest.

Some statistical problems encountered in the application of the above technique are given from a multiple decision point of view.

7. *A class of Riemann surfaces*, by T. A. Atchison, Texas Technological College.

A class of open simply connected Riemann surfaces is considered and the uniformizing function and its derivative are exhibited in an infinite product representation. An infinite product of the form of the uniformizing function is then shown to produce a surface of this class.

8. *On differential equations of generalized order*, by M. A. Al-Bassam, Texas Technological College.

THEOREM. *If $f(x, y)$ is a real-valued, continuous and Lipschitzian function on a domain D of the (xy)-plane and $\sup_{(x,y)\in D}|f(x, y)| = M < \infty$, then there exists a unique and continuous solution $y=y(x)$ which satisfies the differential equation $y^{(\alpha)}=f(x, y)$, $0<\alpha\leqq 1$, together with the condition $y^{(\alpha-1)}(a)=\eta$, in the region $R\subset D$: $a<x\leqq a+h$, $|(x-a)^{1-\alpha}y-b|\leqq k$, where $b=\eta/\Gamma(\alpha)$, and $k>Mh/\Gamma(\alpha+1)$.* Also other existence theorems have been established.

9. *On a system of nonlinear differential equations in Hilbert space*, by A. D. Stewart, Prairie View A. and M. College.

H denotes a Hilbert Space, and E^1 denotes the number-axis, $I=[0, 1]$ on E^1. Let $Y(x) = [y^1(x), y^2(x), \cdots, y^n(x), \cdots]$ be a function in H with a continuous derivative on I. Let $f(x, y)$ be a continuous function in H^+, where H^+ means $y(x)$ is in H and x on I. Furthermore, let $f(x, y)$ satisfy a Lipschitz condition, that is, $|f(x, z)-f(x, y)| < M|z-y|$, where (x, z) and (x, y) are in H^+ and $M>0$. Then the system of nonlinear differential equations, $DY(x)=f(x, y)$ has a unique solution in H such that $Y(0)=Y_0$, where Y_0 is in H.

10. *Relations between two classes of singular integral operators*, by Chang-Char Tu, William Marsh Rice University.

It is shown that certain singular integral operators considered by Calderón and Zygmund are, in a sense, special cases of a class of singular operators considered by Jones. The relation between these two types of operators is studied, the main distinction being the method of defining principal value integrals.

11. *Some reproducing kernels for the unit disk*, by G. S. Innis, Jr., William Marsh Rice University.

12. *A predictor-corrector adaptation of Runge-Kutta methods*, by A. F. Burger and H. A. Luther, Texas A and M University.

To every one-step method of approximating the solution of ordinary differential equations there corrresponds a "backward" formula. The two formulas can be paired to yield a convergent predictor-corrector method.

In this preliminary discussion the technique is exemplified for Runge-Kutta methods involving systems of ordinary equations. Accumulated error proves easy to discuss. Results resemble closely those for the more usual forward and backward technique. There is reason to hope that the present method will permit larger step-sizes.

13. *Generalized second order recurring sequences*, by C. R. Wall, Texas Christian University.

Let (1) $W_{n+2}=gW_{n+1}+hW_n$, with $W_0=q$, $W_1=p$ arbitrary; (2) $Y_n=W_{n+1}+hW_{n-1}$; α, $\beta = [g^2\pm\sqrt{(g^2+4h)}]/2$, and $A=p-q\beta$. For $g=h=1$, (1) and (2) are generalized Fibonacci sequences. Lucas studied the case $q=0$, $p=1$, where (1) is u_n and (2) is v_n. Then $[W_{n+r}+(-h)^rW_{n-r}]/W_n=v_r$ and $[W_{n+r}-(-h)^rW_{n-r}]/Y_n=u_r$ independent of p, q, n. The following de Moivre-type identity was also given: $[(Y_n+(\alpha-\beta)W_n)/2A]^m=(Y_{nm}+(\alpha-\beta)W_{nm})/2A$.

14. *Concerning the center of the doubly stochastic operators on a Hilbert space*, by R. D. Sinkhorn, University of Houston.

Let $\mathcal{H}$ be a complex Hilbert space and let $u_0\in\mathcal{H}$ be such that $\|u_0\|=1$. Denote by $\mathfrak{D}$ the set of doubly stochastic mappings in $[\mathcal{H}]$, i.e. those $T\in[\mathcal{H}]$ such that $Tu_0=T^*u_0=u_0$. Then if $T_0\in\mathfrak{D}$ has a one dimensional range, the center of $\mathfrak{D}$ is precisely $\mathfrak{A}(T_0)$ where $\mathfrak{A}$ is the collection of complex functions f analytic on $\{0\}\cup\{1\}$ with $f(1)=1$.

15. *A space of connectivity functions*, by S. K. Hildebrand, Texas Technological College.

Let $[X, d]$ be a metric space where X represents the set of elements and d the metric on the set X. A metric ϕ is defined on X_X, where X_X represents the set of all connectivity functions on X into X. Additional properties of this structure are investigated.

16. *The inversion of a class of linear operators on QC_{OL}*, by J. A. Dyer, Southern Methodist University.

Let k denote the topology for QC generated by the uniformity which has as a basis the family of all sets of the form $N_{r,s} = \{(f, g): |f(t) - g(t)| < r, -s \leqq t \leqq s, r \text{ rational}, s \text{ a positive integer}\}$, and let k' denote the relative topology for QC_{OL}. Then each k'-continuous P-operator, $\mathfrak{L}$, on QC_{OL} has a unique representation of the form $\mathfrak{L}f(s) = (\sigma M)\int_0^s f(\xi)dL(\xi, s)$, where $L(t, s) = -\{2\mathfrak{L}\tau_t(s) - \mathfrak{L}\tau_t(s^+)\}$, and $\tau_t(s) = -J_L(s-t)$. Furthermore a k'-continuous P-operator, $\mathfrak{L}$, on QC_{OL} has an inverse which is a P-operator only if for each positive integer $\bar{s}$, the P-operator on $\overline{Q}_L[0, \bar{s}]$ generated by the restriction of L to $0 \leqq {}^t_s \leqq \bar{s}$ has an inverse. Lane's condition for the invertability of an operator in T_{OL} is obtained as a special case of this theorem.

17. *Complete continuity conditions in spaces of type $C(K)$—Preliminary Report*, by H. E. Lacey, Abilene Christian College.

X and Y denote Banach spaces and $C(K)$ denotes the Banach algebra of all continuous real (complex) valued functions on the compact Hausdorff space K. $T: X \to Y$ denotes a bounded linear operator. The operator $T: X \to Y$ is said to be completely continuous if and only if it maps weakly Cauchy sequences into norm convergent sequences. Theorem: *The following are equivalent:* (1) *K is dispersed;* (2) *weak sequential convergence equals norm sequential convergence in* $[C(K)]'$; (3) $[C(K)]'$ *does not contain any infinite dimensional reflexive subspaces;* (4) *each $T: C(K) \to Y$ completely continuous is compact;* (5) *no infinite dimensional homomorphic image of $C(K)$ is reflexive.*

18. *A quadric surface and related identities*, by R. S. Underwood, Texas Technological College.

A particular equation is shown by a brief method to have a cylindrical locus in 3-space. This one problem leads to 2178 different equations in four variables, each of which has an unlimited number of integral solutions. The method works in similar cases involving n variables. The role of intuition in this and related problems is emphasized.

19. *A division algebra for sequences and its associated operational calculus*, by Louis Brand, University of Houston.

This article appears in this issue of the MONTHLY, pp. 719–728.

20. *Implementing CUPM in a school of engineering*, by P. W. Latimer, Lamar State College of Technology.

A discussion of some of the problems arising in putting CUPM recommendations into effect. Problems considered will be such things as getting the cooperation of the school of engineering, what changes must be made in the engineering program, and how the mathematics offering must be altered in order to work in the necessary courses. There will also be a discussion of some of the problems that have arisen since implementation.

21. *Report of the commission to study college programs for the preparation of mathematics teachers*, by Dan Cude, Southwest Texas State College.

22. *A topological characterization of countable functions*, by Arthur W. J. Ullman, William Marsh Rice University.

A functional is said to be countable if its values are determined by a finite amount of information about its arguments. Classes of (extensional) functionals satisfying these finite amounts of information are shown to form a base for a topology. This topology is shown to be metrizable; any functional is continuous on this topology if and only if it is countable. These results are obtained for all functionals provided they are of a certain form (to which any functional can be reduced) and it is shown that this is a necessary condition for a uniform topological characterization.

C. R. SHERER, *Secretary*

unicity of composite or A/N a free join, or A/N algebraic over K implies splitting. The quotient ring A_N of A is also analyzed under the condition that F_0', F_1' exist in A. All the results apply in particular to the case where A is a quasilocal algebra over K (hence I a coefficient field).

5. *An applied mapping problem*, by Major Edwin E. Brown, Offutt AFB.

The theory of Mercator and Polar Sterographic map projections yields techniques presently utilized to produce charts via electronic computers. Mathematical relationships between grids that correspond to the character size of a printer and meshes employed for hemispheric analyses of meteorological parameters are illustrated. A lattice of approximately 37,000 points that covers most of a hemisphere and its mirror image in the opposite hemisphere are discussed.

6. *The Nebraska Mathematics Contest*, by John R. Bolingbroke, University of Nebraska, Henry M. Cox, University of Nebraska, and James M. Earl, University of Omaha.

Some 4448 students from 150 high schools enrolled in the Seventh Nebraska (Fifteenth National) Mathematics Contest, held March 5, 1964; 778 students wrote both the Sixth and Seventh Contest Examination ($r=0.57$ with median scores of 9 and 17, respectively). The name of one Nebraska contestant appears on the National Honor Roll. The Nebraska Report, which can be obtained upon request, includes the Nebraska Honor Roll, tables showing distributions of team and individual scores, and an item analysis of the test.

7. *The straight edge and compass construction of the conchoid of Nicomedes and its practical application to the trisection of the acute angle*, by Helen C. Campuzano, Medical Research Staff, Creighton University.

A discussion of the Conchoid of Nicomedes, a higher plane curve, of quartic dimension, its effect upon the radii of a circle, and its practical application to the trisection of the acute angle.

8. *The MAA Secondary School Lecturer Program in Iowa and Nebraska*, by W. E. Mientka, University of Nebraska.

9. *CUPM recommendations and implications*, A Panel Discussion, by G. A. Hutchison, Creighton University, Mildred Gross, Doane College, L. M. Larsen, Kearney State College, Duane Perry, Coordinator of Mathematics of the Omaha Public Schools, Hubert Schneider, The University of Nebraska, and J. F. Wampler, Nebraska Wesleyan University.

H. M. Cox, *Secretary*

MAY MEETING OF THE WISCONSIN SECTION

The thirty-second annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Wisconsin State College, Whitewater on May 2, 1964. Professor C. E. Flanagan, Chairman of the Section, presided. This meeting was held jointly with the May meeting of the Wisconsin Mathematics Council and there were 137 present, including 88 members of the Association and 53 members of the Wisconsin Mathematics Council.

At the business meeting the following officers were elected for the coming year: Chairman, Professor J. M. Osborn, Jr., University of Wisconsin; Vice-Chairman, Professor H. Glander, Carroll College, Waukesha; Secretary-Treasurer, Professor L. F. Wahlstrom, Wisconsin State College, Eau Claire.

The following papers were presented:

1. *Mathematics, physical problems, and the Undergraduate Curriculum at the University of Wisconsin*, by Professor John A. Nohel, University of Wisconsin.

The mathematics curriculum for undergraduates in engineering and physical sciences recommended by the CUPM is discussed. Details are presented of how this program for freshmen and sophomores is being introduced at the University of Wisconsin, together with some of the reasons for it and the philosophy behind it.

2. *What is set theory?* by Professor R. L. Wilder, University of Michigan, Ann Arbor.

The inception and subsequent evolution of set theory are studied with a view to assess the present status of the theory in mathematics. Pros and cons for formalizing within an axiomatic system are weighed, and the implications of the Godel-Cohen results on the independence of the choice principle and continuum hypothesis discussed.

3. *Mathematics for biology, management, and social science students,* by Professor T. D. Sterling, University of Cincinnati.

E. F. WILDE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-eighth Annual Meeting, Denver-Hilton Hotel, Denver, Colorado, January 28–30, 1965.

Forty-sixth Summer Meeting (Fiftieth Anniversary Celebration), Cornell University, Ithaca, New York, August 30–September 2, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 14–15, 1965.

INDIANA

IOWA, University of Dubuque, Dubuque, April 23, 1965.

KANSAS

KENTUCKY, Eastern Kentucky State College, Richmond, Spring, 1965.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 19–20, 1965.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1965.

MINNESOTA, University of Minnesota at Duluth, November 7, 1964.

MISSOURI, University of Missouri, Columbia, Spring, 1965.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 30–May 1, 1965.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 7, 1964.

NORTHEASTERN, Worcester Polytechnic Institute, Worcester, Mass., November 28, 1964.

NORTHERN CALIFORNIA, College of San Mateo, February 6, 1965.

OHIO

OKLAHOMA, University of Arkansas, Fayetteville, Spring, 1965.

PACIFIC NORTHWEST

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 21, 1964.

ROCKY MOUNTAIN, The Colorado School of Mines, Golden, Spring, 1965.

SOUTHEASTERN, Wake Forest College, Winston Salem, North Carolina, April 9–10, 1965.

SOUTHERN CALIFORNIA, Claremont Men's College, March 13, 1965.

SOUTHWESTERN, Arizona State University, Tempe, Spring, 1965.

TEXAS, Texas Christian University, Fort Worth, April 9–10, 1965.

UPPER NEW YORK STATE

WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN MATHEMATICAL SOCIETY, Denver, Colorado, January 26–29, 1965.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Illinois Institute of Technology, Chicago, June 21–25, 1965.

ASSOCIATION FOR COMPUTING MACHINERY, Cleveland, August 24–26, 1965.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Detroit, November 26–28, 1964.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, November 19–21, 1964.

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GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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INVARIANCE OF PROBABILITIES IN FINITE SAMPLE SPACES UNDER STOCHASTIC OPERATIONS

C. E. LANGENHOP, Southern Illinois University, AND
J. C. MATHEWS, Iowa State University

The classical concept of probability depends strongly on the notion of "equally likely cases" and the valid application of a mathematical probability model of a combinatorial type of problem requires a clear recognition of the appropriateness of the identification of the mathematical "equally likely" with the physical "equally likely." The analysis of several simple combinatorial problems leads one to recognize a vague sort of equivalence between the assignment of probabilities to events based on the intuitively obvious equally likely outcomes and a type of invariance of the probability itself.

For example, suppose an urn contains one white ball and two red balls and the balls are withdrawn "at random" without replacement; then the probability that the white ball is the ith ($i=1, 2, 3$) one withdrawn is $\frac{1}{3}$ regardless of the value of i. To motivate the concepts introduced in this paper we might view this result in another way. Let individual A draw the first ball; the probability of his getting white is $\frac{1}{3}$. On the other hand, if individual B draws the first ball and then individual A the second, the probability A gets white is again $\frac{1}{3}$. (That is, the probability is $\frac{1}{3}$ for A to get white if he draws first and, at the risk of confusing the issue with subjective probabilities, it is $\frac{1}{3}$ for A to get white if he draws second, providing he is not informed of the actual outcome of B's drawing.) Of course, this fact is dependent on B's drawing being random as well as A's for obviously if it is known that B's hand has some peculiar affinity for white balls, then A may have little or no chance of getting the white ball if B draws first.

The type of invariance of the value of a probability as exemplified here seems to be essentially a result of the correct assessment of the equally likely cases and assignment of equal probabilities thereto. Strictly speaking, of course, the events of A getting white on first draw or on second, although having the same probability, are in fact different events. They do have, however, a sort of sameness in so far as the outcome to A is concerned. The results presented below represent some initial explorations into the relationship between this type of invariance and the nature of the probability distribution on the elementary events in combinatorial situations. In particular one might consider the possibilities of using this as a defining criterion for the word "random" in such simple contexts as drawing balls from urns or selecting cards from a bridge deck, etc. It seems preferable, however, at this point to introduce other concepts not so commonly used as "random" and to give precise definitions in abstract terms. It may be of aid to the reader to recognize that the terms *stochastic operation*, *stochastic invariance* and *uniform* defined below are exemplified in the urn model described above by B's drawing first, A's probability of $\frac{1}{3}$ for getting the white ball whether he draws first or second, and random in the context of selection from a finite set with each selection being "equally likely," respectively.

1. Basic definitions. Throughout the sequel we consider a finite set $S=\{s_1, s_2, \cdots, s_n\}$ with $n\geqq 2$. On S there will be assumed a probability measure P and for convenience we let

$$p_i = P(s_i), \qquad i = 1, 2, \cdots, n. \tag{1.1}$$

Of course we require

$$\text{(a)} \quad p_i \geqq 0, \quad i = 1, 2, \cdots, n, \qquad \text{(b)} \quad \sum_{i=1}^{n} p_i = 1, \tag{1.2}$$

but in order that (S, P) exhibit a bone fide stochastic character we shall also assume that

$$p_i < 1, \qquad i = 1, 2, \cdots, n. \tag{1.3}$$

(Conditions (1.2) (b) and (1.3) imply that $n\geqq 2$.) The set $A=\{i \mid p_i>0\}$ is, of course, not void but the set I_n-A, where $I_n=\{1, 2, \cdots, n\}$, may or may not be void. The number of elements in A will be denoted by α and clearly $2\leqq\alpha\leqq n$ by virtue of (1.3).

Definition 1.1. *A finite probability space (F, π), where $F=\{f_1, f_2, \cdots, f_m\}$ and π is a probability measure on F, together with a stochastic matrix $B=(b_{ji})$ $(j=1, 2, \cdots, m, i=1, 2, \cdots, n, b_{ji}\geqq 0, \sum_{i=1}^{n} b_{ji}=1)$ will be called a stochastic operation on (S, P) if the b_{ji}'s are functions $b_{ji}(p)$ of the vector $p=(p_1, p_2, \cdots, p_n)$.*

Definition 1.2. *If (F, π, B) is a stochastic operation on (S, P), then the set $F\times S$ with a probability measure P^* will be called the product of (S, P) by (F, π, B) if for all possible j and i*

$$P^*(f_j, s_i) = \pi_j b_{ji}(p) \quad \textit{where} \quad \pi_j = \pi(f_j), \qquad j = 1, 2, \cdots, m. \tag{1.4}$$

Definition 1.3. *The space (S, P) will be said to be stochastically invariant under (F, π, B) if*

$$\sum_{j=1}^{m} P^*(f_j, s_i) = P(s_i), \qquad i = 1, 2, \cdots, n, \tag{1.5}$$

i.e.

$$\sum_{j=1}^{m} \pi_j b_{ji}(p) = p_i, \qquad i = 1, 2, \cdots, n. \tag{1.6}$$

The element f_j of F may be interpreted as a specific operation on the space (S, P) which is performed with probability π_j and, in view of the joint probabilities $P^*(f_j, s_i)$ assigned in (1.4), $b_{ji}(p)$ may be interpreted as the conditional probability of the event s_i given that f_j acts on (S, P). The collection F with measure π and matrix B is called a stochastic operation since each specific $f_j\in F$ acts on (S, P) only with probability π_j. The sum $\sum_{j=1}^{m} \pi_j b_{ji}(p)$ is, of course, a probability measure on S and may be interpreted as the measure which results

after (F, π, B) acts. Thus stochastic invariance of (S, P) means that this resulting measure is the same as the original measure P.

It is evident that any finite probability space (F, π) together with any fixed stochastic matrix B of the proper number of rows and columns is a stochastic operation on (S, P) according to our Definition 1.1. A finite Markov chain provides a well-known interpretation of this situation. Indeed, if S is the set of states of the chain and B the matrix of transition probabilities then (S, P) is stochastically invariant under (S, P, B) if the measure P is given by a stationary probability distribution on the states of the chain. (See for example Chapter XV of the reference.)

In the two cases considered below the matrix B is a nontrivial function of the probability vector p. The functional form of the $b_{ji}(p)$ is suggested by a type of physical operation on the elements of the set S. The intuitive basis for the particular functions used is discussed in each case and it should then be clear why the terms stochastic removal and stochastic permutation are applied in these situations.

In the statement of the results in the cases considered below we find it convenient to use the following:

Definition 1.4. *If (Q, W) is a finite probability space then W will be said to be uniform on $Y \subset Q$ if*

$$W(y) = c = W(Y)/(\text{number of elements in } Y) \tag{1.7}$$

for all $y \in Y$. If W is uniform on Q it will be said to be uniform.

2. Stochastic removal of one element from S.

Definition 2.1. *(F, π, B) will be called a stochastic removal of one element from S if $F = \{f_1, f_2, \cdots, f_n\}$ and the matrix B is defined by*

$$b_{ji} = p_i(1 - p_j)^{-1}, \quad i \neq j \tag{2.1}$$

$$b_{jj} = 0. \tag{2.2}$$

In this case the element f_j of the set F may be interpreted as the removal of s_j from S and the probabilities b_{ji} for $i \neq j$ are taken proportional to the p_i with $b_{jj} = 0$. This preserves the odds among the remaining elementary events of S after s_j is removed and, of course, if s_j is removed from S this becomes an impossible event.

Theorem 2.1. *Let (F, π, B) be a stochastic removal of one element from S. Then (S, P) is stochastically invariant under (F, π, B) if and only if for all $i \in A = \{i \mid p_i > 0\}$*

$$\pi_i = (1 - p_i)\left(1 - \sum_{j \notin A} \pi_j\right) \Big/ (\alpha - 1), \tag{2.3}$$

where α is the number of elements in A.

Proof. By (1.6), (2.1) and (2.2) (S, P) is stochastically invariant under (F, π, B) if and only if for all $i=1, 2, \cdots, n$

$$p_i = \sum_{j\neq i} \pi_j p_i(1 - p_j)^{-1}. \tag{2.4}$$

Since $p_i=0$ for $i\notin A$ equations (2.4) hold in any case for such i so (S, P) is stochastically invariant under (F, π, B) if and only if for all $i\in A$

$$1 = \sum_{j\neq i} \pi_j(1 - p_j)^{-1} \tag{2.5}$$

or, equivalently,

$$1 + \pi_i(1 - p_i)^{-1} = \sum_{j=1}^{n} \pi_j(1 - p_j)^{-1}. \tag{2.6}$$

Thus (S, P) is stochastically invariant under (F, π, B) if and only if for all $i\in A$ $\pi_i(1-p_i)^{-1}=C$ or

$$\pi_i = C(1 - p_i) \tag{2.7}$$

for some number C. The particular value of C is determined by the fact that π is a probability measure. Thus

$$1 = \sum_{i\in A} \pi_i + \sum_{j\notin A} \pi_j = C \sum_{i\in A} (1 - p_i) + \sum_{j\notin A} \pi_j$$

and since

$$\sum_{i\in A} (1 - p_i) = \sum_{i\in A} 1 - \sum_{i\in A} p_i = \alpha - \sum_{i=1}^{n} p_i = \alpha - 1$$

we find $C=(1-\sum_{j\notin A} \pi_j)(\alpha-1)^{-1}$ which in (2.7) gives (2.3).

COROLLARY 2.1. *Under the hypothesis of Theorem* 2.1 *let* (S, P) *be stochastically invariant under* (F, π, B). *If* P *is uniform on* $A_S=\{s_i\in S\mid i\in A\}$, *then* π *is uniform on* $A_F=\{f_i\in F\mid i\in A\}$; *further, if* π *is uniform and nonzero on* A_F, *then* P *is uniform on* A_S.

Proof. By Theorem 2.1 equations (2.3) or (2.7) hold for all $i\in A$. From these equations we see that if p_i is the same for all $i\in A$, then π_i is the same for all $i\in A$ (although π_i will be zero in case $\sum_{j\notin A} \pi_j=1$) and conversely providing π_i is not zero for $i\in A$.

COROLLARY 2.2. *Under the hypothesis of Theorem* 2.1 *let* (S, P) *be stochastically invariant under* (F, π, B) *and let* $A=I_n$. *Then* π *is uniform if and only if* P *is uniform.*

Proof. In this case $I_n-A=\emptyset$ so $\sum_{j\notin A} \pi_j=0$ and π_i must be positive on A. Thus p_i is independent of i for all $s_i\in S$ (hence P is uniform) if and only if π_i is independent of i for all $f_i\in F$ (π is uniform).

3. Stochastic permutations. In this section g_k will denote a permutation on the integers 1, 2, $\cdots$, n which will be considered as a mapping, i.e. if $j=g_k(i)$ the permutation takes i into j.

DEFINITION 3.1. *Let g_k, $k=1, 2, \cdots, N$, $N=n!$, denote the distinct permutations of the integers 1 through n. (F, π, B) will be called a stochastic permutation of the elements of S if $F=\{f_1, f_2, \cdots, f_N\}$ and the matrix B is defined by*

$$b_{ki} = p_{g_k(i)}, \quad k = 1, 2, \cdots, N; \quad i = 1, 2, \cdots, n. \tag{3.1}$$

The element of f_k of F may be interpreted as the permutation on the elements of S which corresponds to g_k acting on the integers 1 through n. To gain an intuitive view of the situation in this case we might consider the following model. Suppose that we have n urns $U_1, U_2, \cdots, U_n$, and in U_i there is a ball B_i. Let an urn be chosen so that p_i is the probability that urn U_i is selected. Then p_i is also the probability that ball B_i is selected. Suppose, however, that before an urn is chosen the balls are interchanged among the urns so that B_i which was in U_i is placed in U_j, where $j=g_k(i)$. Now to get ball B_i it is necessary that U_j be selected so the probability of getting ball B_i after the balls are switched according to g_k is $p_j=p_{g_k(i)}$. That is, we may interpret b_{ki} as defined in (3.1) as the conditional probability of getting ball B_i given that the permutation g_k acted on the balls assuming that in any case urn U_j is chosen with probability p_j, $j=1, 2, \cdots, n$. Thus the probability that the balls are permuted among the urns according to the permutation g_k, and following that, ball i is obtained when an urn is chosen, is as given in (1.4), i.e.

$$P^*(f_k, s_i) = \pi_k p_{g_k(i)}. \tag{3.2}$$

Now let (F, π, B) be a stochastic permutation of the elements of S. Then by (3.1) and (1.6) (S, P) is stochastically invariant under (F, π, B) if and only if for all $i=1, 2, \cdots, n!$

$$p_i = \sum_{k=1}^{n!} \pi_k p_{g_k(i)}, \quad \text{where} \quad \pi_k = \pi(f_k). \tag{3.3}$$

THEOREM 3.1. *Let (F, π, B) be a stochastic permutation of the elements of S. If P is uniform on S, then (S, P) is stochastically invariant under (F, π, B).*

Proof. Equations (3.3) hold if $p_j=1/n$ for $j=1, 2, \cdots, n$ inasmuch as π is a probability measure and $\sum_{k=1}^{n!} \pi_k=1$.

This result is hardly surprising. Theorem 3.2 and Corollary 3.1 below are partial converses of Theorem 3.1 and are perhaps less intuitively obvious. It is helpful with $\ni$ meaning "such that", to introduce

DEFINITION 3.2. *The matrix $V=(v_{ij})$ where*

$$v_{ij} = \sum_{k \ni g_k(i)=j} \pi_k \tag{3.4}$$

will be called the matrix of the stochastic permutation (F, π, B).

In some of the following proofs we rely strongly on certain results in the theory of Markov chains. These may be found in Chapter XV of the reference.

LEMMA 3.1. *The matrix V of the stochastic permutation (F, π, B) is doubly stochastic.*

Proof. We have for each $i \in I_n$

$$\sum_{j=1}^{n} v_{ij} = \sum_{j=1}^{n} \left(\sum_{k \ni g_k(i)=j} \pi_k \right). \tag{3.5}$$

In the double sum on the right of (3.5) each $k=1, 2, \cdots, n!$ is represented at least once since g_k maps I_n onto I_n so $g_k(i)=j$ for some $j \in I_n$. But each k can appear no more than once since for fixed i $g_k(i)$ is unique since g_k is a function. Thus the sum on the right of (3.5) is the same as $\sum_{k=1}^{n!} \pi_k = 1$; the latter sum is equal to 1 since π is a probability measure.

Similarly for each $j \in I_n$

$$\sum_{i=1}^{n} v_{ij} = \sum_{i=1}^{n} \left(\sum_{k \ni g_k(i)=j} \pi_k \right). \tag{3.6}$$

Again each k is represented on the right at least once since the g_k are mappings onto I_n so $g_k(i)=j$ for some $i=1, 2, \cdots, n$. But again each k can appear no more than once since each g_k has an inverse so that $g_k(i) \neq g_k(i')$ if $i \neq i'$. Thus the sum on the right of (3.6) is also 1 so V is doubly stochastic.

LEMMA 3.2. *Let p denote the column vector whose i-th component is p_i. Then equations* (3.3) *are equivalent to*

$$p = Vp. \tag{3.7}$$

Proof. Equation (3.7) is equivalent to the system

$$p_i = \sum_{j=1}^{n} v_{ij} p_j, \qquad i = 1, 2, \cdots, n. \tag{3.8}$$

For fixed i and any k, $g_k(i)=j$ for some $j=1, 2, \cdots, n$, so if on the right in (3.3) we collect all terms for each j such that $g_k(i)=j$ then (3.3) may be written

$$p_i = \sum_{j=1}^{n} \left(\sum_{k \ni g_k(i)=j} \pi_k p_{g_k(i)} \right) = \sum_{j=1}^{n} \left(\sum_{k \ni g_k(i)=j} \pi_k \right) p_j,$$

which by virtue of (3.4) is the same as (3.8).

LEMMA 3.3. *The Markov chain with matrix V is composed of $\sigma \leq n$ irreducible subchains and there are no transient states.*

Proof. We shall consider the set I_n as the set of states of the Markov chain with matrix V. Suppose the set X of transient states is not void and let $\xi > 0$

be the number of states in X. Then for at least one $i_0 \in X$ and one $j_0 \in I_n - X$ we have $v_{i_0 j_0} > 0$ but for all $i \in I_n - X$ and $j \in X$ we have $v_{ij} = 0$. Since V is doubly stochastic, for all $j \in X$

$$\sum_{i \in X} v_{ij} = 1 - \sum_{i \in I_n - X} v_{ij} = 1,$$

so that

$$\sum_{j \in X} \sum_{i \in X} v_{ij} = \xi.$$

But for all $i \in X - \{i_0\}$

$$\sum_{j \in X} v_{ij} \leqq 1 \quad \text{while} \quad \sum_{j \in X} v_{i_0 j} \leqq 1 - v_{i_0 j_0} < 1,$$

so that $\sum_{i \in X} \sum_{j \in X} v_{ij} < \xi$, which is a contradiction. Hence the set of transient states is void. The structure of the chain then is such that it decomposes into $\sigma \leqq n$ irreducible subchains some of which may be periodic, of course.

DEFINITION 3.3. *Let G denote the group of permutations generated by $\{g_k \mid \pi_k > 0\}$. A nonvoid set $I \subset I_n$ such that $g(I) = I$ for all $g \in G$ and having no proper (nonvoid) subset with this property will be called a minimal invariant set under G.*

LEMMA 3.4. *The minimal invariant sets under G are the sets of states of the irreducible subchains of the Markov chain defined by V.*

Proof. By (3.4) if $j = g_k(i)$ for k such that $\pi_k > 0$ then $v_{ij} > 0$ so j can be reached from i in the Markov process. Thus with $K = \{k \mid \pi_k > 0\}$ we see that if $k_1, k_2, \cdots, k_r \in K$ then the integer

$$\overset{d_1}{g}_{k_1}(\overset{d_2}{g}_{k_2}(\cdots(\overset{d_r}{g}_{k_r}(i))\cdots))$$

can be reached from i in the Markov process. Thus for any i and any $g \in G$, $g(i)$ can be reached from i. Now let i be fixed and consider $I(i) = \{g(i) \mid g \in G\}$. Every such set is invariant under G. For if $j \in I(i)$, then $j = g'(i)$ for some $g' \in G$. Thus for any $g \in G$ we have $g(j) = g(g'(i)) = (g \circ g')(i) \in I(i)$ so $g(I(i)) \subset I(i)$. Moreover, since $(g^{-1} \circ g')(i) \in I(i)$ we have $j = g(g^{-1}(g'(i))) \in g(I(i))$. Thus $g(I(i)) = I(i)$ and this for every $g \in G$. Suppose now that I is a minimal invariant set under G. Then if $i, j \in I$ we have $j = g(i)$ for some $g \in G$. Otherwise $I(i) \subset I$ is a proper invariant subset, contradicting the minimality of I. Thus every element in a minimal invariant set under G is reachable from every other in that set so such a set is a set of states of an irreducible subchain of the Markov chain defined by V.

LEMMA 3.5. *Let p be a probability vector satisfying (3.7) and let $I^{(1)}, I^{(2)}, \cdots, I^{(\sigma)}$ denote the irreducible subchain sets defined by V (i.e. the minimal invariant sets under G). Then for all $i \in I^{(c)}$, $p_i = b_c$.*

Proof. As usual, the possible existence of periodic subchains confounds the issue. If $I^{(c)}$ has period τ_c under V let $J_1^{(c)}, J_2^{(c)}, \cdots, J_{\tau_c}^{(c)}$ denote the decomposition of $I^{(c)}$ into disjoint sets such that the states $J_z^{(c)}$ go into states in $J_{z+1}^{(c)}$ for $z=1, 2, \cdots, \tau_c$, where

$$J_{\tau_c+1}^{(c)} = J_1^{(c)}.$$

Then with V^{τ_c} as stochastic matrix each $J_z^{(c)}$ is an irreducible subchain which is ergodic. If $I^{(c)}$ is ergodic let $\tau_c=1$ and $J_1^{(c)}=I^{(c)}$. Now let τ denote the least common multiple of $\tau_1, \tau_2, \cdots, \tau_\sigma$ and consider the stochastic matrix V^τ. The irreducible subchains are all $J_z^{(c)}$, $c=1, 2, \cdots, \sigma$, $z=1, 2, \cdots, \tau_c$ and they are all ergodic. Hence with w taking on positive integral values only, $V^*=\lim_{w\to\infty}(V^\tau)^w$ exists and if $V^*=(v_{ij}^*)$ then

$$(3.9) \qquad v_{ij}^* = \begin{cases} \eta_{cj} > 0 & \text{for } \ i, j \in J_z^{(c)} \\ 0 & \text{for } \ i \in J_z^{(c)},\ j \in J_{z'}^{(c')} \text{ if } c \neq c', \text{ or } z \neq z'. \end{cases}$$

But from (3.7), i.e. $p=Vp$, it follows that $p=V^xp$ for any positive integer x so $p=(V^\tau)^wp=\lim_{w\to\infty}(V^\tau)^wp=V^*p$. Hence from (3.9) we have for all $i\in J_z^{(c)}$

$$(3.10) \qquad p_i = \sum_{j=1}^n v_{ij}^* p_j = \sum_{j\in J_z^{(c)}} \eta_{cj} p_j = b_z^{(c)}.$$

Now by the definition of the $J_z^{(c)}$ sets it follows that

$$(3.11) \qquad v_{ij} = 0 \quad \text{for} \quad i \in J_z^{(c)},\ j \in J_{z'}^{(c')} \quad \text{if} \quad c \neq c' \quad \text{or} \quad z' \neq z+1,$$

where again $J_{\tau_c+1}^{(c)}=J_1^{(c)}$. Returning to $p=Vp$ we have for $i\in J_z^{(c)}$

$$(3.12) \qquad p_i = \sum_{j=1}^n v_{ij}p_j = \sum_{j\in J_{z+1}^{(c)}} v_{ij}p_j = b_{z+1}^{(c)}\left(\sum_{j\in J_{z+1}^{(c)}} v_{ij}\right) = b_{z+1}^{(c)}$$

by (3.10) (3.11) and the fact that V is stochastic. Now (3.12) and (3.10) together imply $b_{z+1}^{(c)}=b_z^{(c)}$ for all c and z. Thus $b_z^{(c)}$ is independent of z, and if we define $b_c=b_z^{(c)}$ we have $p_i=b_c$ for all $i\in J_z^{(c)}$ for all z, so $p_i=b_c$ for all $i\in I^{(c)}$.

THEOREM 3.2. *Let (F, π, B) be a stochastic permutation of the elements of S, let (S, P) be stochastically invariant under (F, π, B) and let G be the group of permutations generated by $\{g_k \mid \pi(f_k)>0\}$. Then P is uniform over each minimal invariant set under G.*

Proof. This is the essence of the results given as Lemmas 3.1 through 3.5.

COROLLARY 3.1. *Under the hypotheses of Theorem 3.2 if G is such that I_n is the only invariant set under G, then P is uniform. In particular if G is the full group of permutations of n things or if $n>2$ and G is the alternating group of permutations of n things, then P is uniform.*

Proof. The first statement is an immediate consequence of Theorem 3.2 and Definition 1.4. Certainly if G is the full group, a minimal invariant set must be all of I_n since for any i and j, $j=g(i)$ for some $g\in G$. If $n=3$ we may take g such that $g(1)=2$, $g(2)=3$, $g(3)=1$ which is an even permutation and the group generated by g has I_3 as the only invariant set and this group is the alternating group on three elements. If $n\geqq 4$ let i and j be distinct elements of I_n. Then there are distinct elements $i', j'\in I_n-\{i,j\}$ and g defined by $g(i)=j$, $g(j)=i$, $g(i')=j'$, $g(j')=i'$ with $g(i'')=i''$ for $i''\in I_n-\{i,j,i',j'\}$ is an even permutation for which $j=g(i)$. Thus in this case also I_n is the only invariant set under the group generated by all permutations of the type described and this is a subgroup of the alternating group on n elements. Thus I_n is the only invariant set under the groups in the second statement so this statement is proved since it is a special case of the first statement.

4. Concluding remarks. It is clear that other "stochastic operations" can be defined and given suitable interpretations and analogous theorems developed. The authors plan to explore other cases in a later paper. The results for the cases considered here are perhaps not too surprising since they pretty well confirm one's intuition regarding the situations described.

We should like finally to give another interpretation of the concepts introduced here which bears directly on some of the philosophical considerations concerning the meaning of "equally likely." The space (S, P) may be thought of as characterizing each of a sequence of independent experiments from the outcomes of which the measure P might be estimated. The space $(F\times S, P^*)$ similarly characterizes each of a sequence of independent two stage experiments, the first stage being (F, π) and the second the "image" of (S, P) under (F, π, B). Again from the outcomes of the second stages of such a sequence the marginal measure P' defined by $P'(s_i)=\sum_{j=1}^{n} P^*(f_j, s_i)$ might be estimated. Now if an estimate of P' can be made on the basis of these outcomes, then it would be impossible to detect the fact that the stochastic operation was active if (S, P) is stochastically invariant under (F, π, B) and conversely, at least in principle. In the second case (stochastic permutations) this impossibility of detecting the operation implies that the outcomes in (S, P) must in fact be equally likely (P is uniform) providing, of course, sufficiently many elements of the operation are possible (Corollary 3.1). In this case also if the outcomes in (S, P) are equally likely then the operation can't be detected (Theorem 3.1). In the first case (removal of an element from S), if all elements are equally likely to be removed, then the action of the operation can be detected unless they were all equally likely to be selected anyway (Corollary 2.2). Or conversely, if all elements are equally likely to be selected, then the operation can be detected unless all elements are equally likely to be removed.

Reference

1. W. Feller, An Introduction to Probability Theory and Its Applications, Vol. 1, 2nd ed., Wiley, New York, 1957.

THE OPERATOR $(a^x\Delta)^n$ AND STIRLING NUMBERS OF THE FIRST KIND

H. W. GOULD, West Virginia University

1. Introduction. In this paper we develop two novel expansions for the operator $(a^x\Delta)^n f(x) = (a^x\Delta)(a^x\Delta) \cdots (a^x\Delta)f(x)$, both expansions involving the q-binomial coefficients. In the limiting case $q\to 1$ we deduce as a novel result the Stirling numbers of the first kind. Other new results follow.

We shall need some well-known results concerning the difference operator

$$\Delta f(x) = \underset{x,h}{\Delta} f(x) = \frac{f(x+h) - f(x)}{h}. \tag{1.1}$$

It is easily shown by induction that

$$\underset{x,h}{\Delta^n} f(x) = \frac{1}{h^n}\sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} f(x+jh). \tag{1.2}$$

Inversely, one may prove that

$$f(x+jh) = \sum_{k=0}^{j}\binom{j}{k} h^k \underset{x,h}{\Delta^k} f(x). \tag{1.3}$$

Indeed (1.3) depends upon nothing more difficult than the orthogonality relation

$$\sum_{j=k}^{n} (-1)^{n-j}\binom{n}{j}\binom{j}{k} = \begin{cases} 0, & k \neq n \\ 1, & k = n \end{cases} \tag{1.4}$$

as may be seen by substitution of (1.3) into (1.2) and making an interchange of order of summation.

When there is possibility of confusion we shall always adjoin the subscripts x, h in order to be explicit. This will also be important because we later encounter another difference operator which is easily confused with (1.1).

2. The q-binomial coefficients. We shall need some elementary facts about q-binomial coefficients [1], [2], [6], [7], [11]. By a q-number $[x]$ we mean

$$[x] = \frac{q^x - 1}{q - 1} \tag{2.1}$$

so that $[x]\to x$ as $q\to 1$. By a q-binomial coefficient we mean

$$\begin{bmatrix} x \\ k \end{bmatrix} = \frac{[x]_k}{[k]!} = \prod_{j=1}^{k}\frac{q^{x-j+1} - 1}{q^j - 1}, \tag{2.2}$$

where the q-factorials are defined by $[x]_k = [x][x-1] \cdots [x-k+1]$ and $[k]! = [k]_k$, $[0]! = 1$.

In case $x=n$ is a nonnegative integer we may also express (2.2) as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}, \tag{2.3}$$

and we note that

$$\binom{x}{k} = \lim_{q\to 1} \begin{bmatrix} x \\ k \end{bmatrix}.$$

Properly speaking we should adjoin a subscript q to the symbol for the q-binomial coefficients and write $\left[\begin{smallmatrix} x \\ k \end{smallmatrix}\right]_q$ so as to indicate the q-base meant. Indeed we may change the q-base and have the useful transformation, for example,

$$\begin{bmatrix} x \\ k \end{bmatrix}_{1/q} = q^{k(k-x)} \begin{bmatrix} x \\ k \end{bmatrix}_q. \tag{2.4}$$

The q-binomial coefficients satisfy relations somewhat similar to the relations satisfied by ordinary binomial coefficients. It is felt to be important to list a few of these, especially those which are used in some way later in this paper:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix} = 0, \text{ for } k < 0 \text{ or } k > n; \quad \begin{bmatrix} x \\ k \end{bmatrix} = \frac{[x]}{[k]} \begin{bmatrix} x-1 \\ k-1 \end{bmatrix}, \tag{2.5}$$

$$\begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} = \begin{bmatrix} x \\ j \end{bmatrix} \begin{bmatrix} x-j \\ k-j \end{bmatrix}, \tag{2.6}$$

$$\begin{bmatrix} -x \\ k \end{bmatrix} = (-1)^k q^{-kx} \begin{bmatrix} x+k-1 \\ k \end{bmatrix} q^{-k(k-1)/2}, \tag{2.7}$$

and the very important recurrence relation

$$\begin{bmatrix} x+1 \\ k \end{bmatrix} = \begin{bmatrix} x \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} x \\ k \end{bmatrix} = q^{x-k+1} \begin{bmatrix} x \\ k-1 \end{bmatrix} + \begin{bmatrix} x \\ k \end{bmatrix}. \tag{2.8}$$

There are many q-analogues of the binomial theorem, and we mention here only

$$\prod_{j=0}^{n-1} (1 - xq^j) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2} x^k. \tag{2.9}$$

3. The first expansion of $(a^x\Delta)^n$. We are led to formulate the theorem

$$(a^x \underset{x,h}{\Delta})^n f(x) = \frac{a^{nx}}{h^n} \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2} f(x+jh), \tag{3.1}$$

where $q=a^h$. This generalizes relation (1.2) which occurs for $a=1$, $q=1$.

This is readily proved by induction, using the second form of (2.8). Indeed we have

$$
\begin{aligned}
(a^x\Delta)^{n+1}f(x) &= (a^x\Delta)(a^x\Delta)^n f(x) \\
&= \frac{a^x}{h^{n+1}}\left\{a^{nx+nh}\sum_{j=0}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2}f(x+h+jh)\right. \\
&\qquad \left. - a^{nx}\sum_{j=0}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2}f(x+jh)\right\} \\
&= \frac{a^{(n+1)x}}{h^{n+1}}\left\{q^n\sum_{j=1}^{n+1}(-1)^{n-j+1}\begin{bmatrix} n \\ j-1 \end{bmatrix} q^{(j-1)(j-2)/2}f(x+jh)\right. \\
&\qquad \left. + \sum_{j=0}^{n}(-1)^{n-j+1}\begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2}f(x+jh)\right\}
\end{aligned}
$$

Therefore the induction goes through because we only need to show that

$$
\begin{bmatrix} n+1 \\ j \end{bmatrix} q^{j(j-1)/2} = q^n\begin{bmatrix} n \\ j-1 \end{bmatrix} q^{(j-1)(j-2)/2} + \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2}.
$$

Since $(j-1)(j-2)/2=j(j-1)/2-(j-1)$ we must show that

$$
\begin{bmatrix} n+1 \\ j \end{bmatrix} = q^{n-j+1}\begin{bmatrix} n \\ j-1 \end{bmatrix} + \begin{bmatrix} n \\ j \end{bmatrix};
$$

but this is the second form of (2.8), and the proof is complete.

Now let $h=1$ in (3.1) and we have

$$
(q^x \underset{x,1}{\Delta})^n f(x) = q^{nx}\sum_{j=0}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2}f(x+j). \tag{3.2}
$$

This should be carefully distinguished from the expansion

$$
\underset{q}{\Delta^n} f(x) = \sum_{j=0}^{n}(-1)^j\begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2}f(x+n-j), \tag{3.3}
$$

given by Carlitz [1] who defines q-differences by

$$
\begin{aligned}
\underset{q}{\Delta} f(x) &= f(x+1) - f(x), \\
\underset{q}{\Delta^{n+1}} f(x) &= \underset{q}{\Delta^n} f(x+1) - q^n \underset{q}{\Delta^n} f(x)
\end{aligned} \tag{3.4}
$$

Now of course the expansions (3.2) and (3.3) are related in that we may express one in terms of the other. For example we find

$$
q^{n(n-1)/2}q^{-x}(q^x \underset{x,1}{\Delta})^n f(x) = \underset{q}{\Delta^n}\,(q^{(n-1)x}f(x)). \tag{3.5}
$$

We are mainly interested here, however, in our general expansion (3.1) since we may find later some interesting results by letting $h\to 0$.

4. Second expansion and the Stirling numbers. We find a second expansion

by substitution of (1.3) into (3.1) and making an interchange of the order of summation. This gives

$$(4.1)\qquad (a^x \underset{x,h}{\Delta})^n f(x) = \frac{a^{nx}}{h^n}\sum_{k=0}^{n} h^k \underset{x,h}{\Delta^k} f(x) \sum_{j=k}^{n} (-1)^{n-j}\begin{bmatrix} n \\ j\end{bmatrix}\binom{j}{k} q^{j(j-1)/2}$$

where $q=a^h$.

Since $h=(\log q)/(\log a)$, we have $h^{k-n}=(\log a)^{n-k}(\log q)^{k-n}$ so that (4.1) may be written in the form

(4.2)

$$(a^x \underset{x,h}{\Delta})^n f(x) = a^{nx}\sum_{k=0}^{n} (\log a)^{n-k} \underset{x,h}{\Delta^k} f(x)(\log q)^{k-n} \sum_{j=k}^{n} (-1)^{n-j}\cdot \begin{bmatrix} n \\ j\end{bmatrix}\binom{j}{k} q^{j(j-1)/2}.$$

Now, in the limiting case that $h\to 0$ we should have a formula for $(a^xD)^nf(x)$. Considering the first few values of n we find

$$\begin{aligned}(a^xD)f(x) &= a^xDf(x),\\ (a^xD)^2f(x) &= a^{2x}\{(\log a)Df(x) + D^2f(x)\},\\ (a^xD)^3f(x) &= a^{3x}\{2(\log a)^2Df(x) + 3(\log a)D^2f(x) + D^3f(x)\},\end{aligned}$$

and we are led to formulate the theorem

$$(4.3)\qquad (a^x D_x)^n f(x) = a^{nx}\sum_{k=0}^{n} (-1)^{n-k} s(n,k)(\log a)^{n-k} D_x^k f(x),$$

where the coefficients $s(n, k)$ are the Stirling numbers of the first kind in the notation of Riordan ([9] p. 33) and which satisfy the recurrence relation

$$(4.4)\qquad s(n+1, k) = s(n, k-1) - n\cdot s(n, k).$$

We should remark that the Stirling numbers of the first kind are defined by

$$(4.5)\qquad \binom{x}{n} n! = \sum_{k=0}^{n} s(n,k)x^k.$$

Now the inductive proof of (4.3) uses nothing more difficult than (4.4):

$$\begin{aligned}(a^xD)^{n+1}f(x) &= (a^xD)(a^xD)^nf(x)\\ &= a^x\Big\{a^{nx}\sum_{k=0}^{n}(-1)^{n-k}s(n,k)(\log a)^{n-k}D^{k+1}f(x)\\ &\qquad + a^{nx}n(\log a)\sum_{k=0}^{n}(-1)^{n-k}s(n,k)(\log a)^{n-k}D^kf(x)\Big\}\\ &= a^{(n+1)x}\Big\{\sum_{k=1}^{n+1}(-1)^{n-k+1}s(n,k-1)(\log a)^{n-k+1}D^kf(x)\\ &\qquad - n\sum_{k=0}^{n}(-1)^{n-k+1}s(n,k)(\log a)^{n-k+1}D^kf(x)\Big\}\end{aligned}$$

which reduces to $a^{(n+1)x}\sum_{k=0}^{n+1}(-1)^{n+1-k}s(n+1,k)(\log a)^{n+1-k}D^kf(x)$ in virtue of (4.4) and the fact that $s(n,k)=0$ for $k<0$ or $k>n$.

Now since (4.2) must agree with (4.3) in the limiting case $h\to 0$, and since $q=a^h$ implies that $q\to 1$ as $h\to 0$, we are led to the conclusion that

$$(4.6)\qquad (-1)^{n-k}s(n,k) = \lim_{q\to 1}(\log q)^{k-n}\sum_{j=k}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix}\binom{j}{k}q^{j(j-1)/2}.$$

In terms of the q-differences (3.3) we may write this as

$$(4.7)\qquad (-1)^k s(n,k) = \lim_{q\to 1}(\log q)^{k-n}\,\underset{q}{\Delta^n}\binom{-x}{k}\Big|_{x=-n}.$$

Using the $1/q$ transformation (2.4) we may write (4.6) as follows:

$$\begin{aligned} s(n,k) &= \lim_{q\to 1}\left(\log\frac{1}{q}\right)^{k-n}\sum_{j=k}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix}_q\binom{j}{k}q^{j(j-1)/2} \\ &= \lim_{q\to 1}(\log q)^{k-n}\sum_{j=k}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix}_{1/q}\binom{j}{k}q^{-j(j-1)/2} \\ &= \lim_{q\to 1}(\log q)^{k-n}\sum_{j=k}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix}_q\binom{j}{k}q^{j(j-n)}q^{-j(j-1)/2} \end{aligned}$$

so that

$$(4.8)\qquad s(n,k) = \lim_{q\to 1}(\log q)^{k-n}\sum_{j=k}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix}\binom{j}{k}q^{j(j+1)/2}q^{-jn}.$$

These limiting forms for the Stirling numbers should be compared with the limiting cases which would follow from some other relations given by the author [6] which, however, do not involve the appearance of log q. We shall return to this in a later section of this paper.

Of course the limit formula is hardly an effective method for computing the well-tabulated Stirling numbers, but the formula is of theoretical interest.

Now in contrast to expansion (4.3) we should like to point out an alternative relation. Let $z=a^x$. Then $D_xz=a^x(\log a)$, so that

$$(a^xD_x)f(x) = (\log a)(z^2D_z)f(x),$$

and inductively

$$(4.9)\qquad (a^xD_x)^nf(x) = (\log a)^n(z^2D_z)^nf(x).$$

But it is readily shown by induction that

$$(z^2D_z)^n g(z) = \sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}\frac{n!}{k!}z^{k+n}D_z^k g(z),$$

whence we have

$$
\begin{aligned}
(a^xD_x)^nf(x) &= (\log a)^n(z^2D_z)^nf(x) \\
&= a^{nx}(\log a)^n \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k}\frac{n!}{k!}a^{kx}D_z^kf(x),
\end{aligned} \tag{4.10}
$$

where $z=a^x$.

5. A formula of Carlitz. Expansion (4.3) allows us to derive as a byproduct an expansion recently posed by Carlitz [10]. Indeed setting again $z=a^x$ we have $(\log a)(zD_z)f(z)=D_xf(x)$, and inductively

$$D_x^kf(x) = (\log a)^k(zD_z)^kf(x). \tag{5.1}$$

With this we have from (4.3)

$$(a^xD_x)^nf(x) = z^n(\log a)^n \sum_{k=0}^{n} (-1)^{n-k}s(n, k)(zD_z)^kf(x).$$

Then, noting (4.9), we have (also replacing $f(x)$ by $g(z)$)

$$(z^2D_z)^ng(z) = z^n \sum_{k=0}^{n} (-1)^{n-k}s(n, k)(zD_z)^kg(z) \tag{5.2}$$

which is the desired formula.

6. The q-Stirling numbers of the first kind. In a previous paper [6] the author introduced the following notation for a type of Stirling number of the first kind in the q-setting:

$$q^{n(n-1)/2}\begin{bmatrix} x \\ n \end{bmatrix}[n]! = \prod_{k=0}^{n-1}([x]-[k]) = \sum_{k=0}^{n}(-1)^{n-k}S_1(n-1, n-k, q)[x]^k. \tag{6.1}$$

In analogy with (4.5) we may now extend the meaning of Riordan's $s(n, k)$ and define

$$q^{n(n-1)/2}\begin{bmatrix} x \\ n \end{bmatrix}[n]! = \sum_{k=0}^{n} s(n, k, q)[x]^k, \tag{6.2}$$

so that

$$s(n, k, q) = (-1)^{n-k}S_1(n-1, n-k, q). \tag{6.3}$$

It was shown by the author [6, (3.19)] that we may take

$$S_1(n, k, q) = (q-1)^{-k}\sum_{j=0}^{k}(-1)^{k-j}\begin{bmatrix} n \\ j \end{bmatrix}\binom{n-j}{k-j}q^{j(j+1)/2}, \tag{6.4}$$

and consequently one possible extension of $s(n, k)$ is

$$s(n,k,q) = (q-1)^{k-n}\sum_{j=0}^{n-k}(-1)^j\begin{bmatrix} n-1 \\ j \end{bmatrix}\binom{n-1-j}{k-1}q^{j(j+1)/2}. \tag{6.5}$$

7. Inversion of expansion (3.1). We wish next to invert (3.1) in the same way that (1.3) inverts (1.2). To do this we shall need some further results about the q-binomial coefficients.

From the q-binomial theorem (2.9) we have with $x=1$ the orthogonality relation

$$\sum_{k=0}^{n-j}(-1)^j\begin{bmatrix} n-j \\ k \end{bmatrix}q^{k(k-1)/2} = \begin{cases} 0, & j \neq n \\ 1, & j = n \end{cases} \tag{7.1}$$

which we may use to invert a q-series. An alternative proof of (7.1) can be made to rest on the following q-theorem (cf. Carlitz [1]): If $f(x)=\sum_{j=0}^{n} A_j q^{jx}$ is a polynomial in q^x of degree $\leqq n$, then its finite (q-Taylor) series is

$$f(x+y) = \sum_{k=0}^{n}\begin{bmatrix} x \\ k \end{bmatrix}\underset{q}{\Delta^k} f(y), \tag{7.2}$$

with the q-differences defined as in (3.4). Moreover, if $m>n$, then

$$\underset{q}{\Delta^m} f(x) = 0.$$

In view of (2.6) it is evident that (7.1) is equivalent to an analogue of (1.4):

$$\sum_{j=k}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} j \\ k \end{bmatrix}q^{(n-j)(n-j-1)/2} = \begin{cases} 0, & k \neq n, \\ 1, & k = n. \end{cases}$$

We have the pair of inverse relations

$$F(n) = \sum_{j=0}^{n}(-1)^j\begin{bmatrix} n \\ j \end{bmatrix}q^{(n-j)(n-j-1)/2}f(j) \tag{7.3}$$

if and only if

$$f(n) = \sum_{j=0}^{n}(-1)^j\begin{bmatrix} n \\ j \end{bmatrix}F(j). \tag{7.4}$$

To illustrate the proof we show that (7.4) does yield $f(n)$:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}F(k) &= \sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}\sum_{j=0}^{k}(-1)^j\begin{bmatrix} k \\ j \end{bmatrix}q^{(k-j)(k-j-1)/2}f(j) \\ &= \sum_{j=0}^{n}(-1)^j\begin{bmatrix} n \\ j \end{bmatrix}f(j)\sum_{k=0}^{n-j}(-1)^{k+j}\begin{bmatrix} n-j \\ k \end{bmatrix}q^{k(k-1)/2} = f(n),\end{aligned}$$

in virtue of (7.1). Since $(n-j)(n-j-1)/2=n(n-1)/2-j(j-1)/2+j(j-n)$, we

may restate these in the form

$$F(n) = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-n)} q^{-j(j-1)/2} f(j) \tag{7.5}$$

if and only if

$$f(n) = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2} F(j). \tag{7.6}$$

Perhaps we should remark that by means of the $1/q$ transformation (2.4) it is evident that (7.5) may be stated more simply, and we thus have the quite elegant pair of inverse relations:

$$F(n) = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_p p^{j(j-1)/2} f(j) \tag{7.7}$$

if and only if

$$f(n) = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-1)/2} F(j), \tag{7.8}$$

provided that the bases p, q satisfy $pq=1$. When $p=q=1$ this pair reduces to the familiar case for binomial coefficients.

It is evident then from (7.3) and (7.4) that (3.1) inverts to yield

$$f(x + nh) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-n)} q^{-j(j-1)/2} h^j a^{-jx} (a^x \underset{x,h}{\Delta})^j f(x) \tag{7.9}$$

where, of course, $q=a^h$. Or we may say that

$$f(x + nh) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_p p^{j(j-1)/2} h^j a^{-jx} (a^x \underset{x,h}{\Delta})^j f(x) \tag{7.10}$$

with $pq=1$ and $q=a^h$. Since it is easily proved that

$$\sum_{k=j}^{n} \begin{bmatrix} k \\ j \end{bmatrix}_p p^k = \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}_p p^j, \tag{7.11}$$

we have from (7.10) the interesting summation formula

$$\sum_{k=0}^{n} p^k f(x + kh) = \sum_{j=0}^{n} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}_p p^{j(j+1)/2} h^j a^{-jx} (a^x \underset{x,h}{\Delta})^j f(x), \tag{7.12}$$

which reduces to a well-known result when $a=p=q=1$.

As a final result we observe that a relation inverse to (4.1) is readily found. Indeed, substitution of (7.10) into (1.2) and an interchange of order of summation give

$$(7.13)\qquad \mathop{\Delta^n}_{x,h} f(x) = \sum_{k=0}^{n} h^{k-n} p^{k(k-1)/2} a^{-kx} (a^x \mathop{\Delta}_{x,h})^k f(x) \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \begin{bmatrix} j \\ k \end{bmatrix}_p ,$$

where $pq=1$, $q=a^h$. Thus we may, by use of q-binomial coefficients, express ordinary differences in terms of $(a^x\Delta)^n f(x)$ and conversely.

The inner summations in (4.1) and (7.13) may be viewed as generalizations of the orthogonality relation (1.4) noted at the outset. On the other hand we have also seen that a limiting case (4.6) led to the Stirling numbers of the first kind.

There is an extensive literature relating to q-numbers. In addition to the list of references at the end of this paper the reader may consult the many papers of F. H. Jackson in the Messenger of Mathematics, Quarterly Journal of Mathematics, and other British journals. We should also mention the extensive papers of R. Tambs Lyche, e.g. [11]. But perhaps the most interesting application of q-number relations is in the study of Gaussian sums. Instead of taking $q=1$, let q be some primitive root of unity. From a suitable q-identity Gauss was able to determine the (closed) numerical value of a sum such as

$$\sum_{k=0}^{n-1} \exp \frac{2\pi m k^2 i}{n}, \qquad i^2 = -1;$$

it is said, however, that the correct determination of the algebraic sign required him some four years. Since the time of Gauss many shorter derivations have been given. All of this leads one to look for other applications of q-identities beyond the case $q=1$. The reader is also referred to a paper of Carlitz [12].

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References

1. L. Carlitz, q-Bernoulli numbers and polynomials, Duke Math. J., 15 (1948) 987–1000.
2. ———, Expansions of q-Bernoulli numbers, *ibid.*, 25 (1958) 355–364.
3. ———, Note on Nörlund's polynomial $B_n^{(z)}$, Proc. Amer. Math. Soc., 11 (1960) 452–455.
4. H. W. Gould, Stirling number representation problems, *ibid.*, 11 (1960) 447–451.
5. ———, The Lagrange interpolation formula and Stirling numbers, *ibid.*, 11 (1960) 421–425.
6. ———, The q-Stirling numbers of first and second kinds, Duke Math. J., 28 (1961) 281–289.
7. F. H. Jackson, q-Difference equations, Amer. J. Math., 32 (1910) 305–314.
8. C. Jordan, Calculus of finite differences, Budapest, 1939, 2nd ed., New York, Chelsea, 1947.
9. J. Riordan, An introduction to combinatorial analysis, New York, 1958.
10. Problem 505, Math. Mag., 36 (1963) 70.
11. R. Tambs Lyche, En generalisasjon av binomial-koefficienten, Nordisk Mat. Tidskr., 8 (1926) 93–102.
12. L. Carlitz, A note on Gauss's "serierum singularium," Portugal. Math., 15 (1956) 9–12.

EXISTENCE AND UNIQUENESS THEOREMS FOR SOLUTIONS OF DIFFERENCE EQUATIONS

SELMO TAUBER, Portland State College

1. Introduction. Existence and uniqueness theorems for difference equations are known and can be found in [1] for linear equations, in [2], and in [3] for the general equation of order n. On the other hand the same theorems have been proved for differential equations in a unified form by the use of vector spaces in [4], [5], and [6]. The unified proofs include the equation of order n and the general system of simultaneous equations of order n, provided it can be put into "normal form," i.e. can be solved with respect to the highest order derivative for every unknown function.

It is the aim of this paper to give the same unified proofs for difference equations and systems of simultaneous difference equations in normal form.

It is known (see [3]) that equations involving the operators Δ, M, E, can be reduced to equations containing only the E-operator. As is usually done, we limit ourselves to the case where $h=\Delta t=1$, so that $Ef(t)=f(t+1)$.

2. Reduction to a first order vector difference equation. Let $x_j(t)$, $j=1, 2, \cdots, n$, be n functions and let us consider a system of n simultaneous difference equations

$$Ex_m = f_m(t, x_1, x_2, \cdots, x_n), \qquad m = 1, 2, \cdots, n. \tag{1}$$

A solution of (1) is a set of functions $x_1, x_2, \cdots, x_n$, that satisfies (1) for $t_0 \leqq t \leqq t_1$. We consider the n functions $x_1, x_2, \cdots, x_n$, as components of the vector $X=[x_1, x_2, \cdots, x_n]$. By definition of the operation E on the vector X we have

$$EX = [Ex_1, Ex_2, \cdots, Ex_n]. \tag{2}$$

It follows that the system (1) can be written as a unique vector difference equation

$$EX = F(X, t), \tag{3}$$

where F is a vector such that $F=[f_1, f_2, \cdots, f_n]$.

Let us now consider a difference equation of order n of the function $x(t)$, i.e. $\phi(t, x, Ex, E^2x, \cdots, E^nx)=0$. We assume that it can be solved with respect to E^n; thus we can write

$$E^nx = f(t, x, Ex, E^2x, \cdots, E^{n-1}x), \tag{4}$$

which is called the normal form of the equation. Let then $x=x_1$, $Ex=x_2$, $E^2x=x_3, \cdots, E^{n-1}x=x_n$. It follows that

$$\begin{cases} Ex_1 = x_2 \\ Ex_2 = x_3 \\ \cdot\ \cdot\ \cdot\ \cdot \\ Ex_{n-1} = x_n, \end{cases} \tag{5}$$

so that letting $X=[x_1, x_2, \cdots, x_n]$, and, $F=[x_2, x_3, \cdots, x_n, f]$, we can write again

$$EX = F(X, t). \tag{3}$$

Finally we consider an arbitrary system of simultaneous difference equations

$$\begin{aligned}\phi_j(t, x_1, Ex_1, E^2x_1, \cdots, E^{a_{1,j}}x_1, x_2, Ex_2, E^2x_2, \cdots, E^{a_{2,j}}x_2, x_3, Ex_3, \cdots, \\ \cdots, x_n, Ex_n, E^2x_n, \cdots, E^{a_{n,j}}x_n) = 0, \qquad j = 1, 2, \cdots, n,\end{aligned} \tag{6}$$

and where the numbers $a_{k,j}$ are clearly positive integers. Let $\max(a_{k,1}, a_{k,2}, \cdots, a_{k,n})=b_k$. We assume that it is possible to solve (6) with respect to $E^{b_k}x_k$, and to obtain the normal form

$$\begin{aligned}E^{b_k}x_k = f_k(t, x_1, Ex_1, \cdots, x_2, \cdots, x_kEx_k, \cdots, E^{b_k-1}x_k, \cdots, x_n, \\ Ex_n, E^2x_n, \cdots, E^{b_n-1}x_n), \qquad k = 1, 2, \cdots, n.\end{aligned} \tag{7}$$

Taking

$$\begin{aligned}&x_1 = y_1, Ex_1 = y_2, \cdots, E^{b_1-1}x_1 = y_{b_1}, x_2 = y_{b_1+1}, Ex_2 = y_{b_1+2}, \cdots, \\ &\quad E^{b_2-1}x_2 = y_{b_1+b_2}, \cdots, E^{b_{n-1}-1}x_{n-1} = y_{b_1+b_2+\cdots+b_{n-1}}, \\ &\quad x_n = y_{b_1+b_2+\cdots+b_{n-1}+1}, \cdots, E^{b_n-1}x_n = y_{b_1+b_2+\cdots+b_n},\end{aligned}$$

we obtain

$$\begin{aligned}&Ey_1 = y_2, Ey_2 = y_3, \cdots, Ey_{b_1-1} = y_{b_1}, Ey_{b_1} = f_1, \\ &Ey_{b_1+1} = y_{b_1+2}, \cdots, Ey_{b_1+b_2-1} = y_{b_1+b_2}, Ey_{b_1+b_2} = f_2, \\ &\cdot\ \cdot \\ &Ey_{b_1+b_2+\cdots+b_{n-1}+1} = y_{b_1+b_2+\cdots+b_{n-1}+2}, \cdots, Ey_{b_1+\cdots+b_n} = f_n,\end{aligned}$$

where the functions f_k, $k=1, 2, \cdots, n$, are defined in (7).

Taking $X=[y_1, y_2, \cdots, y_{b_1+b_2+\cdots+b_n}]$, we have

$$EX = F(X, t), \tag{3}$$

with $F=[y_2, y_3, \cdots, y_{b_1}, f_1, y_{b_1+2}, y_{b_1+3}, \cdots, y_{b_1+b_2}, f_2, \cdots, \cdots, f_n]$. We can state the following theorem to summarize the preceding results:

THEOREM I. *A difference equation of order n or a system of simultaneous difference equations that can be written in normal form are equivalent to the vector difference equation* (3).

3. Existence theorem for $EX=F(X, t)$. We assume that the function F is defined for all t and X, and we look for a solution X such that for $t=t_0$, $X=X_0=[x_1(t_0), x_2(t_0), \cdots, x_n(t_0)]$. Since $EX_0=F(X_0, t)$, and F is determined for all X and t, we have

$$EX_0 = [x_1(t+1), x_2(t+1), \cdots, x_n(t+1)] = X_1 = F(X_0, t_0),$$

so that X_1 is determined. But $EX_1=[x_1(t_0+2),\ x_2(t_0+2),\ \cdots,\ x_n(t_0+2)]=X_2$ $=F(X_1,\ t_0+1)$, so that X_2 is determined. To complete the proof by induction we assume that $X_k=[x_1(t_0+k),\ x_2(t_0+k),\ \cdots,\ x_n(t_0+k)]$ is determined and since F is defined for all X and t, we have

$$\begin{aligned} EX_k &= [x_1(t_0 + k + 1), x_2(t_0 + k + 1), \cdots, x_n(t_0 + k + 1)] = X_{k+1} \\ &= F(X_k, t_0 + k), \end{aligned}$$

so that X_{k+1} is determined. In conclusion we see that X is determined for t_0+k, where k is an arbitrary positive integer or zero. This however does not prove that X exists for any t since we have limited ourselves to values t_0+k, where k is a positive integer or zero. To determine X for any value of t we need more specific initial conditions.

We assume then that for $t_0\leqq t\leqq t_0+1$, $X=X_0=[f_1(t), f_2(t),\ \cdots, f_n(t)]$, where the $f_j(t)$, $j=1,\ 2,\ \cdots,\ n$, and thus the vector X_0, are defined for $t_0\leqq t\leqq t_0+1$. Then

$$EX_0 = F(X_0, t) = X_1 = [x_1(t), x_2(t), \cdots, x_n(t)],$$

which defines X for $t_0+1\leqq t\leqq t_0+2$,

$$EX_1 = F(X_1, t + 1) = X_2 = [x_1(t), x_2(t), \cdots, x_n(t)],$$

which defines X for $t_0+2\leqq t\leqq t_0+3$,

. . . .

Assuming that $X_k=[x_1(t),\ x_2(t),\ \cdots,\ x_n(t)]$, for $t_0+k\leqq t\leqq t_0+k+1$,

$$EX_k = F(X_k, t + k) = X_{k+1},$$

which defines X for the interval $t_0+k+1\leqq t\leqq t_0+k+2$.

Thus X is determined for $t_0\leqq t\leqq t_0+k$, where k is an arbitrary positive integer or zero. We can state this result as follows:

THEOREM II. *The vector difference equation $EX=F(X,\ t)$, where $X=[x_1(t),$ $x_2(t),\ \cdots,\ x_n(t)]$, has a solution X defined for $t_0\leqq t\leqq t_0+k$, where k is a positive integer or zero, provided that F exists for all X and t, and that for $t_0\leqq t\leqq t_0+1$, $X(t)=X_0=[f_1(t),\ f_2(t),\ \cdots,\ f_n(t)]$, where the functions $f_j(t)$, $j=1,\ 2,\ \cdots,\ n$, are defined for $t_0\leqq t\leqq t_0+1$.*

4. Uniqueness theorem. Suppose that (3) has two solutions X, and Y, satisfying the conditions of Theorem II, then

$$\begin{aligned} EY &= F(Y, t) \\ EX &= F(X, t), \end{aligned}$$

with, $Y=X_0(t)=X$, for $t_0\leqq t\leqq t_0+1$. It follows that

$$
\begin{aligned}
EX &= F(X, t) = F(Y, t) = EY, && t_0 \leqq t \leqq t_0 + 1,\\
E^2X &= F(FX, t+1) = F(EY, t+1) = E^2Y, && t_0 + 1 \leqq t \leqq t_0 + 2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

and, assuming that $E^kX = E^kY$ for $t_0+k-1 \leqq t \leqq t_0+k$, then,

$$E^{k+1}X = F(E^kX, t+k) = F(E^kY, t+k) = E^{k+1}Y,$$

for $t_0+k \leqq t \leqq t_0+k+1$, which completes the proof by induction. We state

THEOREM III. *The solution whose existence has been proved by Theorem II is unique.*

5. Initial conditions. According to Theorems II and III, the solution of (3) whose existence and uniqueness has been proved depends on n functions $f_j(t)$, $j = 1, 2, \cdots, n$, defined for $t_0 \leqq t \leqq t_0+1$. We may assume that the functions f are periodic of period *one*, i.e. $f_j(t) = f_j(t+1)$, which in turn is equivalent to $f_j(t) = f_j(t+k)$, where k is an arbitrary integer.

In fact, periodic functions f_j are defined for the given interval, and thus satisfy the condition imposed on the functions f. On the other hand to every function defined for the given interval which is not periodic there corresponds by extension a periodic function. Thus our assumption that f is a periodic function is legitimate. We can state this result as follows:

THEOREM IV. *The solution of the vector difference equation* (3) *in an n-dimensional vector space depends on n arbitrary functions that are periodic of period* **one.**

6. Remark to "Educators." The preceding proofs performed in a vector space are simple and can be used as a prelude to the existence and uniqueness proofs for differential equations in vector spaces (see [4], [5], and [6]) which can be given in an elementary course in differential equations.

The calculus of finite differences and difference equations can be introduced into a course in differential equations, analogies can be pointed out in many places and the course could be supplemented by the introduction of vector spaces and the proof of existence and uniqueness theorems, first for difference equations and then for differential equations.

References

1. N. E. Nörlund, Vorlesungen über Differenzenrechnung, Chelsea, New York, 1954.
2. G. Boole, The Calculus of Finite Differences, 4th ed., Chelsea, New York, 1958.
3. K. S. Miller, An Introduction to the Calculus of Finite Differences and Difference Equations, Holt, New York, 1960.
4. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.
5. R. Bellman, Stability Theory of Differential Equations, Blackwell, New York, 1953.
6. W. Hurewicz, Lectures on Ordinary Differential Equations, Stechert-Hafner, New York, 1958.

FACTORIZATION OF THE GENERAL POLYNOMIAL BY MEANS OF ITS HOMOMORPHIC CONGRUENTIAL FUNCTIONS

DANIEL B. LLOYD, D. C. Teachers College, Washington, D. C.

The discovery of linear factors of polynomials is ordinarily accomplished by synthetic division, using factors of the constant term as trial divisors. When these factors are few the method is practical, even if not elegant. It is impractical in more general cases, however, when the coefficients are large or the factors are numerous or nonlinear. This implies the need for a general scheme which will discover factors of any degree more quickly and with less indirection.

We advance here a theory and a method to determine the reducibility of the general polynomial P by a study of its reduced analog P_m. The scheme yields a procedure for determining the possible pattern of factors and also an exact determination of those factors, if any.

We propose to set up the correspondence $a \to a_m$ which carries each integer a into its residue class modulo m, where m is a prime. This is a homomorphism of the ring of integers to the finite ring of integers A_m. We then consider the general polynomial P with rational integral coefficients. The correspondence $P \to P_m$ carries P into a finite ring of polynomials, with coefficients in A_m. Use will be made here of the following theorems:

I. If a polynomial P is reducible into rational integral factors, then P_m is similarly reducible in A_m, where A_m is the ring of integers modulo m, and P_m is the polynomial P with its coefficients reduced modulo m, (m a prime).

II. The factors of P_m are congruent to the factors of P modulo m.

III. If P_m is irreducible in A_m, then P is irreducible over the rational integers (contrapositive of I).

The following simplified proof of these principles uses linear factors but may easily be generalized: Let

$$P = \sum_{i=0}^{n} C_i x^i \tag{1}$$

$$= \prod_{i=0}^{n} (x - K_i). \tag{2}$$

Then $P_m = \sum_{i=0}^{n} c_i x^i$. Now each $C_i = mq_i + c_i$, and each $K_i = mr_i + k_i$, where the r_i and q_i are integers, and the integers c_i and k_i have the property (if $m \neq 2$) that $|c_i|,\ |k_i| \leqq (m-1)/2$. Hence

$$P = \sum_{i=0}^{n} (mq_i + c_i)x^i = \prod_{i=0}^{n} [x - (mr_i + k_i)]. \tag{3}$$

Reducing (3) by the modulus m we have the desired result:

$$P_m = \prod_{i=0}^{n} (x - k_i). \tag{4}$$

The truth of principles II and III is now immediate.

Linear factors.

Illustration 1: Since $P=x^3-9x^2-34x+336$ factors into $(x+6)(x-7)(x-8)$, the corresponding polynomial reduced modulo 5: $P_5=x^3+x^2+x+1$ yields the corresponding factors

$$\begin{aligned}(x+1)(x^2+1) &\equiv (x+1)(x^2+5x+6)\\ &= (x+1)(x+3)(x+2),\end{aligned}$$

factors clearly congruent with those of P.

Illustration 2: Similarly, to determine whether the following polynomials are factorable or not, we form their homomorphic congruential polynomials and note from the Factor Tables (page 869) whether or not the latter are prime. Thus, forming $P_m\equiv P$, in the following examples:

(a) $x^3-11x^2+7x-16\equiv x^3+x^2+x$ (mod 2), which is composite. But $P_3=x^3+x^2+x-1$ (mod 3), which is prime; hence P is *prime*.

(b) $x^3-12x^2-17x+25\equiv x^3+x+1$ (mod 3), which is composite. But $P_2=x^3+x+1$ (mod 2), which is prime; hence P is *prime*.

(c) $x^4-13x^3-19x^2+21x+45\equiv x^4+x^3+x^2+x+1$ (mod 2) which is prime; hence P is *prime*.

(d) $x^4+7x^3+7x-35\equiv x^4+x^3+x+1$ (mod 2 and 3).

This is composite in A_2 and A_3, so the test is indecisive from the tables alone. It could be tested for factors, as shown below.

(Actually it is prime by Eisenstein's Theorem [1].)

The efficacy of the new method for factoring the general polynomial will be demonstrated by the following examples of the more difficult type:

Example 1. Let $P=x^3+2x^2-531x+3888$. Form $P_{11}=x^3+2x^2-3x+5$, or better, to facilitate factoring,

$$\begin{aligned}P_{11} &= x^3+2x^2-3x-6\\ &= (x+2)(x^2-3)\\ &= (x+2)(x^2-25)\\ &= (x+2)(x+5)(x-5).\end{aligned}\qquad \text{Also form } \begin{aligned}P_{13} &= x^3+2x^2+2x+1\\ &= (x+1)(x^2+x+1)\\ &= (x+1)(x^2+x-12)\\ &= (x+1)(x+4)(x-3).\end{aligned}$$

One next finds factors of 3888 $(=2^4\cdot 3^5)$ which will be congruent with 2, 5, and -5, respectively (mod 11); and with 1, 4, and -3, respectively (mod 13), —simultaneously. The possibilities are, mod 11: 2, ± 6, -9, ± 16, 24, ± 27, 72, -108, 324; and, mod 13: 1, 4, -3, -9, -12, -16, 27, 36, -81, 108, -324, -432. The intersection of these two sets gives -9, -16, and 27.

$$\therefore\ P = (x-9)(x-16)(x+27).$$

(The reader may practice reducing polynomials to a finite field, and factoring them. For example, x^2+6x+5 (mod 5); $x^2+10x+9$ (mod 7); $x^3-67x+126$ (mod 9 and mod 11); and $x^4-7x^3-78x^2+144x+864$ (mod 13).)

Example 2. Let $P=x^3-3x^2-70x+144$. Trying $P_5=x^3-3x^2+4$, we observe the factor $(x+1)$, and thence by division, $P_5=(x+1)(x-2)(x-2)$. The reader may readily complete the solution. It is suggested that P_7 also be formed, and that the two sets of columns of possible factors be intersected to give the desired unique coefficients. They are $(x-9)(x+8)(x-2)$. The factors should be written down, a separate column for each residue class, and then sets of three made, by selecting one factor from each column, that will produce the product 144.

Instead, a larger modulus such as 23 could have been used, but would prove to be more involved,—as the reader may verify.

Example 3. $P=9x^4-254x^3-4016x^2+56{,}736x-103{,}680$.

Try $P_{19}=9x^4-7x^3-7x^2+2x+3$
$=(x-1)(9x^3+2x^2-5x-3)$.

Trying small factors, we find that dividing the cubic factor by $(x+2)$ yields a remainder of -57. This suggests adding 57 ($\equiv 19$). Hence

$$P_{19} = (9x^3 + 2x^2 - 5x + 54)(x - 1)$$

which now factors into $(x+2)(9x^2-16x+27)(x-1)$. To make the x^2-coefficient unity, add $133x$ $(=7\cdot 19x)$ to the x-term, yielding $9(x^2+13x+3)$. And after adjusting coefficients, we obtain

$$x^2 + 13x + 3 \equiv x^2 - 6x - 16 = (x - 8)(x + 2).$$

As before, we next seek numbers congruent with -1, -8, and 2, which also will be factors of 103,680 $(=2^8\cdot 3^4\cdot 5)$. We readily spot the set 18, -8, and -36. Then by division, the fourth factor is found to be $(9x-20)$. As a check at this stage, we have

$$9(x - 1)(x + 2)^2(x - 8) = 9x^4 - 7x^3 - 7x^2 + 2x + 3.$$

Therefore,

$$P = (x - 8)(x + 18)(x - 36)(9x - 20).$$

At this stage of the game the reader might enjoy trying to factor $x^3-21x^2-5850x-129{,}600$, using P_{19} and P_{31}; and noticing the form of P_3 might narrow the selection, as indeed might also the discovery by synthetic division that all the factors must be between 100 and -100.

Nonlinear factors. The discovery of *nonlinear* factors of polynomials is also attainable by congruential polynomials. Historically, this problem has always been tedious, and prior to our attack here, methods have invariably been highly empirical. Previous methods for decomposing $P=\sum_0^n a_ix^i$ into irreducible polynomials have been presented by Kronecker [4], Runge [8], Mandl [6], Glenn [3], and Frumveller [2]. A summary of these is available in English [5] and will not be repeated here. It should be noted however that each of them, though ingenious, is quite lengthy.

Our principles I–III above state that if P factors in the rational field, then

there are corresponding factors for P_m in A_m. The choice of factors in the finite ring is more limited and hence the experimental aspect is minimized. It is possible to list all the prime factors in a finite ring A_m, and to classify them according to their degree. This the author has done in his Factor Tables. Such tables serve to indicate, though not assure, possible factors of P. Illustrative examples here will serve to clarify the method.

Example 4. Let $P=x^4+14x^3+x^2-15x+16$.

Next, form $P_2=x^4+x^2+x=x(x^3+x+1)$. The cubic factor is prime, by the Factor Table. From this, P may either have a linear factor, or else be prime. But, by trial divisors of 16, we find that P has no linear factor. Hence it is prime.

Example 5. Let $P=7x^4+35x^3+17x^2+15x+6$.

We shall seek a quadratic factor here, whether the same is prime or not (the chances are 2:1 against a quartic having a prime cubic factor). So P would factor into the form $(7x^2+ax+b)(x^2+cx+d)$. By multiplying out and equating coefficients, we have

$$\begin{aligned} 7c + a &= 35 \\ ac + 7d + b &= 17 \\ ad + bc &= 15 \\ bd &= 6. \end{aligned} \tag{5}$$

We note that a is of the form $7k$. So, trying $a=0$, we have $c=5$, $b=3$, $d=2$, and thence $P=(7x^2+3)(x^2+5x+2)$ proves to be the winning combination.

However, the use of congruential functions could have guided us more certainly to the factors. For instance,

$$\begin{aligned} P_5 &= 2x^4 + 2x^2 + 1 \\ &\equiv 2x^4 + 7x^2 + 6 \\ &= (2x^2 + 3)(x^2 + 2) \\ &\equiv (7x^2 + 5kx + 3)(x^2 + 5mx + 2). \end{aligned}$$

On multiplying out and equating coefficients, we have:

$$\begin{aligned} 7(5m) &= 35 \\ 17 + (5k)5m &= 17 \\ 10k + 15m &= 15 \end{aligned}$$

whence $k=0$, $m=1$, and the correct factors emerge at once. Our new method thus obviates the difficulty of solving the set of coefficient equations (5) above, as was done by Frumveller and others.

Further simplification of the procedure is often attained by first transforming the given polynomial into a monic polynomial.

Example 6. Let $P=x^4-3x^3-8x^2-12x-48$.

Form

$$P_3 = x^4 + x^2 = x^2(x^2 + 1), \tag{6}$$

and

$$\begin{aligned} P_5 &= x^4 + 2x^3 + 2x^2 + 3x + 2 \\ &= (x + 1)(x^3 + x^2 + x + 2) \\ &= (x + 1)(x - 1)(x^2 + 2x - 2) \\ &= (x^2 - 1)(x^2 + 2x - 2). \end{aligned} \tag{7}$$

Rather than testing for linear factors by the tedious routine of trying factors of the constant term, we shall play the game for higher stakes, quadratic factors, if any. To identify therefore suitable factors of the constant term, we note that 4 and -12 are the only two numbers whose product is -48 and which are congruent with 1 and 0, respectively, modulo 3 (using (6) above), and simultaneously with -1 and -2, modulo 5 (using (7) above). Therefore, set $P=(x^2+kx+4)(x^2+mx-12)$, which gives, as before, $k=0$ and $m=-3$, yielding the factors

$$(x^2 + 4)(x^2 - 3x - 12).$$

Example 7. Let

$$P=x^5-x^4+7x^3-3x^2+11x-5. \tag{8}$$

Form

$$\begin{aligned} P_2 &= x^5 + x^4 + x^3 + x^2 + x + 1 \\ &= (x + 1)(x^4 + x^2 + 1) \\ &= (x + 1)(x^2 + x + 1)(x^2 - x + 1) \\ &= (x + 1)(x^2 + x + 1)^2. \end{aligned}$$

Also $P_3=x^5-x^4+x^3-x+1$. Trying each of the three prime quadratics of A_3 in the Factor Table, we find that (x^2-x-1) divides this quintic. Hence,

$$P_3 = (x^2 - x - 1)(x^3 - x - 1). \tag{9}$$

Since these factors are both prime (per table), therefore P, if factorable, must factor in the form $(x^2+ax+b)(x^3+cx^2+dx+e)$. Whence, using (8),

$$\begin{aligned} a + c &= -1 \\ ac + b + d &= 7 \\ bc + ad + e &= -3 \\ ae + bd &= 11 \\ be &= -5. \end{aligned}$$

To aid in solving this set of equations, we further construct

$$\begin{aligned} P_5 &= x^5 - x^4 + 2x^3 + 2x^2 + x \\ &= x(x^4 - x^3 + 2x^2 + 2x + 1) \\ &= x(x-1)(x^3 + 2x - 1). \end{aligned}$$

Grouping the factors to agree with P_3:

$$P_5 = (x^2 - x)(x^3 + 2x - 1). \tag{10}$$

Since the cubic is prime, this is the only way the factors could group to give a quadratic times a cubic. The factors of P_2 must also be grouped to correspond with those of P_3 and P_5:

$$\therefore\ P_2 = (x^2 + x + 1)(x^3 + 1). \tag{11}$$

Using (9), (10), and (11), the coefficients of P_2, P_3 and P_5 impose the following congruence conditions on the required coefficients:

	P_2	P_3	P_5
$a \equiv$	1	-1	-1
$b \equiv$	1	-1	0
$c \equiv$	0	0	0
$d \equiv$	0	-1	2
$e \equiv$	1	-1	-1.

To agree therewith, the only values for b and e are 5 and -1. Thence $d=2$ is the only possibility giving reasonable values to c and a, which are 0 and -1, respectively. So $P=(x^2-x+5)(x^3+2x-1)$ and the factorization is complete.

Space limitations here preclude the extension of these examples to polynomials of higher degree. The method is thoroughly general for all polynomials, but additional tables are needed for the higher degrees. The author has applied this method through the ninth degree, however, and has carried the Factor Tables through the eleventh degree, for moduli 2, 3, and 5. They will soon be extended to other moduli. In dealing effectively with the higher degree polynomials the author has found certain additional artifices and devices helpful to shorten the computation. Those interested in this aspect may approach him directly.

After a restful pause the reader may enjoy attacking the following polynomial with renewed vigor (it is reducible):

$$x^4 + 9x^3 + 24x^2 - 18x - 216.$$

May his quest be rewarding as well as pleasurable!

References

1. G. Eisenstein, Über die Irreductibilität und andere Eigenschaften der Gleichung, J. Reine Angew. Math., 39 (1850) 166–179.
2. A. F. Frumveller, Quadratic factors of polynomials, this MONTHLY, 24 (1917) 208–212.
3. O. E. Glenn, Relating to the quadratic factors of a polynomial, *ibid.*, 23 (1916) 313–315.
4. L. Kronecker, Grundzüge einer arithmetischen Theorie der algebraischen Grössen, J. Reine Angew. Math., 92 (1882) 111–114.
5. D. B. Lloyd, Some properties of rational quintic equations, Catholic University Press, Washington, D. C., 1940.
6. M. Mandl, Über die Zerlegung ganzer, ganzzahliger Functionen in irreductible Factoren, J. Reine Angew. Math., 113 (1894) 252–261.
7. C. C. MacDuffee, An introduction to abstract algebra, Wiley, New York, 1940.
8. C. Runge, Irreductibilität der Gleichung, J. Reine Angew. Math., 99 (1886) 89–97.
9. J. V. Uspensky, Theory of equations, McGraw-Hill, New York, 1948.
10. B. L. van der Waerden, Modern algebra, vol. 1, Frederick Unger Publ. Co., New York, 1953.

GENERALIZATIONS OF THE FORMULAS OF RODRIGUES AND SCHLÄFLI

JAMES M. HORNER, University of Alabama

In the study of orthogonal polynomials special attention is given to expressing the polynomials in terms of a Rodrigues' formula. Frequently, when the primary concern is the differential equation satisfied by the polynomials, the Rodrigues' formula is generalized to provide contour integral solutions when the parameter which is the index of differentiation in the Rodrigues' formula takes nonintegral values.

A classic example is the Legendre equation

$$(z^2 - 1)y'' + 2zy' - n(n+1)y = 0. \tag{1}$$

It is well known (e.g. [1] pages 303, 304, 317) that for positive integral values of n two linearily independent solutions to (1) are

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left\{(z^2-1)^n\right\} \tag{2}$$

and

$$Q_n(z) = \frac{(-2)^n n!}{(2n)!}\frac{d^n}{dz^n}\left\{(z^2-1)^n \int_z^\infty (v^2-1)^{-n-1}dv\right\}. \tag{3}$$

For arbitrary values of n a solution to (1) is given by Schläfli's contour integral

$$y(z) = \frac{1}{2\pi i}\int_C \frac{(t^2-1)^n dt}{2^n(t-z)^{n+1}},$$

References

1. G. Eisenstein, Über die Irreductibilität und andere Eigenschaften der Gleichung, J. Reine Angew. Math., 39 (1850) 166–179.
2. A. F. Frumveller, Quadratic factors of polynomials, this MONTHLY, 24 (1917) 208–212.
3. O. E. Glenn, Relating to the quadratic factors of a polynomial, *ibid.*, 23 (1916) 313–315.
4. L. Kronecker, Grundzüge einer arithmetischen Theorie der algebraischen Grössen, J. Reine Angew. Math., 92 (1882) 111–114.
5. D. B. Lloyd, Some properties of rational quintic equations, Catholic University Press, Washington, D. C., 1940.
6. M. Mandl, Über die Zerlegung ganzer, ganzzahliger Functionen in irreductible Factoren, J. Reine Angew. Math., 113 (1894) 252–261.
7. C. C. MacDuffee, An introduction to abstract algebra, Wiley, New York, 1940.
8. C. Runge, Irreductibilität der Gleichung, J. Reine Angew. Math., 99 (1886) 89–97.
9. J. V. Uspensky, Theory of equations, McGraw-Hill, New York, 1948.
10. B. L. van der Waerden, Modern algebra, vol. 1, Frederick Unger Publ. Co., New York, 1953.

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$$y(z) = \frac{1}{2\pi i} \int_C \frac{(t^2 - 1)^n dt}{2^n (t - z)^{n+1}},$$

provided that C is any contour such that $(t^2-1)^{n+1}(t-z)^{-n-2}$ resumes its original value after describing C (see [1], page 306).

The purpose of this paper is to present solutions to a certain class of second order differential equations in terms of generalized Rodrigues' formulas and generalized Schläfli contour integrals.

The differential equation to be discussed is

$$(Ax^2+Bx+C)y''+(Dx+E)y'+Fy=R(x), \tag{4}$$

where A, B, C, D, E, and F are constants, $|A|+|B|+|C|\neq 0$, $|A|+|D|\neq 0$, and (except in Theorem 4) x is a real variable.

For convenience let $P_1(x)=Dx+E$, $P_2(x)=Ax^2+Bx+C$ and

$$W(x, x_0)=\exp\left\{-\int_{x_0}^{x}\frac{P_1(t)}{P_2(t)}dt\right\}, \tag{5}$$

where x_0 is any point for which the integral exists.

According to the Abel identity ([2], page 119), $W(x, x_0)$ is a constant multiple of the Wronskian of any two solutions of (4).

The results to be obtained are dependent on the roots of a quadratic equation associated with (4).

DEFINITION. *The associated equation of the differential equation* (4) *is the quadratic equation*

$$At^2+(A-D)t+F=0. \tag{6}$$

THEOREM 1. *If* $R(x)\equiv 0$ *and the associated equation* (6) *has a positive integral root* $t=n$, *then two solutions to* (4) *are*

$$y_1=\frac{d^{n-1}}{dx^{n-1}}\left\{[P_2(x)]^nW(x, x_0)\right\}$$

and

$$y_2=\frac{d^{n-1}}{dx^{n-1}}\left\{[P_2(x)]^nW(x, x_0)\int_{x_1}^{x}\frac{Q_{n-1}(t)dt}{[P_2(t)]^{n+1}W(t, x_0)}\right\},$$

where $Q_{n-1}(t)$ *is an arbitrary polynomial of degree* $\leqq(n-1)$ *and* x_1 *is any point for which the integral exists.*

Proof. The proof that y_1 satisfies (4) has been given by the author in [3], but a similar proof gives both results simultaneously.

Consider the first order differential equation

$$P_2(x)v'+[P_1(x)-nP_2'(x)]v=k_1Q_{n-1}(x) \tag{7}$$

which has the general solution

$$v=[P_2(x)]^nW(x, x_0)\left\{k_2+\int_{x_1}^{x}\frac{k_1Q_{n-1}(t)dt}{[P_2(t)]^{n+1}W(t, x_0)}\right\},$$

where k_1 and k_2 are constants.

Differentiation of (7) n times gives

$$P_2(x)v^{(n+1)} + P_1(x)v^{(n)} - [An^2 + (A - D)n]v^{(n-1)} = 0.$$

Since $R(x) \equiv 0$ and n is a root of (6) this is exactly (4) with y replaced by $v^{(n-1)}$ so $v^{(n-1)}$ is a solution to (4); y_1 is obtained by taking $k_2 = 1$ and $k_1 = 0$ while y_2 results from taking $k_2 = 0$ and $k_1 = 1$. The definite integrals may, of course, be replaced by indefinite integrals.

Example 1. For positive integers n the Legendre equation (1) has the associated equation $t^2 - t - n(n+1) = 0$ which has the root $t = n+1$. Taking $x_1 = +\infty$, $Q_{n-1}(t) \equiv 1$, and replacing the definite integral in $W(x, x_0)$ by the indefinite integral in Theorem 1, we obtain constant multiples of the solutions (2) and (3).

Example 2. For positive integers n the differential equation

$$(x^2 + 1)y'' - y' - n(n + 1)y = 0$$

has the associated equation $(t-n)(t+n+1) = 0$ so one solution is

$$y_1(x) = \frac{d^{n-1}}{dx^{n-1}} \{(x^2 + 1)^n \exp [\arctan x]\}$$

Example 3. In the Hermite equation (as given in [4])

$$v'' - 2xv' + 2nv = 0$$

put $v = \exp(x^2)y$. This yields $y'' + 2xy' + 2(n+1)y = 0$ which has the associated equation $2(t - n - 1) = 0$. So by Theorem 1 (with $x_0 = 0$) one solution is

$$y_1(x) = \frac{d^n}{dx^n} \{\exp (-x^2)\}.$$

Thus one solution to the Hermite equation is

$$v(x) = \exp (x^2) \frac{d^n}{dx^n} \{\exp (-x^2)\}.$$

Results similar to the first solution in Theorem 1 have been given by Brenke [5], when the differential equation (4) has polynomial solutions. His results depend on a Rodrigues' formula, due to Abramescu [6], for a general class of orthogonal polynomials (also see [7], pages 272–273). The method of Brenke consists essentially of showing that the polynomials given by the Rodrigues' formula satisfy a special case of the homogeneous form of (4).

THEOREM 2. *If the associated equation* (6) *has a positive integer root* $t = n$ *then a particular solution to* (4) *is*

$$y_p = \frac{d^{n-1}}{dx^{n-1}} \left\{ [P_2(x)]^n W(x, x_0) \int_{x_1}^{x} \frac{I_n(t)\,dt}{[P_2(t)]^{n+1} W(t, x_0)} \right\},$$

where $I_n(x)=\int_{a_n}^{x}\int_{a_{n-1}}^{v}\cdots\int_{a_1}^{v}R(v)(dv)^n$ *and* $x_1, a_1, a_2, \cdots, a_n$ *are any constants for which the integrals exist and* $I_n^{(n)}(x)=R(x)$.

Proof. The proof is identical to the previous proof after replacing $k_1Q_{n-1}(x)$ by $I_n(x)$ in (7) and replacing the general solution to (7) by the particular solution (inside the braces of the expression for y_p). Because of the previous theorem any polynomial of degree $\leqq(n-1)$ occurring in $I_n(t)$ may be omitted before computing y_p and as before all of the definite integrals may be replaced by indefinite integrals. If $a_1=a_2=\cdots=a_n=a$, then $I_n(x)$ can be written

$$I_n(x)=\int_a^x\frac{(x-v)^{n-1}}{(n-1)!}R(v)dv.$$

Example 4. The differential equation $x^2y''-2y=x^2$ has the associated equation $(t+2)(t-1)=0$. So, with $n=1$, Theorem 1 (with indefinite integrals and $Q_0=1$) gives as solutions to the homogeneous form of the equation

$$y_1=x^2\quad\text{and}\quad y_2=x^2\int x^{-4}dx=-1/3x.$$

Theorem 2 (with indefinite integrals) gives the particular solution

$$y_p=x^2\int\frac{\int x^2dx}{x^4}dx=\frac{1}{3}x^2\log|x|.$$

So a general solution is $y(x)=C_1x^2+C_2x^{-1}+\frac{1}{3}x^2\log|x|$.

The idea of interpreting "derivatives of negative order" as integrals suggests the following results when the associated equation has a nonpositive integer root.

THEOREM 3. *If n is a nonnegative integer such that $t=-n$ is a root of the associated equation* (6) *and if $R(x)$ is of class C^n, then every solution of* (4) *can be obtained from*

$$y(x)=\int\int\cdots\int[P_2(x)]^{-n}W(x,x_0)\left[k+\int_{x_1}^{x}\frac{R^{(n)}(t)dt}{[P_2(t)]^{1-n}W(t,x_0)}\right](dx)^{n+1}$$

for appropriate values of k, x_1, and the $n+1$ constants of integration.

Proof. Let y be any solution to (4). Differentiation of (4) n times yields

$$P_2(x)y^{(n+2)}+[P_1(x)+nP_2'(x)]y^{(n+1)}+[n(n-1)A+nD+F]y^{(n)}=R^{(n)}(x).$$

Since $-n$ is a root of (6) the coefficient of $y^{(n)}$ is zero and $y^{(n+1)}$ satisfies the differential equation

$$P_2(x)v'+[P_1(x)+nP_2'(x)]v=R^{(n)}(x).$$

So for some constants k and x_1, depending also on x_0,

$$y^{(n+1)} = [P_2(x)]^{-n}W(x, x_0)\left[k + \int_{x_1}^{x} \frac{R^{(n)}(t)\,dt}{[P_2(t)]^{1-n}W(t, x_0)}\right]$$

and the conclusion follows. Again the integrals can be indefinite.

Example 5. The differential equation $xy''-xy'+y=0$ has the associated equation $t+1=0$ so every solution can be obtained from

$$y(x) = k\int\int x^{-1}e^{x}(dx)^2.$$

The case when the associated equation has a negative integer root has been discussed by Abbé Lainé [8]. The solutions presented there, however, are in a considerably different form from the result of Theorem 3, (see also [7] page 481).

All of the previous results can be extended to include functions of a complex variable, but additional restrictions are necessary for the various definite integrals. For example, in the Wronskian $W(x, x_0)$ it is necessary that the integral exist and that the path of integration lie within a region containing x_0 within which $P_1(x)/P_2(x)$ is holomorphic (see [2], page 356). Similar restrictions must be applied to the various other integrals.

Suppose now that the Wronskian $W(x, x_0)$ is interpreted in the complex variable sense. Schläfli's integral solution to the Legendre equation and the contour integral representation of the first solution of Theorem 1 suggest the following result.

THEOREM 4. *If* $R(x)\equiv 0$ *and* r *is a root of the associated equation* (6), *then a solution to* (4) *is*

$$y(x) = \int_{\Gamma} [P_2(t)]^{r}(t - x)^{-r}W(t, x_0)\,dt \tag{8}$$

provided that Γ *is any contour on any Riemann surface of the integrand such that*

$$g(x, t) = [P_2(t)]^{r+1}(t - x)^{-r-1}W(t, x_0)$$

resumes its original value after describing Γ, *and provided that differentiation with respect to* x *under the integral sign is valid for* Γ.

Proof. Substitution of (8) into (4), with $R(x)\equiv 0$, yields

$$P_2(x)y'' + P_1(x)y' + Fy = \int_{\Gamma} [P_2(t)]^{r}(t - x)^{-r-2}W(t, x_0)V(x, t)\,dt, \tag{9}$$

where $V(x, t)=r(r+1)P_2(x)+rP_1(x)(t-x)+F(t-x)^2$. After rearrangement of terms this becomes

$$V(x, t) = x^2[Ar^2 + (A - D)r + F] + F(t^2 - 2tx) + r(r + 1)(Bx + C) + rDtx + rE(t - x).$$

can be transformed to the form (4) by the transformation $y=e^{ax}v$, where a is a root of the quadratic equation $At^2+Ct+E=0$.

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References

1. E. T. Whittaker and G. N. Watson, Modern analysis, 4th Ed., Cambridge University Press, Cambridge, 1927.

2. E. L. Ince, Ordinary differential equations, Dover, New York, 1956.

3. J. M. Horner, A note on the derivation of Rodrigues' formulae, this MONTHLY, 70 (1963) 81–82.

4. E. D. Rainville, Special functions, Macmillan, New York, 1960.

5. W. C. Brenke, On polynomial solutions of a class of linear differential equations of the second order, Amer. Math. Soc. Bull. 36 (1930) 77–84.

6. N. Abramescu, Sulle Serie di Polinomi di una Variabile Complessa. Le Serie di Darboux, Ann. Mat. Pura Appl. (series 3), 31 (1922) 207–249.

7. E. Kamke, Differentialgleichungen, Lösungsmethoden und Lösungen, Vol. I, 4th Ed., Akademische Verlagsgesellschaft, Leipzig, 1951.

8. Abbé Lainé, Sur l'Intégration de Quelques Équations Différentielles de Second Ordre, Enseignement Math. 23 (1923) 163–173.

9. J. M. Horner, Generalizations of the Formulas of Rodrigues and Schläfli, MA thesis, University of Alabama, August, 1962.

SPIRAL KERNELS

LUCIO ARTIAGA, University of Windsor

1. Introduction. It is known that there exist certain classes of functions $k(z)$ and $f(z)$ such that

$$g(z) = \int_D k(wz)f(w)dw, \qquad z \in D \tag{1}$$

implies that

$$f(z) = \int_D k(wz)g(w)dw, \qquad z \in D \tag{2}$$

where D is either the positive real axis or the unit circle. In this paper a pair of equations such as (1) and (2) will be called a reciprocity. In the former case $g(z)$ is the general Fourier transform of $f(w)$ (see [1]). In the latter case $g(z)$ is the circular transform of $f(w)$ (see [2]). This suggests the following question:

For which other curve C can there exist reciprocities such that

$$g(z) = \int_C k(wz)f(w)dw, \qquad z \in C \tag{3}$$

can be transformed to the form (4) by the transformation $y=e^{ax}v$, where a is a root of the quadratic equation $At^2+Ct+E=0$.

Acknowledgment. The author expresses his appreciation to Dr. James L. Howell and to the referee for their many constructive suggestions.

References

1. E. T. Whittaker and G. N. Watson, Modern analysis, 4th Ed., Cambridge University Press, Cambridge, 1927.
2. E. L. Ince, Ordinary differential equations, Dover, New York, 1956.
3. J. M. Horner, A note on the derivation of Rodrigues' formulae, this MONTHLY, 70 (1963) 81–82.
4. E. D. Rainville, Special functions, Macmillan, New York, 1960.
5. W. C. Brenke, On polynomial solutions of a class of linear differential equations of the second order, Amer. Math. Soc. Bull. 36 (1930) 77–84.
6. N. Abramescu, Sulle Serie di Polinomi di una Variabile Complessa. Le Serie di Darboux, Ann. Mat. Pura Appl. (series 3), 31 (1922) 207–249.
7. E. Kamke, Differentialgleichungen, Lösungsmethoden und Lösungen, Vol. I, 4th Ed., Akademische Verlagsgesellschaft, Leipzig, 1951.
8. Abbé Lainé, Sur l'Intégration de Quelques Équations Différentielles de Second Ordre, Enseignement Math. 23 (1923) 163–173.
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For which other curve C can there exist reciprocities such that

$$g(z) = \int_C k(wz)f(w)dw, \qquad z \in C \tag{3}$$

implies that

$$f(z) = \int_C k(wz)g(w)dw, \qquad z \in C? \tag{4}$$

In this paper we shall prove that other such curves do exist and that they are equiangular spirals. The case where C is the positive real axis or the unit circle can be regarded as limiting cases of this more general result. Furthermore, we are able to show that such a theory of spiral kernels can be developed from the theory of general Fourier transforms and from the theory of Watson transforms.

2. Possible contours. For which type of curves do there exist reciprocities such that $g(z) = \int_C k(wz)f(w)dw$, $z \in C$ implies that $f(z) = \int_C k(wz)g(w)dw$, $z \in C$ where $k(z)$ is defined on C?

The curve C must be such that if w and z are on C, so is wz. Let

$$\rho = f(\theta) \tag{5}$$

be the equation of C in polar coordinates.

Let $\arg z = \theta_1$, $\arg w = \theta_2$. Since z, w, wz are on C we must have

$$z = f(\theta_1)e^{i\theta_1}, \qquad w = f(\theta_2)e^{i\theta_2}$$
$$wz = f(\theta)e^{i\theta} = f(\theta_1)f(\theta_2)e^{i(\theta_1+\theta_2)}.$$

This implies that $\theta = \theta_1 + \theta_2$ and hence that

$$f(\theta_1 + \theta_2) = f(\theta_1)f(\theta_2). \tag{6}$$

The only continuous real solutions of equation (3) are of form $f(\theta) = e^{h\theta}$ where h is real (see [**3**]). Therefore equation (5) becomes

$$\rho = e^{h\theta}. \tag{7}$$

For any real h, (7) is the equation of an equiangular spiral.

3. Extension. We extend the above result to the case in which w, z, wz move on different curves. Assume that w is on C_1, z on C_2, and wz on C_3 and that $\rho = f(\theta)$, $\rho = g(\theta)$, $\rho = h(\theta)$ are the equations in polar coordinates of C_1, C_2, and C_3 respectively. Then proceeding as before we would have

$$f(\theta_1)g(\theta_2) = h(\theta_1 + \theta_2). \tag{8}$$

Let $\theta_2 = 0$. Then $f(\theta_1)g(0) = h(\theta_1)$ or

$$h(\theta) = Af(\theta). \tag{9}$$

In the same way

$$h(\theta) = Bg(\theta), \tag{10}$$

where A and B are constants. Now (8) becomes

$$h(\theta_1)h(\theta_2) = ABh(\theta_1 + \theta_2). \tag{11}$$

The only real continuous solutions are

$$h(\theta) = ABe^{a\theta}, \qquad f(\theta) = Be^{a\theta}, \qquad g(\theta) = Ae^{a\theta},$$

where a is real (see [3]). Thus the curves C_1, C_2, and C_3 are equiangular spirals with the same parameter a.

4. Fourier spiral kernels. The following theory is stated formally. A function $k(x)$ is a Fourier kernel [1] if for some class of functions $f(x)$ the equation

$$g(x) = \int_0^\infty k(xt)f(t)dt \tag{12}$$

implies that

$$f(x) = \int_0^\infty k(xt)g(t)dt \qquad (x \text{ real positive}). \tag{13}$$

A characteristic property of a Fourier kernel $k(x)$ is that its Mellin transformation,

$$k^*(s) = \int_0^\infty k(x)x^{s-1}dx$$

satisfies the functional equation [1]

$$k^*(s)k^*(1-s) = 1. \tag{14}$$

Next, let $t=w^\alpha$, $\alpha=a+ib$, and $w=\rho e^{i\theta}$; then

$$t = e^{i(a+ib)\theta}\cdot\rho^{a+ib} = \rho^a e^{-b\theta}\cdot e^{i(b\log\rho+a\theta)}. \tag{15}$$

Thus t will be real and positive if $b\log\rho+a\theta=0 \pmod{2\pi}$, i.e., if w moves along the equiangular spiral

$$\rho = Ae^{-a\theta/b}, \qquad A = e^{2\pi m/b}, \tag{16}$$

where m is an integer.

Arguing in the same way about $x=z^\alpha$ one finds that z moves along the equiangular spiral

$$\rho = Be^{-a\theta/b}, \qquad B = e^{2\pi n/b}. \tag{17}$$

For simplicity we shall assume that w and z move in the same equiangular spiral S

$$\rho = e^{-a\theta/b}. \tag{18}$$

Consider the reciprocity determined by equations (12) and (13). Let t be real and positive and $t=w^\alpha$ where w is on $\rho=e^{-a\theta/b}$.

Let x be real and positive and $x=z^\alpha$, where z is on $\rho=e^{-a\theta/b}$. Then equation (12) becomes

$$g(z^\alpha) = \alpha \int_S k(z^\alpha w^\alpha) f(w^\alpha) w^{\alpha-1} dw, \qquad z \in S. \tag{19}$$

Multiplying both sides of (19) by $z^{(\alpha-1)/2}$ we obtain

$$z^{(\alpha-1)/2} g(z^\alpha) = \int_S \alpha (zw)^{(\alpha-1)/2} k(z^\alpha w^\alpha) w^{(\alpha-1)/2} f(w^\alpha) dw. \tag{20}$$

Now, put $G(z)=z^{(\alpha-1)/2}g(z^\alpha)$, $F(z)=z^{(\alpha-1)/2}F(z^\alpha)$, $K(z)=z^{(\alpha-1)/2}k(z^\alpha)$. Then (20) becomes

$$G(z) = \int_S K(wz) F(w) dw, \qquad z \in S.$$

We obtain similarly the inverse formula

$$F(z) = \int_S K(wz) G(w) dw, \qquad z \in S.$$

Therefore, we have this formal result:

If $k(x)$ is a Fourier kernel for the positive real axis then $K(z)=\alpha z^{(\alpha-1)/2}k(z^\alpha)$ is a Fourier spiral kernel for the spiral $\rho=e^{-a\theta/b}$.

Example. Let $a=b=1$, i.e. $\rho=e^{-\theta}$, and use the Fourier kernel $\sqrt{2/\pi}\cos x$; then

$$K(x) = \sqrt{\frac{2}{\pi}}\,(1+i)x^{i/2}\cos x^{1+i}$$

is a Fourier spiral kernel for the spiral $\rho=e^{-\theta}$.

5. Watson spiral kernels. A general theory of Fourier transform is difficult to obtain due to the fact that one would expect that a Fourier kernel would be of the form (see [1])

$$k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k^*(\tfrac{1}{2}+it) x^{-\frac{1}{2}+it} dt. \tag{21}$$

In this case $k(x)$ is the inverse Mellin transform of $k^*(s)$. Although $k^*(s)$ is assumed to satisfy equation (14) there is no reason to believe that the integral (21) exists. Therefore a different approach was investigated by Watson [4] using an integrated type of kernel $k_1(x)$ called the Watson kernel.

Let $k_1(x)/x \in L^2(0,\infty)$ and let $\int_0^\infty k_1(xu)k_1(yu)u^{-2}du=\min(x,y)$. Let $f(x) \in L^2(0,\infty)$. Then [1]

$$\int_0^a g(x)dx = \int_0^\infty k_1(au)f(u)u^{-1}du \tag{22}$$

Acknowledgment. The author would like to thank Professor A. P. Guinand for his interest in this work.

References

1. E. C. Titchmarsh, Introduction to the theory of Fourier Integrals, Chapter VIII, 2nd ed., Oxford University Press, New York, 1948.
2. L. Artiaga, Circular kernels, to appear elsewhere.
3. J. Aczel, Vorlesungen über Functionalgleichungen und ihre Anwendungen, Birkhäuser Verlag, Basel, 1961, page 47.
4. G. N. Watson, General transforms, Proc. London Math. Soc., (2), 35 (1933) 156–199.

SOME INEQUALITIES FOR A TRIANGLE

L. CARLITZ, Duke University

1. If P is a point inside a triangle ABC whose distances from the vertices are x, y, z and whose distances from the sides are p, q, r, then it is well known that

$$x + y + z \geqq 2(p + q + r) \tag{1}$$

with inequality if and only if the triangle ABC is equilateral and P is its center. The inequality (1) is due to Erdös; it was first proved by Mordell and Barrow [1]. Oppenheim [5] has recently proved a number of inequalities connecting the distances x, y, z, p, q, r; for example he has proved that

$$px + qy + rz \geqq 2(qr + rp + pq), \tag{2}$$

$$yz + zx + xy \geqq 4(qr + rp + pq), \tag{3}$$

$$xyz \geqq 8pqr, \tag{4}$$

$$p^{-1} + q^{-1} + r^{-1} \geqq 2(x^{-1} + y^{-1} + z^{-1}). \tag{5}$$

As above, let P be a point inside the triangle ABC and let AP, BP, CP intersect the sides BC, CA, AB, in L, M, N, respectively. We denote PL, PM, PN by u, v, w. The object of the present note is to find inequalities connecting the distances x, y, z, u, v, w. In particular we shall show that

$$x + y + z > u + v + w, \qquad xvw + ywu + zuv \geqq 6uvw,$$

$$xyz \geqq 8uvw, \qquad uyz + vzx + wuv \geqq 12uvw, \qquad yz + zx + xy \geqq 12(uvw)^{2/3},$$

$$2(\sqrt{yz} + \sqrt{zx} + \sqrt{xy}) \leqq x + y + z + 2(u + v + w).$$

See also (18), (19), (22), (24), (27), (28), (29) below.

2. We recall first (see for example [2] pp. 162–3) that

$$\frac{PL}{AL} + \frac{PM}{BM} + \frac{PN}{CN} = 1 \tag{6}$$

Acknowledgment. The author would like to thank Professor A. P. Guinand for his interest in this work.

References

1. E. C. Titchmarsh, Introduction to the theory of Fourier Integrals, Chapter VIII, 2nd ed., Oxford University Press, New York, 1948.
2. L. Artiaga, Circular kernels, to appear elsewhere.
3. J. Aczel, Vorlesungen über Functionalgleichungen und ihre Anwendungen, Birkhäuser Verlag, Basel, 1961, page 47.
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$$p^{-1}+q^{-1}+r^{-1} \geqq 2(x^{-1}+y^{-1}+z^{-1}). \tag{5}$$

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$$x+y+z > u+v+w, \qquad xvw+ywu+zuv \geqq 6uvw,$$
$$xyz \geqq 8uvw, \qquad uyz+vzx+wuv \geqq 12uvw, \qquad yz+zx+xy \geqq 12(uvw)^{2/3},$$
$$2(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}) \leqq x+y+z+2(u+v+w).$$

See also (18), (19), (22), (24), (27), (28), (29) below.

2. We recall first (see for example [2] pp. 162–3) that

$$\frac{PL}{AL}+\frac{PM}{BM}+\frac{PN}{CN}=1 \tag{6}$$

and

$$\frac{AP}{PL} = \frac{AM}{MC} + \frac{AN}{NB}, \qquad \frac{BP}{PM} = \frac{BN}{NA} + \frac{BL}{LC}, \qquad \frac{CP}{PN} = \frac{CL}{LB} + \frac{CM}{MA}. \tag{7}$$

In terms of x, y, z, u, v, w, (6) becomes

$$\frac{u}{x+u} + \frac{v}{y+v} + \frac{z}{z+w} = 1$$

or what is the same thing

$$xyz = 2uvw + xvw + ywu + zuv. \tag{8}$$

We shall show first that

$$x + y + z > u + v + w. \tag{9}$$

If a, b, c denote the sides of ABC, it is evident that $y+z \geqq a$, $z+x \geqq b$, $x+y \geqq c$. It follows that

$$x + y + z \geqq s = \tfrac{1}{2}(a + b + c). \tag{10}$$

On the other hand, it is known that if c is the longest side of ABC then [**4**, p. 89, Problem **41**]

$$u + v + w < c. \tag{11}$$

Combining (10) and (11), we get

$$x + y + z > \frac{s}{c}\ u + v + w). \tag{12}$$

Since $s > c$, it is evident that (12) implies (9).

We note that (9) cannot be improved; more precisely, if $k > 1$ then the inequality $x+y+z > k(u+v+w)$ cannot be asserted. This can be seen from consideration of the triangle ABC with $AC = BC$ and the angle at C slightly less than 180°; P is taken at the midpoint of AB. Another example is furnished when $AC = BC$ and the angle at C is very small; P is taken on AC and close to A.

3. Returning to (7), put

$$\lambda = \frac{BL}{LC}, \qquad \mu = \frac{CM}{MA}, \qquad \nu = \frac{AN}{NB}.$$

Then (7) becomes

$$\frac{x}{u} = \frac{1}{\mu} + \nu, \qquad \frac{y}{v} = \frac{1}{\nu} + \lambda, \qquad \frac{z}{w} = \frac{1}{\lambda} + \mu. \tag{13}$$

Adding these equations we get

$$\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=\lambda+\frac{1}{\lambda}+\mu+\frac{1}{\mu}+\nu+\frac{1}{\nu},$$

which yields

$$\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\geqq 6 \tag{14}$$

or equivalently

$$xvw+ywu+zuv\geqq 6uvw. \tag{15}$$

Combining (14) with (8), we get

$$xyz\geqq 8uvw. \tag{16}$$

Neither (14) or (16) can be improved. Indeed equality holds only if P is the centroid of ABC. If

$$\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=6, \tag{17}$$

then it is clear from the proof that $\lambda=\mu=\nu=1$. Thus, (13) becomes $x=2u$, $y=2v$, $z=2w$. If $xyz=8uvw$, (8) reduces to $xvw+ywu+zuv=6uvw$, which is identical with (17).

4. Returning to (13), we have

$$x+y+z=\left(\lambda v+\frac{w}{\lambda}\right)+\left(\mu w+\frac{u}{\mu}\right)+\left(\nu u+\frac{v}{\nu}\right).$$

Since

$$\lambda v+\frac{w}{\lambda}\geqq 2\sqrt{vw},$$

it follows that

$$x+y+z\geqq 2(\sqrt{vw}+\sqrt{wu}+\sqrt{uv}). \tag{18}$$

It also follows from (13) that

$$yz-vw=\left(\frac{\mu}{\nu}+\lambda\mu+\frac{1}{\nu\lambda}\right)vw,$$

$$zx-wu=\left(\frac{\nu}{\lambda}+\mu\nu+\frac{1}{\lambda\mu}\right)wu,$$

$$xy-uv=\left(\frac{\lambda}{\mu}+\nu\lambda+\frac{1}{\mu\nu}\right)uv.$$

Since

$$\frac{\mu}{\lambda}vw + \frac{\nu}{\lambda}wu + \frac{\lambda}{\mu}uv \geqq 3(uvw)^{2/3}, \qquad \lambda\mu vw + \frac{wu}{\lambda\mu} \geqq 2w\sqrt{uv},$$

$$u\sqrt{vw} + v\sqrt{wu} + w\sqrt{uv} \geqq 3(uvw)^{2/3},$$

we get

$$yz + zx + xy \geqq vw + wu + uv + 9(uvw)^{2/3}. \tag{19}$$

We have also

$$yz + zx + xy - 3$$
$$= \left(\frac{\mu}{\nu} + \frac{\nu}{\lambda} + \frac{\lambda}{\mu}\right) + \left(\mu\nu + \frac{1}{\mu\nu}\right) + \left(\nu\lambda + \frac{1}{\nu\lambda}\right) + \left(\lambda\mu + \frac{1}{\lambda\mu}\right),$$

which yields

$$\frac{yz}{vw} + \frac{zx}{wu} + \frac{xy}{uv} \geqq 12, \tag{20}$$

or if we prefer

$$uyz + vzx + wuv \geqq 12uvw. \tag{21}$$

Similarly we can show that

$$x^2 + y^2 + z^2 \geqq 2(vw + wu + uv) + 6(uvw)^{2/3}. \tag{22}$$

Note that (19) and (22) imply

$$yz + zx + xy \geqq 12(uvw)^{2/3} \tag{23}$$

and

$$x^2 + y^2 + z^2 \geqq 12(uvw)^{2/3}. \tag{24}$$

In the next place, since $u=\alpha(x+u)$, $v=\beta(y+v)$, $w=\gamma(z+w)$, where $\alpha+\beta+\gamma=1$, it follows from the Cauchy inequality that

$$(\sqrt{u} + \sqrt{v} + \sqrt{w})^2 \leqq x + y + z + u + v + w.$$

This reduces to (18). Since

$$x = (1 - \alpha)(x + u), \qquad y = (1 - \beta)(y + v), \qquad z = (1 - \gamma)(z + w),$$

we have also

$$(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 \leqq 2(x + y + z + u + v + w).$$

This reduces to

$$2(\sqrt{yz} + \sqrt{zx} + \sqrt{xy}) \leqq x + y + z + 2(u + v + w), \tag{25}$$

an inequality of a rather unusual kind. Incidentally (25) reduces to an equality when P is the centroid of ABC.

5. Finally we note that

$$\frac{h_1}{p}=\frac{x+u}{u},\qquad \frac{h_2}{q}=\frac{y+v}{v},\qquad \frac{h_3}{r}=\frac{z+w}{w},\tag{26}$$

where h_1, h_2, h_3 are the altitudes of ABC and p, q, r have the same meaning as in (1). Applying (14) we get

$$\frac{h_1}{p}+\frac{h_2}{q}+\frac{h_3}{r}\geqq 9.\tag{27}$$

Since by (26)

$$\begin{aligned}\frac{h_1h_2h_3}{pqr}&=\left(1+\frac{x}{u}\right)\left(1+\frac{y}{v}\right)\left(1+\frac{z}{w}\right)\\&=1+\frac{x}{u}+\frac{y}{v}+\frac{z}{w}+\frac{yz}{vw}+\frac{zx}{wu}+\frac{xy}{uv}+\frac{xyz}{uvw},\end{aligned}$$

it follows from (14), (16) and (20) that

$$h_1h_2h_3\geqq 27pqr.\tag{28}$$

We may also mention

$$(h_1-p)(h_2-q)(h_3-r)\geqq 8pqr.\tag{29}$$

The inequality (16) is known. See O. J. Ramler and C. W. Trigg, E1043, Property of three concurrent cevians, this MONTHLY, 60(1953) 421.

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References

1. Problem 3740, this MONTHLY, 44 (1937) 252–254.
2. N. A. Court, College geometry, Barnes and Noble, New York, 1952.
3. L. Fejes-Tóth, Lagerungen in der Ebene auf der Kugel und im Raum, Springer, Berlin, 1953.
4. N. D. Kazarinoff, Geometric inequalities, Random House, New York, 1961.
5. A. Oppenheim, The Erdös and other inequalities for a triangle, this MONTHLY, 68 (1961) 226–230.

A CORRECTION FOR "N-GON ROTORS MAKING $N+1$ CONTACTS WITH FIXED SIMPLE CURVES"

MICHAEL GOLDBERG, Washington, D. C.

Professor W. Wunderlich, of the Technische Hochschule in Vienna, has pointed out an error in the writer's note "n-gon rotors making $n+1$ contacts with fixed simple curves," which appeared in volume 69 (1962), pages 486–491 of this MONTHLY. The error appears in Figure 1, in some of the subsequent figures, and in some of the conclusions that may be drawn from these figures. The curve is not as smooth as shown in these figures. Instead, the more complicated curve, shown below, is a truer pictorial representation of the curve for Figure 1.

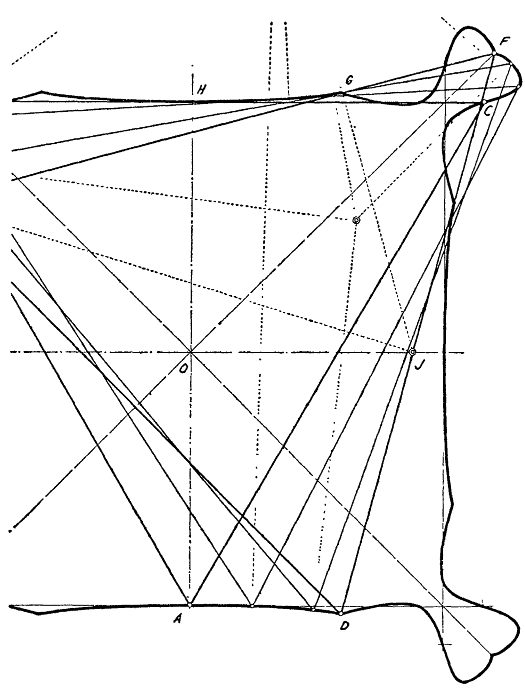

FIG. 1

The vertices of the rotating triangle trace the curve. At each stage, at least one of the sides of the triangle is tangent to the curve. Thus, $n+1$ contacts are made. But the triangle is not confined within the curve as was erroneously assumed from Figure 1.

In Figure 6, however, the triangular rotor is always within the stator curve.

If the set S is not empty, then it has a largest member b. This follows from the left continuity of g and from the fact that, since $g(a) < A$, the set cannot be the whole interval (0, 1). Thus

$$g(b) \geqq A > g(a)$$

so that $b < a$, and

$$m\{x: g(x) \geqq A\} = b < a \leqq m\{x: f(x) \geqq A\}.$$

This completes the proof of the theorem.

The interval (0, 1) of the theorem may, of course, be replaced by the interval $(0, \infty)$.

The concept of rearrangement in decreasing order has also been defined for functions of two variables [1] and an analogous uniqueness theorem may be formulated for the "decreasing" functions occurring in this context. The contour lines used to define such functions form a one-parameter family of curves so that the problem is essentially the one-dimensional one treated above.

References

1. J. C. Burkill, Rearrangements of functions, J. London Math. Soc., 27 (1952) 393–401.
2. G. H. Hardy, J. E. Littlewood and G. Polyá, Some simple inequalities satisfied by convex functions, Messenger of Math., 58 (1929) 145–152.
3. ———, Inequalities, Cambridge, 1929.

INVARIANT UNIFORMITIES ON COSET SPACES

T. S. Wu, Stanford University

It is known that a topological group has a right invariant metric if and only if the topology of the group space is first countable (see [4]). It is also known that if K is a closed subgroup of a topological group G then G acts on the right coset space G/K as the right transformation group in a natural fashion (see [2]). In this note we are interested in the following question. When does G/K have a metric, which is invariant under the action of G (or G-invariant)? More generally, we would like to find the relationship between the uniformity on G and that on G/K, and also the invariance of such uniformities under the action of G. In the following we shall discuss the case when K is a compact subgroup of G.

Standing notations: G is a topological group and K is a compact subgroup of G. Let G/K be the right coset space with decomposition (quotient) topology and $(G/K, G, \pi)$ be the right transformation group, i.e., $(Kg, h)\pi = Kgh \in G/K$. Let $\mathcal{U}$ be the right uniformity of G, and $\mathcal{U}^v$ be the partition uniformity induced by $\mathcal{U}$ on G/K. We say that a uniformity is invariant if it has a basis each member of which is invariant.

PROPOSITION 1. $\mathcal{U}^v$ is *G-invariant and $\mathcal{U}^v$ is compatible with the quotient space G/K.*

If the set S is not empty, then it has a largest member b. This follows from the left continuity of g and from the fact that, since $g(a) < A$, the set cannot be the whole interval $(0, 1)$. Thus

$$g(b) \geqq A > g(a)$$

so that $b < a$, and

$$m\{x: g(x) \geqq A\} = b < a \leqq m\{x: f(x) \geqq A\}.$$

This completes the proof of the theorem.

The interval $(0, 1)$ of the theorem may, of course, be replaced by the interval $(0, \infty)$.

The concept of rearrangement in decreasing order has also been defined for functions of two variables [1] and an analogous uniqueness theorem may be formulated for the "decreasing" functions occurring in this context. The contour lines used to define such functions form a one-parameter family of curves so that the problem is essentially the one-dimensional one treated above.

References

1. J. C. Burkill, Rearrangements of functions, J. London Math. Soc., 27 (1952) 393–401.
2. G. H. Hardy, J. E. Littlewood and G. Polyá, Some simple inequalities satisfied by convex functions, Messenger of Math., 58 (1929) 145–152.
3. ———, Inequalities, Cambridge, 1929.

INVARIANT UNIFORMITIES ON COSET SPACES

T. S. Wu, Stanford University

It is known that a topological group has a right invariant metric if and only if the topology of the group space is first countable (see [4]). It is also known that if K is a closed subgroup of a topological group G then G acts on the right coset space G/K as the right transformation group in a natural fashion (see [2]). In this note we are interested in the following question. When does G/K have a metric, which is invariant under the action of G (or G-invariant)? More generally, we would like to find the relationship between the uniformity on G and that on G/K, and also the invariance of such uniformities under the action of G. In the following we shall discuss the case when K is a compact subgroup of G.

Standing notations: G is a topological group and K is a compact subgroup of G. Let G/K be the right coset space with decomposition (quotient) topology and $(G/K, G, \pi)$ be the right transformation group, i.e., $(Kg, h)\pi = Kgh \in G/K$. Let $\mathcal{U}$ be the right uniformity of G, and $\mathcal{U}^v$ be the partition uniformity induced by $\mathcal{U}$ on G/K. We say that a uniformity is invariant if it has a basis each member of which is invariant.

PROPOSITION 1. $\mathcal{U}^v$ is *G-invariant and $\mathcal{U}^v$ is compatible with the quotient space G/K.*

Proof. Let ϕ be the canonical map from G onto G/K. It is well known that ϕ is an open map. Since K is compact, it is also a closed map. So, by a theorem in [1], $\mathfrak{U}^v$ is compatible with the topology of G/K.

Let $[V_\lambda | \lambda \in \Lambda]$ be a filter base of neighborhoods of the identity in G. Let $\alpha_\lambda = [(Kg, Kf) \in G/K \times G/K | Kg \subset V_\lambda Kf$ and $Kf \subset V_\lambda Kg]$. Then $[\alpha_\lambda | \lambda \in \Lambda]$ is a basis for $\mathfrak{U}^v$. It is clear that $Kgh \subset V_\lambda Kfh$ for all $h \in G$ if and only if $Kg \subset V_\lambda Kf$. So $\mathfrak{U}^v$ is G-invariant.

PROPOSITION 2. *G/K has a G-invariant metric ρ^v if and only if the topology of G/K is first countable. In case G/K has a G-invariant metric ρ^v, then ρ^v can be induced by the partition uniformity $\mathfrak{U}^v$.*

The first part of Proposition 2 is known but the proof presented here is apparently new. See "Invariant Metric on Coset Spaces," by Leif Kristensen, *Scandinavica Mathematica*, (1958) 33–36.

Proof. The necessity is clear. We prove the sufficiency. Since the topology of G/K is first countable and K is compact, there are countably many open sets W_n in G, $n \in N$ (integers) such that (i) $K \subset W_n$, (ii) $[[Kg | g \in W_n] | n \in N]$ is a filter base of neighborhoods of K in G/K, (iii) $W_n = W_n^{-1}$. Let $\beta_n = [(Kg, Kf) \in G/K \times G/K | Kg \subset W_n Kf$ and $Kf \subset W_n Kg]$. We are going to prove that $[\beta_n | n \in N]$ is a basis for $\mathfrak{U}^v$. If V is any symmetric open set in G, i.e., $V = V^{-1}$, then $KV \subset KW_n$ for some $n \in N$ and $W_n K \subset VK$. Hence, if $Kf \subset W_n Kg$, then $Kf \subset VKg$; so $[\beta_n | n \in N]$ is a basis for $\mathfrak{U}^v$. As $[\beta_n | n \in N]$ is countable, it induces a metric ρ^v on G/K, and ρ^v is G-invariant as β_n is G-invariant, (cf. Exercise N, Chapter 6, in [3]).

In a routine manner it can be shown that any two metrics for G/K which are G-invariant and induce the same topology also induce the same uniformity. Thus, the second part of Proposition 2 follows from the first part.

PROPOSITION 3. *If the topology of G is first countable, then G has a right invariant metric ρ, and the right invariant metric ρ^v on G/K is equivalent to the Hausdorff metric induced by ρ on the decomposition space G/K.*

Proof. Since the uniformity defined by ρ is equivalent to the right uniformity $\mathfrak{U}$, the proposition follows from Propositions 1, 2 and the definition of Hausdorff metric.

We should like to express our thanks to R. W. Bagley for his helpful correspondence during our preparation of this paper.

References

1. R. W. Bagley, The uniformizing of partitions, this MONTHLY, 69 (1962) 767–768.
2. W. H. Gottschalk and G. A. Hedlund, Topological dynamics, Amer. Math. Soc. Colloq. Publ., 36 (1955).
3. J. L. Kelley, General topology, Van Nostrand, New York, 1955.
4. D. Montgomery and L. Zippin, Topological transformation groups, Wiley, New York, 1955.

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References

1. C. F. Dunkl and K. S. Williams, A simple norm inequality, this MONTHLY, 71 (1964) 53–54.
2. E. R. Lorch, On certain implications which characterize Hilbert space, Ann. of Math., 49 (1948) 523–532.
3. M. F. Smiley, The proof of the triangle inequality, this MONTHLY, 70 (1963) 546.

Editorial Note. The questions raised by Dunkl and Williams were also answered in letters to Prof. Williams by Gary H. Meisters, University of Colorado, Donald A. Sarason, Institute for Advanced Study, and by James P. Crawford, Lafayette College.

PERIODS OF MEASURABLE FUNCTIONS AND THE STONE-ČECH COMPACTIFICATION

Z. SEMADENI, Poznań, Poland

1. Periods of measurable functions. A number T is called a period of a function f defined for $-\infty<t<\infty$ if $f(t+T)=f(t)$ for all t. Obviously, the set of all periods of f is an additive subgroup of the reals and for any periodic function we have two possibilities: Either there exists the smallest positive period T_0 and all periods are of the form nT_0 with $n=0, \pm 1, \pm 2, \cdots$ or the set of periods is dense. We have two typical examples of functions with dense set of periods: the characteristic function of the set of rationals and the function $f(x)=\phi(x)-\phi(1)x$, where ϕ is any nonmeasurable Hamel solution of the equation $\phi(x+y)=\phi(x)+\phi(y)$; in both cases any rational number is a period. Obviously, a continuous function with dense set of periods must be constant. There exist a nonconstant measurable function with an uncountable set of periods and a measurable function with Darboux property and a countable dense set of periods (cf. [4] pp. 833–836).

The purpose of this note is to give a simple proof of the following theorem due to A. Łomnicki ([4], Theorem 5).

THEOREM 1. *If a measurable function has a dense set of periods, then it is constant almost everywhere.*

The proof is founded on the following well-known

LEMMA. *If T is a positive period of a measurable function f and $\int_0^T|f(t)|\,dt<\infty$, then the limit*

$$\mathfrak{M}(f) = \lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x} f(t)dt$$

exists and is equal to $T^{-1}\int_0^T f(t)dt$.

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References

1. J. C. E. Dekker, The constructivity of maximal dual ideals in certain Boolean algebras, Pacific J. Math., 3 (1953) 73–101.
2. N. Dunford and J. T. Schwartz, Linear operators I, Interscience, New York, 1958.
3. L. Gillman and M. Jerison, Rings of continuous functions, Van Nostrand, Princeton, 1960.
4. A. Łomnicki, Sur les fonctions multipériodiques uniformes d'une variable réelle (Polish; French summary), Sprawozdania Towarzystwa Naukowego Warszawskiego (Comptes Rendus des Séances de la Société des Sciences de Varsovie), Classe III, 11 (1918) 808–846.
5. J. Loś and C. Ryll-Nardzewski, On the application of Tychonoff's theorem in mathematical proofs, Fund. Math., 38 (1951) 233–237.
6. ———, Effectiveness of the representation theory for Boolean algebras, Fund. Math., 41 (1955) 49–56.
7. W. Sierpiński, Fonctions additives non complètement additives et fonctions non mesurables, Fund. Math., 30 (1938) 96–99.
8. R. Sikorski, Boolean Algebras, Ergebnisse Math., Berlin, 1960.

THE NUMBER OF UNITARY DIVISORS OF A GENERALISED INTEGER

E. M. Horadam, University of New England, Australia

The object of this note is to follow the work in [1] for generalised integers which are defined as follows. Suppose there is given a finite or infinite sequence $\{p\}$ of real numbers (generalised primes) such that $1<p_1<p_2< \cdots$. Form the set $\{l\}$ of all possible p-products, i.e., products $p_1^{v_1}p_2^{v_2}\cdots$ where $v_1, v_2, \cdots$ are integers $\geqq 0$ of which all but a finite number are 0. Call these numbers generalised integers and suppose that no two generalised integers are equal if their v's are different. Then arrange $\{l\}$ in an increasing sequence $1=l_1<l_2<\cdots$.

Suppose now that $\tau^*(l_n)$ represents the number of unitary divisors of l_n; that is the number of divisors d of l_n such that d is relatively prime to the complementary divisor l_n/d. Let $[x]$ denote the number of generalised integers $\leqq x$ and assume that

$$[x] = x + R(x) \quad \text{where} \quad R(x) = O(x^\alpha) \quad \text{and} \quad 0 < \alpha < 1. \tag{1}$$

Let $\phi(x, l_n)$ denote the number of generalised integers $\leqq x$ which are prime to l_n and let $\mu(l_n)$ be the Möbius function defined by $\mu(l_n)=0$ if l_n has a square factor; $\mu(l_n)=(-1)^k$, where k denotes the number of prime divisors of l_n and l_n has no square factor; $\mu(1)=1$. Then

$$\phi(x, l_n) = \sum_{d\mid l_n} \mu(d)\left[\frac{x}{d}\right]. \tag{2}$$

This is a simple consequence of Sylvester's theorem (see [2], Theorem 26). Define $\zeta(s)=\sum_{l_n=1}^{\infty} l_n^{-s}(s>1)$. Then it is shown in [3] that $1/\zeta(s)=\sum_{l_n=1}^{\infty}\mu(l_n)l_n^{-s}$ and also that

$$\sum_{l_n\leq x}\frac{1}{l_n^\beta} = \frac{x^{1-\beta}}{1-\beta} + \gamma_\beta + O(x^{\alpha-\beta}), \qquad \begin{cases}\beta \neq 1\\ \beta\neq\alpha\end{cases} \tag{3}$$

References

1. J. C. E. Dekker, The constructivity of maximal dual ideals in certain Boolean algebras, Pacific J. Math., 3 (1953) 73–101.
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3. L. Gillman and M. Jerison, Rings of continuous functions, Van Nostrand, Princeton, 1960.
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6. ———, Effectiveness of the representation theory for Boolean algebras, Fund. Math., 41 (1955) 49–56.
7. W. Sierpiński, Fonctions additives non complètement additives et fonctions non mesurables, Fund. Math., 30 (1938) 96–99.
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$$\sum_{l_n\leqq x}\frac{1}{l_n^\beta} = \frac{x^{1-\beta}}{1-\beta} + \gamma_\beta + O(x^{\alpha-\beta}), \qquad \begin{cases}\beta \neq 1\\ \beta\neq\alpha\end{cases} \tag{3}$$

and γ_β is a constant.

$$\sum_{l_n \leqq x} \frac{1}{l_n} = \log x + \gamma_1 + O(x^{\alpha-1}). \tag{4}$$

$$\sum_{l_n > x} \frac{1}{l_n^\beta} = \zeta(\beta) - \sum_{l_n \leqq x} \frac{1}{l_n^\beta} = O(x^{1-\beta}), \qquad \beta > 1. \tag{5}$$

$$\sum_{l_n > x} \frac{\log l_n}{l_n^2} = O\left(\frac{\log x}{x}\right). \tag{6}$$

Define $T^*(x) = \sum_{l_n \leqq x} \tau^*(l_n)$. Then

THEOREM 1.

$$T^*(x) = \frac{x}{\zeta(2)}\left\{\log x - \frac{2\zeta'(2)}{\zeta(2)} + 2\gamma_1 - 1\right\} + O(x^{(1+\alpha)/2}).$$

Proof. Substitution of (1) in (2) gives

$$\begin{aligned} \phi(x, l_n) &= \sum_{d \mid l_n} \mu(d)\left(\left(\frac{x}{d}\right) + O\left(\frac{x}{d}\right)^\alpha\right) \\ &= x \sum_{d \mid l_n} \frac{\mu(d)}{d} + O\left(x^\alpha \sum_{d \mid l_n} \frac{\mu(d)}{d^\alpha}\right) \\ &= xf(l_n) + O(x^\alpha f_\alpha(l_n)) \text{ say.} \end{aligned} \tag{7}$$

Also

$$\sum_{l_n \leqq x} \frac{1}{l_n^r} \sum_{d \mid l_n} \frac{\mu(d)}{d^s} = \sum_{d\delta \leqq x} \frac{\mu(d)}{d^{r+s}\delta^r} = \sum_{d \leqq x} \frac{\mu(d)}{d^{r+s}} \sum_{\delta \leqq x/d} \frac{1}{\delta^r}. \tag{8}$$

Now

$$T^*(x) = \sum_{l_n \leqq x} \tau^*(l_n) = \sum_{l_n \leqq x} \sum_{\substack{d\delta = l_n \\ (d,\delta)=1}} 1 = \sum_{\substack{d\delta \leqq x \\ (d,\delta)=1}} 1.$$

But if $d\delta \leqq x$, then d and δ cannot simultaneously assume values $> \sqrt{x}$. Hence

$$\begin{aligned} T^*(x) &= 2 \sum_{\substack{d\delta \leqq x, d \leqq \sqrt{x} \\ (d,\delta)=1}} 1 - \sum_{\substack{d \leqq \sqrt{x}, \delta \leqq \sqrt{x} \\ (d,\delta)=1}} 1 \\ &= 2 \sum_{d \leqq \sqrt{x}} \phi\left(\frac{x}{d}, d\right) - \sum_{d \leqq \sqrt{x}} \phi(\sqrt{x}, d) \\ &= 2 \sum_{d \leqq \sqrt{x}} \left\{\frac{x}{d} f(d) + O\left(\frac{x^\alpha}{d^\alpha} f_\alpha(d)\right)\right\} - \sum_{d \leqq \sqrt{x}} \left\{\sqrt{x} f(d) + O(x^{\alpha/2} f_\alpha(d))\right\} \end{aligned}$$

from (7)

$$= 2x \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^2} \sum_{\delta \leq \sqrt{x}/d} \frac{1}{\delta} - x^{1/2} \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d} \sum_{\delta \leq \sqrt{x}/d} 1$$

$$+ O\left(x^{\alpha} \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^{2\alpha}} \sum_{\delta \leq \sqrt{x}/d} \frac{1}{\delta^{\alpha}}\right) + O\left(x^{\alpha/2} \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^{\alpha}} \sum_{\delta \leq \sqrt{x}/d} 1\right)$$

from (8)

$$= 2x \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^2}\left(\log \frac{\sqrt{x}}{d} + \gamma_1 + O\left(\frac{\sqrt{x}}{d}\right)^{\alpha-1}\right)$$

$$- x^{1/2} \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d}\left(\frac{\sqrt{x}}{d} + O\left(\frac{\sqrt{x}}{d}\right)^{\alpha}\right)$$

$$+ O\left(x^{\alpha} \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^{2\alpha}}\left(\frac{\sqrt{x}}{d}\right)^{1-\alpha}\right) + O\left(x^{\alpha/2} \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^{\alpha}} \frac{\sqrt{x}}{d}\right)$$

from (4), (1) and (3)

$$= x(\log x + 2\gamma_1 - 1) \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^2} - 2x \sum_{d \leq \sqrt{x}} \frac{\mu(d) \log d}{d^2} + O\left(x^{1+\alpha/2} \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^{1+\alpha}}\right)$$

$$= x(\log x + 2\gamma_1 - 1)\left(\frac{1}{\zeta(2)} + O\left(\frac{1}{\sqrt{x}}\right)\right) - 2x\left(\frac{\zeta'(2)}{(\zeta(2))^2} + O\left(\frac{\log x}{\sqrt{x}}\right)\right)$$

$$+ O(x^{(1+\alpha)/2})$$

from (5), (6) and (3) (here $\zeta'(s)$ is the derivative of $\zeta(s)$)

$$= \frac{x}{\zeta(2)}\left\{\log x - \frac{2\zeta'(2)}{\zeta(2)} + 2\gamma_1 - 1\right\} + O(x^{(1+\alpha)/2}).$$

This completes the proof of Theorem 1. It will be noted that the form of the answer is very similar to that given in [1] for the average order of the number of unitary divisors of an integer.

References

1. Eckford Cohen, The number of unitary divisors of an integer, this MONTHLY, 67 (1960) 879–880.

2. T. Nagell, Introduction to number theory, Uppsala, 1951.

3. E. M. Horadam, The Euler-ϕ function for generalised integers. Proc. Amer. Math. Soc., to be published.

h is a homotopy of $f|C$ into T. By hypothesis, h can be extended to a homotopy $h^*: X\times I\to T$ with $h^*(x, 0)=f(x)$. We show first that *the existence of this extension implies that C is a G_δ.*

Let $\pi: X\times I\to X$ be the projection $\pi(x, t)=x$. For each positive integer m, let $X_m=\pi\{[X\times[0, 1/m)]\cap h^{*-1}[C\times(0, 1]]\}$. The two intersecting sets are each open in $X\times I$, and π is interior, so that X_m is always open. We show that $\cap X_m=C$. Suppose that x is in $X-C$. By continuity and the definition of f, there is an integer n such that $h^*(x\times[0, 1/n]\cap C\times I)=\varnothing$, since $h^*(x, 0)=x$. It follows that x is not in X_n. Thus C contains $\cap X_m$. If c is in C, then $h^{*-1}(c\times(0, 1])=c\times(0, 1)$, and $\{c\times[0, 1/m)\}\cap c\times(0, 1]=c\times(0, 1/m)$, which projects onto c. Therefore C is in $\cap X_m$. This proves that c is a G_δ.

It is easy to see that *C is a neighborhood deformation retract in our weak sense.* Let U be the set of all points x in X such that $h^*(x, 1)$ is in $C\times(0, 1]$. Then U is open and contains c. The composition $(\pi|T)(h^*|U\times I): U\times I\to X$ is continuous, leaves each point of C fixed at each t, and maps $U\times 1$ into C. This completes the proof.

I will remark that the two conditions, of being a G_δ and of being a neighborhood deformation retract, are independent. Since every closed set in a metric space is a G_δ, it is clear that this does not imply the deformation property. To see the other, let W denote the compact ordered space consisting of all ordinals not greater than the first uncountable ordinal, which I will call w. The space $W\times I$ can be deformed onto $W\times 1$ in the natural way. Let W' be the decomposition space of $W\times I$ formed by collapsing $W\times 1\cup w\times I$ to a point p. The natural map of $W\times I$ onto $W\times 1$ induces a contraction of W' onto p. If p were a G_δ, then w would be a G_δ in W. But it is not; see [2], p. 55.

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References

1. S. T. Hu, Homotopy theory, Academic Press, New York, 1959.
2. J. G. Hocking and G. S. Young, Topology, Addison-Wesley, Reading, Mass., 1961.

RELATION OF ZEROS TO PERIODS IN THE FIBONACCI SEQUENCE MODULO A PRIME

BROTHER U. ALFRED, St. Mary's College, Calif.

In an article in this MONTHLY [1], D. D. Wall has worked out the main theorems regarding the zeros and periods of the Fibonacci sequence modulo m. The purpose of this note is to show the relationship between zero placement and periods for primes in the special Fibonacci sequence: $u_0=0$, $u_1=1$, $u_2=1$, $u_3=2$, $u_4=3$, $u_5=5, \cdots$.

The notation in the article by Wall will be kept. $k=k(p)$ is the period of $u_n \pmod p$, where p is a prime. $d=d(p)$ is the spacing of the zeros in $u_n \pmod p$.

It has been shown in [1] that $d|k$; that for $p=10x\pm 1$, $k(p)|p-1$; and that for $p=10x\pm 3$, $k(p)|2p+2$.

Case B: $k=2^2(2\lambda+1)$. From the fact that $u_{k/2}\equiv 0 \pmod p$ and $u_{(k/2)-1}\equiv -1 \pmod p$ it follows that $u_{(k/2)-t}\equiv(-1)^t u_t \pmod p$. Setting $t=k/4$, $u_{k/4}\equiv -u_{k/4} \pmod p$ so that $u_{k/4}\equiv 0 \pmod p$. Thus in this case $k=4d$.

Case C: $k=2^m(2\lambda+1)$, $m\geqq 3$. If $u_{k/4}\equiv 0 \pmod p$, then by (1), $u_{(k/4)+t}\equiv u_{(k/4)+1}u_t \pmod p$. Setting $t=(k/4)+1$, we get

$$u_{(k/2)+1} \equiv u^2_{(k/4)+1} \pmod p \equiv (-1)^{k/4} \pmod p \text{ by (2)} \equiv 1 \pmod p,$$

since $k/4$ is even. Thus $u_{(k/2)+1}$ would simultaneously be congruent to $+1$ and -1, a contradiction. Hence $u_{k/4}\not\equiv 0 \pmod p$ and $k=2d$.

Summary. The results of the above discussion may be summarized in the following statements which apply to all primes including 2.

(A) If the period $k(p)$ is of the form $2\lambda+1$ or $2(2\lambda+1)$, then $d=k$.
(B) If the period $k(p)$ is of the form $2^2(2\lambda+1)$, then $k=4d$.
(C) If the period $k(p)$ is of the form $2^m(2\lambda+1)$, $m\geqq 3$, then $k=2d$.

For a prime modulus other than 2, the above statements can be reversed to read:

(A) If d is of the form $2(2\lambda+1)$, then $k=d$.
(B) If d is of the form $2\lambda+1$, then $k=4d$.
(C) If d is of the form $2^m(2\lambda+1)$, $m\geqq 2$, then $k=2d$.

The author wishes to thank the referee for various suggestions.

Reference

1. D. D. Wall, Fibonacci series modulo m, this MONTHLY, 67 (1960) 529.

CONGRUENCES FOR THE COEFFICIENTS OF THE k-th POWER OF A POWER SERIES

JOSEPH ARKIN, Spring Valley, New York 10977

Put

$$\left(\sum_{n=0}^{\infty} C_n x^n\right)^k = \sum_{n=0}^{\infty} C_n^{(k)} x^n, \tag{1}$$

where the C_n are integers and k is an integer $\geqq 1$. We define $C_n^{(k)}$ by means of (1) for all integral k. We have the following:

THEOREM. *If* $k\geqq 1$ *then*

$$C_n^{(k)} \equiv 0 \pmod{k/(n,k)} \qquad (n=1,2,3,\cdots) \tag{2}$$

holds for all integral k.

Proof. Differentiation of (1) leads to

$$k\sum_{r=0}^{\infty} rC_r x^r \sum_{n=0}^{\infty} C_n^{(k-1)} x^n = \sum_{n=0}^{\infty} nC_n^{(k)} x^n.$$

Comparing coefficients we get $nC_n^{(k)}=k\sum_{r=0}^{n} rC_rC_{n-r}^{(k-1)}$, and (2) follows immediately.

When k is a prime, we have the familiar congruence

$$\left(\sum_{n=0}^{\infty} C_nx^n\right)^k \equiv \sum_{n=0}^{\infty} C_{nk}x^{nk} \pmod{k},$$

which is evidently in agreement with (2). More generally, when $k=p^r$, where p is a prime and $r\geqq 1$, it follows from (2) that, when $p^s|n$ and $s\leqq r$,

$$C_n^{(k)} \equiv 0 \pmod{p^{r-s}}. \tag{3}$$

COROLLARY. *If* $C_0 = 1$ *and we put* $(\sum_{n=0}^{\infty} C_nx^n)^{-k} = \sum_{n=0}^{\infty} C_n^{(-k)}x^n$, $(k=1, 2, 3, \cdots)$ *then, exactly as above,* $nC_n^{(-k)}=-k\sum_{r=0}^{n} rC_rC_{n-r}^{(-k-1)}$.

It follows, as desired, that

$$C_n^{(-k)} \equiv 0 \pmod{k/(n, k)} \text{ and } (5)\quad C_n^{(-k)} \equiv 0 \pmod{p^{r-s}}. \tag{4}$$

The Ramanujan τ-function is defined by $x\prod_{n=1}^{\infty}(1-x^n)^{24}=\sum_{n=1}^{\infty}\tau(n)x^n$. Applying (2) we get

$$\tau(n+1) \equiv 0 \pmod{24/(n, 24)}. \tag{6}$$

In particular we have

$$\tau(3n) \equiv \tau(3n+2) \equiv 0 \pmod{3}, \tag{7}$$

a congruence due to Lahiri [2]; see also Gandhi [1]. It also follows from (6) that

$$\tau(2n) \equiv 0 \pmod{8}. \tag{8}$$

References

1. J. M. Gandhi, Congruences for $p_r(n)$ and Ramanujan's τ function, this MONTHLY, 70 (1963) 265–274.

2. D. B. Lahiri, On Ramanujan's function $\tau(n)$ and divisor function $\sigma_k(n)$, Bull. Calcutta Math. Soc., 38 (1946) 193–206; 39 (1947) 33–51.

A CORRECTION FOR "CLASSES OF PAIRS OF COMMUTING MATRICES OVER A FINITE FIELD"

L. CARLITZ, Duke University

J. Towber has kindly pointed out to the writer that there is an error in the paper: *Classes of pairs of commuting matrices over a finite field* (this MONTHLY, 70 (1963) 193–195). The error occurs in equation (5) of the paper. The results of the paper remain valid if we redefine $Q(n)$ as equal to the number of pairs of $n\times n$ matrices (A_i, B_i) with elements in $GF(q)$, where A_i runs through a complete set of nonsimilar matrices and for each A_i, B_i commutes with A_i.

Thus the problem of determining the number of classes of pairs of commuting matrices remains open.

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Thus the problem of determining the number of classes of pairs of commuting matrices remains open.

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland 20740.

A COMBINATORIAL PROOF OF THE EXISTENCE OF GALOIS FIELDS

R. C. MULLIN, University of Waterloo, Canada

In a first course in combinatorial mathematics one often desires to show the existence of finite fields of all orders p^n where p is a prime and n a positive integer. One may easily show that the ring of integers modulo p is a field (usually denoted $GF(p)$) but one must show that there exists an irreducible polynomial of degree n over $GF(p)$ in order to construct the nth degree extension $GF(p^n)$. The classical proof of this fact is algebraic in nature and requires a lengthy development of the structure of polynomials over $GF(p)$. The following proof, which I believe is new, is combinatorial in nature and somewhat shorter than the classical proof. For those familiar with Polya's theorems [1] the proof can be made somewhat shorter. By considering the product $(1+x+x^2+\cdots)^n$ or some other elementary method one notes that

$$(1) \qquad (1-x)^{-n} = \sum_{r=0}^{\infty} C(n, r)x^r,$$

where $C(n, r)$ is the number of combinations of n objects taken r at a time, repetitions allowed. Having made this observation we prove the

THEOREM. *The number of irreducible monic polynomials of degree n over any finite field K is strictly positive for any positive integer n.*

Let us assume that K contains s elements. Let f_n represent the number of irreducible monic polynomials of degree n over K. Clearly $f_n \leqq s^n$, the number of monic polynomials of degree n.

Since any polynomial over a field can be factored uniquely into a set of monic irreducible factors, and since there are s^n distinct monic polynomials of degree n,

$$(2) \qquad s^n = \sum_{(\alpha)} \prod_i C(f_i, \alpha_i),$$

where the sum is taken over all nonnegative integer solutions of $\alpha_1+2\alpha_2+\cdots+k\alpha_k=n$. Also

$$(3) \qquad \prod_{n=1}^{\infty} [(1-x^n)f_n]^{-1} = \sum_{n=0}^{\infty}\left(\sum_{(\alpha)}\prod_i C(f_i, \alpha_i)\right)x^n = \sum_{n=0}^{\infty} s^n x^n = (1-sx)^{-1};$$

hence, taking logarithms

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{s^nx^n}{n} &= \ln(1-sx) = \sum_{n=1}^{\infty} f_n \ln(1-x^n)\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_n \frac{x^{mn}}{m} = \sum_{n=1}^{\infty}\sum_{d\mid n} df_d \frac{x^n}{n}.\end{aligned}\tag{4}$$

Equating coefficients yields

$$s^n = \sum_{d\mid n} df_d \quad \text{or} \tag{5}$$

$$nf_n = s^n - \sum_{\substack{d\mid n\\ d<n}} df_d. \tag{6}$$

Since f_d enumerates polynomials, $f_d \geqq 0$, and hence by (6) $f_n \leqq s^n/n$. Therefore

$$\begin{aligned} nf_n &\leqq s^n - \sum_{k=1}^{n-1} kf_k \geqq s^n - \sum_{k=1}^{n-1} s^k\\ &= s^n - \frac{s^{n-1}}{s-1} = \frac{1}{s-1} + s^n\left(1 - \frac{1}{s-1}\right)\end{aligned}\tag{7}$$

which is positive for $s > 1$.

Since $f_k \leqq s^k$ all infinite series and products are absolutely convergent for $|x| < |s^{-1}|$. This completes the proof.

Relation (5) may be used to verify that f_n is a polynomial $f_n(s)$ of degree n in s. The identity

$$\ln(1-sx) = \sum_{n=1}^{\infty} f_n(s)\ln(1-x^n) \tag{4(a)}$$

may be used to show an interesting property of the polynomials $f_n(s)$. Indeed

$$\ln(1-x^m) = \sum_{l=1}^{m}\ln(1-e^{2i\pi l/m}x) = \sum_{n=1}^{\infty}\ln(1-x^n)\sum_{l=1}^{m} f_n(e^{2i\pi l/m}),$$

hence

$$\sum_{l=1}^{m} f_n(e^{2i\pi l/m}) = \delta_{n,m}, \tag{8}$$

where $\delta_{n,m}$ is the familiar Kronecker delta.

Reference

1. G. Polya, Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen, Acta Math., 68 (1937) 145, 253.

$$\Pr\,(T_m < p) = \int_0^p g(t)dt = I_p(m, n - m + 1).$$

We conclude with two additional observations. The density function $g(p)$ can be obtained directly by a somewhat nonrigorous though often helpful argument. If $g(p)$ is the density function of T_m, the probability that T_m takes a value between p and $p+dp$ is equal to $g(p)dp$ (ignoring terms of order smaller than dp). On the other hand, according to the multinomial theorem, this probability (again ignoring terms of order smaller than dp) is equal to

$$(5) \qquad \frac{n!}{(m-1)!(n-m)!1!}\,p^{m-1}(1-p)^{n-m}dp$$

since we must have $m-1$ observations smaller than p, $n-m$ observations greater than $p+dp$, and one observation between p and $p+dp$. It is seen that (5) is simply the right side of (4) multiplied by dp.

In our earlier discussion we assumed for reasons of simplicity that we were sampling from a uniform distribution on the interval (0, 1). Actually we could have considered samples from any continuous distribution $F(x)$. Let P_p satisfy $F(P_p)=p$ so that P_p is a $100p$-percentile of $F(x)$. Let T_m again be the mth smallest observation in a random sample of size n from $F(x)$. Then $\Pr(T_m<P_p) = B(m; n, p)$ since as before the probability that a single observation from $F(x)$ is smaller than P_p is given by $F(P_p)=p$. This result illustrates a property of order statistics that makes them useful in nonparametric statistics. Probability statements like $\Pr(T_m<P_p)$ do not depend on the parent distribution $F(x)$ beyond the assumption that $F(x)$ is continuous.

A MODIFIED DIFFERENTIATION

MARTINUS ESSER, University of Dayton and Aerospace Research Laboratories, Wright-Patterson AFB, Ohio

AND

O. SHISHA, Aerospace Research Laboratories, Wright-Patterson AFB, Ohio

1. We consider a modified form of the concept of derivative ("strong derivative"). Some results are given, particularly continuity properties and necessary and sufficient conditions for "strong differentiability."

2. We assume throughout this section that f is a real function whose domain D is an open subset of the real line.

DEFINITION. *f is strongly differentiable at a point a iff the limit*

$$\lim_{\substack{(x_1,x_2)\to(a,a)\\ x_1\neq x_2}} \{f(x_2)-f(x_1)\}/(x_2-x_1)$$

exists and is finite. If these conditions hold, we denote the limit by $f^*(a)$ and call it the strong derivative of f at a.

$$\Pr(T_m < p) = \int_0^p g(t)dt = I_p(m, n - m + 1).$$

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$$\frac{n!}{(m-1)!(n-m)!1!}p^{m-1}(1-p)^{n-m}dp \tag{5}$$

since we must have $m-1$ observations smaller than p, $n-m$ observations greater than $p+dp$, and one observation between p and $p+dp$. It is seen that (5) is simply the right side of (4) multiplied by dp.

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$$\left|\frac{f(x_2)-f(x_1)}{x_2-x_1}-L\right|=\left|(x_2-x_1)^{-1}\int_{x_1}^{x_2}\{f'(t)-L\}\,dt\right|\leqq\epsilon.$$

Therefore f is strongly differentiable at a.

3. We conclude with two examples.

Example 1. Let $f(x)$, with domain $(-\infty,\infty)$, be defined as $x^2\sin(1/x)$, except that $f(0)=0$ (and so $f'(0)=0$). Then f is absolutely continuous in, say, $[-1,1]$ (for f' is bounded there). On the other hand, $\lim_{x\to 0}f'(x)$ does not exist, and therefore, by Theorem 3, f is not strongly differentiable at 0.

Example 2. This example gives a function which is strongly differentiable at 0, but which is differentiable at no one of the points $1/2, 1/3, 1/4, \cdots$. Define a function g with domain $(-1, 1)$ as follows: $g(t)=0$ if $t\in(-1, 0]$ and if $t\in[1/(n+1), 1/n)$ $(n=2, 4, 6, \cdots)$, and $g(t)=t$ if $t\in[1/(n+1), 1/n)$ $(n=1, 3, 5, \cdots)$. Let $f(x)$ (with domain $(-1, 1)$) be $\int_0^x g(t)dt$ (and so f is continuous in $(-1, 1)$). Obviously

$$\lim_{\substack{(x_1,x_2)\to(0,0)\\ x_1\neq x_2}}\{f(x_2)-f(x_1)\}/(x_2-x_1)=0.$$

However, at every point $1/n$ $(n=2, 3, 4, \cdots)$ f is not differentiable.

FIXED POINTS

IGNACE I. KOLODNER, University of New Mexico

In order to solve a functional equation of the second kind, one frequently proceeds to solve the iterated equation, and concludes that if the latter has a solution which is unique, the former also has a solution. In one case (in [1]), I noticed that such a process requires some effort. The result is, however, a consequence of a very general theorem proven below.

Let X be any set and let f be a mapping of X into X. Let $S_n=\{x\mid x=f^n(x)\}$, the set of fixed points of f^n. Clearly $S_1\subset S_n$ for all n, but S_1 could be empty.

THEOREM. *If for some k, $S_k=\{z\}$, a singleton, then $S_1=\{z\}$.*

Proof. $f^k(f(z))=f^{k+1}(z)=f(f^k(z))=f(z)$. Thus $f(z)\in S_k$, i.e., $f(z)=z$, and $z\in S_1$.

Reference

1. R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. II, Interscience Publ., New York, 1962, pp. 364–365.

CORRECTION FOR "INDEFINITE INTEGRATION BY RESIDUES"

R. P. BOAS, JR., Northwestern University

L. Schoenfeld has pointed out to me that the theorem of this note [this MONTHLY, 71 (1964) page 298] is correct only when the integrand has no poles at all on the unit circumference. He and I are preparing a note dealing with the general case.

$$\left|\frac{f(x_2)-f(x_1)}{x_2-x_1}-L\right| = \left|(x_2-x_1)^{-1}\int_{x_1}^{x_2}\{f'(t)-L\}\,dt\right| \leqq \epsilon.$$

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$$\lim_{\substack{(x_1,x_2)\to(0,0)\\ x_1\neq x_2}} \{f(x_2)-f(x_1)\}/(x_2-x_1) = 0.$$

However, at every point $1/n$ $(n=2, 3, 4, \cdots)$ f is not differentiable.

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MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

A HIGHER REMEDIAL PROGRAM

ROBERT T. STUBBS, Armstrong College, Savannah, Georgia

A somewhat unusual remedial program in algebra was given at Armstrong College to about 200 students during the academic year 1962–63.

Each of the students met the normal admissions requirements, and was placed in a remedial algebra category by placement examination. (The placement criterion used was the American Council on Education Cooperative Mathematics Pre-test, Form Y (1948). Scores less than or equal to 20 were considered remedial.) The object of the new program was to replace a very routine course in remedial algebra by a more valuable course. It was expected that the incorporation of appropriate topics which were new to the student would enhance his interest, and prepare him for a more rigorous treatment of the elementary algebraic concepts. The main purpose of this note is to describe the content of the course, and to give some analysis of the success of the program.

1. Content. The course can be divided into three parts: symbolic logic, set theory, and elementary algebraic concepts; an approximately equal amount of class time was devoted to each part. There were five class meetings per week for one quarter in each section of the course which was offered.

The topical content of the course is listed below.

I. *Symbolic Logic.* Axiomatic systems, mathematical truth, five logical operators, properties of the logical operators, apparent and actual sub-propositions, mathematical proof, propositional functions, and quantification.

II. *Set Theory.* Primitive notions, set equality, subsets, the axiom of specification, set theoretic operators, algebra of sets, complementation, generalized operators, set theory and symbolic logic compared, ordered pairs, Cartesian product, relations, the function concept, and graphical considerations.

III. *Elementary Algebraic Concepts.*

A. THE ELEMENTARY OPERATIONS. Natural numbers, composite and prime numbers, properties of $+$ and $\times$, subtraction, the order relation, division and the set of rational numbers, absolute value, operations with real numbers, and operations with fractions.

B. EXPONENTS. Integral exponents, principal nth root, properties of the absolute value of a real number, properties of real exponents.

C. ALGEBRAIC EXPRESSIONS. Polynomials, operations with polynomials, factoring, operations with rational and algebraic expressions.

D. EQUATIONS AND INEQUALITIES. Linear equations, quadratic equations, graphical considerations, inequalities.

2. Discussion of Content.

I. *Symbolic Logic.* The approach in symbolic logic was naive, but the presentation was careful. The treatment of the implication $p \Rightarrow q$, where p and q denote propositions, was thorough. Some typical results which were established will be listed. For all propositions p, q, r it was shown that:

$$[\text{not-}(p \Rightarrow q)] \Leftrightarrow [p \wedge \text{not-}q],$$
$$[p \Rightarrow q] \Leftrightarrow [\text{not-}q \Rightarrow \text{not-}p],$$
$$[p \Rightarrow q \wedge r] \Leftrightarrow [(p \Rightarrow q) \wedge (p \Rightarrow r)], \text{ etc.}$$

The above results were established by the truth table method, and by the application of appropriate properties of the logical operators. The material on mathematical proof included a treatment of the method of indirect proof with a proof of

$$(p \Rightarrow q) \Leftrightarrow (p \wedge \text{not-}q \Rightarrow r \wedge \text{not-}r).$$

II. *Set Theory.* While the approach was naive, a careful development was given. The axiom of specification was stressed, and a discussion of the paradoxes was not given. The standard results of the algebra of sets were presented. Emphasis was placed upon the relation between set theory and logic. The treatment of the concept of ordered pair included a proof that

$$(a, b) = (c, d) \Leftrightarrow a = c \wedge b = d,$$

where (a, b) denotes the ordered pair whose elements are a and b. The function concept was presented carefully, following the necessary preliminaries on Cartesian product and relations; the latter included equivalence relations, but did not include equivalence classes and partitioning. Into, onto, one-to-one, and inverse functions were discussed. Graphical results were obtained by interpreting an ordered pair of real numbers as a point in Euclidean 2-space.

III. *Elementary Algebraic Concepts.* The role of $+$ and $\times$ as primitive operations was discussed. The presentation of the algebraic part of the course was influenced by a desire to prepare the student for an axiomatic development of the structure of the set of real numbers from the field axioms. The development given was naive; the basic field properties were identified, however, and some proofs were given. Moreover, attention was given to the computational aspects.

In the material on exponents the standard properties were treated. Special attention was given to the concept of the principal nth root of a real number. It would have been difficult for a student to have avoided learning that for all real a it is true that $\sqrt{a^2} = |a|$. The treatment of the remaining topics was standard.

3. Evaluation of the Program. It is felt that the program was very successful; it will be continued in the coming academic year. The text for the course

consisted of material prepared by the author; a revised version of this material was used in 1963–64.

In addition to giving the student the needed review of the elementary algebraic concepts, the program gave him some insight into several areas which were new to him. In the study of symbolic logic students were not particularly hindered by their poor background in mathematics. Furthermore, for many students a good performance in the first part on symbolic logic seemed to set the pace for their work in the balance of the course.

Set theory served as another sort of self-contained study. While some results from logic were applied in the study of set theory, it was not necessary for the student to have any prior detailed knowledge about sets. Moreover, the set theory made it possible to treat the concept of function in a rather sophisticated way.

The students who were successful in the course achieved an insight into some theoretical aspects of the algebra of real numbers, and they also achieved some proficiency with the manipulative aspects. In comparing the new program with the old it was found that there was no significant difference in the grade distributions of the progams. A student's performance in higher courses was significantly better if he came from the new program.

NEW MATHEMATICS TEACHER TRAINING PROGRAMS AT CLEMSON COLLEGE

Recently Clemson College has planned and had approved by the faculty a new mathematics education program at the undergraduate and master's degree level. At the undergraduate level a program leading to a new degree, bachelor of science in science teaching with majors in biology, chemistry, mathematics and physics, is planned at Clemson College which has no College of Education. Many of the students who have selected the teacher training program in mathematics had begun other curricula, primarily engineering. The requirements in mathematics for the major considerably exceed those required by the state and are comparable to the Level III program suggested by the Committee on the Undergraduate Program. In addition, eight hours of general physics are required as well as six hours of the history of civilization. Experience has shown that prospective teachers who take history of civilization do better on the national teachers examination which is used in South Carolina as a basis for determining salary.

The master's degree program is offered under a Master of Education curriculum and includes thirty semester hours, no more than twelve of which are in the field of professional education. Requirements in mathematics are somewhat similar to those of Level III, but also more extensive. It is hoped that both programs will be in full operation in the fall of 1964. This report was submitted by Professor John L. Tilley of Clemson, and a part of the report was given as a paper at the Southeastern section meeting of the Association in Charleston on March 20.

content development in mathematics are already receiving support under this division. Persons interested in submitting proposals should correspond directly with Dr. Ianni.

PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (other than proposers') should be sent to Professor Starke.

ELEMENTARY PROBLEMS

All solutions of Elementary Problems should be sent to A. E. Livingston, Dept. of Math., University of Alberta, Edmonton, Alberta, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before January 31, 1965.

E 1721. *Proposed by J. C. Van Rhijn, Vollenhove, The Netherlands*

Given an ellipse E with foci F_1 and F_2, a point P outside E, the tangents PR_1 and PR_2 from P to E, and a positive number $f(0<f<1)$. Find the locus of P if $PR_1 \cdot PR_2 = f \cdot PF_1 \cdot PF_2$.

E 1722. *Proposed by Norman Schaumberger and Erwin Just, Bronx Community College, New York City*

It is known that $\sum_{j=0}^{n} \cos j\theta$ does not approach a limit as $n \to \infty$ for any θ. Show that as $n \to \infty$, lim $\sum_{j=0}^{n} \binom{n}{j} \cos j\theta$ does not exist or is zero.

E 1723. *Proposed by Franklin C. Smith, Minneapolis, Minn.*

If k is a positive integer and $\theta = 2\pi j/k$, $j = 1, 2, \cdots, k-1$, show that

$$\sum_{n=1}^{[(k-1)/2]} \cos n\theta = -\tfrac{1}{4}\{2 + (-1)^{k+j} + (-1)^{j}\},$$

where $[x]$ denotes the greatest integer not exceeding x.

E 1724. *Proposed by H. W. Guggenheimer, University of Minnesota*

If S is the area of triangle ABC, show that

$$abc \geqq 8S^{3/2}/3^{3/4},$$

with equality for the equilateral triangle.

E 1725. *Proposed by Hwa S. Hahn, Pennsylvania State University*

If $\phi(x)$ is Euler's function and n is a positive integer with the prime decomposition $n=\prod_{p|n} p^h$, prove that

$$\sum_{d|n} d\phi(d) = \prod_{p|n} (p^{2h+1}+1)/(p+1).$$

E 1726. *Proposed by Daniel I. A. Cohen, Princeton University*

Prove (or disprove): every planar map of convex countries can be colored with three colors.

E 1727. *Proposed by Raul Machuca, Fairleigh Dickinson University, Teaneck, N. J.*

If the a_i are real and distinct show that the following equation is true for n real values of x:

$$\frac{a_1}{a_1-x}+\frac{a_2}{a_2-x}+\cdots+\frac{a_n}{a_n-x}=n.$$

E 1728. *Proposed by Jet Wimp, Midwest Research Institute*

For $t\geqq 0$, $0<a\leqq 1$, let $f(t)$ be defined by

$$t=\frac{1}{\sqrt{2a}}\int_1^{f(t)}\frac{dx}{(\ln x-\ln a)^{1/2}}\,.$$

Show that

$$t=\frac{1}{a}\int_0^{f'(t)} e^{x^2/2a}dx.$$

E 1729. *Proposed by Azriel Rosenfeld, The Budd Company and the University of Maryland*

A 3×3 *arithmagic* square is an arrangement of nine of the digits $0, 1, \cdots, 9$ into a 3×3 matrix such that if the rows and columns are regarded as three-digit numbers, then the sum of the first two rows is equal to the third row, and similarly for the columns. Prove there is only one such arithmagic square.

E 1730. *Proposed by Robert Spira, Duke University*

Let $(a, b)=1$. If n is any integer, there is an x such that $(ax+b, n)=1$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Sum of Consecutive Odd Numbers

E 1641 [1963, 1099]. *Proposed by W. C. Waterhouse, Harvard University*

Prove that for any integer $k>1$ and any positive integer n, n^k is the sum of n consecutive odd integers.

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Sum of Consecutive Odd Numbers

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A Characterization of Linear Polynomials

E 1642 [1963, 1099]. *Proposed by Ralph Greenberg, University of Pennsylvania*

Prove that a polynomial $P(x)$ with rational coefficients which assumes rational values for rational x and irrational values for irrational x must be linear.

1. *Solution by S. Chowla, Pennsylvania State University.* We may assume that the polynomial $P(x)$ is monic. If now $P(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, then for any prime p, the polynomial $Q(x)=P(x)-p-a_0$ also has the stated property. Since $Q(0)<0$, there is a real root of $Q(x)=0$. Now the only possible rational roots of $Q(x)=0$ are ± 1, $\pm p$. But if $n>1$, we can choose the prime p so large that none of the numbers ± 1, $\pm p$ satisfies $Q(x)=0$. Thus we must have $Q(x)=0$ for some irrational x, which contradicts the fact that $Q(x)$ is irrational for irrational x.

II. *Solution by A. M. Vaidya, Pennsylvania State University.* Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0(a_n\neq 0)$. We assume, without loss of generality, that the coefficients are integers. Then $a_n^{n-1}P(x)=a_n^nx^n+a_{n-1}a_n^{n-1}x^{n-1}+\cdots+a_0a_n^{n-1}=Q(a_nx)$, say. Since $Q(x)=a_n^{n-1}f(x/a_n)$, $Q(x)$ also has the stated property. If now, $n\geqq 2$, then we can find an integer k such that $Q(k+1)-Q(k)>1$, and hence there is an integer h such that $Q(k)<h<Q(k+1)$. It now follows from the continuity of $Q(x)$ that there is an m such that $k<m<k+1$ and $Q(m)=h$; this real number m is rational because $Q(m)$ is, and it is an integer because $Q(x)$ is monic. But this contradicts $k<m<k+1$. This proves the assertion.

Also solved by Shair Admad, W. R. Becker, J. L. Brenner, Leonard Carlitz, M. J. Cohen, H. L. Montgomery, A. E. Newman, Hugh Noland, Ulrich Seip, and the proposer.

Brenner, Carlitz, and the proposer point out the assumption that $P(x)$ has rational coefficients is redundant.

An Inequality

E 1643 [1963, 1099]. *Proposed by J. E. MacDonald, Jr., International Business Machines Corporation, Poughkeepsie, New York*

Given that $\sum_{i=1}^n b_i=b$ with each b_i a nonnegative integer, prove that $\sum_{j=1}^{n-1} b_jb_{j+1}\leqq b^2/4$.

I. *Solution by Lieutenant Rudolf Tabbe, Department for Mathematical Statistics, Chalmers University of Technology, Gothenberg, Sweden.* The following proof shows that the restriction to nonnegative integers is avoidable. In fact the inequality is valid for arbitrary nonnegative numbers $\{b_i\}_1^n$ with sum $\sum_{i=1}^n b_i=b$.

The function $f(x)=x(b-x)$, $x\in[0,b]$, satisfies $f(x)\leqq b^2/4$. We thus have

$$b^2/4\geqq\left(\sum_{i\text{ odd}} b_i\right)\left(\sum_{j\text{ even}} b_j\right)\geqq\sum_{j=1}^{n-1} b_jb_{j+1}.$$

II. *Solution by Martin J. Cohen, Beverly Hills, California.* Let $b_k=\max\{b_i\}_{i=1}^n$, so that

$$\begin{aligned}\sum_{i=1}^{n-1} b_ib_{i+1} &= \sum_{i=1}^{k-1} b_ib_{i+1} + \sum_{i=k}^{n-1} b_ib_{i+1}\\ &\leqq b_k \sum_{i=1}^{k-1} b_i + b_k \sum_{i=k}^{n-1} b_{i+1}\\ &= b_k(b - b_k)\\ &= \frac{b^2}{4} - \left(\frac{b}{2} - b_k\right)^2\\ &\leqq \frac{b^2}{4}.\end{aligned}$$

Note that there is equality if and only if $b_k = b/2$, in which case $n=2$. Thus for $n \geqq 3$, $\sum_{i=1}^{n-1} b_ib_{i+1} < b^2/4$.

III. *Solution by Miltiades S. Demos, Drexel Institute of Technology.* We have

$$\sum_{i=1}^{n} b_i^2 - \left\{\sum_{i=1}^{n} (-1)^i b_i\right\}^2 = 4\left\{\sum_{k=1}^{[n/2]} b_{2k}\right\}\left\{\sum_{m=1}^{[n/2]} b_{2m-1}\right\} \geqq 4\sum_{j=1}^{n-1} b_jb_{j+1}$$

since every term on the right is included in the product in the middle. Since

$$\left(\sum (-1)^i b_i\right)^2 \geqq 0, \qquad \text{we have } b^2 = \left(\sum_{i=1}^{n} b_i\right)^2 \geqq 4\sum_{j=1}^{n-1} b_jb_{j+1}.$$

Note that the statement is true if each b_i is any nonnegative real number.

Also solved by Shair Ahmad, K. F. Bailie, Raymond Balbes, J. E. Barger, R. J. Bridgman, Robert Burton, Leonard Carlitz, D. H. Carlson, D. I. A. Cohen, G. C. Dodds, Michael Goldberg, Mark Hayamizu, Stephen Hoffman, J. L. Howell, R. F. Jackson, R. A. Jacobson, Norman Schaumberger and Erwin Just (jointly), E. S. Langford, W. W. Leutert, Tung-Po Lin, Jiang Luh, E. L. Magnuson, D. C. B. Marsh, H. L. Montgomery, C. B. A. Peck, Stanton Philipp, L. J. Pratte, George Purdy, B. E. Rhoades, Dennis C. Russell, L. Sauvé, P. H. Schottler, Richard Sinkhorn, R. A. Smith and A. M. Vaidya (jointly), Guy Torchinelli, R. P. Soni, R. L. Syverson, Simon Vatriquant, Charles Wexler, K. S. Williams, K. L. Yocom, and the proposer.

A Well-Known Inequality

E 1644 [1963, 1099]. *Proposed by T. R. Curry, College on Long Island, State University of New York*

Prove that the sum of the sines of a triangle never exceeds $3\sqrt{3}/2$, with equality when and only when the triangle is equilateral.

I. *Solution by Andrew N. Aheart, West Virginia State College.* Let A, B, C be the angles of a triangle opposite the sides a, b, c. Let R = circumradius, r = inradius, and s = semi-perimeter. Then $2R = a/\sin A = b/\sin B = c/\sin C$. Hence,

$$\sin A + \sin B + \sin C = (a+b+c)/2R = s/R.$$

$$\begin{aligned}\sum_{i=1}^{n-1} b_ib_{i+1} &= \sum_{i=1}^{k-1} b_ib_{i+1} + \sum_{i=k}^{n-1} b_ib_{i+1}\\ &\leqq b_k \sum_{i=1}^{k-1} b_i + b_k \sum_{i=k}^{n-1} b_{i+1}\\ &= b_k(b - b_k)\\ &= \frac{b^2}{4} - \left(\frac{b}{2} - b_k\right)^2\\ &\leqq \frac{b^2}{4}.\end{aligned}$$

Note that there is equality if and only if $b_k = b/2$, in which case $n=2$. Thus for $n \geqq 3$, $\sum_{i=1}^{n-1} b_ib_{i+1} < b^2/4$.

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$$\sum_{i=1}^{n} b_i^2 - \left\{\sum_{i=1}^{n} (-1)^i b_i\right\}^2 = 4\left\{\sum_{k=1}^{[n/2]} b_{2k}\right\}\left\{\sum_{m=1}^{[n/2]} b_{2m-1}\right\} \geqq 4 \sum_{j=1}^{n-1} b_jb_{j+1}$$

since every term on the right is included in the product in the middle. Since

$$\left(\sum (-1)^i b_i\right)^2 \geqq 0, \qquad \text{we have } b^2 = \left(\sum_{i=1}^{n} b_i\right)^2 \geqq 4 \sum_{j=1}^{n-1} b_jb_{j+1}.$$

Note that the statement is true if each b_i is any nonnegative real number.

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$$\sin A + \sin B + \sin C = (a + b + c)/2R = s/R.$$

A Unimodular Matrix

E 1645 [1963, 1099]. *Proposed by E. R. Barnes, Morgan State College*

Let r be a nonnegative integer and let $A=[a_{ij}]$ be an n by n matrix where $a_{ij}=(i+j+r-2)!/(i-1)!(j+r-1)!$. Show that $|A|=1$.

Solution by William M. Stone, Oregon State University. Problem 3468, reprinted in the Dunkel memorial pamphlet, and E 1600 amount to the statement that

$$\left|\binom{r+s}{s}\right| = 1, \qquad 0 \leqq r, \quad s \leqq n,$$

n a nonnegative integer. A more general statement is that any determinant "hung" from the hth one at either side of the Pascal triangle is unity,

$$F(n,h) = \left|\binom{r+s+h}{s}\right| = 1,$$

h a nonnegative integer. Replace each row of the $F(n, h)$ by the difference between itself and the preceding row; do likewise with columns of the resulting determinant, arriving at

$$\binom{r+s+h}{s} \to \binom{r+s+h}{s} - \binom{r+s+h-1}{s} - \binom{r+s+h-1}{s-1} + \binom{r+s+h-2}{s-1} = \binom{r+s+h-2}{s-1}.$$

That is, $F(n, h)=F(n-1, h)$ with $F(0, h)=1$.

Also solved by M. T. L. Bizley, A. W. Brunson, Robert Burton, Leonard Carlitz, David Carlson, D. I. A. Cohen, R. F. Jackson, R. A. Jacobson, P. G. Kirmser, E. S. Langford, E. L. Magnuson, D. C. B. Marsh, S. G. Mohanty, J. W. Moon, D. E. Nixon, F. D. Parker, C. B. A. Peck, Stanton Philipp, L. J. Pratte, George Purdy, V. K. Rohatgi, Perry Scheinok, R. P. Soni, Guy Torchinelli, R. Vande Velde, Simon Vatriquant, David Zeitlin, and the proposer.

Zeitlin points out that this problem and E 1600 are special cases of Example 729 in Muir, *A Treatise on the Theory of Determinants*, Dover, p. 679.

Derived Sets of a Finite Topological Space

E 1646 [1963, 1099]. *Proposed by L. J. Green, Case Institute of Technology*

Exhibit a finite topological space and a subset A such that A, A' (the set of all limit points of A), A'', $\cdots$, $A^{(k)}$ are all different, none of them closed.

Solution by Jack Williamson, University of Wisconsin. Let $X=\{0, 1, 2, \cdots, k+1\}$ and consider the following topology for X:

$$T = \{1, 2, \cdots, j \mid j \leqq k+1\} \cup \{X\} \cup \{\theta\}.$$

Let $A=\{1, 2, \cdots, k+1\}$. Then, A is clearly not closed, since $0 \in A'$ and $0 \in X-A$. Now 1 is not an accumulation point of A since $t=\{1\}$ is a neighborhood of 1 disjoint from $A-\{1\}$; further, if $j \in A$ $(j \neq 1)$, then j is an accumulation point of A since every neighborhood of j contains 1. Hence,

$$A' = \{2, \cdots, k+1\}$$

and clearly A' is not closed. Similarly, $A''=\{3, \cdots, k+1\}, \cdots, A^{(j)} =\{j+1, \cdots, k+1\}, \cdots, A^{(k)}=\{k+1\}$.

Also solved by M. D. Mavinkurve and the proposer.

Sum of Exponentials

E 1647 [1963, 1099]. *Proposed by P. R. Chernoff, Harvard University*

Sum the infinite series $\sum_{n=0}^{\infty} 1/(kn)!$, where k is any positive integer.

Solution by D. I. A. Cohen, Princeton University. We have

$$\frac{1}{k}\sum_{j=1}^{k} e^{e^{2\pi ij/k}} = \frac{1}{k}\sum_{j=1}^{k}\sum_{l=1}^{\infty}\frac{e^{2\pi ijl/k}}{l!} = \frac{1}{k}\sum_{l=1}^{\infty}\frac{1}{l!}\sum_{j=1}^{k} e^{2\pi ijl/k} = \sum_{l=1}^{\infty}\frac{1}{(kl)!}$$

since if $k \nmid l$, then $\sum_{j=1}^{k} e^{2\pi ijl/k}=0$.

Also solved by E. R. Barnes, M. T. L. Bizley, Robert Burton, S. Chowla and R. A. Smith and A. M. Vaidya (jointly), C. A. Church, Jr., and D. P. Roselle (jointly), M. J. Cohen, John de Pillis, D. T. Dwyer, J. A. Faucher, H. E. Fettis, Michael Goldberg, Ralph Greenberg, S. H. Greene, J. E. Hafstrom, Eldon Hansen, Stephen Hoffman, R. F. Jackson, D. E. Johnson, Erwin Just and Norman Schaumberger (jointly), R. N. Kesarwani, P. G. Kirmser, E. S. Langford, W. W. Leutert, D. C. B. Marsh, Imanuel Marx, H. L. Montgomery, M. G. Murdeshwar, Hugh Noland, C. S. Ogilvy, Walter Penney, Stanton Philipp, George Purdy, D. L. Ragozin, John Raleigh, D. C. Russell, C. M. Sandwick, Jr., F. C. Smith, R. A. Smith and A. M. Vaidya (jointly), R. P. Soni, W. M. Stone, L. K. Tolman, Guy Torchinelli, Robert Weinstock, K. S. Williams, Kung-Wei Yong, David Zeitlin, and the proposer.

The distinct solutions to this problem were too many to reproduce in their entirety: Writing $y=f(x)=\sum_{n=0}^{\infty} x^{kn}/(kn)!$ several solvers observed that the series in the problem is the solution of the initial-value problem

$$y^{(k)} = y; \quad f(0) = 1, \quad f^{(1)}(0) = \cdots = f^{(k-1)}(0) = 0;$$

Stone used Laplace transforms and Tolman a combination of MacLaurin series and Mikusinski operators; Kesarwani obtained the solution as a hypergeometric function; and so on.

The problem is not new: references were given to A. J. Carr, Mathematical Gazette, vol. XLVII (1963); A. Erdelyi, et al., *Higher Transcendental Functions*, vol. 3, McGraw-Hill (1953); I. J. Schwatt, *An Introduction to the Operations with Series*, 2nd ed., Chelsea (1924); Problem E844, this MONTHLY, 56 (1949) 474.

Rings of Order Four

E 1648 [1963, 1099]. *Proposed by David Singmaster, University of California at Berkeley*

(1) What is the order of the smallest nontrivial ring with identity which is not a field? Find two such rings with this minimal order. Are there more?

Let $A=\{1, 2, \cdots, k+1\}$. Then, A is clearly not closed, since $0 \in A'$ and $0 \in X-A$. Now 1 is not an accumulation point of A since $t=\{1\}$ is a neighborhood of 1 disjoint from $A-\{1\}$; further, if $j \in A$ $(j \neq 1)$, then j is an accumulation point of A since every neighborhood of j contains 1. Hence,

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since if $k \nmid l$, then $\sum_{j=1}^{k} e^{2\pi ijl/k}=0$.

Also solved by E. R. Barnes, M. T. L. Bizley, Robert Burton, S. Chowla and R. A. Smith and A. M. Vaidya (jointly), C. A. Church, Jr., and D. P. Roselle (jointly), M. J. Cohen, John de Pillis, D. T. Dwyer, J. A. Faucher, H. E. Fettis, Michael Goldberg, Ralph Greenberg, S. H. Greene, J. E. Hafstrom, Eldon Hansen, Stephen Hoffman, R. F. Jackson, D. E. Johnson, Erwin Just and Norman Schaumberger (jointly), R. N. Kesarwani, P. G. Kirmser, E. S. Langford, W. W. Leutert, D. C. B. Marsh, Imanuel Marx, H. L. Montgomery, M. G. Murdeshwar, Hugh Noland, C. S. Ogilvy, Walter Penney, Stanton Philipp, George Purdy, D. L. Ragozin, John Raleigh, D. C. Russell, C. M. Sandwick, Jr., F. C. Smith, R. A. Smith and A. M. Vaidya (jointly), R. P. Soni, W. M. Stone, L. K. Tolman, Guy Torchinelli, Robert Weinstock, K. S. Williams, Kung-Wei Yong, David Zeitlin, and the proposer.

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(1) What is the order of the smallest nontrivial ring with identity which is not a field? Find two such rings with this minimal order. Are there more?

(2) How many rings of order four are there?

Solution by D. M. Bloom, University of Massachusetts. There are 11 rings of order 4:

(A) 1. $I/(4)$ (the integers mod 4).
(A) 2. The 4-element subring of $I/(8)$.
(A) 3. The 4-element subring of $I/(16)$ (null ring).
4. $GF(4)$.
5. $GF(2)\times GF(2)$.
6. $GF(2)\times N$, where N is the null ring of two elements.
7. $N\times N$.

8. Matrices $\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$ over $GF(2)$.

9. Matrices $\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}$ over $GF(2)$.

(B) 10. Matrices $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ over $GF(2)$.

(B) 11. Matrices $\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$ over $GF(2)$.

(A) additive group is cyclic.
(B) ring is noncommutative.

Proof. Ring No. 10 has a left identity whereas ring No. 11 does not. The three multiplication tables for rings 1, 2, 3 yield different numbers of zero products in each. The same is true for the five rings 4, 5, 6, 7, 9. Each of these latter five rings contains no element M such that $M^3=0\neq M^2$, but ring 8 does contain such an element. From all of these observations, we conclude that *the* 11 *rings listed are mutually nonisomorphic.*

Conversely, suppose that R is any ring of four elements; we wish to show that R is one of the 11 rings in the list.

If the additive group of R is cyclic with generator e, then the structure of R is uniquely determined by the product ee, and we see that R is one of the rings 1, 2, 3.

Assume on the other hand that *R is the four-group under addition.* In each row or column of the multiplication table for R, we must either have one element appearing 4 times, 2 elements appearing twice each, or 4 elements appearing once each; from this, the following observations easily follow:

1. If R has a left annihilator a and a right annihilator b, but no two-sided annihilator, then the multiplication table is

Postulates for a Group

E 1649 [1963, 1100]. *Proposed by R. A. Jacobson, South Dakota State College*

Let "$*$" be a binary operation on a set S. Do the following axioms define a group?

A1. There exists an element $e \in S$ such that $x * e = x$ for all $x \in S$, $x \neq e$.

A2. For all $x, y, z \in S$, $z \neq e$, $x * (y * z) = (x * y) * z$.

A3. $x * w = y$ has a unique solution $w \in S$, for each ordered pair $x, y \in S$, $x \neq y$.

Solution by E. S. Langford, Autonetics Division, North American Aviation. We replace A3 by the weaker

A3$'$. Every $x \neq e$ has a unique right inverse x^R relative to e.

Let us examine the product $e^2 = w$. (We write ab for $a * b$ throughout). Assume that $w \neq e$. Then $w^R \neq e$, since $w^R = e$ implies that $e = ww^R = we = w$. Likewise $w^{RR} \neq e$, etc. Consider

$$w^R = w^R e = w^R(w^{RR}w^{RRR}) = (w^R w^{RR})w^{RRR} = ew^{RRR}.$$

Thus $e = ww^R = w(ew^{RRR}) = (we)w^{RRR} = ww^{RRR}$. By A3$'$, we have that $w^R = w^{RRR}$, and so $w^R = ew^R$. Hence $e = w^R w^{RR} = (ew^R)w^{RR} = e(w^R w^{RR}) = e^2 = w$, a contradiction. Thus $e^2 = e$. It is clear that the associative law now holds for $z = e$. But it is well-known (Jacobson, vol. I, problem, p. 24) that a semigroup with a right identity and right inverses is a group.

REMARK. *If we replace* A1 *by "$ex = x$ for all $x \neq e$," then S need not be a group; for example, we can define $xy = y$ for all $x, y \in S$.*

Also solved by R. D. Bird and G. W. Lofquist (jointly), Robert Burton, J. M. Cardoso, W. Fairchild, Francis Florey, W. E. Gould, Stan Hales, Stephen Hoffman, D. C. B. Marsh, Paul Pang, Richard Paul and Raymond Whitney (jointly), J. R. Porter, H. L. Rolf, R. P. Soni, Guy Torchinelli, Simon Vatriquant, Jim Wahab, and the proposer.

Cardoso calls attention to E. Artin, *Modern Higher Algebra. Galois Theory* (mimeo notes), New York (1956).

Simultaneous Approximation

E 1650 [1963, 1100]. *Proposed by J. L. Brenner, Stanford Research Institute*

Let α, β be any two real numbers and ϵ any positive quantity. Prove that there are integers a, b, m, q such that the inequalities

$$|q\alpha - a - b\sqrt{m}| < \epsilon q, \qquad |q\beta - a + b\sqrt{m}| < \epsilon q$$

hold simultaneously, and m is not a perfect square.

Solution by Leo J. Pratte, Norman, Oklahoma. We prove the following more general result: Let α, β be any two real numbers and γ any nonzero real number. Let $\epsilon > 0$ be given. Then there are integers a, b, q such that the inequalities

$$\text{(A)} \qquad |q\alpha - a - b\gamma| < \epsilon q, \qquad |q\beta - a + b\gamma| < \epsilon q$$

hold simultaneously. To this end, choose integers p_1, $q_1>0$, p_2, $q_2>0$, such that $|(\alpha-\beta)/2\gamma-p_1/q_1|<\epsilon/2|\gamma|$, $|(\alpha+\beta)/2-p_2/q_2|<\epsilon/2$.

Let $b=p_1q_2$, $a=p_2q_1$, $q=q_1q_2$. Then $b/q=p_1/q_1$, $a/q=p_2/q_2$, and

$$|(\alpha-\beta)/2-b\gamma/q|<\epsilon/2,\qquad |(\alpha+\beta)/2-a/q|<\epsilon/2.$$

Now,

$$\begin{aligned}|\alpha-a/q-b\gamma/q| &= |(\alpha+\beta)/2+(\alpha-\beta)/2-a/q-b\gamma/q|\\ &\leqq |(\alpha-\beta)/2-b\gamma/q|+|(\alpha+\beta)/2-a/q|\\ &<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Hence,

$$|\alpha q-a-b\gamma|<\epsilon q. \tag{1}$$

Also,

$$\begin{aligned}|\beta-a/q+b\gamma/q| &= |(\alpha+\beta)/2-(\alpha-\beta)/2-a/q+b\gamma/q|\\ &\leqq |-(\alpha-\beta)/2+b\gamma/q|+|(\alpha+\beta)/2-a/q|\\ &<\epsilon/2+\epsilon/2=\epsilon.\end{aligned}$$

Hence,

$$|\beta q-a+b\gamma|<\epsilon. \tag{2}$$

Equations (1) and (2) give (A). In (A) we choose $\gamma=\sqrt{m}$, where m is a positive integer that is not a perfect square, to get the result in this problem.

Also solved by J. W. Baldwin, Robert Burton, D. I. A. Cohen, M. J. Cohen, E. S. Langford, W. W. Leutert, D. C. B. Marsh, Stanton Philipp, R. A. Smith and A. M. Vaidya (jointly), R. Stroud, and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08900. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before April 30, 1965.

5227. *Proposed by Jiang Luh, Indiana State College, Terre Haute*

Following E. H. Feller, a ring A is called a duo ring if every one-sided ideal of A is a two-sided ideal. Prove that a ring is duo and right (or left) primitive if and only if it is a division ring.

hold simultaneously. To this end, choose integers p_1, $q_1>0$, p_2, $q_2>0$, such that $|(\alpha-\beta)/2\gamma-p_1/q_1|<\epsilon/2|\gamma|$, $|(\alpha+\beta)/2-p_2/q_2|<\epsilon/2$.

Let $b=p_1q_2$, $a=p_2q_1$, $q=q_1q_2$. Then $b/q=p_1/q_1$, $a/q=p_2/q_2$, and

$$|(\alpha-\beta)/2-b\gamma/q|<\epsilon/2, \qquad |(\alpha+\beta)/2-a/q|<\epsilon/2.$$

Now,

$$\begin{aligned}|\alpha-a/q-b\gamma/q| &= |(\alpha+\beta)/2+(\alpha-\beta)/2-a/q-b\gamma/q|\\ &\leqq |(\alpha-\beta)/2-b\gamma/q|+|(\alpha+\beta)/2-a/q|\\ &<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Hence,

$$(1) \qquad |\alpha q-a-b\gamma|<\epsilon q.$$

Also,

$$\begin{aligned}|\beta-a/q+b\gamma/q| &= |(\alpha+\beta)/2-(\alpha-\beta)/2-a/q+b\gamma/q|\\ &\leqq |-(\alpha-\beta)/2+b\gamma/q|+|(\alpha+\beta)/2-a/q|\\ &<\epsilon/2+\epsilon/2=\epsilon.\end{aligned}$$

Hence,

$$(2) \qquad |\beta q-a+b\gamma|<\epsilon.$$

Equations (1) and (2) give (A). In (A) we choose $\gamma=\sqrt{m}$, where m is a positive integer that is not a perfect square, to get the result in this problem.

Also solved by J. W. Baldwin, Robert Burton, D. I. A. Cohen, M. J. Cohen, E. S. Langford, W. W. Leutert, D. C. B. Marsh, Stanton Philipp, R. A. Smith and A. M. Vaidya (jointly), R. Stroud, and the proposer.

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5235. *Proposed by Hans Schneider, University of Wisconsin*

Let R be an associative ring with unity, and let I be a transfinite cardinal. If F is a left R-module which is the (weak) direct sum of I copies of R, then it is known that every other representation of F as a direct sum of copies of R contains I summands.

Now let I be an arbitrary cardinal. Show that there exists a ring R (depending on I) with unity, which is isomorphic as a left R-module to the (complete) direct product of I copies of R.

SOLUTIONS OF ADVANCED PROBLEMS

Arithmetic for x^n Modulo n

5128 [1963, 764]. *Proposed by E. A. Franz, Culver-Stockton College, Canton, Mo.*

For a given positive integer n consider the function $f(x)=x^n$. If the domain of f is $\{0, 1, 2, \cdots, n-1\}$ and the arithmetic is mod n, what is the range of f?

Solution by L. Carlitz, Duke University. Let $\Omega(n)$ denote the range of $x^n \pmod n$; also let $\Omega_0(n)$ denote the range of $x^n \pmod n$, where $(n, x)=1$. Let $\omega(n)$ denote the number of distinct members of $\Omega(n)$ and $\omega_0(n)$ the number of distinct members of $\Omega_0(n)$.

First suppose $n=p^r$ where p is prime and $r\geqq 1$. Since

$$(ap)^{p^r} \equiv 0 \pmod{p^r}$$

it follows that $\Omega(p^r)-\Omega_0(p^r)$ consists of the single number 0. Again if $a\equiv b \pmod p$, then

$$a^{p^r} \equiv b^{p^r} \pmod{p^r}$$

and conversely. Since

$$a^{p^r} \equiv a^{p^{r-1}} \pmod{p^r}$$

it follows that $\Omega_0(p^r)$ consists of the $p-1$ numbers that are prime to p and of the form

$$a^{p^{r-1}} \pmod{p^r}.$$

In particular we have $\omega_0(p^r)=p-1$, $\omega(p^r)=p$ $(r\geqq 1)$.

Now suppose

$$n = p_1^{r_1} \cdots p_k^{r_k}$$

and put

$$P_j = p_j^{r_j}, \qquad n_j = n/P_j \quad (j = 1, \cdots, k).$$

If $a^{P_j}\equiv \bar{a}_j \pmod{P_j}$, where $\bar{a}_j \in (P_j)$, it follows that

$$\text{(1)} \qquad a^n \equiv \bar{a}_j^{n_j} \pmod{P_j} \qquad (j = 1, \cdots, k).$$

Put $d_j=(n_j, p_j-1)$, $(j=1, \cdots, k)$. Then if p_j is not a divisor of a it follows that the number of incongruent numbers in (1) is $(p_j-1)/d_j$; if $p_j|a$ we get the single number 0. Applying the Chinese Remainder Theorem we get

$$\omega_0(n) = \prod_{j=1}^{k} \frac{p_j - 1}{d_j}, \qquad \omega(n) = \prod_{j=1}^{k}\left(1 + \frac{p_j - 1}{d_j}\right).$$

It is worth noting that the reduced range $\Omega_0(n)$ evidently constitutes an abelian group relative to multiplication (mod n) of order $\omega_0(n)$.

Counting Triangles

5129 [1963, 765]. *Proposed by D. S. Mitrinovic, Belgrade, Yugoslavia*

I. Let π be a plane, O a point of π, and $L_k(k=1, 2, \cdots, s)$ a finite sequence of lines passing through O and lying in π. Let $E_k=\{S_k^1, S_k^2, \cdots, S_k^{n_k}\}$ be a set of n_k points of line L_k.

(a) How many triangles are there whose vertices belong to the set

(1) $E_1 \cup E_2$; (2) $E_1 \cup E_2 \cup E_3$; (3) $E_1 \cup E_2 \cup \cdots \cup E_s$?

(b) How many triangles are there whose vertices are O and two other points belonging to two distinct lines E_k?

II. Let L_k $(k=1, 2, \cdots, s)$ be s lines passing through O such that no three lines are coplanar. Let E_k have the same significance as in I above.

(a) Determine the cardinal number of the set of tetrahedrons whose vertices belong to

(1) $E_1 \cup E_2 \cup E_3$; (2) $E_1 \cup E_2 \cup \cdots \cup E_s$ $(s \geqq 4)$.

(b) How many tetrahedrons are there whose vertices are O and three points belonging to three distinct E_k?

III. Generalize the problem for space of $N(\geqq 4)$ dimensions.

Solution by D. I. A. Cohen, Princeton University. A generalization of the problem is equivalent to determining the number of ways in which n points can be taken from $s \geqq \frac{1}{2}n$ sets E_k of n_k points $(k=1, 2, \cdots, s)$, subject to the restriction that no three points come from the same set; (part (b) restricts any two of $n-1$ points to come from different sets).

If there are h pairs of points which come from the same sets then the number of figures possible is

$$\sum \binom{n_{a_1}}{2}\binom{n_{a_2}}{2} \cdots \binom{n_{a_h}}{2} (n_{a_{h+1}}) \cdots (n_{a_{n-h}}),$$

where the summation is extended over all sets $\{a_i\}$ $(i=1, 2, \cdots, n-h)$ which are subsets of $\{1, 2, \cdots, s\}$. If we sum this number over h from 0 to $[\frac{1}{2}n]$ we have the answer to part (a).

The answer to part (b) is simply $\sum \prod_{i=1}^{n} n_{a_i}$, where the summation is extended over all sets $\{a_i\}$ $(i=1, 2, \cdots, n)$ which are subsets of $\{1, 2, \cdots, s\}$. This is just the term corresponding to $h=0$ in part (a).

Editorial Note. The solution tacitly assumes that three different points (s) chosen from three different lines form a proper triangle, with similar assumptions in the other cases. It is clear that the problem as stated requires such an assumption.

Maximal Conjugate Classes in Finite Groups

5130 [1963, 766]. *Proposed by Barbara L. Osofsky, Douglass College, Rutgers, The State University*

Let G be a finite group in which any two elements that have the same period are conjugate. If G has a nontrivial character of degree $\leqq 3$, then G is either S_2 or S_3.

Solution by the proposer. Since all generators of each cyclic subgroup of G are conjugate, every character of G is a rational integer. Let p be a prime dividing the order of G, $p>3$. All rational characters of a cyclic p group of degree less than $p-1$ are multiples of the identity. Hence all p elements of G lie in the kernel of any character χ of degree $\leqq 3$, and the order of $G/\text{kernel } \chi = 2^i 3^j \neq 1$.

$G/\text{kernel } \chi$, having order divisible by at most two primes, is solvable, hence it has a homomorphic image of prime order. All characters of any homomorphic image of G are rational, so G must have a homomorphic image of order 2. Call the kernel of this homomorphism H.

If H is of even order, all involutions of G lie in H, so if $x \in G$, $x \notin H$, then $x \neq x^{-1}$, but there is a $y \in G$ such that $yxy^{-1} = x^{-1}$. Then for $m = 2n+1$, $y^m x y^{-m} = x^{-1}$, so there is an element $z \in G$ of order 2^i, such that $zxz^{-1} = x^{-1}$.

$(zx)^2 = zxzxz^{-1}z = zxx^{-1}z = z^2$. Since the order of z^2 is 2^{i-1} and neither zx nor z is the identity of G, both z and zx are of order 2^i. Hence z is conjugate to zx in G. But if $zx \in H$, $z \notin H$; and if $zx \notin H$, $z \in H$; so z cannot be conjugate to zx.

Hence H is of odd order. An involution of G must take every element of H into its inverse, so H is abelian. Then an involution must take every element of H into each generator of its cyclic subgroup. Thus H is elementary, abelian, and of order 3^i. Since all three elements of H are conjugate, $i=0$ or $i=1$; that is, $G=S_2$ or $G=S_3$.

Atomless Boolean Algebras

5131 [1963, 897]. *Proposed by J. L. Pietenpol, Columbia University*

Let an element A of a Boolean algebra be called a "point" if and only if $A \neq 0$ and $A = B \cup C$ implies $B = A$ or $C = A$. Show that there exists a Boolean algebra having no points.

Solution by Robert Bowen, Fairfield, California. Let S be the family of sets consisting of

(i) all arithmetic sequences of integers,
(ii) all unions of finitely many such sequences,
(iii) the empty set.

It is easily verified that S forms a Boolean algebra without points under the

usual set theoretical operations, for (1) the complement of an arithmetic sequence relative to the integers is the finite union of arithmetic sequences and (2) each arithmetic sequence may be expressed as the union of two other distinct arithmetic sequences merely by taking every other term of the sequence.

Also solved by Andrew Astromoff, P. S. Bloch, W. S. Brainerd, G. W. Day, Jim Dombek, D. P. Giesy, E. D. Goodrich, G. A. Heuer, Charles Himmelberg, R. D. Horowitz, R. R. Korfhage, Jeanne LaDuke, E. S. Langford, Brockway McMillan, M. D. Mavinkurve, R. A. Melter, T. T. Raghunathan, D. L. Silverman, James Singer, R. T. Sandberg, W. A. O'N. Waugh, Alan Weinstein, A. B. Willcox, and the proposer.

It was noted by several solvers that a point as defined in the problem is equivalent to an "atom" and that examples of Boolean algebras without atoms are well known. See, e.g., *The Elements of Mathematical Logic*, P. C. Rosenbloom, p. 27, ex. 4; *Lectures on Boolean Algebras*, P. R. Halmos, p. 69. Day suggests the diplomatic advisability of referring to such examples as "atomless" rather than "pointless."

Zeros of the Derivatives of a Complex Polynomial

5132 [1963, 897]. *Proposed by H. T. Croft, University of California, Berkeley*

Let $f(z)$ be a complex polynomial of the complex variable z and let $f(1) = f(-1) = 0$. Prove or disprove: $f'(z) = 0$ has a root in the strip $-1 \leqq \mathrm{Re}(z) \leqq 1$.

1. *Solution by A. J. Casson, Trinity College, Cambridge, England.* The proposition is false, for consider

$$f(z) = \int_{-1}^{z} (w - (a + bi))^4 (w - (-a + bi))^4 dw,$$

where a, b are real and $a > 1$. Then $f'(z)$ has no zeros in the strip $-1 \leqq \mathrm{Re}(z) \leqq 1$, and $f(-1) = 0$. It remains to satisfy the condition $f(+1) = 0$, i.e.

$$I(a, b) = \int_{-1}^{1} (z - a - bi)^4 (z + a - bi)^4 dz = 0.$$

$$\begin{aligned} \overline{I(a, b)} &= \int_{-1}^{1} (z - a + bi)^4 (z + a + bi)^4 dz \\ &= \int_{-1}^{1} (-z - a + bi)^4 (-z + a + bi)^4 dz \qquad [z \to -z] \\ &= \int_{-1}^{1} (z + a - bi)^4 (z - a - bi)^4 dz = I(a, b). \end{aligned}$$

Therefore $I(a, b)$ is real for all real a, b.

We show that $I(a, b)$ takes positive and negative values in the domain $a > 1$; it then follows that I vanishes at some point of this domain. Since I is a continuous function of (a, b), it is sufficient to show that $I(1, 0) > 0 > I(1, 1)$. Now $I(1, 0) = \int_{-1}^{1} (z^2 - 1)^4 dz > 0$ obviously. Further,

$$I(1, 1) = \int_{-1}^{1}(z - i - 1)^4(z - i + 1)^4dz = \int_{-1}^{1}((z - i)^2 - 1)^4dz$$

$$= \int_{-1-i}^{1-i}(z^2 - 1)^4dz = \frac{1}{i}\int_{1-i}^{1+i}(z^2 + 1)^4dz$$

$$= \frac{1}{i}\int_{1-i}^{1+i}(z^8 + 4z^6 + 6z^4 + 4z^2 + 1)dz$$

$$= \frac{1}{i}\left[\frac{z^9}{9} + \frac{4z^7}{7} + \frac{6z^5}{5} + \frac{4z^3}{3} + \frac{z}{1}\right]_{1-i}^{1+i}$$

$$= 2\left[\left(\frac{16}{9} - \frac{32}{7}\right) + \left(-\frac{24}{5} + \frac{8}{3} + 1\right)\right] < 0.$$

2. *Solution by Gabor Szegö, Stanford University.* It is possible to show that there is no necessary bound to the real part of $f'(z)$. Let $f(z) = \int_{-1}^{z}(1 - \lambda^2t^2/n)^n dt$, where λ is to be so chosen that $\int_{-1}^{1}(1 - \lambda^2t^2/n)^n dt = 0$. The zeros of $f'(z)$ are $\pm n^{1/2}/\lambda$ and it is sufficient to show that λ may be found in a fixed neighborhood (independent of n) which does not intersect the imaginary axis.

a) We see first that $h(\lambda) = \int_{-1}^{1}\exp(-\lambda^2t^2)dt$ has zeros and these can not be pure imaginary. If $h(\lambda)$ had no zeros then we could write $h(\lambda) = \exp(g(\lambda))$ where $g(\lambda)$ must be an even quadratic in λ with leading coefficient -1. Setting $\lambda = 0$ yields the representation $h(\lambda) = 2e^{-\lambda^2}$ and $e^{\lambda^2}h(\lambda) \to 2$ as $\lambda \to \infty$. However, $e^{\lambda^2}h(\lambda) \geqq e^{\lambda^2}\int_0^{1/2}\exp(-\lambda^2t^2)dt \to \infty$ as $\lambda \to \infty$. Thus $h(\lambda)$ must have zeros.

b) Designate by λ_0 one of the zeros of $h(\lambda)$. Choose a neighborhood of λ_0: $|\lambda - \lambda_0| \leqq \delta$ which contains no other zero of $h(\lambda)$ and which does not intersect the imaginary axis. Now, as $n \to \infty$, $(1 - t^2\lambda^2/n)^n \to \exp(-\lambda^2t^2)$ uniformly for λ in this neighborhood and for $-1 \leqq t \leqq 1$. Therefore, for n sufficiently large

$$\left|\int_{-1}^{1}(1 - \lambda^2t^2/n)^n dt - h(\lambda)\right| < \min_{|\lambda - \lambda_0| = \delta} |h(\lambda)|.$$

Therefore by Rouché's theorem $\int_{-1}^{1}(1 - \lambda^2t^2/n)^n dt$ has, for all n sufficiently large, a zero in $|\lambda - \lambda_0| \leqq \delta$ and this completes the proof.

Counting Inversions in a Permutation

5133 [1963, 898]. *Proposed by D. S. Mitrinovic, Belgrade, Yugoslavia*

Let k, n be positive integers and $E = \{1, 2, \cdots, kn\}$. For every $r \in \{0, 1, \cdots, k-1\}$ let E_r denote the ordered set of all numbers in E having r as remainder upon dividing by k. The sets $E_0, E_1, \cdots, E_{k-1}$ yield a set of $k!$ permutations. If k is even, one of them is $(*)E_2E_4 \cdots E_{k-2}E_1E_3 \cdots E_{k-1}$.

Determine the number of inversions in the permutation (*). Answer the same question for other elements of P.

Solution by Oswald Wyler, University of New Mexico. We change the notation slightly, replacing E_0 by E_k. Then the "word" E_iE_j has $\binom{n}{2}$ inversions if $i<j$, and $\binom{n}{2}+n$ inversions if $i>j$. Thus for a permutation $i_1, \cdots, i_k$ of $\{1, \cdots, k\}$ with r inversions, the number of inversions in the permutation $E_{i_1}E_{i_2}\cdots E_{i_k}$ of $\{1, \cdots, kn\}$ is $\binom{k}{2}\binom{n}{2}+rn$. Replacing E_0 with E_k in the permutation (*) and setting $k=2h$, we have $r=\binom{h}{2}+h-1=(k-2)(k+4)/8$.

Subsets of Finite Groups

5134 [1963, 898]. *Proposed by Joseph R. Landau, University of California, Berkeley*

Let G be a finite group of order mn, and let K be a subset of G containing m elements. If $g\in G$, the set of all kg for $k\in K$ may be called a coset of K. Prove: if K is not empty, then if K has just n cosets, one of these cosets is a subgroup of G; in particular, if K contains the identity and has just n cosets, then K itself is a subgroup.

Solution by M. G. Murdeshwar, University of Alberta. Since every element of G belongs to some coset, and there are exactly n cosets, it follows that if two cosets are not identical then they are disjoint. Let $H=Kk^{-1}$ where $k\in K$. If a and b are elements of H, then $a=k_1k^{-1}$, $b=k_2k^{-1}$ for some $k_1, k_2\in K$. This implies $ab^{-1}=k_1k_2^{-1}\in Kk_2^{-1}$. But $Kk_2^{-1}=Kk^{-1}$ since they have a common element e. Therefore, $ab^{-1}\in H$. Thus H is a subgroup. If K contains identity, then $H=K$.

Also solved by R. W. Ball, T. B. Berger, Zwi Birnbaum and U. J. Shild, Robert Bowen, John Bramsen, Martin Broekhuysen, H. L. Chow, T. S. Frank, Stephen Fisk, D. P. Giesy, Ralph Greenberg, G. A. Heuer, R. D. Horowitz, Victor Keiser, H. Kestelman, B. M. Kiernan, Jr., E. S. Langford, C. C. Lindner, M. D. Mavinkurve, Stephen Montague, Jim Morrow, Barbara L. Osofsky, Veselin Perić, Hermann Simon, John Stout, W. C. Waterhouse, Oswald Wyler, and the proposer.

Greenberg and Montague observed that the critical situation was the disjointedness of the cosets of K, and given such a hypothesis, it is not necessary to have G finite.

Quadratics Irreducible over Fields of Characteristic 2

5135 [1963, 898]. *Proposed by Seth Warner, Duke University*

Give an example of a quadratic over a field of characteristic 2 which is not solvable by radicals.

Solution by Joseph Schoenfield and the proposer.

THEOREM. *If K is a field of characteristic 2 which contains a primitive n-th root of unity for every odd n and if $f(X)=X^2+aX+b$ is an irreducible quadratic over K such that $a\neq 0$, then $f(X)$ is not solvable by radicals.*

Proof. Suppose the contrary, and let F be an extension of K obtained by successive adjunction of radicals in which $f(X)$ has a root c. Then there is a subfield L of F containing K such that $c\notin L$ but $c\in L(u)$ where for some prime p, $u^p=d\in L$.

Case 1: p is odd. By a well-known theorem, $g(X)=X^p-d$ is either irreducible over L or has a root z in L. The former case is impossible, since the degree of c over L divides the degree of u over L, and therefore if $g(X)$ were irreducible over L, we would have $2|p$. The latter case is also impossible, for if z is a root of $g(X)$ in L, then g factors completely in $L[X]$. Hence $L(u)=L$, since $z, \zeta z, \cdots, \zeta^{p-1}z$ are p distinct roots of g in L, where $\zeta\in K$ is a primitive pth root of unity.

Case 2: $p=2$. Let $c=x+yu$ where $x, y\in L$. Then

$$0 = c^2 + ac + b = (x^2 + y^2d + ax + b) + ayu;$$

so $ay=0$ and hence $y=0$ since $\{1, u\}$ is a basis of $L(u)$ over L. But then $c=x\in L$, a contradiction.

For the desired example, let Ω be an algebraically closed field of characteristic 2, and let $K=\Omega(X)$ be the field of all rational functions in one indeterminate over Ω. Then Ω has a primitive nth root of unity for every odd integer n, and consequently the same is true for K. If $g(X)\in K$ is a polynomial of odd degree then the quadratic $Y^2+Y+g(X)\in K[Y]$ is easily seen to be irreducible over K and hence by the theorem is not solvable by radicals.

Exponential Approximation and Euler's Constant

5136 [1963, 898]. *Proposed by A. V. Boyd, University of Witwatersrand, Johannesburg, South Africa*

Prove that, for $x>0$,

$$\text{(1)}\quad \int_0^\infty \left\{e^{-t} - 1 + \frac{t}{1!} - \frac{t^2}{2!} + \cdots + \frac{(-1)^n t^{n-1}}{(n-1)!} - \frac{(-t)^n}{n!(tx+1)}\right\} t^{-n-1}dt = \frac{(-1)^{n+1}}{n!}\left\{\gamma - \log x - \sum_{r=1}^{n} \frac{1}{r}\right\}.$$

$$\text{(2)}\quad \int_0^\infty \left\{e^{-t} - 1 + \frac{t}{1!} - \frac{t^2}{2!} + \cdots + \frac{(-1)^n t^{n-1}}{(n-1)!} - \frac{(-t)^n}{n!(t^2x^2+1)}\right\} t^{-n-1}dt = \frac{(-1)^{n+1}}{n!}\left\{\gamma - \log x - \sum_{r=1}^{n} \frac{1}{r}\right\}.$$

Solution by M. Laurence Glasser, Battelle Memorial Institute, Columbus, Ohio. Call the first integral I_1, the second I_2. We first note that

$$I_1 - I_2 = \frac{(-1)^n}{n!}\int_0^\infty \frac{(1-u)du}{(u+1)(u^2+1)} = 0.$$

Using a formula due to Cauchy and Saalschütz (*Zeit. für Math. u. Physics*, 33 (1888))

$$\Gamma(\epsilon - n) = \int_0^\infty \left\{e^{-t} - 1 + t - \frac{t^2}{2!} + \cdots + \frac{(-1)^n t^{n-1}}{(n-1)!}\right\} t^{-n-1+\epsilon}dt$$

with $0<\epsilon<1$. From the integral for the Beta function we obtain

$$\int_0^\infty \frac{t^{\epsilon-1}dt}{tx+1} = x^{-\epsilon}\Gamma(\epsilon)\Gamma(1-\epsilon).$$

Now using the recursion relation for the Γ-function we find

$$I_1 = \lim_{\epsilon\to 0^+} \frac{\{[1/(\epsilon-1)\cdots(\epsilon-n)] - (-1)^n x^{-\epsilon}\Gamma(1-\epsilon)/n!\}}{\{1/\Gamma(\epsilon)\}}.$$

This is of the form 0/0. Using l'Hospital's rule, in view of the well-known results

$$\lim_{\epsilon\to 0^+} \frac{d}{d\epsilon}\{1/\Gamma(\epsilon)\} = 1, \qquad \lim_{\epsilon\to 0^+} \frac{d}{d\epsilon}\Gamma(1-\epsilon) = \gamma;$$

we obtain the stated result after noting that

$$\lim_{\epsilon\to 0^+} \frac{d}{d\epsilon}\left\{\frac{1}{(\epsilon-1)\cdots(\epsilon-n)}\right\} = \frac{(-1)^n}{n!}\sum_{r=1}^{n}\frac{1}{r}.$$

Also solved by J. Boersma, P. J. de Doelder, J. Koekoek, Eldon Hansen, and the proposer. Koekoek and de Doelder prove that the formula remains unaltered when the denominator t^2x^2+1 is replaced by t^kx^k+1, k a positive real number.

Topologies on Finite Sets

5137 [1963, 898]. *Proposed by R. A. Rankin, Glasgow University, Scotland*

Let S_n be a set consisting of n different points and let $T_0(n)$ be the number of different T_0-topologies that can be formed on S_n; let $T_c(n)$ be the number of different connected topologies on S_n; and let $T^*(n)$ be the number of different connected T_0-topologies on S_n. For example, $T_0(1)=T_c(1)=T^*(1)=1$; $T_0(2)=T_c(2)=3$, $T^*(2)=2$; $T_0(3)=T_c(3)=19$, $T^*(3)=12$; $T_0(4)=219$, $T_c(4)=233$, $T^*(4)=146$.

Prove that $T_0(n)$ and $T_c(n)$ are odd for all $n\geqq 1$, and that $T^*(n)$ is even for $n\geqq 2$.

Solution by J. L. Pietenpol, Columbia University. With each topology on a finite set we can associate a dual topology by interchanging the notion of open and closed. The dual topology has the T_0 or connected property if and only if the original topology has the same property. Thus the topologies of the three types occur in pairs, and we need count only those which are their own duals. The only self-dual connected topology is the trivial topology (i.e. only two open sets), and therefore $T_c(n)$ is odd. The only self-dual T_0 topology is discrete, so that $T_0(n)$ is odd. For $n\geqq 2$, the trivial and the discrete topology are different, so that there is no self-dual connected T_0 topology, and thus $T^*(n)$ is even.

Also solved by M. D. Mavinkurve, Henry Sharp, Jr., and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, University of California, Berkeley, and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457. *Programmed Materials: K. O. May,* Univ. of California, Berkeley. *Films: E. P. Vance, Oberlin College, Oberlin, Ohio 44074.*

Introduction to Statistical Inference. By Harold Freeman. Addison Wesley, Reading, Mass., 1963. 445 pp. $7.75.

This book is divided into five parts: (I) Probability and Random Variables, (II) Specific Probability Distributions, (III) Sampling Theory, (IV) Statistical Inference, and (V) An Application to Regression.

Part I is a fairly routine, but effective, presentation of basic notions in probability. The chapter on change of variable is especially good, with both distribution function and Jacobian techniques used.

Of the distributions presented in Part II, the Poisson, Hypergeometric, and Negative Binomial are most interestingly given.

Sampling from partitioned populations and from finite populations (without replacement) are included in Part III, but the emphasis is on sampling from a normal population. The necessary distribution theory for developing topics in statistical inference are clearly presented.

The climax of the text is Part IV—statistical inference. Properties of point estimators—unbiasedness, invariance, consistency, and sufficiency—are developed and used in discussing maximum likelihood estimates that follow. Testing statistical hypothesis and interval estimation receive lucid and thorough treatment. Decision theory is not introduced.

Part V contains a short chapter on regression.

This excellent book (about on the level of Brunk) can be used for a one-semester course in probability, a one-semester course in statistical inference, or a two-semester course in which both areas are covered. Its strengths lie in its clarity, depth of presentation, and choice of topics and methods of current interest to the statistical community.

Problems are few in number but well chosen. It is the author's intent that the student work through derivations and examples in the text and make use of the 211 references to relevant journal articles and books.

ANDREW STERRETT, Denison University

Variationsrechnung und ihre Anwendung in Physik und Technik (Grundlehren der mathematischen Wissenschaften, Band 94). By Paul Funk. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1962. 676 pp. D.M. 92,60.

The author of this book thought it desirable that the pure mathematician should become acquainted with applications that stimulated the development of the calculus of variations. On the other hand it seemed desirable that the applied mathematician, the physicist, and technologist who is looking for methods

of solving practical problems should find theorems and general principles explained in the clearest way. The material is presented with the historical development of the calculus of variations in mind.

This is a treatise containing a vast amount of material, rather than a classroom text book. The following list of a few of the many names of those whose contributions are mentioned will give some idea of the variety of topics discussed: Baire, Bernoulli, Bliss, Bolza, Born, Carathéodory, Courant, Dirichlet, Du Bois-Reymond, Euler, Haar, Hamel, Hilbert, Jacobi, Lagrange, Legendre, Lorentz, Morse, Maxwell, Noether, Rayleigh, Ritz, Schrödinger, Schwarzschild, Tonelli, Weierstrass, Weyl. The following list of some of the applications will give an idea of the variety to be found here, too: Maxwell's fish eye problem, theory of beams, electron index of refraction, cylindrical coordinates in electron optics, Carathéodory foundation of thermodynamics, plane problems of elastostatics, stability of separation surfaces between fluids, Rayleigh-Ritz method of computing eigenvalues for vibrating string and tuning fork, principle of Friedrichs and its application to elastostatic problems. A chapter is devoted to Finsler spaces.

Glancing through the book one might get the impression of a vast mêlée of theory and applications of all sorts. But careful reading rewards one with an insight into the ideas that led to the development of the calculus of variations.

ALINE H. FRINK, Pennsylvania State University

An Introduction to Digital Computing. By B. W. Arden. Addison-Wesley, Reading, Mass., 1963. 389 pp. $9.75.

The title of this book is an accurate description of its contents, and of its weakness. This would have been a far better book had the author not attempted to cover all aspects of the digital computing field in less than 400 pages. The major part of the book is devoted to an introduction to numerical analysis. This section is clearly written, and within the scope of the student with a knowledge of elementary calculus.

The section of the book on programming, however, is another matter. The style of writing here is simply not clear; too often the author uses complicated phrasing where simple phrasing is indicated. The discussion of programming is based on the MAD (Michigan Algorithmic Decoder) compiler, a system which appears to have some definite advantages as far as ease of programming is concerned.

There are a few brief illustrations of assembly language programming, most of which are lumped together with a discussion of computer hardware in a single chapter. The reviewer believes that assembly language programming should either be discussed in detail, or omitted entirely.

The chapter on nonnumerical problems, particularly the discussion of recursive definitions, will be of great interest to the more advanced programmer, as will the chapter entitled "A Simple Compiler."

E. J. SELIGMAN, United Aircraft Research Laboratories

Programming and Coding Digital Computers. By Philip M. Sherman. Wiley, New York, 1963. 444 pp. $11.00.

This is a complete and well-written book for the beginning or prospective programmer. The book starts at the very beginning—the first chapter, in fact, would make a good article for a popular monthly magazine. The gentle introduction, however, soon gives way to plenty of good, solid programming and coding information. Several chapters, particularly those entitled "Non-numerical Problems," "Macro-instructions," and "Interpreters and Simulation" will be of value to the experienced programmer. The major emphasis is on assembly language programming, but FORTRAN and ALGOL are also discussed in detail. Each chapter is followed by a set of questions and problems.

There are a few minor faults. A hypothetical computer ("DELTA 63") is used for coding; in the reviewer's experience, students are better motivated by using a real (and widely-used) computer for learning purposes. The author states that several different supplemental booklets will be published with all examples in this book coded in the language of each of several computers in common use today. There are a few places where the author uses terms which are not defined until later in the book. The style is generally very clear, however, and, with the exception of a few coding examples, the book is within the grasp of the reader with a good high-school mathematics background.

The book is recommended as a text for a programming course, as a self-teacher, or as a reference volume for a computing laboratory library.

E. J. SELIGMAN, United Aircraft Research Laboratories

Introduction to the Theory of Games. By Ewald Burger, translated by John E. Freund. Prentice Hall, N. J., 1963, 202 pp. $7.95.

This is a remarkably self-contained and strictly mathematical treatment of the subject in which, as the author states in the Preface, "intuitive considerations have been reduced to a minimum." On the other hand a fortunate selection of applications and examples, most of them from mathematical economics, help to clarify the crucial ideas very successfully. The result is a book which will be highly useful to any reader with a reasonable degree of mathematical training, who may wish to acquire a solid enough knowledge of the theory of games to be able to start reading research papers. It is possibly less useful as a text, and the fact that no exercises or problems are made available to the student is perhaps a not too minor handicap that should be recorded.

The work is divided into four chapters and one Appendix. Chapter I, after several introductory considerations and examples, states the basic definitions on games in normal form. Chapter II contains the theory of equilibrium points for noncooperative n-person games followed by applications to the oligopoly model and to Gale's model for economic equilibrium of exchange. Chapter III is devoted to zero-sum two-person games with sections on matrix games, infinite games, applications (expansion and production equilibria in economics), and an independent exposition of the essentials of linear programming. Chapter IV

discusses the cooperative theory and ends with a study and comparison of the von Neumann and Shapley definitions of solution in this case. A short Appendix on fixed point theorems closes the book.

A precise style and a careful translation contribute to make this book a pleasant reading for a mathematician.

A. G. AZPEITIA, University of Massachusetts

Notes on Intrinsic Calculus, Parts I and II. By Bernard Friedman. Wiley, New York, 1963. iii+329 and iii+321 pp. (Prepublication edition.)

These volumes of photographically reproduced notes appear to be aimed at science and engineering students with one or two years of collegiate mathematics. The title stems from the coordinate-free treatment of vectors and derivatives of vector-valued functions in the early chapters of Volume I. The other major topics are: transformations and matrices, wedge products, set functions and multilinear forms, the differential and integral calculus of functions of several variables (including change of coordinates via Jacobians, Stokes' theorem, etc.) convergence, series and differential equations.

The approach is mainly heuristic, but in the sections on convergence, series, and differential equations, a fair degree of rigor is maintained for such a course. Physical and geometrical explanations are readable and leisurely, and the section on circulation of a vector field is noteworthy. The main effect of the coordinate-free treatment is the underscoring of the concept of a scalar or vector-valued function defined on some set. That is, $f(P)$ rather than e.g. $f(x, y, z)$. This paves the way for the use of additive set functions in discussing "change of variables" in a multiple-integral, but certain abstract topics, such as groups, receive insufficient discussion. Although the definition is given early, the student must wait until the last section, "Groups and Differential Equations," to get any reason for the formulation of the concept of a group.

A number of problems are interspersed in the notes, but few are deep, and some which could be slightly challenging have hints which relieve the student of the need to think.

Typewritten letters are satisfactorily reproduced, but a number of figures do not fare so well. There is no index, but a number of chapters have summaries.

There are some minor inaccuracies, and the definition of "vector" in Chapters 1 and 10 are different. This could aid the student's intuition, as well as hinder him.

On the whole, the ordinary student would probably be swamped with the vast number of concepts and be left mainly with terminology to impress his associates and potential employers. The notes could serve as collateral reading for able students taking a modern advanced calculus course covering relatively few topics but covering them rigorously. That is, the notes would provide an easy way to learn techniques covered in the traditional advanced calculus course, but passed over in some present day analysis sequences.

MALCOLM GOLDMAN, New York University

Morse Theory. By J. Milnor. Annals of Mathematics Studies, Number 51. Princeton University Press, Princeton, N. J., 1963. 153 pp. \$3.00.

This beautiful book makes available to the mathematician with some knowledge of homology theory the spectacular developments and applications of Morse theory. This theory concerns a smooth real valued function f on a manifold M and the sets $M^a = \{x \in M; f(x) \leqq a\}$.

Part I (39 pages) presents the relation of the critical points of f to the homotopy type of M (which is that of a CW-complex with one cell of dimension λ for each critical point of f of index λ, provided no critical point of f is degenerate and each M^a is compact) and derives the original inequalities of Morse relating the Betti numbers of M and the number of critical points of f of index λ.

Part II (24 pages) is a brief discussion of contemporary Riemannian geometry which leads in Part III (42 pages) via the calculus of variations to the fundamental theorem of Morse Theory: If M is a complete Riemannian manifold and $p, q \in M$ are not conjugate along any geodesic, then the space $\Omega(M; p, q)$ of smooth paths from p to q (in a suitable topology) has the homotopy type of a countable CW-complex which contains one cell of dimension λ for each geodesic from p to q of index λ. Part IV (40 pages) treats the results of R. Bott obtained by applying Morse Theory to Lie groups and symmetric spaces. For example, if G is a compact, simply connected Lie group, then $\Omega(G; p, p)$ has the homotopy type of a CW-complex having only finitely many cells in each dimension and no cells in odd dimensions. A brief appendix indicates an alternative proof of the fundamental theorem.

Many other related results are discussed and adequate citations of the relevant literature are included.

This book is highly recommended not only for its expert presentation of known results but for the sake of future applications of the fundamental ideas of Morse.

M. F. SMILEY, University of California, Riverside

Fundamentals of Banach Algebras. By Kenneth Hoffman. Monografias Matematicas da Universidade do Parana, Vol. 3. Instituto de Matematica da Universidade do Parana, Curitiba, 1962. 116 pp. \$2.00.

Each mathematician has his own pet example of a "typical" Banach algebra: $C(X)$, $L_1(G)$, functions continuous on a closed disc and analytic in the interior, operator algebras. The present book is a revised and enlarged version of M.I.T. lecture notes, cleanly multilithed and adequately bound, though unfortunately not indexed. It touches upon all aforementioned aspects of Banach algebras. The preparation required of the reader, if elementary, is yet fairly extensive, including familiarity with ring theory, Banach spaces (especially the theorems of Hahn-Banach, Krein-Mil'man, and Stone-Weierstrass), integration, and analytic functions. Two (only apparently contradictory) consequences are the ease and rapidity with which important results are obtained (see below, bearing

in mind the number of pages) and the relative inaccessibility of the presentation to the mathematical novice.

The book begins with the spectral radius theorem, the Gelfand representation for commutative algebras, and some of their applications (e.g., Gelfand-Mazur theorem on Banach fields, uniqueness of the norm-topology for commutative semi-simple algebras, joint spectrum and polynomial convexity for e.g. commutative algebras). Then an operational calculus is presented, using analytic functions on spectral neighborhoods (in the last chapter, the calculus is extended to functions of several complex variables analytic in a neighborhood of the joint spectrum). Next, spectral synthesis is touched upon, after which the Shilov boundary is defined, its representation by measures obtained, and Bishop's theorem on strong boundary points (elsewhere sometimes called peak points) proved. Finally, in two stages, the Gelfand-Neimark theorem ($B^* = C^*$) is proved: first the commutative case, using (not the operational calculus but) the Shilov boundary to see that the Gelfand representation preserves the conjugation, and then the general case, adding together all representations due to states and using the commutative case to get the square root lemma.

F. E. J. LINTON, Wesleyan University

Theory of Formal Systems. By Raymond Smullyan. Rev. ed. *Ann. of Math. Studies*, *47*. Princeton University Press, Princeton, N. J., 1961. 147 pp. $3.00.

Some of the results contained in this book, and some of the novel techniques used in it, have made their appearance before, as papers presented at meetings of the ASL and the AMS, but they have appeared in print only in the form of abstracts. The novelty of the approach, the elegance of the presentation, and the fact that many of the results are new or are extensions of known results or are proved on a far slenderer basis than usual, combine to make this an exciting book.

The basic notion is that of representation in an *elementary formal system;* recursive enumerability is defined as representability in a particular elementary formal system. The first chapter concludes with a very short and simple proof of Church's unsolvability theorem. The second chapter is devoted to developing results needed for later chapters, and in particular for Chapter 3 in which Gödel's and Rosser's incompleteness theorems are treated. Gödel's theorem is presented in an extended form: (Theorem 19, p. 57) *If a formal system is strong enough so that every recursive set is representable in it, then it must be either inconsistent or incomplete.* The last section of this chapter deals with the relation between undecidability and recursive inseparability. Chapter 4 presents recursive function theory, using the technique of relating everything to elementary formal systems; and the final chapter presents new results on creativity and inseparability, and on the theory of universal sets and double universal pairs. The applications of the results of Chapter 3 to mathematical logic are treated separately in a supplementary section.

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Problems in Differential Equations. By J. L. Brenner. Adapted from *Problems in Differential Equations* by A. F. Filippov. Freeman, San Francisco, 1963. ix+157 pp. $2.00.

From the preface: "At the beginning of every paragraph, the basic ideas needed for solving the problems are given . . . . The explanations are so extensive that this book is almost a complete course in itself."

The major contribution, I think, is to subjects which are not standard in the American colleges: stability, singular points, theory of oscillations. But even in the standard topics there are many ingenious and demanding exercises.

HARRY POLLARD, Purdue University

Probability Theory and Mathematical Statistics. 3rd ed. By Marek Fisz. Wiley, New York, 1963. 667 pp. $15.75.

The first two editions of this book appeared in Polish and German, and although this new edition is the first appearance of the book in English, it does manage to attain the quality expected of a third edition. It is not, however, simply a translation of the earlier editions, for a number of sections have been revised and brought up to date.

The book is divided into two main parts, with the first one devoted to probability theory. It contains chapters on random variables, distributions, characteristic functions, Markov chains and stochastic processes. This is very definitely the strongest part of the book; it is well written, the ideas follow along in an orderly fashion, and very good sets of problems follow each chapter. This part of the book contains a considerable amount of asymptotic theory. For example, Chapter 6 entitled "Limit Theorems" has the most pages (75) and the most problems (50) of any chapter in the book. This could be either an advantage or a disadvantage depending upon the user's point of view.

Part 2 is an introduction to mathematical statistics and contains chapters on sample moments, order statistics, significance tests, estimation, sampling theory and the analysis of variance. It should be pointed out that the author's objectives include stressing "the intuitive approach as well as the applicability of the concepts and theorems" in addition to introducing probability theory and mathematical statistics. But unfortunately, the book is not too well suited "for nonmathematicians," and by the nature of the material this shows mostly in Part 2. The chapters on sampling theory and analysis of variance are short, and have only eight and four problems respectively. Furthermore, some of the discussion and the examples are open to question as presented; for instance, the definition of random sampling and the related example could create a considerable amount of confusion in the minds of readers.

In conclusion, I believe this book is a very good introduction to probability theory and the mathematical aspects of statistics. Unfortunately the attempted appeal to nonmathematicians has added to the length and price of the book, and this may deter from its use in an appropriate course in mathematical statistics. It would not be a good book for other than a curriculum in mathematics or mathematical statistics.

W. H. WILLIAMS, Bell Telephone Laboratories, Inc.

Compact Calculus. By Philip Franklin, McGraw-Hill, New York, 1963. 245 pp. $6.50.

The choice of title is accurate: the print is large and uncrowded on pages of medium size, the tersely phrased text is divided into sections of about one page, the total number of pages is small for the ground covered, which is differential and integral calculus of one variable, infinite series, and partial derivatives and multiple integrals (eight pages plus problems). The book is a challenge to the thick all inclusive texts which dominate today's scene.

The author states that he has attempted to provide a sequel to present-day reformed high school mathematics courses. In so far as the philosophy of these courses is to motivate, to stimulate creativity, and to educate without shrouding mathematics in mystery, the compactness of this text opposes this philosophy. On the other hand, do instructors who use thick texts take advantage of their fullness and let the students work and perform, or do they make them listen?

The problem sets at the end of chapters are sometimes too brief, but there are challenging problems.

N. D. KAZARINOFF, University of Michigan

Logic, Computing Machines, and Automation. By Alice Mary Hilton. Spartan Books, Washington, D. C., 1963. 427 pp.

The author states: "It is the purpose of this book to tie together the basic principles gathered from various disciplines that form the foundations for the development of automation and its tools, the computing machines, and to explain the fundamental laws of logic without which one cannot understand the real power of these devices. . . . There are no prerequisites. . . . Training in mathematics beyond simple calculus and elementary algebra is not required."

As it turns out, this book is weak on fundamentals, but does contain a lot of information about topics related to computers. It is eloquently but unevenly written, and generously sprinkled with the author's sometimes quite opinionated comments on all the ills of "our detail-ridden world." At the same time it conveys the author's enthusiasm and some feeling for the magnitude of computer based "cyberculture revolution" which our generation is witnessing. Since the computer revolution is not very well understood by most people (including mathematicians) who are not directly connected with it, and since its scientific parts have not yet found their proper academic setting, this book may serve a limited purpose by informing the reader about computers and some computer related topics.

The mathematical parts of this book are quite disappointing and contain many mistakes. For example, five pages (!) are dedicated to the discussion of "The Σ-Notation for Sums" and "The Π-Notation for Products." It is easy to find more than five errors in these five pages, some of them quite basic (p. 138). A mathematician's nightmare occurs on page 188 where the "Stone Representation Theorem" and the distributive laws are massacred.

J. HARTMANIS, G.E. Research Laboratory, Schenectady, N. Y.

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Logic, Computing Machines, and Automation. By Alice Mary Hilton. Spartan Books, Washington, D. C., 1963. 427 pp.

The author states: "It is the purpose of this book to tie together the basic principles gathered from various disciplines that form the foundations for the development of automation and its tools, the computing machines, and to explain the fundamental laws of logic without which one cannot understand the real power of these devices. . . . There are no prerequisites. . . . Training in mathematics beyond simple calculus and elementary algebra is not required."

As it turns out, this book is weak on fundamentals, but does contain a lot of information about topics related to computers. It is eloquently but unevenly written, and generously sprinkled with the author's sometimes quite opinionated comments on all the ills of "our detail-ridden world." At the same time it conveys the author's enthusiasm and some feeling for the magnitude of computer based "cyberculture revolution" which our generation is witnessing. Since the computer revolution is not very well understood by most people (including mathematicians) who are not directly connected with it, and since its scientific parts have not yet found their proper academic setting, this book may serve a limited purpose by informing the reader about computers and some computer related topics.

The mathematical parts of this book are quite disappointing and contain many mistakes. For example, five pages (!) are dedicated to the discussion of "The Σ-Notation for Sums" and "The Π-Notation for Products." It is easy to find more than five errors in these five pages, some of them quite basic (p. 138). A mathematician's nightmare occurs on page 188 where the "Stone Representation Theorem" and the distributive laws are massacred.

J. HARTMANIS, G.E. Research Laboratory, Schenectady, N. Y.

Calculus for Students of Engineering and the Exact Sciences. By Hugh A. Thurston. Prentice Hall, Englewood Cliffs, N. J., 1963, vol. I, 193 pp. $4.95, vol. II, 208 pp. $5.95.

A review of this two-volume, two-year calculus text must come to grips with the fundamental question, "What is the place of rigor in a beginning calculus course?" The author believes, quite rightly, that the beginning student is not ready for a mathematically rigorous treatment from the start. However, he believes that rigor should be introduced within the first two years since he gives proofs about half way through the second volume and proves, using ϵ's and δ's, the earlier theorems which up to this point have been discussed only intuitively. At the beginning he substitutes "practical rigor." What then is practical rigor? The author defines it in the preface as follows: "Practical rigor entails the *proper* statement of all theorems and definitions . . . ." Let us see how this turns out. The closest thing to a proper definition of function we can find is the following statement on page 5: "To define a function ψ we have to say, for each and every x, whether $\psi(x)$ is defined, and if so what it is." On page 42, we learn that "If $\phi(a)=\lim_{h\to 0}\phi(a+h)$ we say that ϕ is continuous at a." Yet the very next theorem depends on a more general definition of continuity. On page 76, we learn that the definite integral is "the net area under the graph" with no reference to the Riemann sums. On page 133, we learn that " . . . if $\phi'(x)$ is negative whenever $x<2$ and positive when $x>2$, then $\phi(x)$ $\cdots$ is least when $x=2$." We might add, "What if ϕ is discontinuous at $x=2$?" The "mean-value theorem" is stated on page 143 in such a way that a counter example can easily be constructed using a function defined on a closed interval $a\leqq x\leqq b$, discontinuous at a and b, but differentiable in the open interval $a<x<b$. This reviewer feels that this kind of "rigor" is far from practical, for it tends to confuse the good students and to delude the poor students into thinking that they understand the calculus when they really do not. It may turn out that the rigor in the advanced calculus-like material of volume two is impractical for a different reason. The nonmathematics major, who is not interested in the details of the mathematical proofs, will probably not find the latter parts interesting and will end up wishing that there had been more rigor at the beginning and less at the end.

J. W. DETTMAN, Oakland University

Numerical Methods of Curve Fitting. By P. G. Guest. Cambridge University Press, New York, 1961. xii+422 pp. $15.00.

This is a working manual for applied statisticians and others concerned with processing observational data. It consists of theoretical discussions, informal derivations of formulas, detailed calculating schemes and numerical examples. There are adequate reference notes and a bibliography. Though the author discusses iterative methods of solving the normal equations and touches briefly on the use of orthogonal polynomials in curve fitting, the book is written from the standpoint of desk computation and is not addressed to users of high speed computers.

NATHANIEL MACON, Auburn University

BRIEF MENTION

Theory of Probability, 2nd ed. By B. V. Gnedenko. Chelsea, New York, 1963. 471 pp. $8.75.

This book differs from the first edition primarily in the inclusion of answers to the exercises.

Neue topologische Methoden in der algebraischen Geometrie. By Friedrich Hirzebruch. Zweite ergänzte Auflage. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1962. viii+181 pp. DM 30.80.

Cargèse Lectures in Theoretical Physics. By M. Lévy. Benjamin, New York, 1963. ix+456 pp. $13.00.

Lectures on the many-body problem and elementary particle physics given at the 1962 Cargese Summer School in Corsica.

Deductive Geometry. By E. A. Maxwell. Macmillan (Pergamon Press, The Commonwealth and International Library), New York, 1963. viii+180 pp. $1.95.

Designed for courses in the United Kingdom for General Certificate of Education at advanced and scholarship levels.

Elementary Particle Physics and Field Theory, 1962 Brandeis Lectures, vol. 1. Benjamin, New York, 1963. vii+532 pp. $5.95 (paper) $11.00 (cloth).

Astrophysics and the Many-Body Problem, 1962 Brandeis Lectures, vol. 2. By E. N. Parker, J. S. Goldstein, A. A. Maradudin, and V. Ambegaokar. Benjamin, New York, 1963. vii+438 pp. $5.95 (paper) $11.00 (cloth).

Statistical Physics, 1962 Brandeis Lectures, vol. 3. By G. E. Uhlenbeck, N. Rosenzweig, A. J. F. Steigert, E. T. Jaynes, and S. Fujita. Benjamin, New York, 1963. vii+252 pp. $4.95 (paper) $10.00 (cloth).

Theory of Linear Physical Systems. By E. A. Guillemin. Wiley, New York, 1963. xvii +586 pp. $12.50.

Theory of physical systems from the viewpoint of classical dynamics, including Fourier methods.

Mathematics and the Physical World. By Morris Kline. Doubleday and Company, New York, 1963. x+548 pp. $1.95.

A republication of a book first published in 1959.

Cases in Management Statistics. By Norbert Lloyd Enrick. Holt, Rinehart, and Winston, New York, 1963. xvi+158 pp. $2.50.

A collection of problems, mostly actual cases, designed to supplement any standard introductory statistics textbook.

Measurement in Economics, Studies in Mathematics, Economics, Econometrics in Memory of Yehuda Grunfeld. By Christ, Friedman, Goodman, Griliches, Harberger, Liviatan, Mundlak, Nerlove, Patinkin, Lester, and Theil. Stanford University Press, Stanford, California, 1963. xiii+320 pp. $10.00.

Scientific Change. Edited by A. C. Crombie. Basic Books, New York, 1963. xii+896 pp. $17.50.

Historical studies in the intellectual, social and technical conditions for scientific discovery and technical invention, from antiquity to the present.

Nonlinear Control Systems Analysis. By R. H. Macmillan. The Commonwealth and International Library of Science, Technology and Liberal Studies, Macmillan (Pergamon Press) New York, 1963. x+174 pp. $3.75.

The contents of this book are compiled from articles recently published in the journal *Process Control and Automation*.

Light, Principles and Experiments, 2nd ed. By George S. Monk. Dover, New York, 1963. xi+489 pp. $2.45.

First edition published in 1937.

The collected works of John von Neumann, vol. 5, Design of Computers, Theory of Automata and Numerical Analysis. Edited by A. H. Taub. Macmillan (Pergamon Press) New York, 1963. x+784 pp. $14.00.

Republication of von Neumann's articles in these fields.

Collected Works of John von Neumann, vol. 6. Edited by A. H. Taub, Macmillan (Pergamon Press) New York, 1963. x+536 pp. $14.00.
Papers on the theory of games, astrophysics, hydrodynamics and meteorology.

The Scientist Speculates, An anthology of partly-baked ideas. I. J. Good, editor. Basic Books, New York, 1963. xvii+413 pp. $6.95.

An anthology of articles in which leading scientists divulge some of their wilder brainstorms about every field from physics to sociology.

Readings in Mathematical Psychology, vol. 1. By R. Duncan Luce, Robert R. Bush, and Eugene Galanter, editors. Wiley, New York, 1963. ix+535 pp. $8.95.

Handbook of Mathematical Psychology, vol. I and II. By R. Duncan Luce, Robert R. Bush, and Eugene Galanter, editors. Wiley, New York, 1963. x+490 pp. $10.50.

The *Handbook* consists of chapters written by experts in various subfields of mathematical psychology. Vol. I includes measurement, psychophysics, reaction time, computers, and statistics; Vol. II, learning models, formal theories of language, and models for social interaction. A third volume is planned.

The *Readings* is a companion to the *Handbook* containing research papers reprinted from journals. Vol. I treats measurement, psychophysics, reaction time, learning and stochastic processes. A second volume is planned.

Operational Methods in Applied Mathematics, 2nd ed. By H. S. Carslaw and J. C. Jaeger. Dover, New York, 1963. xvi+359 pp. $2.25.

Republication of the second edition (1948) of the work first published in 1941.

Probability and Statistics for Everyman. By Irving Adler. John Day, New York, 1963. 256 pp. $5.95.

Mathematics for Science and Engineering. By Philip L. Alger. McGraw-Hill, New York, 1963. xi+366 pp. $2.95.

Based on "Engineering Mathematics" by Steinmetz. First published in 1957.

50 Brain-Twisters, A Book of Mathematical and Reasoning Problems. By D. St. P. Barnard. Van Nostrand, Princeton, N. J., 1962. 107 pp. $2.50.

Selections from the author's column in the *Observer*.

Fundamental Electromagnetic Theory, 2nd ed. By Ronold W. P. King, Dover, New York, 1963. xvi+580 pp. $2.75.

First edition, 1945. Formerly titled *Electromagnetic Engineering.*

Introduction to the Theory of Statistics, 2nd ed. By Alexander M. Mood and Franklin A. Graybill. McGraw-Hill, New York, 1963. xv+445 pp. $8.95.

Modernization of a text first published in 1950.

Probability Theory, 3rd ed. By Michel Loeve. Van Nostrand, Princeton, N. J., 1963. xvi+685 pp. $14.95.

Revision of a standard reference and graduate text first published in 1955.

Tables of Integral Error Functions and Hermite Polynomials. (Volume 19 of Mathematical Table Series.) By O. S. Berlyand, R. I. Gavrilova, and A. P. Prudnikov. Translated by Prasenjit Basu from the Russian. Macmillan (Pergamon Press) New York, 1962. 163 pp. $15.00

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Herbert Busemann, University of Southern California, has been elected as a foreign member of the Royal Danish Academy of Arts and Sciences.

Dean Mina S. Rees, City University of New York, has been awarded honorary Doctor of Science degrees by Oberlin College, Wheaton College (Massachusetts), and Wilson College.

Professor Alice T. Schafer, Wellesley College, has been awarded an honorary Doctor of Science degree by the University of Richmond.

Professor J. N. Eastham, Queensborough Community College, represented the Association at the inauguration of Alan Simpson as President of Vassar College on October 16.

Western Washington State College: Associate Professor F. H. Young, Oregon State University, has been appointed Professor; Dr. Ling-Erl Eileen T. Wu, University of Washington, has been appointed Assistant Professor.

Dr. Paul Brock, Hughes Aircraft Company, Fullerton, California, has accepted a position with the senior staff of the Logistics Department of the Rand Corporation, Santa Monica, California.

Dr. W. O. Buschman, California State Polytechnic College, has been appointed Coordinator of the Computer Center.

Assistant Professor Torcom Chorbajian, University of Alaska, has been promoted to Associate Professor.

Dr. J. R. Durbin, University of Kansas, has been appointed Assistant Professor at the University of Texas.

Visiting Professor Robert Ellis, Wesleyan University, has been appointed Professor.

Dr. E. C. Kennedy, Senior Research Engineer at the Ordnance Aerophysics Laboratory, has been appointed Professor at Arlington State College.

Professor M. S. Klamkin, SUNY at Buffalo, has been appointed Visiting Professor at the University of Minnesota.

Dr. J. P. Mayberry, Office of the Vice Chief of Staff, U. S. Air Force, has been promoted to Chief of the Research Team.

Professor C. L. Riggs, on leave of absence at Occidental College, has returned to Texas Technological College.

Dr. R. T. Sandberg, University of Arizona, has been appointed Assistant Professor at California State College at Fullerton.

Associate Professor D. R. Sudborough, San Jose State College, has been appointed Associate Professor at Southern Oregon College.

Dr. L. A. Hazeltine, Maplewood, New Jersey, died on May 24, 1964. He was a member of the Association for 31 years.

Dr. E. C. Molina, Newark College of Engineering, died on April 29, 1964. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE IOWA SECTION

The fifty-first regular meeting of the Iowa Section of the Mathematical Association of America was held at Luther College, Decorah, on April 17, 1964. Chairman C. H. Lindahl presided. Total attendance was 95, including 48 members of the Association. Routine business was considered during the afternoon meeting.

A report of the Iowa 1964 high school mathematics contest was given by T. A. Moilien of the Des Moines Actuaries Club, which sponsors the contest. A treasurer's report was given and a balance of $251.92 was indicated. The following officers were elected: Chairman, R. V. Hogg, State University of Iowa, Iowa City; Vice-Chairman, D. E. Sanderson, Iowa State University, Ames; Secretary-Treasurer, E. L. Canfield, Drake University.

The following papers completed the program:

1. *Engel conditions on groups*, by D. H. Pilgrim, Luther College, introduced by the Chairman.

Let g, c denote positive integers. A group is said to have type $(g\to c)$ if every subgroup which can be generated by g elements is nilpotent of class at most c. A result of R. H. Bruck shows that groups of type $(4\to 5)$ without elements of order 2 are nilpotent of class at most 7. In the present paper the following result is reported: If G is a $(4\to 5)$ group on 5 generators without elements of order 2, then G is nilpotent of class at most 6.

2. *Relations between local and global properties*, by D. E. Sanderson, Iowa State University.

Two-neighborhood and one-neighborhood definitions of a local property and their relation to one another and to the corresponding global property are discussed. In particular, four definitions of locally compact are shown to be equivalent for Hausdorff spaces but to be distinct in gen-

general linear k-step operator for the numerical solution of ordinary differential equations. The existence of various families of stable and unstable operators for a given k and a given order is discussed.

12. *Some results using either stable or unstable linear operators*, by Ronald Mehaffey, Iowa State University, presented by the Chairman.

The formulae presented show that an unstable explicit linear operator, used as a predictor for a stable implicit linear operator, may reduce the number of iterations necessary to converge to the solution of the difference equation, when the procedure is used in the numerical solution of ordinary differential equations. By examining the non-homogeneous difference equation satisfied by the error of the combined predictor-corrector method, an appropriate choice of the unstable explicit linear operator can be found. An example is provided to demonstrate this reduction of the number of iterations.

E. L. CANFIELD, *Secretary*

APRIL MEETING OF THE KANSAS SECTION

The forty-ninth annual meeting of the Kansas Section of the MAA was held at Kansas State University, Manhattan, Kansas, on April 18, 1964, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. There were 275 persons registered including 108 members of the Association. Chairman Robert Thompson presided at the morning and afternoon sessions.

The following officers were elected: Chairman, Laura Z. Greene, Washburn University, Topeka; Vice-chairman, Gilbert Ulmer, University of Kansas, Lawrence; Secretary-Treasurer, Helen Kriegsman, Kansas State College, Pittsburg.

The following papers were presented:

1. *A variation on Steiner's problem*, by Dale Brownawell and Victor Goodman, University of Kansas.

The problem of finding the point P' which maximizes the expression OP-AP-BP, where A, B, and O are three arbitrary points in the plane is considered. If $\measuredangle OAB \geqq 60°$ and $\measuredangle OBA \geqq 60°$, then P' is given by $\measuredangle OP'A = \measuredangle OP'B = 60°$. Then, letting $OA = \alpha$, $OB = \beta$, $AB = \gamma$, $\Sigma\alpha^r = \alpha^r + \beta^r + \gamma^r$, and $\Sigma\alpha^2\beta^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$,

$$OP' - AP' - BP' = \sqrt{\frac{\Sigma\alpha^2 - \sqrt{6\Sigma\alpha^2\beta^2 - 3\Sigma\alpha^4}}{2}}.$$

In any other case, P' is either A or B, whichever is farther from O, and $OP' - AP' - BP' = \max\{\alpha, \beta\} - \gamma$.

2. *A coordinate approach to 25-point geometry*, by Martha Heidlage, Mount St. Scholastica College.

The twenty-five point, thirty-line miniature coordinate geometry (so called "Mini-Co Geometry") was developed by associating an ordered pair of numbers from the set of residue classes mod 5 with each of the twenty-five letters of a miniature, synthetic geometry. The author compared and contrasted these miniature coordinate systems in terms of point, line, parallelism, perpendicularity and distance, and general equations of lines and conic sections were developed.

3. *Translations of infinite subsets of a group*, by W. R. Scott and L. M. Sonneborn, University of Kansas.

An infinite Abelian group G has the property that for every infinite subset H whose complement $\tilde{H}$ is infinite there is an element $x \in G$ such that $(x+H) \cap \tilde{H}$ is infinite if, and only if, G is uncountable or G is countable and possesses an element which generates an infinite subgroup of infinite index. For more general results, see the paper with the same title and authors in *Colloquium Mathematicum*, 10 (1963) 217–220.

4. *Some thoughts on groups and orthogonal matrices*, by L. J. Dixon, Kansas State University.

An introductory course in modern algebra normally includes a discussion of elementary groups and numerous examples, usually selected from various well-known numerical examples. A survey of texts currently available revealed lack of examples from matrix theory. The set of 2×2 orthogonal matrices represents an example of a multiplicative group with many interesting finite subgroups. A limited knowledge of matrix theory is needed in the presentation of these examples of groups of orthogonal matrices.

5. *Negative base numeration systems*, by G. E. Bartel, Kansas State Teachers College, Emporia.

To express a number in a negative base it is first necessary to determine the largest exponent to which the base must be raised to express the number. The digits are then determined from left to right. Polynomial expansion and repeated division can also be used to change from one base to another. In some addition problems the regrouping process continues indefinitely, but this can be avoided by subtracting the additive inverse of what is carried. In subtraction the digits in the minuend are increased instead of decreased when regrouping is necessary. After proving the theorem that a number and the sum of the digits in the numeral that represent the number are congruent, modulo base-minus-one, the operations of addition, subtraction, and multiplication can be checked by casting out base-minus-one.

6. *On set theoretic matrices*, by R. H. Lohman, Kansas State College of Pittsburg.

A system of matrices whose elements are subsets of a universal set is defined. Definitions of equality, addition, and multiplication of these matrices are presented. Some elementary properties of these operations are stated. The determinant of a matrix, partitioning, and a system of linear homogeneous set equations are defined. These concepts are said to be useful in proving a theorem concerning necessary and sufficient conditions for the existence of an inverse of a matrix of this type.

HELEN KRIEGSMAN, *Secretary*

APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the MAA was held at New Mexico State University, University Park, New Mexico on April 10–11, 1964. There were 79 persons in attendance including 53 members of the Association. E. A. Walker, Chairman of the Section, presided. P. D. Lax, AEC Computing and Applied Mathematics Center, New York University, gave the banquet address on "Scattering Theory."

At the business meeting the following officers were elected: Chairman, E. D. Nering, Arizona State University; Vice-Chairman, Jorg W. P. Mayer-Kalkschmidt, University of New Mexico.

The following papers were presented:

1. *The number of coprime chains with largest member n* by R. C. Entringer, University of New Mexico.

An increasing sequence $\{a_1, \cdots, a_k\}$ of integers greater than 1 is a *coprime chain* iff it contains exactly one multiple of each prime equal to or less than a_k. If $s(n)$ is the number of coprime chains with largest member n, and p is a prime, then $s(p)=\sum s(n)$ for $n<p$ and n not prime, and $\log s(n)\sim\sqrt{n}$.

2. *Quotient categories of modules* by Carol Walker, New Mexico State University.

3. *Concordant and harmonic functors* by Elbert Walker, New Mexico State University and the Institute for Advanced Study.

4. *The algebra of functions* by Berthold Schweizer, University of Arizona.

5. *Abstract characterization of the mapping algebra of Schweizer and Sklar* by H. Ian Whitlock, University of Arizona (introduced by Professor Schweizer).

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group and of algebraic and transcendental group extensions were augmented by defining conjugacy (over a subgroup) of pairs of elements in an algebraically closed abelian group. Normal and separable group extensions were defined and analogies of several theorems from the field theory were stated for abelian groups and their extensions.

E. L. WALTER, *Secretary*

MAY MEETING OF THE ILLINOIS SECTION

The forty-third annual meeting of the Illinois Section of the MAA was held at Bradley University, Peoria, Illinois, on May 8 and 9, 1964. Professor Eugene Hellmich, vice chairman of the Section, presided due to the illness of the chairman, Professor Charles Moulton. Sessions were held Friday afternoon and Saturday morning. There were 119 persons in attendance, including 88 members of the Association.

The following officers were elected: Chairman, Eugene Hellmich, Northern Illinois University; Vice Chairman, Amos Black, Southern Illinois University; Secretary-Treasurer, Arnold Wendt, Western Illinois University.

The business meeting opened with a eulogy for the late Professor T. E. Rine of Illinois State University, who passed away suddenly of a heart attack on April 23, 1964. The eulogy was read by Professor Francis Brown of Illinois State University. Professor Rine was a past chairman of the Illinois Section and was the regional representative of the Secondary Lecturers Program in Illinois-Indiana at the time of his death. In his memory the section voted to send a check for $100 to Mrs. Rine along with heartfelt condolences.

Professor Douglas Bey of Illinois State University, the former chairman of the Committee on Secondary School Lecturers, gave the report for the committee. He reported that the NSF grant for the Illinois-Indiana sections amounted to $4500. In addition to this there was the allocation of $500 made by the Illinois Section toward support of the program in Illinois. Forty-eight of the sixty-five Illinois schools applying were granted lecturers. Each of the sixteen Indiana schools applying was granted a lecturer. Twenty-six lecturers from Illinois and fourteen from Indiana participated.

The Membership Committee under the chairmanship of John Schumaker reported sending membership materials to the 37 nonmembers attending last year's meeting and sending copies of the Section Newsletter to 15 college and junior college mathematics departments having no known members in the Association. The committee is pleased to learn of the MAA Representative plan since the committee has been conducting a similar activity the past year.

Hiram Paley, chairman, gave the report of the Undergraduate Participation Committee. The committee recommended that increased undergraduate participation in section activities be encouraged by having an invited address specifically for undergraduates by a prominent mathematician from academic or industrial ranks, by instructors being alert for students capable of presenting acceptable papers, and by the Illinois Section encouraging participation in the Putnam competition by offering prizes to those Illinois students scoring highest.

The Section voted to grant the committee $200 to implement its third recommendation, and the first and second recommendations were commended to the attention of the Program Committee.

The Contest Committee report was presented by its chairman, Walter McCurdy. He reported another highly successful year, both from the standpoint of participation and finances. 16,664 students from 316 schools participated. Lane Technical High School for the second year was the winning team and also had, for the third consecutive year, the highest scoring individual in Richard Schroeppel.

The Section realized a profit of $607 from conducting the contest, indicating a large amount of volunteer work on the part of the committee.

An *ad hoc* committee, the Committee on Subject Matter Organizations, under the

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6. *Repeated quadrature methods for the numerical solution of differential equations*, by L. D. Gates, Jr., Southern Illinois University.

A one-step method for numerical solution of differential equations uses a first order difference equation in place of a first order differential equation, the best known being the Runge-Kutta methods. A new family, the methods of repeated quadratures, is described. A method having order of accuracy n is obtained by using a quadrature formula of nth order accuracy with one-step methods whose accuracy is at least of order $n-1$. The methods of repeated quadrature are much easier to derive than the Runge-Kutta methods, and are somewhat better with respect to error propagation at the cost of being less efficient in computing time.

7. *Groups with nilpotent commutator subgroup*, by E. E. Shult, Southern Illinois University.

The following results are presented:

(a) Let V be a group of order pq acting as a fixed-point-free group of automorphisms on a solvable group G. If pq does not divide $o(G)$, the order of G, and if neither p nor q are Fermat primes when $o(G)$ is even, then G has nilpotent length at most 2.

(b) If, in (a), V is the symmetric group S_3, then G has nilpotent commutator subgroup.

(c) Let V be a solvable group with the property that whenever V is a fixed-point-free group of operators on a solvable group, G, for which $(o(G), o(V))=1$, G' is nilpotent. Then V is either cyclic of prime order, has order 4, or is S_3.

ARNOLD WENDT, *Secretary*

MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the MAA was held on Saturday, May 2, 1964, at Butler University, Indianapolis. Ninety-eight persons attended of whom 60 were members of the Association. Chairman Harley Flanders of Purdue University presided. The meeting consisted of a symposium on *Probability and Statistics*. Discussions centered around the following hour lectures:

1. *The Zero-One laws of probability theory*, by D. L. Burkholder, University of Illinois.

2. *The statistical basis of decision under uncertainty*, by Leo Katz, Michigan State University.

3. *Industrial applications of quality control statistics*, by Irving W. Burr, Purdue University.

4. *Statistical inference in a problem of disputed authorship*, by D. L. Wallace, University of Chicago.

The authorship referred to is that of the *Federalist Papers*. This lecture has been published under the title "Inference in an Authorship Problem" in the Journal of the American Statistical Association, 58 (1963) 275–309.

The meeting also included a period for discussion of the role of the Section in high school contests. In the past, two types of contest have been held in the state, that of the Mathematical Association of America and a special contest sponsored by Indiana University. A crisis has been created by the recent decision of Indiana University to discontinue its contest, which had enjoyed considerable popularity, especially for use at the more elementary levels. Several school teachers expressed the opinion that the MAA contest cannot fill the void thus created and that a new contest is needed to replace the Indiana University contest. The question considered was whether the Section should assume the responsibility for this new contest. There was general agreement that it should not, since by so doing it would be putting itself in competition with its national parent organization. It was the consensus rather that the Section should actively encourage state schools to use the MAA contest and also to seek means within the MAA for giving the contest wider appeal.

At the business meeting there was a discussion of the Visiting Lecturer Program of

the Indiana Academy of Science by Prof. W. G. Kessel of Indiana State College, director of the program.

Officers elected for the coming year are: R. E. Dowds, Butler University, Chairman; Robert Troyer, Indiana University, Vice-Chairman; and Paul Mielke, Wabash College, Secretary-Treasurer.

P. T. MIELKE, *Secretary*

MAY MEETING OF THE KENTUCKY SECTION

The Kentucky Section of the MAA met May 1–2, 1964 at the University of Kentucky, Lexington, Kentucky. Professor J. C. Eaves, Chairman of the Section, presided. The first session dealt with mathematical education. Participants in this program were Professor J. C. Eaves, University of Kentucky, Chairman of the Kentucky Section; Dr. Sidney Simandle, Kentucky State Department of Education; Dr. Allan Anderson, Western Kentucky State College; Professor Alvin McGlasson, Eastern Kentucky State College; Brother Edward Daniel, St. Xavier High School; Dr. T. J. Pignani, University of Kentucky and Dr. Leland Scott, University of Louisville.

The following officers were elected for the coming year: Chairman R. S. Park, Eastern Kentucky State College; Secretary-Treasurer, W. C. Royster, University of Kentucky.

Papers presented at the second session were:

1. *Finite difference formulae for the Laplacian operator*, by W. S. Krogdahl, University of Kentucky.

For the purposes of numerical computation, the Laplacian operator is customarily given by some suitable finite difference operator. Such operators are generally represented by symmetric stencils which represent the array of coefficients of the values of the function at the points of a square grid. These stencils are not unique. It was shown how suitable stencils might be generated and a method was suggested for eliminating the ambiguity in a systematic way.

2. *Restricted convergence of multiple series*, by Henry Spragens, University of Louisville.

A discussion of various schemes for summing multiple series was given along with several examples.

3. *Quasi-conformal mappings by the Grotzsch definition*, by Harold Robertson, University of Kentucky.

An expository talk on the Grotzsch definition and quasi-conformal mappings was given. A generalization of the Schwarz lemma was obtained for K-quasi-conformal mappings.

4. *Simple applications of functional analysis*, by Casper Goffman, Purdue University (by invitation).

Two examples, one concerning summability and one concerning universal series, were used to indicate how functional analysis can be applied to give simple and precise results in analysis.

5. *Remarks on product integrals*, by Raymond Cox, University of Kentucky.

The notion of a *product integral* for a function A, from the real line to a set of $n\times n$ matrices, was discussed.

In particular, necessary and sufficient conditions on A were given to insure the existence of the product integral; and several properties of the integral, such as its plane series expansion, were given.

6. *On matrix representation of cubic forms*, by J. C. Eaves, University of Kentucky.

The multiplication of matrices is extended to include three dimensional matrices in such a way that the expression for the general cubic in n variables is given as a product of matrices, one of which is of dimension $n\times n\times n$.

W. C. ROYSTER, *Secretary*

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W. C. ROYSTER, *Secretary*

MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual spring meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at the Westinghouse Defense Center, Friendship International Airport, Glen Burnie, Maryland, on May 2, 1964. Dr. John W. Wrench, Jr. Chairman of the section, presided. There were 101 persons in attendance, including 89 members of the Association. The invited address on "Information Theory" was delivered by Dr. Brockway McMillan, Under Secretary of the Air Force for Research and Development.

At the business meeting the following officers were elected: Chairman Daniel Shanks, Consultant, Applied Mathematics Laboratory, David Taylor Model Basin; Vice Chairmen, J. W. Brace, University of Maryland; and J. E. Shockley, College of William and Mary; Secretary, S. S. Saslaw, U. S. Naval Academy; Treasurer, S. B. Jackson, University of Maryland.

The following program was presented:

1. *Calculus of variations solution for minimum energy transfer between two coplanar circular orbits*, by W. H. Land, Jr., IBM Federal Systems Division, Rockville, Maryland.

This paper presents a variational formulation, whereby the optimum thrust magnitude and thrust direction of a satellite are programmed in such a way that the satellite will traverse between two coplanar circular orbits with a minimum expenditure of fuel. Also included are the Weierstrass Excess function, the first integral, the transversality conditions and the dynamical and kinematical equations of motion as they apply to this problem. A method for solving the two-point boundary problem by iterative means is developed along with the computer flow chart for the complete system solution.

2. *A generalization of Rennie's inequality*, by A. J. Goldman, Operations Research Section, National Bureau of Standards.

A useful inequality due to Kantorovich, giving an upper bound for the product of a weighted mean and the corresponding weighted mean of reciprocals, was extended by Rennie (this MONTHLY, 70) to an additive inequality. This is now generalized to a bound for a certain linear combination of an rth order moment and an sth order moment. The generalization yields the Cargo-Shisha bounds on ratios of weighted means (J. Res. Nat. Bur. Stand., 66B) in the same way that Rennie's inequality yields Kantorovich's. To appear in J. Res. Nat. Bur. Stand., 68B.

3. *Computer assisted instruction in mathematics*, by Joseph Hilsenrath, National Bureau of Standards.

The present method of using computers in the undergraduate program in mathematics and other departments requires the student to learn to program. This time-consuming activity leaves room for little more than one or two simple applications. Although it may motivate some students to become programmers, this system does not help the student obtain a deeper insight into mathematical and physical concepts being treated in the normal curriculum.

The use of a general purpose interpretive program like OMNITAB or COGO which instructs the machine via simple English words permits the teacher to plan dozens of exercises to be solved by the students on the machine. Examples are given from elementary mathematics, statistics, and physics, in which the computer can give insight and experience which would ordinarily be ruled out by time limitations.

4. *A note on the use of the Laplace transform and initial conditions in the solution of a system of linear differential equations*, by F. Marion Clarke-Carroll, Computer and Data Systems Technology Group, Westinghouse Electric Corporation.

The easy application of the Laplace transformation to the use of nonzero initial conditions in the solution of a system of linear differential equations with constant coefficients, together with its preservation of the jump capability of the system, and the adaptability of the result to high speed digital techniques are compared with the hazards of classical integration procedures through a simple example.

and Huntington. Report of special committee of National Academy of Sciences. Detailed discussion of method of equal proportions which has been used beginning with the reapportionment after the 1940 Census. Discussion of minimum range solution proposed by Burt and Harris in paper published in Operations Research, July–August 1963. Reference to papers by Richard Bellman on Dynamic programming.

S. S. SASLAW, *Secretary*

MAY MEETING OF THE MINNESOTA SECTION

The annual spring meeting of the Minnesota Section of the MAA was held on May 9, 1964, at the College of St. Thomas, in St. Paul, Minnesota. Hubert Walczak, College of St. Thomas, presided at the morning session, and the Section Chairman, Seymour Schuster, University of Minnesota, presided at the afternoon session. There were 129 persons registered for the meeting, of whom 108 were members of the Association.

At the business meeting copies of the report of the 1964 Minnesota High School Mathematics Contest were distributed. The Minnesota Section is one of the sponsoring agencies for this contest. Ten thousand four hundred and fifty Minnesota students participated this year.

The following officers were elected to serve during the school year of 1964–5: Chairman, Robert Cameron, University of Minnesota; Secretary-Treasurer, Walbert Kalinowski, St. John's University; Members of the Executive Committee: Seymour Schuster, University of Minnesota, Frank Arena, North Dakota State University, and Stanley Dice, Carleton College.

At the close of the regular session, for any who wished to participate, Dr. Robert Smith, of Control Data Corporation, and Professor James Lindsay, College of St. Thomas, conducted a short seminar on computer operation, in St. Thomas's new computing center. There were about twenty-five participants, each of whom had an opportunity to compose programs and try them on the computer.

The following papers were presented:

1. *Near rings on groups of prime order*, by R. A. Jacobson, South Dakota State University.

This note develops the structure and, consequently, the exact number of left near rings on groups of prime order.

2. *On the derived set operator*, by Shair Ahmad, South Dakota State University.

The four axioms for a derived set operator as given by Harvey are shown to be equivalent to three somewhat simpler axioms. A slight modification of these axioms renders them absolutely independent.

3. *New identities for Chebyshev polynomials and Fibonacci numbers*, by David Zeitlin, Minneapolis.

4. *Solution of a boundary value problem for the biharmonic difference operator*, by Charles Turner, Macalester College.

A formula was discovered for writing explicitly the solution to a particular boundary value problem for the biharmonic partial difference operator over any n by m region.

5. *A note and a query on* Σ *and* Π, by Daihachiro Sato, University of Saskatchewan, Regina.

6. *Directed graphs and the structure of powers of nonnegative matrices*, by A. L. Dulmage, University of Manitoba (by invitation).

The connection between the structure of powers of a nonnegative irreducible matrix and the directed graph of the matrix is well known. In this paper the structure of powers of a reducible matrix is discussed. The structure of the sub-diagonal blocks in powers of such a matrix is seen to depend strongly on the sets of imprimitivity of the directed graphs of the constituents of the matrix.

7. *A Fourier-type shift theorem for Walsh Functions*, by F. R. Ohnsorg, Honeywell Corporation, Minneapolis.

In a Fourier series representation of a function the amplitudes of the harmonic components are invariant under the shift transformation $f(x) \to f(x+x_0)$. It is shown that similar though not identical invariant functionals exist for finite sets of Walsh functions under the corresponding shift transformation where now x_0 is dyadically rational. The proof and generation of these functionals are obtained through Hadamard matrices corresponding to the Walsh sets.

8. *Absolutely independent axioms for groups*, by K. L. Yocom, South Dakota State University.

Recently, Harary introduced the notion of absolute independence of axiom systems. In this note, such axiom systems are exhibited both for a semi-group with right identity, and for a group.

9. *Etymology of mathematical terms*, by Margaret W. Perisho, Mankato State College.

The origin of words used in mathematics is an interesting study and aids in understanding the present meaning of terms. The etymology of some words with interesting origins was given, and non-mathematical related words were discussed.

10. *Independence and quantification*, by Kenneth O. May, Carleton College.

The word "independent" is often used in analysis to emphasize order of quantification, e.g., in the definition of uniform convergence: For every epsilon . . . there exists a delta independent of x . . . . This usage is redundant, falsely suggests that lack of "independence" means functional dependence, and may lead to error when used to indicate the order of universal quantification. The reality of this danger is illustrated by E. D. Rainville's treatment of Saalschütz' theorem (*Special Functions*, New York 1960, pp. 86 ff.), where the order of universal quantification of four variables in the proof is reflected in a restriction that the first three must be "independent" of (not functionally dependent on) the fourth.

11. *The first course in computer programming*, by C. B. German, College of St. Thomas.

Over 500 colleges and 100 high schools now offer courses in computer programming. These courses for the most part are of four types: (1) Covering only machine language. These courses are most common at the high school level and account for about 5% of the students who receive instruction in computing. (2) Covering only Fortran and intended to enable the freshman engineer to use the computer much as he now uses a slide rule. About 75% of the students fall into this category. (3) Covering machine language, symbolic programming, and an introduction to Fortran. About 8% of the students, mostly in business courses, fall into this category. (4) Covering machine language, Fortran, symbolic programming and sometimes numerical analysis. About 12% of the students fall into this category.

12. *The image of the set where the derivative is zero*, by D. E. Varberg, Hamline University.

For a real valued function f defined on $[a, b]$, let $D = \{x \mid f(x)$ is a maximum or a minimum$\}$ and $E = \{x \mid f'(x) = 0\}$. It is shown that: $f(D)$ is at most countable, $f(E)$ may be uncountable (cf. problem 5114 this MONTHLY) but the Lebesgue measure of $f(E)$ is zero.

MURRAY BRADEN, *Secretary*

MAY MEETING OF THE OHIO SECTION

The forty-eighth annual meeting of the Ohio Section of the MAA was held at the University of Akron, Akron, Ohio, on Saturday, May 9, 1964. Professor Charles Capel, Chairman of the Section, presided at the general sessions, and Professors Robert Roberts and John Warner presided at the sectional sessions. One hundred fifty-one registered in attendance, including one hundred seventeen members of the Association.

Officers elected for the following year are: Chairman, Andrew Sterrett, Denison University; Chairman-elect, W. T. Fishback, Ohio University; Secretary-Treasurer, Foster Brooks, Kent State University. Program Committee: Robert Roberts, Denison

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6. There are just five known Fermat primes $2^{2^t}+1$, $(t=0, 1, 2, 3, 4)$.
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8. Kuratowski's Theorem on nonplanar graphs.
9. Heawood's Theorem that every planar graph is 5-chromatic.

7. *Computation of elliptic integrals using Gauss' transformation*, by H. E. Fettis, Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Wright-Patterson Air Force Base.

By means of Gauss' transformation, the computation of the three kinds of elliptic integral may be reduced to routine operations involving only elementary functions, without any further restrictions on the modulus and parameter. The resulting formulae are easily programmed to provide subroutines for a digital computer.

8. *Projective invariants of a curvilinear element*, by Rodney Angotti, University of Akron.

The projective invariants of certain configurations associated with a regular third order differential element, i.e., expansions including the third degree terms in a projective three space are discussed; in particular, a construction of one such invariant is exhibited.

9. *The construction of the real numbers*, by L. D. Rodabaugh, Ohio Northern University.

An extension and refinement of the author's earlier work on this subject as reported to the Illinois Section in May 1951, (see this MONTHLY, 59 (1952) 286).

10. *Some theorems about simple semigroups*, by C. E. Aull, Kent State University.

The following are proved: A semigroup S is a simple semigroup iff for $a, b\in S$, the equation $xay=b$ has at least one solution $x, y\in S$. A simple semigroup S, with identity is a group if any of the following conditions is satisfied: (a) S is commutative, (b) S is left (right) cancellative, (c) S is finite.

11. *Statistical hypothesis modification —a new point of view for statistical inference*, by Thaddeus Dillon, Youngstown University.

Instead of accepting or rejecting the hypothesis $\theta=\theta'$, it is suggested that the hypothesis be modified to $\theta=(Q+\lambda\theta')/(1+\lambda)$, where Q is a statistic and λ is a nonnegative real function of three variables: (1) sample size, (2) population size, and (3) the probability used for comparison to decide whether to accept or reject the hypothesis. Under rather general conditions on the power function such procedures are self-correcting in the sense that the worse of two theories is likely to receive less weight and a really bad theory θ' is likely to receive negligible weight.

FOSTER BROOKS, *Secretary*

MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The forty-seventh annual meeting of the Rocky Mountain Section of the MAA was held at Colorado College, Colorado Springs, Colorado, on Friday and Saturday, May 1 and 2, 1964.

The following officers were elected for 1964–65: Chairman, F. M. Carpenter, Colorado School of Mines; Vice-Chairman, F. M. Stein, Colorado State University; and Secretary-Treasurer, W. N. Smith, University of Wyoming. E. R. Deal, continues, in his second year of a three-year term, as coordinator of High School Mathematics Contests.

The 1965 spring meeting will be held at the Colorado School of Mines, Golden, Colorado.

Changes in the By-Laws for the section were considered and discussed and a new draft approved for presentation to the Board of Governors for approval.

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The Friday evening guest speaker was Professor W. J. LeVeque, visiting professor at the University of Colorado; his topic was *Probability and Number Theory.*

The following papers were presented:

1. *Idempotent matrices* (mod p^a), by J. H. Hodges, University of Colorado.

For positive integers m, a and prime p, the number $N(m, p, a)$ of idempotent matrices (mod p^a) of order m is determined. First, the number for $a=1$ is determined by using canonical forms, involving elementary divisors, for matrices under similarity. Then it is shown that $N(m, p, a+1) = N(m, p, 1)$ for all $a \geqq 1$. The method employed in the second step is the standard recursive one in number theory of using solutions mod p^a to generate solutions mod p^{a+1}. $N(m, p, 1)$ can be expressed as a simple sum involving the number g_r of nonsingular matrices of order r (mod p).

2. *An application of symmetric functions to statistics*, by P. W. Mielke, Colorado State University.

It is well known that symmetric functions have desirable statistical estimation properties. Methodology for treating the two-way classification finite model with disproportionate population subcell sizes is discussed. In particular some symmetric function variance component estimators are introduced which can be applied to this present model even if the sample subcell sizes are disproportionate. An immediate consequence of the use of symmetric functions is the unbiased estimation of the sampling variance for these variance component estimators.

3. *Estimation with some prior information*, by M. M. Siddiqui, Colorado State University.

4. *Chebyshev lines*, by B. L. Foster, Denver Research Center, Marathon Oil Company.

The best fitting line for a set of data points depends on what is meant by best. According to Chebyshev, that line is best which minimizes the worst data deviation. The x-Chebyshev line is the one minimizing the worst x-deviation; the y-Chebyshev line minimizes the worst y-deviation. With uninteresting exceptions, these lines are the same. Using the over-under-over theorem discussed by Scheid (this MONTHLY, 68 (1961) 862), this can be proved by a simple geometrical argument that extends to oblique coordinate systems. A different proof was announced at this meeting of the Association by Professor M. M. Siddiqui.

5. *Some results on T-fractions*, by B. W. Jones and W. J. Thron, University of Colorado.

A T-fraction is a continued fraction of the form $(1+d_0z)+z/(1+d_1z)+z/(1+d_2z)+\cdots$, where z is a complex variable and the d_n are complex numbers. Among convergence criteria for T-fractions given by W. J. Thron (Bull. Amer. Math. Soc., 54 (1948) 206–218) is the following: if $d_n>0$ for $n\geqq 0$, the T-fraction converges for all z, not on the negative real axis, to a function $f(z)$ which is holomorphic in the interior of this region. In the present work the authors show that if $d_n > 0$ for $n \geqq 0$, there exists a bounded, nondecreasing function $\psi(t)$ such that $f(z) = 1 + d_0z + z\int_{-\infty}^{0} d\psi(t)/(z-t)$. If t_0 is the largest point of increase of $\psi(t)$ then $z=t_0$ is a singular point of $f(z)$ or if $\{d_n\}$ is unbounded then $f(z)$ has a singularity at $z=0$.

6. *Hankel transforms and entire functions II*, by K. R. Unni, Utah State University.

7. *The value of a coalition in applied games*, by W. C. White, Cadet, USAFA.

8. *The University of Colorado Computer Center for secondary schools*, by R. L. Albrecht, Control Data Corporation.

A center for exploring methods of secondary school computer education has been established by the University of Colorado, College of Engineering, Denver Center, with the cooperation of the Control Data Corporation and the Denver Chamber of Commerce. During the 1963–64 school year, 144 high school students and 26 high school teachers were enrolled in an experimental program. The main objective is the development of methods for using a computer to reinforce classroom training in secondary school mathematics and science. The computer is regarded as a "mathematics laboratory" with which students solve textbook problems, perform mathematical experiments, and process scientific data.

CALENDAR OF FUTURE MEETINGS

Forty-eighth Annual Meeting, Denver-Hilton Hotel, Denver, Colorado, January 28–30, 1965.

Forty-sixth Summer Meeting (Fiftieth Anniversary Celebration), Cornell University, Ithaca, New York, August 30–September 2, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Inst. of Tech. Pittsburgh, Pa., May 1, 1965.

ILLINOIS, Southern Illinois University, Carbondale, May 14–15, 1965.

INDIANA

IOWA, University of Dubuque, Dubuque, April 23, 1965.

KANSAS, Washburn University, Topeka, April 10, 1965.

KENTUCKY, Eastern Kentucky State College, Richmond, Spring, 1965.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 12–13, 1965.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Towson, Maryland, November 21, 1964.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1965.

MINNESOTA, University of Minnesota at Duluth, November 7, 1964.

MISSOURI, University of Missouri, Columbia, Spring, 1965.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 30–May 1, 1965.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 7, 1964.

NORTHEASTERN, Worcester Polytechnic Institute, Worcester, Mass., November 28, 1964.

NORTHERN CALIFORNIA, College of San Mateo, February 6, 1965.

OHIO

OKLAHOMA, University of Arkansas, Fayetteville, Spring, 1965.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 18, 1965.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 21, 1964.

ROCKY MOUNTAIN, The Colorado School of Mines, Golden, Colorado, Spring, 1965.

SOUTHEASTERN, Wake Forest College, Winston Salem, North Carolina, April 9–10, 1965.

SOUTHERN CALIFORNIA, Claremont Men's College, March 13, 1965.

SOUTHWESTERN, Arizona State University, Tempe, Spring, 1965.

TEXAS, Texas Christian University, Fort Worth, April 9–10, 1965.

UPPER NEW YORK STATE, Colgate University, Hamilton, May 15, 1965.

WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Montreal, December 26–31, 1964.

AMERICAN MATHEMATICAL SOCIETY, Denver, Colorado, January 26–29, 1965.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Illinois Institute of Technology, Chicago, June 21–25, 1965.

ASSOCIATION FOR COMPUTING MACHINERY, Cleveland, August 24–26, 1965.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Detroit, November 26–28, 1964.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, November 19–21, 1964.

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The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2} \times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

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NONORTHOGONAL IDEMPOTENTS WHOSE SUM IS IDEMPOTENT

J. G. MAULDON, Corpus Christi College, Oxford

Throughout this paper $\mathfrak{R}$ will denote an associative ring with a unit element e, whose additive group is torsion-free. The main result is

THEOREM 1. *For every integer $n>2$, there exists a ring $\mathfrak{R}$, whose additive group is torsion-free, containing a set $\{a_1, a_2, \cdots, a_n\}$ of n idempotents whose sum is idempotent, such that $a_ia_j\neq 0$ for all $i, j=1, 2, \cdots, n$.*

Two interesting special cases, used in the proof of Theorem 1, are

LEMMA 1. *There exists a ring $\mathfrak{R}_1$, whose additive group is torsion-free, containing a set of four mutually nonorthogonal idempotents whose sum is the unit e.*

LEMMA 2. *There exists a ring $\mathfrak{R}_0$, whose additive group is torsion-free, containing a set of four mutually nonorthogonal idempotents whose sum is zero.*

Any three of the idempotents in Lemma 1 yield a negative answer to the question raised in Problem 5082 [5], namely:

Let $\mathfrak{R}$ be a ring in which, if either $x+x=0$ or $x+x+x=0$, it follows that $x=0$. Suppose that a, b, c and $a+b+c$ are all idempotents in $\mathfrak{R}$. Does it follow that $ab=0$?

A rather more general question is answered in Section 15, where there are also some conjectures on uniqueness of the solution of this and related problems. In Section 14 it will be seen that, for certain fairly general types of ring, the answer to [5] is affirmative.

As a result of careful and valuable criticism by a referee, hereby gratefully acknowledged, the presentation of the results has been entirely recast, clarifying both the motivation of the method and the possibility of its wider applicability —see, in particular, the end of Section 1 and (13) and (14) in Section 10.

1. An outline of the construction. We start in Sections 2 and 3 by attempting a familiar method of attack on Lemma 1. Our primary object is to exhibit a ring $\mathfrak{R}$, with a torsion-free additive group, containing elements A, B, C satisfying the conditions

$$(1)\quad A^2=A,\quad B^2=B,\quad C^2=C,\quad (A+B+C)^2=A+B+C,\quad AB\neq 0.$$

We therefore construct a ring $\mathfrak{B}$ containing elements a, b, c and $d=e-a-b-c$ such that the ideal $\mathfrak{a}$ generated by a^2-a, b^2-b, c^2-c and $(a+b+c)^2-(a+b+c)$ $(=d^2-d)$ does not contain the element ab. Then the residue class ring $\mathfrak{R}=\mathfrak{B}/\mathfrak{a}$ clearly satisfies the required conditions, provided that its additive group is torsion-free.

Looking at the matter in a slightly different way, we may say that the ring $\mathfrak{B}$ is to admit an equivalence relation $(x\equiv y\Leftrightarrow x-y\in\mathfrak{a})$ such that (i) the set of equivalence classes forms a ring in a natural way, (ii) $a^2\equiv a$, etc., (iii) $ab\not\equiv 0$ and (iv) $x\not\equiv 0\Rightarrow x+x+\cdots+x\not\equiv 0$. The advantage of using a ring $\mathfrak{B}$ larger

than the required ring $\mathfrak{R}$ (in fact, having $\mathfrak{R}$ as a homomorphic image) is that we can weaken our equivalence relation beyond the trivial one ($x \equiv y \Leftrightarrow x - y = 0$), giving more flexibility in our attack on (iii) and (iv).

Unfortunately, while (as will be seen in Section 3) it is easy to find rings $\mathfrak{B}$ for which (iii) holds, and also those for which (iv) holds, we have not succeeded in finding a ring $\mathfrak{B}$ admitting a direct proof of (iii) and (iv) simultaneously. The main novelty of the present method is that, in order to weaken our equivalence relation still further, we introduce in Section 4 an even larger algebraic system $\mathfrak{S}$, admitting $\mathfrak{B}$ as a homomorphic image. We then find that a direct proof of (i), (ii), (iii) and (iv) is possible for the equivalence relation in $\mathfrak{S}$ corresponding to the above equivalence relation in $\mathfrak{B}$ (and to the identity relation in $\mathfrak{R}$). This enables us to prove Lemma 1, which is a substantial step in the proof of Theorem 1.

The author owes to the referee the observation that this technique can be applied to other problems of the type:

Given an ideal $\mathfrak{a}$ *generated by certain elements of the semigroup algebra* $\mathfrak{B}$ *of a free semigroup relative to a ring* Λ, *and an element* $z \in \mathfrak{B}$, *show that* $z \notin \mathfrak{a}$.

Two such problems are solved in Section 10—see (13), (14).

2. The free semigroup M. The *free multiplicative semigroup* or *monoid* generated by a given set E may be defined [2, p. 18] as the set of all finite sequences of elements of E with multiplication defined by juxtaposition, so that the *product* of the sequences $a_1a_2 \cdots a_m$ and $b_1b_2 \cdots b_n$ (a_i, $b_j \in E$) is simply the sequence $a_1a_2 \cdots a_mb_1b_2 \cdots b_n$. (Such sequences, with or without a scalar coefficient, may be referred to as *monomials*.) The empty sequence is conventionally included and (by the above definition) it is in fact the unit element of the monoid (a *monoid* is a semigroup with a unit).

If the additive notation is used, so that the *sum* of the sequences $a_1+a_2+ \cdots +a_m$ and $b_1+b_2+ \cdots +b_n$ is $a_1+a_2+ \cdots +a_m+b_1+b_2+ \cdots +b_n$, we have the *free additive semigroup* generated by E.

We may identify sequences differing merely in the order of their terms. More precisely, we may regard such sequences as *equivalent* and study the *quotient semigroup* [1], [2], whose elements are the equivalence classes and whose combining operation is defined by asserting that the natural map is a homomorphism. In this case we shall have a commutative semigroup—the *free commutative semigroup* generated by the set E [2, p. 20]. We shall make use of this concept in Section 4, but for the present we define M to be the free multiplicative semigroup generated by the set $\{a, b, c\}$, noticing that M has a unit element.

3. The ring $\mathfrak{B}$ and the ideal $\mathfrak{a}$. Let Λ be any ring with a unit, whose additive group is torsion-free. Λ will have the rôle of a coefficient ring and, as far as this paper is concerned, nothing would be lost by assuming Λ to be a commutative ring, thus simplifying the definition of the ring $\mathfrak{B}$ below. For future purposes, however, we present the slightly more general theory in which Λ need not be

commutative. Let M be the multiplicative semigroup defined in Section 2 and let $\mathfrak{B}$ be the *semigroup ring* determined by the semigroup M and the ring Λ. The definition of $\mathfrak{B}$ is exactly like that ([2], Ch. 4, Section 1, e) of the semigroup algebra of a semigroup over a commutative ring with identity, except that the coefficient ring Λ need not be commutative and hence we regard $\mathfrak{B}$ as a ring, not as a Λ-algebra. Thus M is identified with a basis of the unitary Λ-module $\mathfrak{B}$, and multiplication in $\mathfrak{B}$ is the unique bilinear extension of multiplication in M, so that

$$\left(\sum_{i=1}^{p} \lambda_i m_i\right)\left(\sum_{j=1}^{q} \mu_j m_j'\right) = \sum_{i=1}^{p}\sum_{j=1}^{q} (\lambda_i\mu_j)(m_i m_j') \qquad (\lambda_i, \mu_j \in \Lambda, m_i, m_j' \in M). \tag{2}$$

$\mathfrak{B}$ may also be regarded as the ring of all noncommutative polynomials in the three indeterminates a, b, c with coefficients in the noncommutative ring Λ. If ϵ is the unit element of Λ it is convenient to denote, for example, the element $\epsilon m^2+(-\epsilon)m$ by $m^2-m\in\mathfrak{B}$ $(m\in M)$.

Following the plan of Section 1 we now define $\mathfrak{a}\subset\mathfrak{B}$ as the ideal generated by the four elements a^2-a, b^2-b, c^2-c and $bc+cb+ca+ac+ab+ba$. Taking the residue classes as equivalence classes, we have

$$a^2 \equiv a, \quad b^2 \equiv b, \quad c^2 \equiv c, \quad cb \equiv -bc-ca-ac-ab-ba. \tag{3}$$

Now if, in particular, Λ is the ring of integers, let $\mathfrak{b}$ be the set of all $x\in\mathfrak{B}$ such that the sum of the coefficients of x, expanded as a linear combination of the elements of M, is even. Clearly $\mathfrak{b}$ is an ideal containing $\mathfrak{a}$ but not ab, so that $ab\notin\mathfrak{a}$, $ab\neq 0$, which is (iii) of Section 1. If, on the other hand, Λ is a field, the property (iv) of Section 1 is trivial. The difficulty facing us is to satisfy (iii) and (iv) simultaneously.

Premultiplying the last equivalence in (3) by c and using (3) to eliminate factors of the form a^2, b^2, c^2 and cb, we find

$$cab \equiv 2ac+2ba+2bc+ca+ab+abc+bca+bac+aba+aca. \tag{4}$$

Using (3) and (4), it would be easy to show that every equivalence class contains at least one element with no factor a^2, b^2, c^2, cb or cab in any of its terms. If we could then show that each equivalence class contains *only one* element of this "canonical" type, we should have an immediate proof of properties (i), (ii), (iii) and (iv), and hence of Lemma 1.

It is in fact true, for any coefficient ring Λ, that each residue class (mod $\mathfrak{a}$) in $\mathfrak{B}$ contains a unique canonical representative (see Lemma 10), but in order to prove this fact we study an enlarged system $\mathfrak{S}$, the *semigroup semiring* determined by the semigroup M and the ring Λ.

4. The semiring $\mathfrak{S}$. Let M and Λ be as defined in Sections 2 and 3, let Λ^* be the set of nonzero elements of Λ, and let $\mathfrak{S}$ be the free commutative additive semigroup (see Section 2) generated by the set

$$\Lambda^*\times M = \{\lambda m : \lambda\in\Lambda^*, m\in M\}.$$

Thus any element of $\mathfrak{S}$ is either zero or a finite commutative sum of the form $\sum_1^p \lambda_i m_i$ $(0 \neq \lambda_i \in \Lambda,\ m_i \in M)$, where there is no combination or cancellation of similar terms, so that $\lambda m + \mu m \neq (\lambda + \mu)m$. We now introduce into $\mathfrak{S}$ the associative left- and right-distributive multiplication defined by (2), where any terms on the right for which $\lambda_i \mu_j = 0$ are simply omitted from the sum. Then $\mathfrak{S}$ can properly be described as the *semigroup semiring* determined by the semigroup M and the ring Λ. The unit element ϵ of Λ will again be omitted, so that (for example) $\epsilon m + (-\epsilon)m$ will be denoted by $m - m \in \mathfrak{S}$, which is not equal to the zero element of $\mathfrak{S}$, even when $m \in M$ is the empty sequence.

5. Reducibility in $\mathfrak{S}$. Referring to (3) and (4) for our motivation, we pick out ten elements $\{X_\alpha,\ Y_\alpha : \alpha = 1, \cdots, 5\}$ of $\mathfrak{S}$ by writing

$$\begin{aligned} &X_1 = a^2, \quad Y_1 = a; \quad X_2 = b^2, \quad Y_2 = b; \quad X_3 = c^2, \quad Y_3 = c; \\ (5)\quad &X_4 = cb, \quad Y_4 = -bc - ca - ac - ab - ba; \\ &X_5 = cab, \quad Y_5 = 2ac + 2ba + 2bc + ca + ab + abc + bca + bac + aba + aca, \end{aligned}$$

noticing that $2 \neq 0$, since Λ is torsion-free.

A nonzero element $\sum_1^p \lambda_i m_i \in \mathfrak{S}$ will be said to be *reducible* if either (i) $m_i = m_j$ for some $i \neq j$ or (ii) at least one of the m_i contains as a factor one of the X_α—more precisely if (ii) $\exists\ i \in \{1, \cdots, p\}$ and $\exists\ \alpha \in \{1, \cdots, 5\}$ such that $m_i = uX_\alpha v$, where $u, v \in M$. In the first of these two cases $(m_i = m_j)$ we write $y\mathbf{R}_\sigma x$ $(x, y \in \mathfrak{S})$ if the expression y is obtained from x by replacing $\lambda_i m_i + \lambda_j m_j$ by $(\lambda_i + \lambda_j)m_j$, omitting this term altogether if $\lambda_i + \lambda_j = 0$; in the second case $(m_i = uX_\alpha v)$ we write $y\mathbf{R}_\pi x$ if y is obtained from x by replacing m_i by $uY_\alpha v$ and expanding by the distributive laws; finally we introduce into $\mathfrak{S}$ the binary relation $\mathbf{R}$ by the definition

(D1) $$y\mathbf{R}x \text{ iff either } y\mathbf{R}_\sigma x \text{ or } y\mathbf{R}_\pi x \qquad (x, y \in \mathfrak{S}).$$

Intuitively $y\mathbf{R}x$ means "y is a reduced form of x" and the subscripts σ, π refer to the words *sum* and *product*. An element $x \in \mathfrak{S}$ is *irreducible* iff $\{y : y\mathbf{R}x\}$ is the empty set.

6. The equivalence relation in $\mathfrak{S}$. We now introduce into $\mathfrak{S}$ two more binary relations, $\leq$ and $\equiv$, by the definitions

(D2) $y \leq x$ *iff* $\exists n \geqq 0$ *and* $x_0, \cdots, x_n \in \mathfrak{S}$ *such that* $x = x_0$, $x_n = y$ *and* $x_j\mathbf{R}x_{j-1}$ *for* $j = 1, 2, \cdots, n$.

(D3) $x \equiv y$ *iff* $\exists z \in S$ *such that* $z \leq x$ *and* $z \leq y$.

The notation is justified by

LEMMA 3. $\leq$ *is an order relation*, and

LEMMA 4. $\equiv$ *is an equivalence relation.*

The proofs are given in the next two sections, together with the proof of

LEMMA 5. *Each equivalence class in* $\mathfrak{S}$ (*under* (D3)) *contains exactly one irreducible element.*

Since ab and 0 are distinct irreducible elements of $\mathfrak{S}$, Lemma 5 yields

$$ab \not\equiv 0. \tag{6}$$

Since Λ has a torsion-free additive group it follows that, if $x = \sum_1^p \lambda_i m_i$ is nonzero and irreducible, so is $\sum_1^p (n\lambda_i)m_i \equiv x + x + \cdots + x$ (n terms, $n \geqq 1$). Hence another corollary of Lemma 5 is that, if x is irreducible, then

$$x \not\equiv 0 \quad \text{and} \quad n \geqq 1 \Rightarrow x + x + \cdots + x \not\equiv 0 \ (n \text{ terms}). \tag{7}$$

Finally, it is immediate from the definition of $\equiv$ that

$$a^2 \equiv a, \quad b^2 \equiv b, \quad c^2 \equiv c, \quad bc + cb + ca + ac + ab + ba \equiv 0, \tag{8}$$

$$\sum_1^p \lambda_i m_i + \sum_1^p (-\lambda_i) m_i \equiv 0. \tag{9}$$

7. Proof of Lemma 3. The relation $\leq$ is obviously reflexive and transitive, and it only remains to prove that it is antisymmetric—that is, that $y \leq x$ and $x \leq y$ together imply $x = y$. We define the *height* $h(x)$ of any element x of $\mathfrak{S}$ by $h(0) = 0$ and

$$x = \sum_i \lambda_i m_i \Rightarrow h(x) = \sum_i (d_i + 1)(10)^{q_i}, \tag{10}$$

where d_i is the number of elements in the sequence $m_i \in M$, q_i is the total number of times in that sequence that a factor c precedes (at any distance) a factor b, and it will be remembered that $\lambda_i \neq 0$. The only important properties of this definition are that $h(x)$ is a nonnegative integer and that, as is easily verified, $h(y) < h(x)$ whenever $y\mathrm{R}x$. Hence, if $y \leq x$, either $y = x$ or $h(y) < h(x)$, the latter alternative being impossible if $x \leq y$.

8. Proofs of Lemmas 5 and 4. We start with Lemma 5. The fact that every element x of $\mathfrak{S}$ is equivalent to some irreducible element z (indeed that there is an irreducible $z \leq x$) is by induction on $h(x)$. If two irreducible elements x and y are equivalent, we have $z \leq x$, $z \leq y$ and hence $x = z = y$, so that there cannot be two distinct irreducible elements in the same equivalence class, and this completes the proof of Lemma 5.

For the proof of Lemma 4 we need

LEMMA 6. *If* $x\mathrm{R}t$ *and* $y\mathrm{R}t$, *then* $\exists z \in \mathfrak{S}$ *with* $z \leq x$ *and* $z \leq y$.

Proof. If $x\mathrm{R}_\sigma t$ and $y\mathrm{R}_\sigma t$, or if the reductions $t \to x$ and $t \to y$ do not interact, we can actually find $z \in \mathfrak{S}$ with $z\mathrm{R}x$ and $z\mathrm{R}y$, so that the (D2)-chains $x \to z$ and $y \to z$ are of length $n = 1$. If $x\mathrm{R}_\sigma t$ and $y\mathrm{R}_\pi t$ with interaction, we may assume without loss of generality that

$$t = \lambda u X_\alpha v + \mu u X_\alpha v \quad (\lambda, \mu \in \Lambda;\ u, v \in M),$$

with $x=(\lambda+\mu)uX_\alpha v$ and $y=\lambda uX_\alpha v+\mu uY_\alpha v$. Then, taking $z=(\lambda+\mu)uY_\alpha v$, we find that the (D2)-chain $x\rightarrow z$ is of length $n=1$ and the (D2)-chain $y\rightarrow z$ is obvious and inevitable, though (if $\alpha=5$) it may have to be as long as $n=11$.

If $t=\lambda uX_\alpha vX_\beta w$ $(u, v, w\in M)$, with $x=\lambda uY_\alpha vX_\beta w$ and $y=\lambda uX_\alpha vY_\beta w$, we may obviously take $z=\lambda uY_\alpha vY_\beta w$.

Hence finally it only remains to prove the result when t contains a term $\lambda m=\lambda uZv(u, v\in M)$, where $Z=c^2b$ or cb^2 or c^2ab or cab^2 and the reductions are effected on Z. By symmetry the cases $t=c^2b$ and c^2ab suffice, with $x=X_\alpha$, $y=cY_\alpha$, $z=Y_\alpha\leq x$ ($\alpha=4$ or 5). Then, although the minimal (D2)-chains $y\rightarrow z$ contain respectively 18 and 41 steps, yet repeated reduction in any manner of the element $y=cY_4[cY_5]$ of $\mathfrak{S}$ leads inevitably to the irreducible element $z=Y_4[Y_5]$ as required, completing the proof of Lemma 6.

LEMMA 7. *If $x\leq t$ and $y\leq t$, then $x\equiv y$.*

The result is trivial if $h(t)=0$ and we proceed by induction on $h(t)$. Ignoring the trivial cases $x=t$ and $y=t$, we have $x\leq u$, $u\mathbf{R}t$, $y\leq v$ and $v\mathbf{R}t$. By Lemma 6 $\exists w$ with $w\leq u$, $w\leq v$. Since $h(u)<h(t)$, $x\leq u$ and $w\leq u$, our inductive hypothesis shows that $x\equiv w$ and so $\exists r$ with $r\leq x$, $r\leq w\leq v$. Since $h(v)<h(t)$, $r\leq v$ and $y\leq v$, we similarly have $r\equiv y$. Hence, finally $\exists z$ with $z\leq r\leq x$, $z\leq y$, showing that $x\equiv y$ as required.

Referring to (D3) we see that the relation $\equiv$ is certainly reflexive and symmetric, so for the proof of Lemma 4 it will suffice to establish

LEMMA 8. *If $x\equiv y$ and $y\equiv z$, then $x\equiv z$.*

Proof. $\exists u, v$ with $u\leq x$, $u\leq y$, $v\leq y$, $v\leq z$. Lemma 7 shows that $u\equiv v$ so that $\exists w$ with $w\leq u\leq x$, $w\leq v\leq z$, and this completes the proof of Lemmas 8 and 4.

9. Proof of Lemma 1. The ring $\mathfrak{R}_1$. Since $\mathfrak{S}$ admits no cancellation, it follows from (D1) and (D2) that if $y\leq x$ then $y+z\leq x+z$, $yz\leq xz$ and $zy\leq zx$. That is to say, the relation $\leq$ is *compatible* [1] with the operations in $\mathfrak{S}$ and hence the relation $\equiv$ is also compatible, so that

$$(11)\qquad x\equiv x' \quad\text{and}\quad y\equiv y'\Rightarrow x+y\equiv x'+y' \quad\text{and}\quad xy\equiv x'y'.$$

It is worth emphasizing here that these assertions, valid for the semigroup semiring $\mathfrak{S}$, fail for the corresponding theory in the semigroup ring $\mathfrak{B}$, in which $a\leq a^2$ but $a-a^2\nleq 0$. This, in essence, is the reason for replacing $\mathfrak{B}$ by the more unfamiliar structure $\mathfrak{S}$.

It follows from (11) that our equivalence relation yields a *quotient semiring* $\mathfrak{S}/\{x: x\equiv 0\}$ (see [1] and [2]) whose elements are the equivalence classes, with the natural definition of addition and multiplication induced by the structure of $\mathfrak{S}$. Since $\mathfrak{S}$ has a unit element, it follows from (9) that this quotient semiring is in fact a ring with a unit element E, from (7) and Lemma 5 it follows that this ring $\mathfrak{R}_1$ has torsion-free additive group, and from (8) and (6) that the equivalence classes A, B, C containing respectively the elements a, b, c satisfy (1).

Hence, writing

$$D = E - A - B - C, \tag{12}$$

we see that $\{A, B, C, D\}$ is a set of four mutually nonorthogonal idempotents whose sum is E, and this completes the proof of Lemma 1.

Since every equivalence class contains a unique irreducible polynomial in a, b, c with coefficients in Λ, it follows that the additive group of $\mathfrak{R}_1$ is the direct sum of an enumerably infinite set of copies of the additive group of Λ.

10. Consequences for the rings $\mathfrak{B}$ and $\mathfrak{B}/\mathfrak{a}$. From the method of construction it is plausible to suppose that the rings $\mathfrak{R}_1$ and $\mathfrak{B}/\mathfrak{a}$ are isomorphic. That this is so is the content of

THEOREM 2. $\mathfrak{B}/\mathfrak{a} \approx \mathfrak{S}/\{x: x \equiv 0\} = \mathfrak{R}_1$.

This has the immediate corollaries

$$ab \notin \mathfrak{a} \qquad (a, b \in \mathfrak{B}), \tag{13}$$

$$x \notin \mathfrak{a} \Rightarrow x + x + \cdots + x \notin \mathfrak{a} \ (n \textit{ terms}, n \geqq 1) \qquad (x \in \mathfrak{B}). \tag{14}$$

For the proof of Theorem 2 we refer to Section 3 and define a *canonical* element of $\mathfrak{B}$ as a polynomial in a, b, c (with coefficient in Λ) none of whose terms contains a factor a^2 or b^2 or c^2 or cb or cab. We now prove

LEMMA 9. *The ideal* $\mathfrak{a} \subset \mathfrak{B}$ *contains no nonzero canonical element.*

Proof. By an abuse of notation, we may take X_α, Y_α $(\alpha = 1, 2, 3, 4)$ defined in (5) to be elements of $\mathfrak{B}$ not $\mathfrak{S}$. Then the general element of $\mathfrak{a}$ is of the form

$$\sum_{\alpha=1}^{4} \sum_{i=1}^{p_\alpha} u_{\alpha i}(X_\alpha - Y_\alpha)v_{\alpha i} \qquad (u_{\alpha i}, v_{\alpha i} \in \mathfrak{B}). \tag{15}$$

This is not in canonical form as it stands, unless $p_\alpha = 0$ $(\alpha = 1, 2, 3, 4)$, but *prima facie* it may, by cancellation of terms, be equal to a canonical polynomial x say, where we may further suppose that the polynomial x has no zero coefficients and no repetitions of similar monomial terms (that is, any pair of terms $\lambda m + \mu m$ is replaced by $(\lambda + \mu)m$).

We now re-interpret the symbols x, X_α, Y_α, $u_{\alpha i}$ and $v_{\alpha i}$ as elements of $\mathfrak{S}$ not $\mathfrak{B}$. Then x is an irreducible element of $\mathfrak{S}$, x is equivalent $(\equiv)$ to (15) and (15) is clearly equivalent to zero. It now follows from Lemma 5 that $x = 0$, and this completes the proof of Lemma 9.

LEMMA 10. *Each residue class* (mod $\mathfrak{a}$) *in* $\mathfrak{B}$ *contains a unique canonical element.*

Proof. The fact that any element of $\mathfrak{B}$ can be reduced, within its own residue class (mod $\mathfrak{a}$) to a canonical element, is proved by induction on $h(x)$, defined in (10) using the representation of $x \in \mathfrak{B}$ which has no zero coefficients and no

repetitions of monomial terms. If a residue class contained two distinct canonical elements, their difference would be a nonzero canonical element in $\mathfrak{a}$, contradicting Lemma 9. This completes the proof of Lemma 10, and Theorem 2 is an immediate deduction.

11. The case $n \not\equiv 2 \pmod 3$. We have

LEMMA 11. *The conclusion of Theorem* 1 *holds if* $n \not\equiv 2 \pmod 3$.

Proof. Let $\mathfrak{R}_1$ be the ring defined in Section 9 and let $\mathfrak{R}_k$ be the tensor product of k copies of $\mathfrak{R}_1$ regarded as an algebra over the ring of integers [**1**, Ch. 2, Section 3, No. 1]. Then the additive group of $\mathfrak{R}_k$ is torsion-free and in fact [**1**, Ch. 3, Section 1, Corollary 2 of Prop. 7], regarded as a module over the integers, it has a basis, so that

$$(16) \qquad x_1 \otimes x_2 \otimes \cdots \otimes x_k = 0 \Leftrightarrow x_j = 0 \qquad \text{for some } j = 1, 2, \cdots, k.$$

Multiplication in $\mathfrak{R}_k$ is defined by

$$(17) \quad (x_1 \otimes x_2 \otimes \cdots \otimes x_k)(y_1 \otimes y_2 \otimes \cdots \otimes y_k) = x_1y_1 \otimes x_2y_2 \otimes \cdots \otimes x_ky_k.$$

Let A, B, C and D be the idempotents in $\mathfrak{R}_1$ defined in Section 9 and, for each $j=1, 2, \cdots, k$, let

$$A_j = x_1 \otimes x_2 \otimes \cdots \otimes x_k \qquad (j = 1, 2, \cdots, k,)$$

where

$$x_i = \begin{cases} A & \text{if } i = j, \\ D & \text{if } i \neq j. \end{cases}$$

Define $B_1, \cdots, B_k$ and $C_1, \cdots, C_k$ in a similar manner and take

$$F = D \otimes D \otimes \cdots \otimes D.$$

Then these elements $\{F, A_j, B_j, C_j : j=1, 2, \cdots, k\}$ constitute a set of $3k+1$ mutually nonorthogonal idempotents in $\mathfrak{R}_k$, while $\Omega = \{F, (A_j+B_j+C_j): j=1, 2, \cdots, k\}$ is a set of $k+1$ mutually orthogonal idempotents in $\mathfrak{R}_k$. Thus the sum of any subset of Ω is idempotent, proving the required result in the case $n \leqq 3k+1$ and, since k is arbitrary, completing the proof of Lemma 11.

12. Proof of Lemma 2. The ring R_0. Replace the expressions for Y_4 and Y_5 in (5) by

$$(18) \quad \begin{aligned} Y_4 &= -2a-2b-2c-bc-ca-ac-ab-ba, \\ Y_5 &= 8a+6b+6c+6ac+6ba+6bc+3ca+3ab+abc+bca+bac+aba+aca, \end{aligned}$$

and replace (8), (10) and (1) by

$$(19) \quad a^2 \equiv a, \quad b^2 \equiv b, \quad c^2 \equiv c, \quad (a+b+c)^2 + (a+b+c) \equiv 0,$$

$$(20) \quad x = \sum_i \lambda_i m_i \Rightarrow h(x) = \sum_i (d_i + 1)(13)^{q_i},$$

$$(21)\quad A^2 = A, \quad B^2 = B, \quad C^2 = C, \quad (A + B + C)^2 = -(A + B + C), \quad A \neq 0.$$

Then we find that the work of Sections 5, 6, 7, 8 and 9 goes through verbatim as far as (12), except that we read $\mathfrak{R}_0$ for $\mathfrak{R}_1$ and we need longer (finite) (D2)-chains in the proof of Lemma 6. From (21) it follows that the four idempotents A, B, C and $D = -(A+B+C)$ satisfy the conditions of Lemma 2 as required—mutual nonorthogonality is a consequence of the theory, or can be deduced directly from (21). Adding the unit element E to this set of four idempotents, forming the set $\{A, B, C, D, E\}$, we have

LEMMA 12. *The conclusion of Theorem* 1 *holds if* $n=5$.

13. Proof of Theorem 1. Let $\mathfrak{R}_0$ and $\mathfrak{R}_k$ be the rings defined in Sections 12 and 11 and let $\mathfrak{R} = \mathfrak{R}_0 \otimes \mathfrak{R}_k$ be the tensor product of $\mathfrak{R}_0$ and $\mathfrak{R}_k$, so that $\mathfrak{R}$ is a torsion-free ring with a unit. Remember that $\mathfrak{R}_0$ contains a unit E and four mutually nonorthogonal idempotents A, B, C, D whose sum is zero, while $\mathfrak{R}_k$ contains a set $\{F_i : i = 0, 1, 2, \cdots, 3k\}$ of $3k+1$ mutually nonorthogonal idempotents F_i whose sum is idempotent. It follows that

$$(22)\qquad \begin{array}{llll} A \otimes F_0, & B \otimes F_0, & C \otimes F_0, & D \otimes F_0, \\ E \otimes F_i & (i = 0, 1, 2, \cdots, 3k) & (k \geqq 1), & \end{array}$$

is a set of $3k+5$ mutually nonorthogonal idempotents in $\mathfrak{R}$ whose sum is idempotent. Since $k \geqq 1$ is arbitrary, we see by reference to Lemmas 11 and 12 that the proof of Theorem 1 is complete.

14. Qualified affirmatives to Problem 5082. For certain rings the question proposed in the preamble to this note can be answered affirmatively. We have

THEOREM 3. *If the additive group of a commutative ring contains no nonzero elements of order* $\leqq n$, *then any set of* n (*or fewer*) *idempotents in the ring whose sum is idempotent must be mutually orthogonal.*

THEOREM 4. *If* $\mathfrak{R}$ *is any ring of finite matrices over a field of characteristic zero, then any set of idempotents in* $\mathfrak{R}$ *whose sum is idempotent must be mutually orthogonal.*

Theorem 4 is certainly not new. A proof is indicated in [**3**, Section 56, Exercise 5(a)]. Theorem 3 also is at least 17 years old. For a proof let the given idempotents (not all zero) be $a_1, a_2, \cdots, a_n$ and let p be the maximum possible degree of a nonvanishing product of distinct a_i, so that $1 \leqq p \leqq n$. Re-order the a_i so that $P = \prod_1^p a_i \neq 0$. Then multiplication of the equation $(\sum_1^n a_i)^2 = \sum_1^n a_i$ by $\prod_1^p a_i$ yields $p^2 P = pP$, so that $p(p-1)P = 0$, $(p-1)P = 0$, $p = 1$ and so (as required) $a_i a_j = 0$ whenever $i \neq j$.

15. A further result and some conjectures. The answer to a generalization of the problem [**5**] is provided by

References

1. N. Bourbaki, Algèbre, Hermann, Paris, 1942.
2. C. Chevalley, Fundamental Concepts of Algebra, Stechert-Hafner, New York, 1956.
3. P. R. Halmos, Finite-dimensional Vector Spaces, Van Nostrand, Princeton, N. J., 1958.
4. H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, 1956.
5. Advanced Problem 5082, this MONTHLY, 70 (1963) 335.

GENERALIZED BASES FOR THE INTEGERS

J. L. BROWN, JR., Ordnance Research Laboratory, Pennsylvania State University

Let $\{f_i\}_1^\infty$ be a given sequence of positive integers and consider two fixed sequences, $\{k_i\}_1^\infty$ and $\{m_i\}_1^\infty$ of nonnegative integers. Here the terms of the sequence $\{f_i\}$ are to play the role of a basis for the integers, while the auxiliary sequences $\{k_i\}$ and $\{m_i\}$ represent bounds for the coefficients appearing in the basis expansion of an arbitrary integer. That is, for a given integer n (positive, negative, or zero), we wish to consider the possibility of expanding n in the form $n = \sum_1^N \alpha_i f_i$, where each α_i is an integer satisfying $-m_i \leqq \alpha_i \leqq k_i$ for $i=1, 2, \cdots, N$.

In the classical case of representing positive integers in terms of a given base (radix) b, where b is an integer greater than unity, the identification is as follows: $f_i = b^{i-1}$ for $i \geqq 1$, $k_i = b-1$ and $m_i = 0$ for all $i \geqq 1$. Note, in the general case to be considered, that the f_i need not be powers of a given number, both the coefficient bounds k_i and m_i may vary with i, and the m_i need not be zero.

Previous work ([1]–[4]) has considered only the expansion of *positive* integers with various restrictions on the coefficient bounds.

Our purpose here is to extend this work to include representations of negative integers as well and to examine conditions under which such integer representations are unique. Since the terms of both the sequences $\{k_i\}$ and $\{m_i\}$ are assumed nonnegative in the general case, we may choose, for example, $k_i = 0$ for i odd and $m_i = 0$ for i even, so that only negative values are permitted for some of the coefficients. The negative sign appearing with any such coefficient can be regarded, however, as attached to the corresponding basis element; in this fashion the general theory subsumes expansions in which both positive and negative integers appear as basis elements. This result will be applied to show that every integer n (positive, negative or zero) has a unique expansion in terms of a negative base; that is, $n = \sum_1^N \alpha_i(-b)^{i-1}$, where $-b<0$ is the negative integer base in question and the coefficients α_i are integers satisfying the same restriction, $0 \leqq \alpha_i \leqq b-1$, as in the case of a positive base.

We introduce first the concept of a generalized base:

References

1. N. Bourbaki, Algèbre, Hermann, Paris, 1942.
2. C. Chevalley, Fundamental Concepts of Algebra, Stechert-Hafner, New York, 1956.
3. P. R. Halmos, Finite-dimensional Vector Spaces, Van Nostrand, Princeton, N. J., 1958.
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We introduce first the concept of a generalized base:

DEFINITION 1. *A sequence $\{f_i\}_1^\infty$ of positive integers is said to be a generalized base with respect to the sequences $\{k_i\}_1^\infty$ and $\{m_i\}_1^\infty$ of nonnegative integers (briefly, a generalized base) iff every integer n satisfying*

$$(1) \qquad -\sum_1^N m_if_i \leqq n \leqq \sum_1^N k_if_i$$

for some $N>0$ has a representation in the form

$$(2) \qquad n = \sum_1^N \alpha_if_i,$$

where each α_i is an integer satisfying

$$(3) \qquad -m_i \leqq \alpha_i \leqq k_i \quad for \quad i = 1, 2, \cdots, N.$$

We remark that in the case where $k_i>0$ for all $i \geqq 1$ and the representation is required to hold for positive integers only, a generalized base is a quasi-complete sequence as defined in [3]. From the definition, it follows that if the sequences of coefficient bounds $\{k_i\}$ and $\{m_i\}$ each contain an infinite number of nonzero terms, then every integer (positive, negative and zero) has a representation of the form (2) involving the elements of a generalized base. Note that the number of terms allowed in the expansion is controlled by the number of terms required for the validity for inequality (1). A generalized base $\{f_i\}$ possesses the following characterization:

THEOREM 1. *For given sequences $\{k_i\}_1^\infty$ and $\{m_i\}_1^\infty$ of nonnegative integers, the sequence $\{f_i\}_1^\infty$ of positive integers with $f_1=1$ is a generalized base iff*

$$(4) \qquad f_{p+1} \leqq 1 + \sum_1^p (k_i + m_i)f_i \qquad (p = 1, 2, 3 \cdots).$$

Before giving the proof of Theorem 1, we recall a lemma due to Alder [2]:

LEMMA. *Let $\{\gamma_i\}_1^\infty$ be a given sequence of positive integers and let $\{f_i\}_1^\infty$ be a sequence of positive integers with $f_1=1$ such that $f_{p+1} \leqq 1+\sum_1^p \gamma_if_i$ for $p=1, 2, 3, \cdots$. Then for any integer n satisfying the inequality $0 \leqq n \leqq \sum_1^N \gamma_if_i$, there exist integers $\{\beta_i\}_1^N$ such that $n=\sum_1^N \beta_if_i$, with $0 \leqq \beta_i \leqq \gamma_i$ for $i=1, 2, \cdots, N$.*

REMARK. We may assume without loss of generality that $k_i+m_i>0$ for all $i \geqq 1$ in the statement of Theorem 1; for, if $k_{i_0}+m_{i_0}=0$ then $k_{i_0}=m_{i_0}=0$ and in view of (3), the corresponding term f_{i_0} may be dropped from the sequence $\{f_i\}$ with impunity.

Proof of Theorem 1. Assume (4) is satisfied. Then, for n satisfying

$$-\sum_1^N m_if_i \leqq n \leqq \sum_1^N k_if_i,$$

we have $0 \leqq n + \sum_1^N m_i f_i \leqq \sum_1^N (k_i + m_i) f_i$, so that the Lemma implies

$$n + \sum_1^N m_i f_i = \sum_1^N \beta_i f_i,$$

where $0 \leqq \beta_i \leqq k_i + m_i$ and therefore $n = \sum_1^N (\beta_i - m_i) f_i$, which is an expansion in the required form (2) with the definition $\alpha_i = \beta_i - m_i$ for each i.

We prove the necessity of (4) by contradiction. Assume $\{f_i\}_1^\infty$ is a generalized base for the given sequences $\{k_i\}$ and $\{m_i\}$ and that (4) is not satisfied for all p. Then there exists $r > 0$ such that

$$f_{r+1} > 1 + \sum_1^r (k_i + m_i) f_i. \tag{5}$$

Now, $k_{r+1} + m_{r+1} > 0$ as remarked earlier. We may assume $k_{r+1} > 0$ without loss of generality. (For if $k_{r+1} = 0$ and $m_{r+1} > 0$, an analogous argument suffices.) Then

$$0 < f_{r+1} - \sum_1^r m_i f_i - 1 < f_{r+1} \leqq \sum_1^{r+1} k_i f_i,$$

so that by the property of a generalized base

$$f_{r+1} - \sum_1^r m_i f_i - 1 = \sum_1^{r+1} \alpha_i f_i \tag{6}$$

with $-m_i \leqq \alpha_i \leqq k_i$. If $\alpha_{r+1} \leqq 0$, then $f_{r+1} \leqq 1 + \sum_1^r (k_i + m_i) f_i$, contrary to (5). Thus $\alpha_{r+1} \geqq 1$ and rearrangement of (6) yields

$$(\alpha_{r+1} - 1) f_{r+1} = -1 - \sum_1^r (\alpha_i + m_i) f_i, \tag{7}$$

an obvious contradiction since the right hand side of (7) is negative. This completes the proof of the theorem.

Next, we investigate conditions under which representations of the form (2) with coefficients satisfying (3) are unique.

DEFINITION 2. *For fixed sequences of nonnegative integers $\{k_i\}_1^\infty$ and $\{m_i\}_1^\infty$ a sequence of positive integers $\{f_i\}_1^\infty$ will be said to possess Property U (uniqueness property) iff the equality $\sum_1^N \alpha_i f_i = \sum_1^N \beta_i f_i$, with α_i and β_i integers satisfying $-m_i \leqq \alpha_i \leqq k_i$ and $-m_i \leqq \beta_i \leqq k_i$ for $i = 1, 2, \cdots, N$, implies that $\alpha_i = \beta_i$ for $i = 1, 2, \cdots, N$.*

THEOREM 2. *Let $\{f_i\}_1^\infty$ be an arbitrary sequence of positive integers satisfying*

$$f_{p+1} \geqq 1 + \sum_1^p (k_i + m_i) f_i \tag{8}$$

for $p = 1, 2, 3, \cdots$. Then $\{f_i\}$ possesses Property U.

Proof. For a proof by contradiction, assume that $\{f_i\}$ does not possess Property U. Then there exists a least integer $N \geqq 1$, call it N_0, such that

$$\sum_1^N \alpha_i f_i = \sum_1^N \beta_i f_i \tag{9}$$

with $-m_i \leqq \alpha_i \leqq k_i$, $-m_i \leqq \beta_i \leqq k_i$, and

$$\sum_1^N |\alpha_i - \beta_i| \neq 0, \tag{10}$$

(α_i and β_i represent integers throughout). Clearly, $N_0 > 1$; for if $N_0 = 1$, (9) would imply $\alpha_1 f_1 = \beta_1 f_1$ or $\alpha_1 = \beta_1$ contradicting (10). Further, $\alpha_{N_0} \neq \beta_{N_0}$; otherwise we would have from (9)

$$\sum_1^{N_0-1} \alpha_i f_i = \sum_1^{N_0-1} \beta_i f_i$$

with $-m_i \leqq \alpha_i \leqq k_i$, $-m_i \leqq \beta_i \leqq k_i$, and from (10), $\sum_1^{N_0-1} |\alpha_i - \beta_i| \neq 0$, thus contradicting the choice of N_0 as the least value of N for which (9) and (10) are possible.

From (9), with $N = N_0$, $(\beta_{N_0} - \alpha_{N_0}) f_{N_0} = \sum_1^{N_0-1} (\alpha_i - \beta_i) f_i$, so that

$$f_{N_0} \leqq |\beta_{N_0} - \alpha_{N_0}| f_{N_0} \leqq \sum_1^{N_0-1} |\alpha_i - \beta_i| f_i \leqq \sum_1^{N_0-1} (k_i + m_i) f_i \leqq f_{N_0} - 1, \tag{11}$$

where the last inequality follows from assumption (8). Inequality (11) provides the contradiction needed and the theorem is established.

We have shown that condition (8) is sufficient for unique representations. If $\{f_i\}$ is a generalized base, then (8) is also a necessary condition.

THEOREM 3. *Let $\{f_i\}_1^\infty$ with $f_1 = 1$ be a generalized base corresponding to the nonnegative sequences $\{k_i\}_1^\infty$ and $\{m_i\}_1^\infty$. Then condition* (8) *is necessary in order that $\{f_i\}_1^\infty$ possess Property U.*

Proof. Assume $\{f_i\}_1^\infty$ is a generalized base possessing Property U but not satisfying (8) for all $p \geqq 1$. Then there exists $r > 0$ such that

$$f_{r+1} \leqq \sum_1^r (k_i + m_i) f_i. \tag{12}$$

From Theorem 1, we have $f_{p+1} \leqq 1 + \sum_1^p (k_i + m_i) f_i$ for all $p \geqq 1$, so that (12) in combination with the Lemma implies

$$f_{r+1} = \sum_1^r \alpha_i f_i \tag{13}$$

with α_i integral and $0 \leqq \alpha_i \leqq k_i + m_i$ for $i = 1, 2, \cdots, r$. Now, either $k_{r+1} > 0$ or

$m_{r+1}>0$ since $k_{r+1}+m_{r+1}>0$. If $k_{r+1}>0$, we have from (13),

$$f_{r+1} - \sum_1^r m_i f_i = \sum_1^r (\alpha_i - m_i) f_i, \tag{14}$$

and Property U may be invoked to equate coefficients of f_{r+1} on the two sides of (14), giving $1=0$, a contradiction. A similar argument is valid for the case $k_{r+1}=0$, $m_{r+1}>0$.

THEOREM 4. *Let the sequence of positive integers* $\{f_i\}_1^\infty$ *with* $f_1=1$ *be a generalized base corresponding to the nonnegative sequences* $\{k_i\}$ *and* $\{m_i\}$. *Then* $\{f_i\}_1^\infty$ *possesses Property U iff*

$$f_i = \phi_i, \tag{15}$$

where the sequence $\{\phi_i\}_1^\infty$ *is defined by*

$$\begin{cases} \phi_1 = 1 \\ \phi_{p+1} = 1 + \sum_1^p (k_i + m_i)\phi_i \qquad p = 1, 2, \cdots, \end{cases} \tag{16}$$

or equivalently

$$\begin{cases} \phi_1 = 1 \\ \phi_{p+1} = \prod_1^p (1 + k_i + m_i) \quad \textit{for} \quad p \geqq 1. \end{cases} \tag{17}$$

Proof. By Theorem 1, $f_{p+1} \leqq 1+\sum_1^p (k_i+m_i)f_i$ for $p\geqq 1$, while by Theorems 2 and 3, Property U is equivalent to $f_{p+1}\geqq 1+\sum_1^p (k_i+m_i)f_i$ for $p\geqq 1$. Thus the condition $f_{p+1}=1+\sum_1^p (k_i+m_i)f_i$ for $p\geqq 1$ is necessary and sufficient for the generalized base $\{f_i\}_1^\infty$ to possess Property U.

Applications. Let I denote the set consisting of all integers and consider two disjoint subsets A and B of I such that $A\cup B=I$. Further, let $\{f_i\}_1^\infty$ be a set of positive integers with $f_1=1$ and satisfying

$$f_{p+1} \leqq 1 + \sum_1^p \gamma_i f_i \qquad (p \geqq 1) \tag{18}$$

for some fixed sequence $\{\gamma_i\}_1^\infty$ of positive integers. Define

$$\begin{cases} k_i = C_A(i)\gamma_i & \text{for} \quad i \geqq 1 \\ m_i = C_B(i)\gamma_i & \text{for} \quad i \geqq 1, \end{cases} \tag{19}$$

where $C_A(x)$ and $C_B(x)$ are the characteristic functions of sets A and B respectively; that is, $C_A(i)=1$ if $i\in A$ and $C_A(i)=0$ if $i\notin A$. Note that (19) implies that $k_i+m_i=\gamma_i$ for all $i\geqq 1$.

THEOREM 5. *Under the conditions set forth above, every integer n satisfying the inequality*

$$-\sum_{i\in B}^{N}\gamma_i f_i \leqq n \leqq \sum_{i\in A}^{N}\gamma_i f_i$$

has an expansion in the form

$$n = \sum_{i=1}^{N}\alpha_i[(-1)^{C_B(i)}f_i],$$

where α_i is an integer satisfying $0\leqq\alpha_i\leqq\gamma_i$ for $i=1, 2, \cdots, N$.

Proof. Theorem 1 implies that every n satisfying the inequality

$$-\sum_{i\in B}^{N}\gamma_i f_i \leqq n \leqq \sum_{i\in A}^{N}\gamma_i f_i$$

has an expansion $n=\sum_{i=1}^{N}\beta_i f_i$, where β_i is integral and

$$-C_B(i)\gamma_i \leqq \beta_i \leqq C_A(i)\gamma_i \quad \text{for} \quad i = 1, 2, \cdots, N.$$

The present theorem then follows on defining

$$\alpha_i = (-1)^{C_B(i)}\beta_i.$$

It is clear, therefore, that by allowing nonnegative values for the terms of the sequences $\{k_i\}$ and $\{m_i\}$, we are able effectively to treat expansions in which the basis elements are themselves both positive and negative.

COROLLARY. (Negative base expansions): *Let $b>1$ be an integer and define $f_i=b^{i-1}$ for $i\geqq 1$ and $\gamma_i=b-1$ for $i\geqq 1$ independently of i. Then every integer n satisfying*

$$-\sum_{1}^{[N/2]}(b-1)b^{2i-1} \leqq n \leqq \sum_{1}^{[(N+1)/2]}(b-1)b^{2i-2}$$

has a **unique** *expansion in the form*

$$n = \sum_{1}^{N}\alpha_i(-b)^{i-1},$$

where α_i is integral and $0\leqq\alpha_i\leqq b-1$ for $i=1, 2, \cdots, N$.

Proof. In Theorem 5, take A as the set of all odd integers and B as the set of all even integers. Theorem 5 then establishes the existence of the expansion; uniqueness follows from Theorem 4 on observing that $f_1=1$ and

$$f_{p+1} = b^p = 1 + \sum_{1}^{p}(b-1)b^{i-1} = 1 + \sum_{1}^{p}\gamma_i f_i = 1 + \sum_{1}^{p}(k_i+m_i)f_i \quad \text{for} \quad p \geqq 1.$$

where each α_i is either 0 or 1.

Since any increasing complete sequence [1] satisfies (20), representations in the form of (21) are possible for any such sequence.

The assumption that $f_1=1$ for a generalized base (e.g. in Theorems 3 and 4) may be omitted since $f_1=1$ is a direct consequence of the definition. From Definition 1, if $\{f_i\}$ is a generalized base, then $-m_1f_1\leqq n\leqq k_1f_1$ implies $n=\alpha_1f_1$ with $-m_1\leqq\alpha_1\leqq k_1$. Since at most k_1+m_1+1 distinct integers can be represented in the latter manner, it follows that the range $-m_1f_1\leqq n\leqq k_1f_1$ must contain no more than k_1+m_1+1 numbers. This, in turn, implies $f_1=1$.

References

1. J. L. Brown, Jr., Note on complete sequences of integers, this MONTHLY, 68 (1961) 557–560.
2. H. L. Alder, The number system in more general scales, Mathematics Magazine, 35 (1962) 145–151.
3. J. L. Brown, Jr., A generalization of semi-completeness for integer sequences, Fibonacci Quart., 1 (1963) 3–15.
4. N. S. Mendelsohn, Representations of positive real numbers by infinite sequences of integers, Trans. Royal Soc. of Canada, (3 Ser.) Sec. III, 47 (1952) 45–55.

PRIMARY IDEALS IN RINGS OF CONTINUOUS FUNCTIONS

CARL W. KOHLS, Syracuse University

The Lasker-Noether decomposition theorem states that in a commutative ring satisfying the ascending chain condition every ideal is the intersection of a finite number of primary ideals [7, p. 209]. Krull [6] considered rings without finiteness assumptions and obtained, under certain conditions, a representation in which an infinite collection of primary ideals is allowed. The following special case is given by Bochner in [1, 4.5.5]: If R is a commutative ring with unit in which every proper prime ideal is maximal, then every ideal in R is the intersection of primary ideals. It is known that in a general commutative ring an ideal may not be the intersection of primary ideals [7, p. 208], but there seems to be no simple example in the literature. We give an example in Section 1, using two facts about ideals in rings of continuous real-valued functions on a topological space. Actually, we describe a large class of such ideals, since this is just as easy to do. Let $C(X)$ denote the ring of all continuous real-valued functions on any topological space X. The first result needed was obtained by L. Gillman and the author in [3, 4.5]: Let $p\in X$, let Q be a nonmaximal, prime z-ideal contained in $\boldsymbol{M}_p=\{f\in C(X):f(p)=0\}$, and let $f\in\boldsymbol{M}_p-Q$; then the ideal (Q,f) in $C(X)$ is not primary. The second fact was noted by the author in [5]: Every primary ideal in $C(X)$ is absolutely convex.

Now let P be any prime ideal in $C=C(X)$. In the second section of the paper, we give a complete description of the primary ideals in C/P that are not prime.

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References

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Now let P be any prime ideal in $C=C(X)$. In the second section of the paper, we give a complete description of the primary ideals in C/P that are not prime.

It was observed in [3, 4.6] that a set of the form

$$P|^a = \bigcap_n \{b \in C/P : |b| < a^{1-1/n}\},$$

where a is any positive nonunit in C/P and n ranges over the set of positive integers, is a primary ideal that is not prime. Every nonprime primary ideal turns out to be either of this form, or a sort of dual form, namely,

$$P|_a = \bigcup_n \{b \in C/P : |b| \leqq a^{1+1/n}\}.$$

Now it is known that every primary ideal in C contains a prime ideal, since the radical of a primary ideal is prime [3, 4.1]. So the theorem about primary ideals in C/P yields a description of the primary ideals in C as well. The proof of the first lemma used in obtaining the characterization also leads to an improvement in the theorem of Gillman and the author mentioned above.

We list here some standard facts, terminology and notation for reference. The reader should consult [2] for further background.

The set of all positive integers is denoted by $\mathbf{N}$. In any ring $C(X)$, the constant function whose value is r is designated by $\boldsymbol{r}$. For any $f \in C(X)$, we write $\mathbf{Z}(f)$ for $\{x \in X : f(x) = 0\}$.

An ideal I in C is said to be fixed if $\bigcap\{\mathbf{Z}(f) : f \in I\}$ is nonempty. A proper ideal will mean one that is neither the whole ring nor the zero ideal. For any ideal I in C and $f \in C$, the residue class of f modulo I is written $I(f)$. The ideal I is called a z-ideal if $\mathbf{Z}(f) = \mathbf{Z}(g)$ and $g \in I$ implies that $f \in I$.

Unions and intersections of chains of primary ideals are primary. When P is a prime ideal in C, the ring C/P is totally ordered, and the canonical homomorphism of C onto C/P is a lattice homomorphism. For each positive nonunit $a \in C/P$, there is a smallest prime ideal P^a containing a and a largest prime ideal P_a not containing a. In P^a, $\{a^{1/n} : n \in \mathbf{N}\}$ is a cofinal subset.

1. The example. Let P be a prime ideal in C. The primary ideals in C containing P are all convex [5], so by [3, 2.2], each primary ideal in C/P is convex. As has been noted before ([3], 4.1 and [2], 14.3), a convex ideal in C/P is an interval that is symmetric about zero, and hence the convex ideals in C/P form a chain. But the intersection of a chain of primary ideals is primary; so the statement that an ideal I in C/P is not the intersection of primary ideals is equivalent to the statement that I is not primary. Thus, to obtain our example, we need only exhibit an ideal in C/P that is not primary.

It follows immediately from [3, 4.5] that if Q is a fixed prime z-ideal in C, then no proper principal ideal in C/Q is primary. This already provides a large class of examples. We now show that the restriction to z-ideals is easily removed. (It will be shown later that the restriction to fixed ideals is unnecessary also.) Let P be any fixed prime ideal in C. Then P contains a minimal prime ideal Q, and Q is a (fixed) z-ideal [2, 14.7]. Now any proper principal ideal in C/P is the homomorphic image of an ideal of the form $(P/Q, a)$, where $a \in C/Q - P/Q$,

$a>0$. But P/Q is contained in the ideal Q_a of C/Q; and every element of Q_a is a multiple of a [2, 14.6 and 1D.3]. Hence $(P/Q, a)=(a)$. It follows that the image of $(P/Q, a)$ in C/P is not primary, because any primary ideal in C/P comes from a primary ideal in C/Q containing P/Q.

2. The theorems. We shall now describe precisely the primary ideals in rings of the form C/P, where P is a prime ideal in C. First we prove two lemmas about primary ideals in C/P.

LEMMA 1. *Let P be a prime ideal in C, and let a be a positive non-unit in C/P. Then $P|^a$ is the smallest primary ideal in C/P containing a.*

Proof. It was observed in [3, 4.6] that $P|^a$ is a primary ideal in C/P, and clearly $a\in P|^a$. Since the intersection of a chain of primary ideals is primary, there is a smallest primary ideal containing a; to prove that it is $P|^a$, it obviously suffices to show that if I is a primary ideal in C/P containing a, then $P|^a\subset I$.

Evidently, we may assume that $I\subset P^a$. Choose $f\in C$ such that $P(f)=a$ and $\mathbf{0}\leqq f\leqq \mathbf{1}$. Given $c\in P|^a$, with $c\geqq a$, let $g\in C$ be such that $P(g)=c$ and $f\leqq g\leqq f^{1/2}$. Clearly $\mathbf{Z}(g)=\mathbf{Z}(f)$. Set $h=\sum_{n\in\mathbf{N}} 2^{-n}f^{1/n}$; then $h\in C$, and $\mathbf{Z}(f)=\mathbf{Z}(h)$.

Consider the function k defined by $k(x)=g(x)/h(x)$ if $x\notin\mathbf{Z}(h)$, $k(x)=0$ if $x\in\mathbf{Z}(h)$. Since $g\leqq f^{1/2}$ and $h\geqq \mathbf{2}^{-4}f^{1/4}$, we have $g(x)/h(x)\leqq 16(f(x))^{1/4}$ for $x\notin\mathbf{Z}(h)$; hence $k\in C$. Set $d=P(k)$; then we have $dP(h)=c$. Now $h\leqq\mathbf{1}$, so $g\leqq k$, and $c\leqq d$.

Next consider the function l defined by $l(x)=h(x)f(x)/g(x)$ if $x\notin\mathbf{Z}(g)$, $l(x)=0$ if $x\in\mathbf{Z}(g)$. Since $f(x)/g(x)\leqq 1$ for $x\notin\mathbf{Z}(g)$, we have $l(x)\leqq h(x)$ for $x\notin\mathbf{Z}(g)$, whence $l\in C$. Clearly $kl=f$, so $dP(l)=a$. Furthermore, since $P(h)\geqq 2^{-4n}a^{1/4n}$, $c<a^{1-1/4n}$ and $cP(l)=aP(h)$, we have $P(l)>2^{-4n}a^{1/2n}$, or $a^{1/2n}<2^{4n}P(l)$. Because $2^{4n}a^{1/2n}<1$ [2, 14.5(a)], it follows that $a^{1/n}<2^{4n}a^{1/2n}P(l)<P(l)$, for all $n\in\mathbf{N}$. Hence $P(l)\notin P^a$, so $(P(l))^n\notin P^a$ for all n. The assumption that $I\subset P^a$ implies $(P(l))^n\notin I$ for all $n\in\mathbf{N}$. But $dP(l)=a\in I$, and I is primary; hence $d\in I$. Now $c\leqq d$ and I is convex; so we conclude finally that $c\in I$. Thus, because both sets in question are symmetric, $P|^a\subset I$.

REMARK. It is known that many but not all rings of the form C/P are valuation rings with the property that a is a multiple of b whenever $0<a<b$ (see [4], Section 3, and [3], 3.6 for details). Now if C/P is a valuation ring, the proof of Lemma 1 can be greatly simplified. For in that case, we can assert that there exists $e\in C/P$ such that $a=ec$. If $e\in P^a$, then $e<a^{1/n}$ for some n, and $a=ec<a^{1/n}a^{1-1/n}=a$, which is absurd. So $e^n\notin I$ for all $n\in\mathbf{N}$, and $c\in I$.

LEMMA 2. *Let P be a prime ideal in C, and let a be a positive nonunit in C/P. Then $P|_a$ is the largest primary ideal in C/P not containing a.*

Proof. It is easily seen that $P|_a$ is an ideal in C/P and that $a\notin P|_a$. To show that $P|_a$ is primary, suppose that $bc\in P|_a$ while $b\notin P|_a$. Then $|bc|\leqq a^{1+1/n}$ for some $n\in\mathbf{N}$. But $|b|>a^{1+1/2n}$, so $|c|\leqq a^{1/2n}$ and hence $|c|^{4n}\leqq a^2$. Therefore $c^{4n}\in P|_a$. Since the union of a chain of primary ideals is primary, there is a

unit. But $\mathbf{Z}(f) = \mathbf{Z}(h)$, and f belongs to the maximal ideal containing P, which is a z-ideal, so h does also; this implies that $P(h)$ is not a unit. It follows that $d \notin (a)$, and hence $(a) \neq P|^a$. Therefore (a) is not primary.

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References

1. S. Bochner, Lectures on commutative algebra, Edwards Bros., Ann Arbor, 1938.
2. L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, 1960.
3. L. Gillman and C. W. Kohls, Convex and pseudoprime ideals in rings of continuous functions, Math. Z., 72 (1960) 399–409.
4. C. W. Kohls, Prime ideals in rings of continuous functions, Illinois J. Math., 2 (1958) 505–536.
5. ———, Problem 5141, this MONTHLY, 71 (1964) 1050.
6. W. Krull, Idealtheorie in Ringen ohne Endlichkeitsbedingung, Math. Ann., 101 (1929) 729–744.
7. O. Zariski and P. Samuel, Commutative Algebra, I, Van Nostrand, Princeton, 1958.

SEMIGROUPS OF CONTINUOUS FUNCTIONS

K. D. MAGILL, JR., State University of New York at Buffalo

1. Introduction. A source of interesting and illustrative examples of semigroups is the collection $S(X)$ of all continuous functions from a topological space X into itself where, for f and g in $S(X)$, $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$ for all $x \in X$. Some interesting results concerning algebraic properties of $S(X)$ have been obtained recently for certain spaces. For example, Nadler [4] has shown in the case $X = [0, 1]$ that the subsemigroup of $S(X)$ consisting of all differentiable functions has no idempotent elements (g is idempotent if $g \circ g = g$) other than the identity mapping and the constant functions. He also gives an example to show that $S(X)$ does contain idempotent elements not of this form.

Here too, we are interested in the algebraic structure of $S(X)$ but primarily in its relation to the topological structure of X. It is obvious that if X and Y are homeomorphic then $S(X)$ and $S(Y)$ are isomorphic; specifically, if h is a homeomorphism from X onto Y, then $f \to h \circ f \circ h^{*}$ is an isomorphism from $S(X)$ onto $S(Y)$ (for a 1-1 function h, we use the symbol h^{*} to denote the function defined by $h^{*}(y) = x$ iff $h(x) = y$). This paper is devoted to the converse question: if $S(X)$ and $S(Y)$ are isomorphic, must X and Y be homeomorphic? In general, the answer is no. Let X denote any set (with more than one element) with the discrete topology. Let Y denote the same set with the indiscrete topology. Then $S(X)$ consists of all functions which map X into X and $S(Y)$ consists of all functions which map Y into Y. Evidently $S(X)$ and $S(Y)$ are isomorphic but X and

unit. But $\mathbf{Z}(f)=\mathbf{Z}(h)$, and f belongs to the maximal ideal containing P, which is a z-ideal, so h does also; this implies that $P(h)$ is not a unit. It follows that $d\notin(a)$, and hence $(a)\neq P|^{a}$. Therefore (a) is not primary.

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References

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3. L. Gillman and C. W. Kohls, Convex and pseudoprime ideals in rings of continuous functions, Math. Z., 72 (1960) 399–409.
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5. ———, Problem 5141, this MONTHLY, 71 (1964) 1050.
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Y are not homeomorphic. Thus, if isomorphism is to imply homeomorphism, the spaces involved must be restricted. In Section 2, we define an S-space and obtain several results showing, among other things, that the class of S-spaces includes some familiar topological spaces. Section 3 is devoted to a proof of the main result of this paper: two S-spaces X and Y are homeomorphic if and only if $S(X)$ and $S(Y)$ are isomorphic.

We take this opportunity to thank the referee for several suggestions which resulted in simplifications of some of the proofs.

2. S-spaces.

DEFINITION (2.1). *Let X be a topological space and x a point of X. An open set G containing x is an S-neighborhood of x if it consists of x alone or if there exists a continuous function f mapping* cl G *(the closure of G—also denoted by* $\mathrm{cl}_X G$*) into X such that $f(x) \neq x$ but $f(y) = y$ for each $y \in \mathrm{cl}\, G - G$.*

DEFINITION (2.2). *A topological space is an S-space if it is Hausdorff and every point has a basis of S-neighborhoods. Such a basis will be referred to as an S-basis.*

THEOREM (2.3). *Let X be a Hausdorff space and suppose that each $x \in X$ has a basis B_x of open sets such that* cl $G - G$ *consists of at most one point for each $G \in B_x$. Then X is an S-space.*

Proof. The result is immediate if X consists of only one point so assume X has more than one point. If $\mathrm{cl}\, G - G = \emptyset$, choose any $y \neq x$ and define $f(z) = y$ for all $z \in \mathrm{cl}\, G$. In the event $\mathrm{cl}\, G - G$ consists of a point p, define $f(z) = p$ for all $z \in \mathrm{cl}\, G$.

A 0-dimensional space is defined as one which has a basis of sets which are both open and closed. Thus, the following corollary is a consequence of Theorem (2.3).

COROLLARY (2.4). *Every 0-dimensional Hausdorff space is an S-space.*

THEOREM (2.5). *A regular Hausdorff space which is the union of a collection of open subspaces which are S-spaces is an S-space.*

Proof. Let x be a point of the space X. Denote by H an open subset of X which is an S-space and contains the point x. Let B_x be an S-basis for x in H. Since X is regular, there exists an open subset V of X and a closed subset F of X such that $x \in V \subset F \subset H$. Let

$$B_x^* = \{G \in B_x : G \subset V\}.$$

Then B_x^* is a basis for x in X. Let us consider an arbitrary $G \in B_x^*$. If G contains a point other than x, there exists a continuous function f from $\mathrm{cl}_H G$ into H such that $f(x) \neq x$ and $f(y) = y$ for each $y \in \mathrm{cl}_H G - G$. Since $\mathrm{cl}_X G \subset F \subset H$, $\mathrm{cl}_X G = \mathrm{cl}_H G$ Thus, f maps $\mathrm{cl}_X G$ into X which implies G is an S-neighborhood of x. Therefore B_x^* is an S-basis for x in X and it follows that X is an S-space.

We refer to a Hausdorff space X as being locally Euclidean if for each $x \in X$

there exists an open neighborhood of x which is homeomorphic to the Euclidean N-space E^N for some N (N depends on x).

THEOREM (2.6). *Every locally Euclidean space is an S-space.*

Proof. In view of Theorem (2.5), the fact that a locally Euclidean space is a regular Hausdorff space, and the fact that a homeomorphic image of an S-space is an S-space, it will be sufficient to show that E^N is an S-space for each positive integer N. For $p \in E^N$, let

$$S_n(p) = \{q \in E^N: d(p, q) < 1/n\},$$

where d denotes Euclidean distance. Then

$$\operatorname{cl} S_n(p) - S_n(p) = \{q \in E^N: d(p, q) = 1/n\}.$$

For each integer i such that $1 \leqq i \leqq N$, define a function f_i from cl $S_n(p)$ into E^1 by $f_i(q) = q_i + d(p, q) - 1/n$. Each f_i is continuous and hence the function f defined by $f(q) = (f_1(q), f_2(q), \cdots, f_N(q))$ is a continuous mapping from cl $S_n(p)$ into E^N. Moreover, $f(q) = q$ if $q \in \operatorname{cl} S_n(p) - S_n(p)$ and

$$f(p) = (p_1 - 1/n, p_2 - 1/n, \cdots, p_N - 1/n).$$

Therefore, $\{S_n(p)\}_{n=1}^{\infty}$ is an S-basis for p in E^N and the proof is complete.

(2.7). *An example of a Hausdorff space which is not an S-space.* Let X denote the collection of points in the plane E^2. For a point x whose coordinates x_1 and x_2 are not both zero, let

$$S_n(x) = \{y \in X: d(x, y) < 1/n\},$$

and let $\{S_n(x)\}_{n=1}^{\infty}$ be a neighborhood basis for x. For the point $p = (0, 0)$, let

$$Q_n(p) = \{y \in X: d(y, p) < 1/n \text{ and } y_2 \neq 0\} \cup \{p\}.$$

Thus, $Q_n(p)$ contains no point whose second coordinate is zero other than p. Let $\{Q_n(p)\}_{n=1}^{\infty}$ be a neighborhood basis for p. The resulting topology for X is Hausdorff. We will show that no basis at p is an S-basis. Let B_p be any neighborhood basis at p. Then there exists a set $H \in B_p$ such that $H \subset Q_1(p)$. Now let f be a continuous function from cl H into X such that $f(p) \neq p$. Choose a positive integer M such that $1/M < d(p, f(p))$. Then $S_M(f(p))$ is an open set containing $f(p)$ and since f is continuous, there exists an integer N such that

$$\operatorname{cl} S_M(f(p)) \cap \operatorname{cl} Q_N(p) = \varnothing, \qquad Q_N(p) \subset H \quad \text{and} \quad f[Q_N(p)] \subset S_M(f(p)).$$

Then

$$f[\operatorname{cl} Q_N(p)] \subset \operatorname{cl} [f[Q_N(p)]] \subset \operatorname{cl} S_M(f(p)).$$

Choose any number z such that $0 < |z| < 1/N$. Then $(z, 0) \in \operatorname{cl} Q_N(p)$ and hence $(z, 0) \in \operatorname{cl} H$. Moreover, $(z, 0) \notin H$ since $(z, 0) \notin Q_1(p)$. Thus $(z, 0) \in \operatorname{cl} H - H$. But $f(z, 0) \in \operatorname{cl} S_M(f(p))$ which does not contain the point $(z, 0)$. Therefore

Therefore h is a function from X into Y with the property that $h(x)=y$ if and only if $\phi(\mathbf{x})=\mathbf{y}$. In a similar manner, there exists for each $y\in Y$ a unique $x\in X$ such that $\phi^{\leftarrow}(\mathbf{y})=(\mathbf{x})$. This implies that h is a one-to-one mapping from X onto Y. Now we show that for each $f\in S(X)$, $h[H(f)]=H(\phi(f))$. In fact, let $y=h(x)$, i.e., $\mathbf{y}=\phi(\mathbf{x})$. In view of Lemma (3.4), the following are successively equivalent: $y\in h[H(f)]$, $x\in H(f)$, $f\circ\mathbf{x}=\mathbf{x}$, $\mathbf{y}=\phi(f\circ\mathbf{x})$, $\mathbf{y}=\phi(f)\circ\phi(\mathbf{x})$, $\mathbf{y}=\phi(f)\circ\mathbf{y}$, $y\in H(\phi(f))$. We now appeal to Lemma (3.2) and conclude that h is a homeomorphism.

The homeomorphism h and the isomorphism ϕ are rather closely related. Recalling that $\phi(\mathbf{x})=\mathbf{h}(\mathbf{x})$ for each $x\in X$, we let y be any element of Y and f any function of $S(X)$ and we see that

$$\begin{aligned}(h\circ f\circ h^{\leftarrow})(y) &= h(f(h^{\leftarrow}(y))) = \mathbf{h}(\mathbf{f}(\mathbf{h}^{\leftarrow}(\mathbf{y})))(y) = \phi(\mathbf{f}(\mathbf{h}^{\leftarrow}(\mathbf{y})))(y) = \phi(f\circ\mathbf{h}^{\leftarrow}(\mathbf{y}))(y)\\ &= (\phi(f))\circ(\phi(\mathbf{h}^{\leftarrow}(\mathbf{y})))(y) = ((\phi(f))\circ\mathbf{y})(y) = (\phi(f))(y).\end{aligned}$$

Thus we have the additional bit of information that

(3.5). *If X and Y are S-spaces and ϕ is an isomorphism from $S(X)$ onto $S(Y)$, there exists a homeomorphism h from X onto Y such that $\phi(f)=h\circ f\circ h^{\leftarrow}$ for all $f\in S(X)$.*

References

1. A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, Math. Surveys, No. 7, Amer. Math. Soc., 1961.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, 1955.
3. K. Kuratowski, Introduction to set theory and topology, Pergamon Press Ltd., London, 1962.
4. S. B. Nadler, Jr., The idempotents of a semigroup, this MONTHLY, 70 (1963) 996–997.

A CANONICAL FORM FOR LINEAR TRANSFORMATIONS OF E_n UNDER NONLINEAR SUBSTITUTIONS

HOWELL K. WILSON, Georgia Institute of Technology

1. Statement of the problem. It is shown that every nonsingular linear transformation of the real vector space E_n, which has no eigenvalue of modulus one, is homeomorphically equivalent to one of $4n$ canonical linear transformations. A similar result is first proved in Theorem 2.2 for the one-parameter family of mappings corresponding to the solutions of a linear differential system with constant coefficients and this is then generalized to the main Theorem 3.4.

2. Canonical forms for the system $\dot{x}=Ax$. Let x and u denote real m-vectors and let A be a real $m\times m$ matrix with eigenvalues $\lambda_j=\alpha_j+i\beta_j$ $(j=1,\cdots,m)$.

Therefore h is a function from X into Y with the property that $h(x)=y$ if and only if $\phi(\mathbf{x})=\mathbf{y}$. In a similar manner, there exists for each $y\in Y$ a unique $x\in X$ such that $\phi^{\leftarrow}(\mathbf{y})=(\mathbf{x})$. This implies that h is a one-to-one mapping from X onto Y. Now we show that for each $f\in S(X)$, $h[H(f)]=H(\phi(f))$. In fact, let $y=h(x)$, i.e., $\mathbf{y}=\phi(\mathbf{x})$. In view of Lemma (3.4), the following are successively equivalent: $y\in h[H(f)]$, $x\in H(f)$, $f\circ\mathbf{x}=\mathbf{x}$, $\mathbf{y}=\phi(f\circ\mathbf{x})$, $\mathbf{y}=\phi(f)\circ\phi(\mathbf{x})$, $\mathbf{y}=\phi(f)\circ\mathbf{y}$, $y\in H(\phi(f))$. We now appeal to Lemma (3.2) and conclude that h is a homeomorphism.

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References

1. A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, Math. Surveys, No. 7, Amer. Math. Soc., 1961.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, 1955.
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2. Canonical forms for the system $\dot{x}=Ax$. Let x and u denote real m-vectors and let A be a real $m\times m$ matrix with eigenvalues $\lambda_j=\alpha_j+i\beta_j$ $(j=1,\cdots,m)$.

LEMMA 2.1. *If $\alpha_j<0$ for $j=1, \cdots, m$, then there is a homeomorphism $T: x\to u = Tx$ of E_m such that the system*

$$\dot{x} = Ax \tag{1}$$

becomes

$$\dot{u} = -u \tag{2}$$

under the substitution $u=Tx$. Moreover T is of class C^∞ on $E_m-\{0\}$.

Proof. We assume without loss of generality that A is in such a real canonical form with respect to the x-coordinates such that, if $\phi(t, x)$ denotes the solution of system (1) through the point $(0, x)\in E_{m+1}$, the Euclidean norm $\|\cdot\|$ of $\phi(t, x)$ decreases steadily with increasing t (cf. [1]).

Let $\mathcal{B}=\{x: \|x\|<1\}$. Then, for each $x\in E_m-\{0\}$, there is exactly one time $t(x)$ and exactly one point $p(x)\in\partial\mathcal{B}$ such that

$$p(x) = \phi(t(x), x) \quad \text{or} \quad x = \phi(-t(x), p(x)).$$

Define

$$\begin{aligned} Tx &= p(x)e^{t(x)} \qquad \text{for } x \neq 0, \\ T0 &= 0. \end{aligned}$$

It is clear that T is a one-one mapping of E_m onto itself. To establish differentiability on a neighborhood of an arbitrary point $x_0\in E_m-\{0\}$, note that the C^∞ function $f(t, x)=\|\phi(t, x)\|^2-1$ satisfies

$$\frac{\partial}{\partial t}f(t(x_0), x_0) = \frac{\partial}{\partial t}\|\phi(t(x_0), x_0)\|^2 \neq 0$$

and

$$f(t(x_0), x_0) = 0.$$

Thus $t(x)$ and, consequently, $p(x)$ are C^∞ functions on a neighborhood of x_0. It follows then that T is of class C^∞ on $E_m-\{0\}$.

For arbitrary $\epsilon>0$, $\mathcal{G}=\phi(\epsilon^{-1}, \mathcal{B})$ is a neighborhood of the origin with the property that $t(x)<-\epsilon^{-1}$ for all $x\in\mathcal{G}$. Thus $t(x)\to-\infty$ as $x\to 0$. Then T is continuous at the origin since $\|Tx-T0\|=e^{t(x)}$ for all $x\in E_m-\{0\}$. Since the substitution $u=Tx$ maps the solutions of system (1) onto those of system (2) by construction, the proof is complete.

We note here that the lemma remains true when $\alpha_j>0$ for $j=1, \cdots, m$ if the system (2) is replaced by

$$\dot{u} = u. \tag{2'}$$

In what follows, transformations T corresponding to several systems of the form (1) will be used. To avoid confusion, we write T_A for T, where A is the

matrix of the system. Now let z, w denote real n-vectors and let C be a real $n\times n$ matrix with eigenvalues $\lambda_j=\alpha_j+i\beta_j$ $(j=1,\cdots,n)$.

THEOREM 2.2. *If m of the α_j's are negative and the remaining $n-m$ are positive, then there is a homeomorphism $T: z\to w=Tz$ of E_n which is of class C^∞ on $E_n-\{0\}$ such that*

$$\dot{z} = Cz \tag{3}$$

becomes

$$\dot{w} = \begin{bmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{bmatrix} w \tag{4}$$

under the substitution $w=Tz$, where I_ν denotes the $\nu\times\nu$ identity matrix.

Proof. Assume that C is in real canonical form with respect to the z-coordinates and that $\alpha_j<0$ for $j=1,\cdots,m$. Then system (3) is of the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \tag{5}$$

where x is an m-vector, A has the negative α_j's,

$$z = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Apply Lemma 2.1 to the decoupled system (5) to obtain the homeomorphisms T_A of E_m and T_B of E_{n-m}. Extend these functions to E_n by defining

$$T_A z = \begin{bmatrix} T_A x \\ 0 \end{bmatrix} \quad \text{and} \quad T_B z = \begin{bmatrix} 0 \\ T_B y \end{bmatrix}.$$

Then the homeomorphism T_C defined by $T_C z=T_A z+T_B z$ has the desired properties. This completes the proof.

Notice that if T_C is of class C^1 on a neighborhood of the origin, then the mean value theorem implies that $C=\operatorname{diag}\{-I_m, I_{n-m}\}$, and T_C is necessarily the identity homeomorphism. Thus differentiability on all of E_n can be expected in the trivial case only.

P. Hartman [2] has considered the nonlinear system

$$\dot{\zeta} = C\zeta + f(\zeta), \tag{6}$$

where f is of class C^2 near $\zeta=0$ with $f(0)=0$ and $f(\zeta)=o(\zeta)$ as $\zeta\to 0$, and has shown that if no eigenvalue of C has zero real part, then there is a homeomorphism U of a neighborhood of the origin such that, under the substitution $z=U\zeta$, the system (6) becomes

$$\dot{z} = Cz. \tag{7}$$

Theorem 2.2 in conjunction with this result provides a simple method for classifying all such systems near the origin.

3. Canonical forms for real linear transformations. Let x and u be real m-vectors and let y and v be real $(n-m)$-vectors. Write

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} u \\ v \end{bmatrix}.$$

We consider first a linear transformation

$$L\colon x \to Ax \tag{8}$$

of E_m with eigenvalues μ_j which satisfy $0<|\mu_j|<1$ for $j=1, \cdots, m$. If no μ_j is real and negative, then there is a real matrix A such that $\overline{A}=e^A$. It follows from Lemma 2.1 that

$$T_A L T_A^{-1}\colon u \to e^{-1}u, \quad \text{i.e.,} \quad T_A L T_A^{-1} u = T_A\phi(1, x) = p(x)e^{t(x)-1} = e^{-1}u. \tag{9}$$

Suppose now that $\overline{A}$ has real, negative eigenvalues. We can assume without loss of generality that $-1<\mu_j<0$ for $j=1, \cdots, k$, where each such μ_j has multiplicity ν_j, that the remaining μ_j's are either positive or complex, and that, relative to the x-coordinates,

$$A = \begin{bmatrix} \overline{D}_1 & & & 0 \\ & \ddots & & \\ & & \overline{D}_k & \\ 0 & & & \overline{\mathrm{P}} \end{bmatrix}, \tag{10}$$

where P is in real canonical form and each $\overline{D}_j$ is a $\nu_j \times \nu_j$ matrix of the form

$$\overline{D}_j = \begin{bmatrix} \mu_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \ddots \\ 0 & & & \mu_j \end{bmatrix} \qquad j = 1, \cdots, k.$$

LEMMA 3.1. *For each fixed* $j=1, \cdots, k$, *there is a real matrix* Δ_j *such that* $-\overline{D}_j = \exp \Delta_j$ *and if* $q \in E_{\nu_j}$ *then*

$$T_{\Delta_j}\overline{D}_j T_{\Delta_j}^{-1}\colon q \to -e^{-1}q. \tag{11}$$

Proof. For convenience, omit the subscript j in the proof and define $\bar{\Delta} = -\overline{D}$. It is known that there is a real matrix Δ such that $\bar{\Delta} = e^{\Delta}$.

Now let $q \in E_\nu - \{0\}$ be arbitrary and define $\tau = \ln \|q\|$. From Lemma 2.1,

$$T_\Delta^{-1} q = e^{-\tau\Delta}\frac{q}{\|q\|}.$$

Thus

$$\overline{D}T_\Delta^{-1}q = -e^{\Delta}e^{-\tau\Delta}\frac{q}{\|q\|}. \tag{12}$$

To find $T_\Delta\overline{D}T_\Delta^{-1}q$, the equations $p=e^{t\Delta}\overline{D}T_\Delta^{-1}q$, $\|p\|=1$ must be solved for p and t. Using equation (12), we obtain

$$p = -e^{(t+1-\tau)\Delta}\frac{q}{\|q\|}, \qquad \|p\| = 1.$$

The unique solution is, by inspection,

$$t = \tau - 1, \qquad p = -\frac{q}{\|q\|}.$$

Then $T_\Delta\overline{D}T_\Delta^{-1}q=(-q/\|q\|)e^{\tau-1}=-e^{-1}q$. This completes the proof.

Observe that Lemma 3.1 remains true if we assume that $1<|\mu_j|$ for $j=1,\cdots,m$, provided that equations (9) and (11) are replaced by

$$T_A L T_A^{-1}\colon u \to eu, \tag{9'}$$

and

$$T_{\Delta_j}\overline{D}_j T_{\Delta_j}^{-1}\colon q \to -eq. \tag{11'}$$

LEMMA 3.2. *There is a homeomorphism F_ℓ of E such that*

$$F_\ell S F_\ell^{-1}\colon \rho \to a\begin{bmatrix}(-1)^\ell & 0\\ 0 & I_{\ell-1}\end{bmatrix}\rho,$$

where S is the linear transformation of E_ℓ defined for $a>0$ by $S\colon \sigma\to -a\sigma$. Moreover, F_ℓ is of class C^∞ on $E_\ell-\{0\}$.

Proof. It is sufficient to prove the lemma for the case $\ell=2$ as this easily leads to the general case. Let $\hat{S}\colon \rho\to a\rho$ and write $b=a^{1/\pi}$. Define F_2 by

$$F_2(r,\theta) = (r, \theta+\log_b r) \quad \text{for} \quad r\neq 0,$$
$$F_2(0,0) = 0.$$

For an arbitrary point (r,θ) with $r\neq 0$, $F_2^{-1}\hat{S}F_2(r,\theta)=(ar,\theta-\pi)=S(r,\theta)$. This completes the proof.

By applying Lemmas 3.1 and 3.2 with $\ell=\nu_1+\cdots+\nu_k$ to the transformation (8), we obtain

LEMMA 3.3. *There is a homeomorphism G of E_m which is of class C^∞ on $E_m-\{0\}$ such that*

$$GLG^{-1}\colon u \to e^{-1}\begin{bmatrix}(-1)^\ell & 0\\ 0 & I_{m-1}\end{bmatrix}u.$$

Proof. Define

$$G = \left[\begin{array}{c|c} F & 0 \\ \hline 0 & I_{m-\ell} \end{array}\right] \left[\begin{array}{c|c} \begin{matrix} T_{\Delta_1} & & 0 \\ & \ddots & \\ 0 & & T_{\Delta_k} \end{matrix} & 0 \\ \hline 0 & T_{\mathrm{P}} \end{array}\right], \quad \bar{\mathrm{P}} = e^{\mathrm{P}}.$$

Note that if $|\mu_j| > 1$ for $j = 1, \cdots, m$ we have

$$G_A L G_A^{-1}: u \to e \begin{bmatrix} (-1)^{\ell} & 0 \\ 0 & I_{m-1} \end{bmatrix} u.$$

We combine the lemmas above to obtain the main result of this section, the proof of which is now clear.

THEOREM 3.4. *Let L: $z \to Cz$ be a linear transformation of E_n with eigenvalues μ_j ($j = 1, \cdots, n$). Assume that m of the μ_j's satisfy $0 < |\mu_j| < 1$ and ℓ of these satisfy $-1 < \mu_j < 0$. If the remaining $n - m$ μ_j's satisfy $1 < |\mu_j|$ and r of these satisfy $\mu_j < -1$, then there is a homeomorphism Ψ of E_n which is of class C^∞ on $E_n - \{0\}$ such that*

$$\Psi L \Psi^{-1}: w \to \left[\begin{array}{c|c} \begin{matrix} e^{-1}(-1)^{\ell} & 0 \\ 0 & e^{-1} I_{m-1} \end{matrix} & 0 \\ \hline 0 & \begin{matrix} e(-1)^r & 0 \\ 0 & e I_{n-m-1} \end{matrix} \end{array}\right] w,$$

where $w = \Psi z$.

One easily counts that there are $4n$ such canonical forms for $\Psi L \Psi^{-1}$, and of course none of these are homeomorphically equivalent. Thus one may say that each linear transformation of the type considered is characterized (as far as topological properties are concerned) by the dimension of its stability manifold and its orientation preserving properties on the stability and instability manifolds.

References

1. E. A. Coddington and N. Levinson, Theory of ordinary differential equations, McGraw-Hill, New York, 1955, pp. 106–107.

2. P. Hartman, A lemma in the theory of structural stability of differential equations, Proc. Amer. Math. Soc., 11 (1960) 619–620.

CONVERGENCE OF SERIES WHOSE TERMS ARE DEFINED RECURSIVELY

M. K. FORT, JR., University of Georgia AND
SEYMOUR SCHUSTER, University of Minnesota

Let f be a real-valued function such that:

(1) f is differentiable on an interval $[0, a]$ for some $a>0$,

(2) $0<f(x)<x$ for $0<x\leqq a$,

(3) there exists $c>0$ such that $f'(x)\geqq c$ for $0\leqq x\leqq a$, and

(4) if $0<x_1<x_2\leqq a$, then $f(x_1)/x_1\geqq f(x_2)/x_2$.

In this paper we investigate the convergence of the infinite series $\sum_n u_n$, where $u_0=a$ and $u_{n+1}=f(u_n)$ for all n. Assumption (1) implies that f is continuous, and this fact together with assumption (2) implies that $u_n>u_{n+1}$ for each n and $\lim_{n\to\infty} u_n=0$. The existence of $f'(0)$ implies that $\lim_{n\to\infty} u_{n+1}/u_n$ exists.

REMARK 1. Consider the conditions:

(4′) f' is a nonincreasing function on $[0, a]$, and

(4″) f'' exists and is nonpositive on $(0, a)$.

In the presence of (1), (2), and (3), (4″) implies (4′), and it is not difficult to show that (4′) implies (4). It is often easy to verify indirectly that a function f satisfies condition (4) by showing that f satisfies (4′) or (4″).

THEOREM 1.

$$a+c\int_0^a \frac{x}{x-f(x)}\,dx \leqq \sum_{n=0}^{\infty} u_n \leqq \int_0^a \frac{x}{x-f(x)}\,dx,$$

and hence $\sum_n u_n$ *converges if and only if the integral* $\int_0^a x\,dx/[x-f(x)]$ *converges.*

Proof. Let $g(x)=x/[x-f(x)]$, $\phi(x)=f(x)/x$, and $\psi(x)=1-\phi(x)$ for $0<x\leqq a$. Then $g(x)=1/[1-\phi(x)]=1/\psi(x)$. Condition (4) states that ϕ is nonincreasing on $(0, a]$. It follows that ψ is nondecreasing and g is nonincreasing on $(0, a]$.

For each n,

$$u_0+\sum_{i=1}^{n}[u_i/(u_i-u_{i+1})]\cdot[(u_i-u_{i+1})/(u_{i-1}-u_i)]\cdot(u_{i-1}-u_i)$$
$$=\sum_{i=0}^{n} u_i = u_0+\sum_{i=1}^{n}[u_i/(u_{i-1}-u_i)](u_{i-1}-u_i).$$

We have:

$$u_i/(u_i-u_{i+1}) = u_i/(u_i-f(u_i)) = g(u_i),\ u_i/(u_{i-1}-u_i) = f(u_{i-1})/[u_{i-1}-f(u_{i-1})]$$
$$= -1+u_{i-1}/(u_{i-1}-f(u_{i-1})) = -1+g(u_{i-1}),$$

and

$$(u_i-u_{i+1})/(u_{i-1}-u_i) = (f(u_{i-1})-f(u_i))/(u_{i-1}-u_i) = f'(t_i)$$

for some t_i, $u_i<t_i<u_{i-1}$. We now substitute to obtain

$$u_0+\sum_{i=1}^{n} g(u_i)\cdot f'(t_i)\cdot(u_{i-1}-u_i) = \sum_{i=0}^{n} u_i = u_0+\sum_{i=1}^{n}[-1+g(u_{i-1})]\cdot(u_{i-1}-u_i).$$

Since $f'(t_i) \geqq c$ for each i, and $\sum_{i=1}^{n} -(u_{i-1}-u_i) = u_n - u_0$, we see that

$$u_0 + c\sum_{i=1}^{n} g(u_i)(u_{i-1} - u_i) \leqq \sum_{i=0}^{n} u_i = u_n + \sum_{i=1}^{n} g(u_{i-1})(u_{i-1} - u_i).$$

We now use the fact that g is a nonincreasing function to obtain

$$\sum_{i=1}^{n} g(u_i)(u_{i-1} - u_i) \geqq \int_{u_n}^{u_0} g(x)dx \geqq \sum_{i=1}^{n} g(u_{i-1})(u_{i-1} - u_i).$$

Thus,

$$u_0 + c\int_{u_n}^{u_0} g(x)dx \leqq \sum_{i=0}^{n} u_i \leqq u_n + \int_{u_n}^{u_0} g(x)dx.$$

We now take the limit as $n \to \infty$ and obtain

$$u_0 + c\int_0^a g(x)dx \leqq \sum_{i=0}^{\infty} u_i \leqq \int_0^a g(x)dx.$$

Since $c > 0$, it follows that $\sum_{i=0}^{\infty} u_i < \infty$ if and only if $\int_0^a g(x)dx < \infty$. This is what we were to prove.

Example 1. Let $f(x) = rx$ for $0 \leqq x \leqq a$, where $0 < r < 1$. In this case $u_n = ar^n$ and $\sum_n u_n$ is the geometric series with ratio r. It is obvious that (1), (2), (3) and (4) are satisfied, and we may let $c = r$. A simple computation shows that

$$\int_0^a \frac{x}{x - f(x)} dx = \frac{a}{1 - r},$$

and

$$a + c\int_0^a \frac{x}{x - f(x)} dx = \frac{a}{1 - r}.$$

Hence $\sum_n u_n$ converges and $\sum_{n=0}^{\infty} u_n = a/(1-r)$.

Example 2. Choose a, h and p satisfying $0 < a$, $0 < h$, $p > 1$, and $pha^{p-1} < 1$. For $0 \leqq x \leqq a$ we define $f(x) = x - hx^p$.

The function f obviously satisfies conditions (1), (2), (3), and (4)

$$\int_0^a \frac{x}{x - f(x)} dx = h^{-1} \int_0^a x^{-p+1} dx = \begin{cases} a^{2-p}/h(2 - p) & \text{for } 1 < p < 2 \\ \infty & \text{for } p \geqq 2. \end{cases}$$

Thus, by Theorem 1, $\sum_n u_n$ converges for $1 < p < 2$ and diverges for $p \geqq 2$.

REMARK 2. Let ϕ and ψ be functions on $[0, a]$ such that

$$0 < \phi(x) \leqq \psi(x) < x$$

for $0 < x \leqq a$. Let ϕ and ψ generate series $\sum_n v_n$ and $\sum_n w_n$ respectively $[v_0 = a = w_0,\ v_{n+1} = \phi(v_n),\ w_{n+1} = \psi(w_n)]$. If at least one of ϕ, ψ is nondecreasing then $v_n \leqq w_n$ for each n and thus $\sum_{n=0}^{\infty} v_n \leqq \sum_{n=0}^{\infty} w_n$.

Example 3. Suppose $p>0$ and $0<a<1$. We define $u_0=a$, and for each nonnegative integer n we define $u_{n+1}>0$ by the equation $u_{n+1}+u_{n+1}^p=u_n$. We investigate the convergence of $\sum_n u_n$. For $p=2$, this problem is equivalent to one which appeared in the 1954 William Lowell Putnam Competition.

If $0<p\leqq 1$, then $u_{n+1}^p \geqq u_{n+1}$ and hence $2u_{n+1}\leqq u_n$. In this case it is obvious that $\sum_n u_n$ converges.

Now suppose that $p>1$. We define a nonnegative function f implicitly on $[0, a]$ by $f(x)+f(x)^p=x$. Obviously $0<f(x)<x$ for $0<x\leqq a$ and f is continuous. It follows that $f(x)^p<f(x)$ and hence $x<2f(x)$. Thus $x/2<f(x)<x$ and consequently $2^{-p}x^p<f(x)^p<x^p$. It now follows that

$$x - 2^{-p}x^p > x - f(x)^p > x - x^p.$$

We define $\phi(x)=x-x^p$ and $\psi(x)=x-2^{-p}x^p$. Since $x-f(x)^p=f(x)$, we obtain $0<\phi(x)<f(x)<\psi(x)<x$ for $0<x\leqq a$.

If $1<p<2$, then Example 2 shows that the series generated by ψ converges, and it follows from Remark 2 that $\sum_n u_n$ also converges.

If $p\geqq 2$, then Example 2 shows that the series generated by ϕ diverges, and it follows from Remark 2 that $\sum_n u_n$ also diverges.

We have now shown that $\sum_n u_n$ converges for $0<p<2$ and diverges for $p\geqq 2$.

COROLLARY 1. *If $f'(0)<1$, then $\sum_n u_n$ converges.*

Proof. If $f'(0)<1$, then

$$\lim_{x\to 0+} \frac{x}{x-f(x)} = \frac{1}{1-f'(0)} < \infty.$$

Hence $\int_0^a x\,dx/(x-f(x))$ is finite and $\sum_n u_n$ converges.

THEOREM 2. *If $f'(0)=1$ and f'' exists and is bounded below on $[0, a]$, then $\sum_n u_n$ diverges.*

Proof. There exists a positive number p such that $f''(x)>-p$ for $0\leqq x\leqq a$. It follows from Taylor's Theorem that if $0<x\leqq a$, then $f(x)=f(0)+xf'(0)+x^2f''(t)/2$, where $0<t<x$. Thus, $f(x)=x+x^2f''(t)/2>x-px^2/2$ for $0<x\leqq a$. We obtain $1/(x-f(x))>2/px^2$ and hence

$$\int_0^a \frac{x}{x-f(x)}\,dx \geqq \int_0^a \frac{2}{px}\,dx = \infty.$$

By Theorem 1, $\sum_n u_n$ diverges.

Example 4. Theorem 2 may be used to prove that each of the following functions generates a divergent series:

(i) $0<a<\pi/2$, $f(x)=\sin x$ for $0\leqq x\leqq a$,

(ii) $a>0$, $f(x)=\log(x+1)$ for $0\leqq x\leqq a$.

(iii) $0<a<\pi/2$, $f(x)=x-1+\cos x$ for $0\leqq x\leqq a$.

(iv) $0<a<1$, $f(x)=xe^{-x}$ for $0\leqq x\leqq a$.

REMARK 3. Let $a=1$ and $f(x)=x/(x+1)$ for $0\leqq x\leqq 1$. It is easy to see that $u_n=1/(n+1)$ for each n. If we expand f in a power series about 0, we see that Corollary 2 applies with $c=1$ and $p=2$. We thus obtain the well-known fact that $\sum_n (n+1)^{-s}$ converges for $s>1$ and diverges for $0<s\leqq 1$.

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References

1. J. Ouspensky, Arch. d. Math. u. Phys. Series 3, Basel and Stuttgart, 20 (1913).
2. G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis I, Die Grundlehren der mathematischen Wissenschaften, Vol. 19, New York 1945.

ON FERMAT'S LAST THEOREM

J. M. GANDHI, University of Rajasthan, Jaipur, India

1. Introduction. According to Fermat's Last Theorem (FLT) the equation

$$x^n + y^n = z^n \qquad \text{for } n > 2, \tag{1.1}$$

has no integral solution. The purpose of the present paper is to extend Fermat's conjecture and study the equations of the form

$$x^n \pm y^n = pz^n.$$

On the basis of a large number of theoretical results and some computational data, the following two conjectures have been made:

Conjecture 1. The equation $x^n+y^n=pz^n$ has no integral solution.

Conjecture 2. Probably the equation $x^n-y^n=pz^n$ has no integral solution.

Here x, y, z are nonzero, unequal coprime integers, $n>2$, and p is a positive integer $\leqq n$.

It is evident that conjectures 1 and 2 include Fermat's Last Theorem as a special case when $p=1$ and that the two conjectures are identical when n is odd.

The following results from the literature support the above conjectures. E. Maillet ([17], and [3] p. 761), using Kummer's method, proved that $x^a+y^a=az^a$, $a>2$, has no integral solution $\neq 0$ if a is divisible by 4 or if a is even and divisible by a prime $4n+3$; or if $2<a\leqq 100$, $a\neq 37, 59, 67, 74$; or if a has no prime factor >17. He has also conjectured the impossibility of the equation $x^n+y^n=nz^n$, which is the special case of Conjecture 1 when $p=n$. For the proof of Maillet's conjecture for some other cases see [16] and [8]. In this paper we give (Theorem 2) the proof of Maillet's conjecture for all powers of the form $n^c(n-1)$, n being an odd prime, with $c\geqq 0$.

REMARK 3. Let $a=1$ and $f(x)=x/(x+1)$ for $0\leqq x\leqq 1$. It is easy to see that $u_n=1/(n+1)$ for each n. If we expand f in a power series about 0, we see that Corollary 2 applies with $c=1$ and $p=2$. We thus obtain the well-known fact that $\sum_n (n+1)^{-s}$ converges for $s>1$ and diverges for $0<s\leqq 1$.

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References

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The impossibility of the equations $x^n \pm y^n = 2z^n$, $n > 2$, which is contained in the above conjectures, appears to be justified, since assuming the impossibility of these equations, the proof of FLT for all even powers follows at once (see [11] and [14]).

We will show that conjectures 1 and 2 are true for $n=3$ and $n=4$, while for $n=5, 6, 7$, they are found to be true except for the following cases:

Conjecture 1. For $n=5$, $p=2$; $n=7$, $p=2$.

Conjecture 2. For $n=5$, $p=2$; $n=6$, $p=6$; $n=7$, $p=2$.

It was proved by Legendre ([12], and [3] p. 573) that $x^3+y^3=2z^3$ implies that $x=\pm y$, hence, if x, y, z are unequal coprime integers, then the equation is inconsistent. It is also known that the equation $x^3+y^3=3z^3$ has no integral solution ([22] p. 412, Prob. 7). Thereby conjectures 1 and 2 are found to be true for $n=3$.

It was proved by R. D. Carmichael ([1], and [3] p. 634) that the equations $x^4 \pm y^4 = 4z^2$ are inconsistent, while Schopis ([19], and [3] p. 618) proved that $x^4 \pm y^4 = 2z^2$ are impossible. In view of the fact that the equation $x^2+y^2=mz^2$ can have solutions only if m is the sum of two squares ([3] p. 427), the equation $x^4+y^4=3z^4$ is inconsistent. In this paper the impossibility of the equation $x^4-y^4=3z^4$ has been proved, establishing conjectures 1 and 2 for $n=4$.

Combining the results of G. L. Dirichlet ([4], and [3] p. 735) and J. G. van der Corput ([23], and [3] p. 773), the truth of the conjectures can be verified for $n=5$, except for the case $p=2$.

Since the equations $x^3 \pm y^3 = pz^3$ are impossible for $p \leqq 3$, it follows that $x^6 \pm y^6 = pz^6$ are also impossible for $p \leqq 3$.

E. Swift ([20], and [3] p. 773) proved that $x^6 \pm y^6$ are not squares and hence the equations $x^6 \pm y^6 = 4z^6$ are impossible.

J. G. van der Corput ([24], and [3] p. 578) proved that the equations $x^3 \pm y^3 = pz^3$ are impossible if p is a prime $\equiv 2$ or 5 (mod 9) which implies that the equations $x^6 \pm y^6 = 5z^6$ are impossible.

E. Maillet [17] proved that the equation $x^a+y^a=az^a$ has no integral solution if a is even and divisible by a prime $4n+3$, which implies that the equation $x^6+y^6=6z^6$ is impossible (or see Theorem 2 below) and thereby Conjecture 1 has been verified for $n=6$ also, while for Conjecture 2, the only equation which is to be proved to be impossible is $x^6-y^6=6z^6$.

It was proved by E. Maillet ([16], and [3] p. 759) that the equation $x^7+y^7=cz^7$ has no integral solution when c is of the form $49k \pm 3$, ± 4, ± 5, ± 6 etc., which means that the above equation is impossible for $c = \pm 1$, ± 3, ± 4, ± 5 and ± 6. In the same paper he proves the impossibility of the equation $x^7+y^7=7z^7$; to justify Conjectures 1 and 2 for $n=7$, therefore, the only equations which are to be proved to be impossible in unequal coprime integers are $x^7 \pm y^7 = 2z^7$.

In the text of the paper the following special cases of the conjectures have been proved.

THEOREM 1. *The equation* $px^R+y^R=z^R$ *has no integral solution with* x *prime to* n, n *being an odd prime when* (a) $p=R=n$; (b) $R=k\cdot n^c(n-1)$, $c\geqq 0$, $p\not\equiv 0$, 1 *or* $-1 \pmod{n^{c+1}}$.

THEOREM 2. *If* $n^c(n-1)$ *is a factor of* $2m$, $c\geqq 0$, *and* n *is an odd prime, then the equation* $x^{2m}+y^{2m}=pz^{2m}$ *has no integral solution if* $p\not\equiv 1$ *or* $2 \pmod{n^{c+1}}$.

THEOREM 3. *The equation* $x^{k(n-1)/2}+y^{k(n-1)/2}=pz^{k(n-1)/2}$ *has no integral solution if* $p\not\equiv 0, -1, -2, 1$ *or* $2 \pmod{n}$ n *being an odd prime* >5, k *odd, and* z *prime to* n.

It is evident that Theorem 1 proves the Conjecture 2 for some powers when x is prime to n, while Theorem 3 proves Conjectures 1 and 2 for a large number of odd and even powers with the restriction that z is prime to n. Theorem 2 proves the truth of Conjecture 1 for all powers of the form $n^c(n-1)$, n being an odd prime except when $p=1$ and 2; it is interesting that these are the cases of FLT and its corollary respectively.

Using Theorem 2, we can give the proof of Conjecture 1 for a large number of even powers other than $n^c(n-1)$, n being an odd prime. For example it has been shown that Conjecture 1 is true for the values of $n=8$, 20, 24, 32, 44, 48, 56, 64, 80, 84, etc. The author has listed all solutions of

$$x^n+y^n=pz^n<10^7 \qquad \text{for } n>2 \tag{1.2}$$

and of

$$pz^n+y^n=x^n<10^7 \qquad \text{for } n>2. \tag{1.3}$$

For $n=3$, the solutions of (1.2) and (1.3) are too numerous to be included here. For the range specified, however, no counter example has been found for the above conjectures, as was expected on theoretical grounds.

For $n=4$, Cunningham ([**2**], and [**3**] p. 634) listed all solvable equations $a^4+b^4=mc^2<10^7$ and $1+y^4=mc^2$, $y<1000$, and E. Lucas ([**15**], and [**3**] p. 630) listed all solvable equations $ax^4+by^4=cz^2$ in which 2 and 3 are the only primes dividing a, b or c, while the author has listed all solutions of $z^4=pz^4+y^4<10^7$, and no counterexample was found in these lists for the above conjectures as was expected.

For $n>4$, the only solutions for equations (1.2) and (1.3) for the range specified are

$$7^5+25^5=11\cdot 27791\cdot 2^5, \qquad 13^5+19^5=881\cdot 101\cdot 2^5, \qquad 11^5+21^5=132661\cdot 2^5.$$

Thus we find that the available computational data also support the above conjectures.

It is of some interest to note that, by using Theorem 2, it can be shown that

$$x^n+y^n=(n+1)z^n \tag{1.4}$$

has no integral solution for a large number of even values of n. All such values

of $n<1000$ have been listed in the paper. Using the same theorem, it has also been shown that the equation

$$x^n + y^n = (n+2)z^n \tag{1.5}$$

has no integral solution for a large number of even values of n. For even values of $n<2000$, equation (1.5) is proved to be impossible except when $n=284$, 1244, 1604, 1784, or 1844.

Probably equations (1.4) and (1.5) are also impossible; if that is so then the condition $p\leqq n$ for Conjecture 1 can be extended to $p\leqq n+2$. It may be noted that since

$$17^3 + 37^3 = 6\cdot 21^3, \qquad 71^3 + (-17)^3 = 7\cdot 38^3, \quad \text{and} \quad 3^4 - 1^4 = 5\cdot 2^4,$$

the condition $p\leqq n+2$ for Conjecture 1 and $p\leqq n$ for Conjecture 2 can not be further extended.

2. In this section, we make some more observations regarding equations of the form $x^n \pm y^n = pz^n$.

It is trivial to prove that if p is allowed to vary with x, y, z, then the equation

$$pz^n = x^n - y^n \tag{2.1}$$

has infinitely many solutions. The equation

$$x^n + y^n = pz^n \tag{2.2}$$

also has infinitely many solutions when (1) n is odd, and (2) n is of the form $2(2k+1)$.

To prove the above statements, one has to factor the expressions $x^n \pm y^n$, put one factor equal to z^n, so that the resulting equation can easily be solved, and equate the other factor with p.

Example: $44^6+117^6=164634913\cdot 5^6$.

It is an interesting problem to attempt the solutions of the equation (2.2) when n is divisible by 4, for when $z>1$ and $x^4+y^4<10^7$, it has no integral solution ([2] and [3] p. 634, 620 and 663). It is of real interest, of course, to find the solutions of the equations $x^n \pm y^n = pz^n$ for given values of both n and p, but it is not easy to find such solutions. On the other hand, we have shown that for given n there are infinitely many values of p for which equations (2.1) and (2.2) are solvable. It is then quite natural to ask the question: Given n, what is the smallest value of p such that the above equations have nontrivial solutions? By nontrivial solution, we mean one such that $z>1$ and that none of the integers x, y, z is zero, nor are any two of them equal.

For $n=2$, if p is given, (2.2) has either an infinite number of coprime solutions or none, depending on the value of p. For $n=3$, it is probable that both (2.1) and (2.2) have a finite number of coprime solutions for any given p. Thus for $n>3$, the smallest value of p for which nontrivial solution exists probably

varies wildly with n; on the basis of the theoretical results and some computational data it is quite natural to arrive at the above conjectures.

3. In this section it will be proved that the equation

$$3x^4 + y^4 = z^4 \tag{3.1}$$

has no integral solution.

Now x and y cannot both be odd, since then x^4 and y^4 would each be of the form $8m+1$, and hence $3x^4+y^4$ would be of the form $8k+4$, which can never be a fourth power of an even number.

Therefore, let x be odd and y be even; then z is odd. In this case $3x^4+y^4$ will be of the form $8S+3$, which can never be a fourth power of an odd number and hence x cannot be odd.

Now let x be even and y odd, so that z is odd. Rewrite (3.1) as

$$3x^4 = z^4 - y^4. \tag{3.2}$$

Let $z=p+q$, $y=p-q$, with p and q coprime and of different parity because z and y are coprime and both are odd. Whence from (3.2) we get

$$3x^4 = 8pq(p^2 + q^2). \tag{3.3}$$

Since p and q are coprime so also are p, q and p^2+q^2. Let p then be even and q odd (it is immaterial if p is odd and q even) so that p^2+q^2 is odd.

Let us first assume that $p\equiv 0 \pmod 3$ or $p=3A$, so that from (3.3) we get

$$x^4 = (8A)q(9A^2 + q^2). \tag{3.4}$$

Since all factors on the right of (3.4) are coprime, each must be a perfect fourth power, so that

$$\text{(a) } 8A = M^4, \qquad \text{(b) } q = L^4, \qquad \text{(c) } 9A^2 + q^2 = D^4. \tag{3.5}$$

From (3.5b) and (3.5c) we have

$$9A^2 + L^8 = D^4 \quad \text{or} \quad D^4 - L^8 = (3A)^2.$$

Since $x^4-y^4=z^2$ is impossible ([**3**] p. 618) it follows that $p\not\equiv 0 \pmod 3$. Similarly it can be proved that $q\not\equiv 0 \pmod 3$.

Then consider that $p^2+q^2\equiv 0 \pmod 3$, so that from (3.3) we get

$$x^4 = (8p)q((p^2 + q^2)/3) \tag{3.6}$$

since all the factors on the right of (3.6) are coprime, each must be a perfect fourth power. Therefore

$$\text{(a) } 8p = B^4, \qquad \text{(b) } q = V^4, \qquad \text{(c) } (p^2 + q^2)/3 = t^4. \tag{3.7}$$

Since (3.7c) is impossible ([**3**] p. 427), $p^2+q^2\not\equiv 0 \pmod 3$. As we have proved that $p\not\equiv 0 \pmod 3$, $q\not\equiv 0 \pmod 3$ and $p^2+q^2\not\equiv 0 \pmod 3$, therefore (3.3) or (3.1) is impossible.

4. In this section the following theorems will be proved:

THEOREM 1. *The equation* $px^R+y^R=z^R$ *is impossible if* x *is prime to* n, n *being an odd prime when* (a) $p=R=n$, (b) $R=k\cdot n^c(n-1)$, $c\geqq 0$, $p\not\equiv 0$, -1 *or* 1 (mod n^{c+1}).

THEOREM 2. *If* $n^c(n-1)$ *is a factor of* $2m$, $c\geqq 0$ *and* n *is an odd prime, then the equation* $x^{2m}+y^{2m}=pz^{2m}$ *has no integral solution if* $p\not\equiv 1$ *or* 2 (mod n^{c+1}).

THEOREM 3. *The equation* $x^{k(n-1)/2}+y^{k(n-1)/2}=pz^{k(n-1)/2}$ *has no integral solution if* $p\not\equiv 0$, 1, 2, -1 *or* -2 (mod n), n *being an odd prime* >5, k *odd and* z *prime to* n.

Theorem (1a) was proved by J. Westlund ([**21**], [**3**] p. 766).

Proof (1b). If $R=k\cdot n^c(n-1)$, n an odd prime, $c\geqq 0$, then

$$px^{k\cdot n^c(n-1)} = z^{k\cdot n^c(n-1)} - y^{k\cdot n^c(n-1)} \tag{4.1}$$

x being assumed to be prime to n. By a generalized Fermat's theorem it is known that if a is prime to n, n an odd prime, then

$$a^{k\cdot n^c(n-1)} \equiv 1 \pmod{n^{c+1}}. \tag{4.2}$$

Let us consider that both z and y are prime to n. Then, using (4.2), from (4.1) we get $p\equiv 0$ (mod n^{c+1}), but this case has already been excluded in the theorem and hence z and y cannot be simultaneously prime to n.

It can also be seen that neither z nor y is divisible by n, since x and p are each prime to n. Then, according as z or y is divisible by n, from (4.1) we have $p\equiv -1$ or 1 (mod n^{c+1}); since both these cases have been excluded, the theorem follows.

Proof of Theorem 2. Let

$$x^{k\cdot n^c(n-1)} + y^{k\cdot n^c(n-1)} = pz^{k\cdot n^c(n-1)} \tag{4.3}$$

where n is an odd prime and $c\geqq 0$.

Let us first consider that p is prime to n. Let x, y, z be each prime to n; then, using (4.2), from (4.3) we get $2\equiv p$ (mod n^{c+1}), but this congruence has already been excluded.

Then let us consider the case when x and y are each prime to n and z is divisible by n, so that using (4.2), from (4.3) we get $2\equiv 0$ (mod n^{c+1}), which is impossible since $n>2$. Let then x or y be divisible by n, so that from (4.3) we get $1\equiv 0$ (mod n^{c+1}) or $1\equiv p$ (mod n^{c+1}), according as z is divisible by n or prime to it. The last congruence has been excluded in the theorem and the former is impossible for $n>1$. Finally let us assume that both x and y are divisible by n. Let $x=n^m t$, $y=n^s u$, and assume $m\geqq s$, t and u being coprime to n. Then from (4.3) we get

$$n^{(m-s)k\cdot n^c(n-1)}t^{k\cdot n^c(n-1)} + u^{k\cdot n^c(n-1)} = p(z/n^s)^{k\cdot n^c(n-1)}. \tag{4.4}$$

Since u and t are coprime to n, dividing (4.4) by n^{c+1} we have

$$1 \text{ or } 2 \equiv p(z/n^s)^{k\cdot n^c(n-1)} \pmod{n^{c+1}} \tag{4.5}$$

according as $m>s$ or $m=s$. Since $p\not\equiv 1$ or 2 (mod n^{c+1}), (4.5) can never be satisfied.

Then, assuming p to be divisible by n and proceeding as before, it can be shown that (4.3) leads to impossible congruences, and hence the theorem follows.

Using the well-known result that

$$a^{(n-1)/2} \equiv \pm 1 \pmod{n}, \tag{4.6}$$

where a is prime to n, n being an odd prime, the sign in (4.6) being $+$ or $-$ according as $k^2\equiv a$ (mod n) has or has not integral solutions, Theorem 3 can be easily proved.

As stated before, Theorem 2 proves Conjecture 1 for all powers of the form $n^c(n-1)$, $c\geqq 0$, n being an odd prime except for the values of $p=1$ and 2, but these are the cases of FLT and its corollary respectively.

Using Theorem 2, the Conjecture 1 can be verified for a large number of even powers other than $n^c(n-1)$, n and c being already defined. As before, it is to be remembered that the cases with $p=1$ and 2 are to be excluded in the following discussions except when the Conjecture 1 is found to be true for these cases too. For example consider the equation

$$x^{24} + y^{24} = pz^{24}. \tag{4.7}$$

Since 12 is a factor of 24 and 13 is a prime the above equation is impossible for all values of p other than $p\equiv 1$ or 2 (mod 13), and such values of $p\leqq 24$ are $p=1$, 2, 14 and 15.

Also, since 4 is a factor of 24 and 5 is a prime the above equation is impossible for all values of p except when $p\equiv 1$ or 2 (mod 5), and as such the cases with $p=14$ and 15 are to be eliminated. Hence (4.7) will have no integral solution for $p\leqq 24$, except when $p=1$ or 2.

Since Conjecture 1 is true for the power 4, and (4.7) is also inconsistent for the values of $p=1$ and 2, we have established the truth of Conjecture 1 for the power 24.

It is to be noted that (4.7) is also impossible for all values of p for which $p\not\equiv 1$ or 2 (mod 3) and $p\not\equiv 1$ or 2 (mod 7). Since $3(3-1)$ is a factor of 24, (4.7) is also impossible for all values of p for which $p\not\equiv 1$ or 2 (mod 3^2).

Using this method, Conjecture 1 (including the values of $p=1$ and 2) has been shown to be true for the powers 8, 20, 24, 32, 44, 48, 56, 64, 80, 90, 92, etc.

Using the above method it has been shown that the equation

$$x^n + y^n = (n+1)z^n \tag{4.8}$$

has no integral solution for a large number of even values of n. For even values of $n \leqq 1000$, equation (4.8) is found to be impossible except for the following values of n:

$$n = 34, 94, 118, 142, 202, 226, 262, 274, 334, 378, 394, 436, 454, 514, 526, 538, 582, 622, 634, 694, 706, 766, 778, 802, 922, 934, 958, 982 \text{ and } 994.$$

Using the result that the equation

$$x^{4n} + y^{4n} = 4pz^{4n} \tag{4.9}$$

with $p \leqq n+1$ has no integral solution (proof to be discussed elsewhere) and Theorem 2, it has been shown that the equation

$$x^n + y^n = (n + 2)z^n \tag{4.10}$$

has no integral solution for a large number of even n's. For even values of $n < 2000$, (4.10) is found to be impossible except for the values of $n = 284$, 1244, 1604, 1784 or 1844.

Probably equations (4.9) and (4.10) are also impossible.

In passing, we may mention an interesting application of Theorem 3. In view of this theorem, the necessary condition for the solution of the equation $x^{68}+y^{68}=pz^{68}$, for the values of $p \leqq 68$ (the values of $p=1$ and 2 are also included, since for those values of p's the above equation is impossible in view of the fact that Conjecture 1 is true for the power 4), is that z must be divisible by 137. Hence, $x^{68}+y^{68}=pz^{68}<(137)^{28}\approx 10^{146}$, has no integral solution for $p \leqq 68$. Similar limits can be obtained for other powers too.

Finally we state the following corollaries to the above theorems. Their proofs, being on the same lines as the proofs of the theorems themselves, are omitted. Consider the equation

$$z^{2m} - y^{2m} = px^{2m}, \tag{4.11}$$

where x, y, z are prime to each other, $n^c(n-1)$ is a factor of $2m$ and n is an odd prime.

COROLLARY 1. *If* $p \equiv 0 \pmod{n^{c+1}}$ *then* z *and* y *must be coprime to* n.

COROLLARY 2. *If* x *is divisible by* n, *then for all* p'*s*, z *and* y *must be coprime to* n.

COROLLARY 3. *If* x *is prime to* n, *then of* z *and* y, *one must be divisible by* n, *the other prime to* n, *according as*

$$p \equiv -1 \pmod{n^{c+1}} \quad \textit{or} \quad p \equiv 1 \pmod{n^{c+1}}.$$

Now consider the equation

$$x^{2m} + y^{2m} = pz^{2m} \tag{4.12}$$

with x, y, z being prime to each other.

COROLLARY 4. *The necessary condition for the solution of* (4.12) *is* $p \not\equiv 0 \pmod 3$, $p \not\equiv 0 \pmod 4$. *The latter condition follows from the fact that the sum of two odd squares is not divisible by* 4.

Now let $n^c(n-1)$ be a factor of $2m$ such that n is an odd prime.

COROLLARY 5. *If* $p \equiv 2 \pmod{n^{c+1}}$, *then* x, y, z *each must be prime to* n.

COROLLARY 6. *If* $p \equiv 1 \pmod{n^{c+1}}$, *then* z *must be prime to* n *and either* x *or* y *divisible by* n *and other being prime to* n.

This corollary implies that FLT is true for the power $2m$, $n^c(n-1)$ being a factor of $2m$ and x, y, z are all coprime to n.

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References

1. R. D. Carmichael, Diophantine Analysis, Mathematical Monographs, No. 16, New York, (1915) 77–79.
2. A. Cunningham, L'intermédiaire des math., 18 (1911) 45–46.
3. L. E. Dickson, History of the Theory of Numbers, vol. 2, Chelsea, New York, 1952.
4. G. L. Dirichlet, Werke I, 21–46. J. Reine Angew. Math., 3 (1828) 354–375.
5. A. Desboves, Nouv. Ann. Math. (2), 18 (1879) 400, 491; (3), 5 (1886) 577.
6. ———, *ibid.* (2), 18 (1879) 408 and 481–489.
7. Fermat, Oeuvres, I, 291; French transl. III, 241. *Diophanti Alexandrini Arith.*, libri sex, Ed. by S. Fermat, Tolosae (1670) 61. (Ref. 3, p. 731.)
8. T. Hayashi, J. Indian Math. Soc., 3 (1911) 16–22; 111–114.
9. E. E Kummer, S.-B. Deutsch. Akad. Wiss. Berlin, (1847) 132–139, (Ref. 3, p. 741) and Abh. Deutsch. Akad. Wiss. Berlin, (Math.) for 1857, 1858, 41–74 (Ref. 3, p. 744).
10. ———, Letter to L. Kronecker, Jan. 2, 1852 (Ref. 3, p. 742).
11. ———, J. Reine Angew. Math., 17 (1837) 203–209.
12. A. M. Legendre, Théorie des Nombres, Paris, (1798) 409.
13. D. N. Lehmer, Factor Table for the First Ten Millions, Carnegie Inst. Wash., Publ. No. 105, 1909.
14. J. Liouville, J. Math. Pures Appl., 5 (1840) 360.
15. E. Lucas, Recherches sur l'analyse indéterminée, Moulins, 1873; extract from Bull. Soc. d'Emulation du département de l'Allier, 12 (1873) 441–532.
16. E. Maillet, Comptes Rendus, Ac. Sci. de Paris, 129 (1899) 198–199. Proofs in Acta Math., 24 (1901) 247-256.
17. ———, Ann. Mat. Pura Appl., (3), 12 (1906) 145–178.
18. L. J. Mordell, Three lectures on Fermat's Last Theorem, Chelsea, New York, 1955.
19. Schopis, Einige Sätze aus der unbestimmten Analytik, Progr. Gumbinen, 1825 (Ref. 3, p. 618).
20. E. Swift, this MONTHLY, 21 (1914) 238–239; 23 (1916) 261.
21. J. Westlund, *ibid.*, 16 (1909) 3–4.
22. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill, New York, 1939.
23. J. G. van der Corput, Nieuw Arch. Wisk., 11 (1915) 68–75.
24. ———, *ibid.*, (2) 11 (1915) 64–68.
25. H. S. Vandiver, Trans. Amer. Math. Soc., 31 (1929) 613–642.
26. ———, this MONTHLY, 53 (1946) 555–578.

ENUMERATION OF ROOTED TRIANGULAR MAPS

R. C. MULLIN, University of Waterloo

It is only recently that much has been done with respect to the enumeration of planar maps; much of the pioneering work in this region has been done by Tutte in his census series appearing in the Canadian Journal of Mathematics. We demonstrate a method whereby results from these papers may be combined with results of Brown [1] to enumerate maps in which every country is essentially a topological triangle; the exterior or surrounding "ocean," however, may meet more than three countries. More precisely, for the purposes of this paper we define a *rooted triangular map* T (henceforth abbreviated to "map") as the dissection of the interior of a convex (topological) polygon J (other than the loop polygon) in the Euclidean plane E^2 into topological triangles by means of a set S of Jordan curves, subject to the following conditions:

(1) S contains the edges of J;

(2) no vertex of any triangle is an interior point of the edge of another;

(3) the ends of each edge are distinct;

(4) one vertex of J is distinguished as the *root vertex*,

and one of the edges of J incident with the root vertex is distinguished as the *root edge*.

The vertices and edges of J are referred to as external; remaining vertices and edges of T are internal. Two maps T_1 and T_2 are isomorphic if there exists a homeomorphism of E^2 into itself which carries T_1 into T_2 and preserves the rooting. As usual, we enumerate classes of isomorphic triangulations. We note that a map T may be interpreted as a graph $G=G(T)$ by defining $V(G)$ as the set of vertices of T and admitting the pair $(v_1, v_2)(v_1, v_2 \in V(G))$ to $E(G)$ if and only if v_1 and v_2 are the vertices of a triangle in T. If $G(T)$ is a simple graph, that is, if no pair of edges have the same ends, T is said to be a 2-connected triangulation. (The phrase, simple triangulation, has another meaning, cf. Tutte [2] p. 22.) If T is 2-connected and no interior edge has both ends in the boundary J, T is said to be 3-connected. If J is a triangle, the terms are equivalent. A triangulation with $m+3$ exterior vertices and n interior vertices is said to be of type $[n, m]$.

Three-connected triangulations of type $[n, m]$ $m, n \geqq 0$ have been enumerated by Tutte [2] who shows their number to be

$$\frac{3(m-1)!(m+2)!}{(3n+3m+3)!} \sum_{j=0}^{\min(m,n-1)} \frac{(m-3j)(m+j+2)(4n+3m-j+1)!}{j!(j+1)!(m-j+2)!(m-j)!(n-j-1)!}$$

for $m>0$, and

$$\frac{2(4n+1)!}{(3n+2)!(n+1)!} \qquad \text{for } m = 0.$$

Brown [1] has shown that the number of 2-connected triangulations of type $[n, m]$ is

$$\frac{2(2m+3)!(4n+2m+1)!}{(m+2)!m!n!(3n+2m+3)!}, \qquad m, n \geqq 0.$$

Tutte ([2] pp. 410–411) has also enumerated 2-connected triangulations of type $[n, 0]$ (in their dual form) and obtains the formula

$$\frac{2^{n+2}(3n+3)!}{(n+1)!(2n+4)!}.$$

We shall cross the results of Tutte and Brown, optimistic that the hybrid produced will yield an enumeration of maps of type $[n, m]$. The results will differ from those quoted in that for the first time we obtain a non-zero result for the number of maps of type $[n, -1]$, $n>0$.

We dispose of this important case first. From any such map we may obtain a map of type $[n-1, 0]$ by expunging the nonroot exterior edge, and using a root preserving homeomorphism if necessary to render the resulting boundary triangle convex. The process is clearly reversible. Hence, if we use the symbol p_n to denote the number of maps of type $[n, -1]$, $n>0$, then

$$p_n = \frac{2^{n+1}(3n)!}{n!(2n+2)!}.$$

If $P(x)=\sum_{n=1}^{\infty} p_n x^n$, then (as shown in [3] p. 411), $P(x)$ is defined parametrically by the equations

$$(1.1) \qquad u = x(1+2u)^3, \qquad P(x) = u(1-2u).$$

We note for future reference that

$$(1.2) \qquad 1 + P(x) = (1-u)(1+2u).$$

For $m \geqq 0$, every map can be derived from a 2-connected triangulation by replacing some of the edges by maps of type $[n, -1]$. If $d_{n,m}$ denotes the number of 2-connected triangulations of type $[n, m]$, we define $D(x, y)$ by

$$D(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} d_{n,m} x^n y^m.$$

Similarly if $h_{n,m}$ denotes the number of maps of type $[n, m]$, we define

$$H(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n,m} x^n y^m.$$

Since any 2-connected triangulation contains $3n+2m+3$ edges, any of which may be replaced by a map of type $[n, -1]$ in the construction of a triangular map, we have the relation

and, after simplification,

$$h_{n,m} = \frac{2^{n+1}(2m+3)!(3n+2m+2)!}{(m+1)^2 n!(2n+2m+4)!} \qquad m, n \geqq 0.$$

The formula evidently holds for $m = -1$. For fixed m we see, by Stirling's formula, that as $n \to \infty$

$$h_{n,m} \sim \frac{e^2 2^{n+1}(2m+3)!(3n+2m+2)^{3n+2m+2}}{(m+1)^2 n^n (2n+2m+4)^{2n+2m+4}} \sqrt{\frac{3n+2m+2}{2\pi n(2n+2m+4)}}$$

$$\sim \frac{1}{4} \frac{(2m+3)!}{(m+1)!^2} \left(\frac{27}{2}\right)^n \left(\frac{9}{4}\right)^{m+1} n^{-5/2} \sqrt{\frac{3}{\pi}}.$$

References

1. W. G. Brown, Enumeration of triangulations of the disc, Proc. London Math. Soc. (To appear.)
2. W. T. Tutte, A census of planar triangulations, Canad. J. Math., 14 (1962) 21–38.
3. ———, A census of Hamiltonian polygons, Canad. J. Math., 14 (1962) 708–722.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146.

THE STRUCTURE OF PRE-p^k-RINGS AND GENERALIZED PRE-p-RINGS

ADIL YAQUB, University of California, Santa Barbara

In [1] Abian and McWorter defined a *pre-p-ring*, where p is a prime integer, to be a *commutative* ring of characteristic p which satisfies $ab^p = a^p b$ for all elements a, b in the ring. They also announced the structure of such a ring and, in addition, specialized their results to the Boolean case ($p=2$). Our present object is to generalize their results in two directions. First, we define and determine the structure of *pre-p^k-rings* (p prime). The case $k=1$ then recovers the pre-p-ring case of the above authors. Secondly, we define a *generalized pre-p-ring* to be a ring which lacks being a pre-p-ring by the commutative law, and obtain the structure of such a generalized pre-p-ring (*without any assumption of commutativity*).

We begin with the following

DEFINITION. *Let p be a prime integer and let k be a positive integer. A commutative ring A is called a pre-p^k-ring if and only if*

(I) $ab^{p^k} = a^{p^k} b$, *and* (II) $pa = 0$, *for all* a, b *in* A.

and, after simplification,

$$h_{n,m} = \frac{2^{n+1}(2m+3)!(3n+2m+2)!}{(m+1)^2 n!(2n+2m+4)!} \qquad m, n \geqq 0.$$

The formula evidently holds for $m=-1$. For fixed m we see, by Stirling's formula, that as $n \to \infty$

$$h_{n,m} \sim \frac{e^2 2^{n+1}(2m+3)!(3n+2m+2)^{3n+2m+2}}{(m+1)^2 n^n (2n+2m+4)^{2n+2m+4}} \sqrt{\frac{3n+2m+2}{2\pi n(2n+2m+4)}}$$

$$\sim \frac{1}{4} \frac{(2m+3)!}{(m+1)!^2} \left(\frac{27}{2}\right)^n \left(\frac{9}{4}\right)^{m+1} n^{-5/2} \sqrt{\frac{3}{\pi}}.$$

References

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We begin with the following

DEFINITION. *Let p be a prime integer and let k be a positive integer. A commutative ring A is called a pre-p^k-ring if and only if*

(I) $ab^{p^k} = a^{p^k} b$, *and* (II) $pa = 0$, *for all a, b in A.*

LEMMA 1. *In a pre-p^k-ring, we have,* (i) $a^{2p^k+1}=a^{p^k+2}$; (ii) $a^{p^{2k}}=a^{p^{3k}}$; (iii) $a-a^{p^{2k}}$ *is nilpotent.*

Proof. By (I) of the definition of a pre-p^k-ring, $a(a^2)^{p^k}=a^{p^k}(a^2)$ which proves (i). To prove (ii), observe that, by (i),

$$a^{2p^k+1}a^{p^k-1} = a^{p^k+2}a^{p^k-1} = a^{2p^k+1}.$$

Multiplying the last equation by a^{p^k-1} a suitable number of times, we obtain,

$$a^{2p^k+1}(a^{p^k-1})^j = a^{2p^k+1}$$

for *any* positive integer j. Hence, $a^{p^k+2}(a^{p^k-1})^j=a^{p^k+2}$. Therefore,

$$a^{p^{2k}-p^k-2}a^{p^k+2}(a^{p^k-1})^j = a^{p^{2k}-p^k-2}a^{p^k+2} = a^{p^{2k}}.$$

Hence, in particular, $a^{p^{2k}}(a^{p^k-1})^{p^{2k}}=a^{p^{2k}}$, $a^{p^{3k}}=a^{p^{2k}}$, which proves (ii). Now (iii) follows at once from (ii) and (II) of the above definition, since, $(a-a^{p^{2k}})^{p^{2k}}=a^{p^{2k}}-a^{p^{4k}}=0$. This completes the proof of the lemma.

We recall that a p^k-ring, in the sense of [3], is a ring A satisfying $a^{p^k}=a$, and $pa=0$, for all a in A. Here p is a given *prime*, while k is a given positive integer. A p-ring is simply a p^1-ring ($k=1$).

THEOREM 2. *A pre-p^k-ring A is the direct sum of a p^k-ring and a nil pre-p^k-ring. Conversely, any such direct sum is a pre-p^k-ring.*

Proof. Since A is commutative with prime characteristic p, we see from Lemma 1 (ii) that

$$(a^{p^k} + b^{p^k})^{p^{2k}} = (a^{p^k})^{p^{2k}} + (b^{p^k})^{p^{2k}} = a^{p^{2k}} + b^{p^{2k}},$$

$$(a^{p^k} \cdot b^{p^k})^{p^{2k}} = (a^{p^k})^{p^{2k}} \cdot (b^{p^k})^{p^{2k}} = a^{p^{2k}} \cdot b^{p^{2k}}.$$

Now let S be the set of elements of the form $a^{p^{2k}}$ with a in A. The above equations then show that S is a *subring* of A which, by (ii) of the above Lemma, is clearly a p^k-ring. The proof is completed by observing the decomposition $a=a^{p^{2k}}+(a-a^{p^{2k}})$ (see Lemma 1). The proof of the converse is trivial.

COROLLARY 3. *A pre-p-ring is the direct sum of a p-ring and a nil pre-p-ring. Conversely, any such direct sum is a pre-p-ring.*

This is the case $k=1$ of the above theorem (cf. [1]).

We shall now take a closer look at the case $k=1$ of Theorem 2. Indeed, we shall prove that, for $k=1$, Theorem 2 is essentially true even *without any assumption of commutativity*. First, we make the following

DEFINITION. *Let p be a prime integer, and let A be an arbitrary ring. The ring A is called a generalized pre-p-ring if and only if*
(I′) $ab^p=a^pb$, *and* (II′) $pa=0$, *for all a, b in A.*

LEMMA 4. *Let A be a generalized pre-p-ring. Then a^p is in the center of A for every a in A.*

Proof. The proof is quite similar to the proof given in ([2] pp. 145–146) of the theorem that a p-ring is commutative. Thus, by (I′), we have, $a(a+b)^p = a^p(a+b)$. Hence, simplifying by using (I′), we get,

$$a[A_1 + A_2 + \cdots + A_{p-1}] = 0,$$

where A_i is the sum of all terms in $(a+b)^p$ which contain b exactly i times and a exactly $(p-i)$ times. Now, replacing b by $2b$, then replacing b by $3b, \cdots$, and finally, replacing b by $(p-1)b$, in the above equation, we obtain

$$\begin{aligned}
&a[A_1 + A_2 + \cdots + A_{p-1}] = 0\\
&a[2A_1 + 2^2A_2 + \cdots + 2^{p-1}A_{p-1}] = 0\\
&a[3A_1 + 3^2A_2 + \cdots + 3^{p-1}A_{p-1}] = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&a[(p-1)A_1 + (p-1)^2A_2 + \cdots + (p-1)^{p-1}A_{p-1}] = 0.
\end{aligned} \tag{1}$$

Let us now set

$$m = \begin{vmatrix} 1 & 1 \cdots 1 \\ 2 & 2^2 \cdots 2^{p-1} \\ \cdots & \cdots\cdots \\ (p-1) & (p-1)^2 \cdots (p-1)^{p-1} \end{vmatrix},$$

and denote the co-factors of the elements in the first column by $m_1, m_2, \cdots, m_{p-1}$, respectively. If we multiply the equations (1) by these respective co-factors and add, an elementary property of determinants shows that $m(aA_1) = 0$. Furthermore, the determinant m is a Vandermonde determinant, and it is well known that m is equal to a product of positive integers, each of which is less than the *prime* p. Hence m is relatively prime to p, and there therefore exist integers r, s such that $pr+ms=1$. Since A has characteristic p and $m(aA_1)=0$, $aA_1=(pr+ms)aA_1=0$. Similarly, $A_1a=0$. Hence $aA_1-A_1a=0$. Since $A_1=ba^{p-1}+aba^{p-2}+\cdots+a^{p-1}b$, we see that $0=aA_1-A_1a=a^pb-ba^p$, and the lemma is proved.

A careful examination of the proof of Lemma 1 shows that this proof does *not* depend on the commutativity of the ring under consideration. Hence, we have the following

COROLLARY 5. *In a generalized pre-p-ring, we have,* (i) $a^{2p+1}=a^{p+2}$; (ii) $a^{p^2}=a^{p^3}$: (iii) $a-a^{p^2}$ *is nilpotent.*

LEMMA 6. *The set N of all nilpotent elements in a generalized pre-p-ring A is a subring of A.*

THEOREM 10. *A generalized pre-p-ring A is the direct sum of a p-ring and a nil generalized pre-p-ring. Conversely, any such direct sum is a generalized pre-p-ring.*

Proof. An easy combination of Lemmas 6, 7, 8, and 9, shows that the decomposition $a=a^{p^2}+(a-a^{p^2})$ renders A as a direct sum of the desired kind. The proof of the converse is trivial.

In conclusion, I with to express my indebtedness to Dr. Glen J. Culler for his generous counsel.

References

1. A. Abian and W. A. McWorter, On the structure of pre-p-rings, Abstract, Notices, A. M. S., 10 (1963) 290.
2. N. H. McCoy, Rings and Ideals, Math. Assoc. of Amer., Carus Math. Monog., 8 (1948).
3. N. H. McCoy and D. Montgomery, A representation of generalized Boolean rings, Duke Math. J., 3 (1937) 455–459.

FUNCTIONAL INEQUALITIES

GEORGE BRAUER, University of Minnesota

Ewing and Utz obtained in [2] a characterization of the real valued solutions of the functional equation $f^{(n)}(x)=f(x)$, where n is a given natural number, and $f^{(n)}$ denotes the nth iterate of f. As a variation of this problem, we may consider that of finding the real valued continuous functions which satisfy the inequality $f^{(n)}(x)\geqq x$.

I have obtained a characterization of solutions of this inequality in [1], and I have also obtained results on the inequality

$$f^{(n)}(x) \geqq \Phi(x), \tag{1}$$

where $\Phi(x)$ is a continuous function increasing from minus infinity to infinity. In this note, I obtain results on solutions of (1) where $\Phi(x)$ is a continuous nondecreasing function.

Throughout the paper f denotes a continuous function.

THEOREM 1. *If f is a solution of* (1) *with* $\lim_{x\to-\infty}\Phi(x)>-\infty$ *then* $\liminf_{x\to-\infty}f(x)>-\infty$. *If* $\limsup_{x\to-\infty}f(x)=\infty$ *or* $\limsup_{x\to\infty}f(x)=\infty$ *then* $\liminf_{x\to\infty}f(x)>-\infty$.

Proof. If $\liminf_{x\to-\infty}f(x)=-\infty$, then, if A is a positive number, there exists an arbitrarily large number x_1 such that $f(-x_1)=-A$. There is an arbitrarily large number x_2 such that $f(-x_2)=-x_1$ or $f^2(-x_2)=-A$. More generally, there exists an arbitrarily large number x_n such that $f^{(n)}(-x_n)=-A$. Since A is arbitrary, $\liminf_{x\to-\infty}f^{(n)}(x)=-\infty$ for each n. This situation is impossible if f is a solution; hence the first part of the theorem is proved.

A technique similar to the one used in the preceding paragraph shows that if $\limsup_{x\to-\infty}f(x)=\infty$, $\liminf_{x\to\infty}f(x)=-\infty$, then $\liminf_{x\to\infty}f^{(n)}(x)=-\infty$, when n is odd while $\liminf_{x\to-\infty}f^{(n)}(x)=-\infty$ when n is even. Thus f cannot be a solution of (1) if Φ is bounded below. If $\limsup_{x\to\infty}f(x)=\infty$, $\liminf_{x\to\infty}f(x)=-\infty$, then $\liminf_{x\to\infty}f^{(n)}(x)=-\infty$ for all values of n, which is again impossible.

THEOREM 10. *A generalized pre-p-ring A is the direct sum of a p-ring and a nil generalized pre-p-ring. Conversely, any such direct sum is a generalized pre-p-ring.*

Proof. An easy combination of Lemmas 6, 7, 8, and 9, shows that the decomposition $a = a^{p^2} + (a - a^{p^2})$ renders A as a direct sum of the desired kind. The proof of the converse is trivial.

In conclusion, I with to express my indebtedness to Dr. Glen J. Culler for his generous counsel.

References

1. A. Abian and W. A. McWorter, On the structure of pre-p-rings, Abstract, Notices, A. M. S., 10 (1963) 290.
2. N. H. McCoy, Rings and Ideals, Math. Assoc. of Amer., Carus Math. Monog., 8 (1948).
3. N. H. McCoy and D. Montgomery, A representation of generalized Boolean rings, Duke Math. J., 3 (1937) 455–459.

FUNCTIONAL INEQUALITIES

GEORGE BRAUER, University of Minnesota

Ewing and Utz obtained in [2] a characterization of the real valued solutions of the functional equation $f^{(n)}(x) = f(x)$, where n is a given natural number, and $f^{(n)}$ denotes the nth iterate of f. As a variation of this problem, we may consider that of finding the real valued continuous functions which satisfy the inequality $f^{(n)}(x) \geqq x$.

I have obtained a characterization of solutions of this inequality in [1], and I have also obtained results on the inequality

$$f^{(n)}(x) \geqq \Phi(x), \tag{1}$$

where $\Phi(x)$ is a continuous function increasing from minus infinity to infinity. In this note, I obtain results on solutions of (1) where $\Phi(x)$ is a continuous nondecreasing function.

Throughout the paper f denotes a continuous function.

THEOREM 1. *If f is a solution of* (1) *with* $\lim_{x\to-\infty}\Phi(x) > -\infty$ *then* $\liminf_{x\to-\infty} f(x) > -\infty$. *If* $\limsup_{x\to-\infty} f(x) = \infty$ *or* $\limsup_{x\to\infty} f(x) = \infty$ *then* $\liminf_{x\to\infty} f(x) > -\infty$.

Proof. If $\liminf_{x\to-\infty} f(x) = -\infty$, then, if A is a positive number, there exists an arbitrarily large number x_1 such that $f(-x_1) = -A$. There is an arbitrarily large number x_2 such that $f(-x_2) = -x_1$ or $f^2(-x_2) = -A$. More generally, there exists an arbitrarily large number x_n such that $f^{(n)}(-x_n) = -A$. Since A is arbitrary, $\liminf_{x\to-\infty} f^{(n)}(x) = -\infty$ for each n. This situation is impossible if f is a solution; hence the first part of the theorem is proved.

A technique similar to the one used in the preceding paragraph shows that if $\limsup_{x\to-\infty} f(x) = \infty$, $\liminf_{x\to\infty} f(x) = -\infty$, then $\liminf_{x\to\infty} f^{(n)}(x) = -\infty$, when n is odd while $\liminf_{x\to-\infty} f^{(n)}(x) = -\infty$ when n is even. Thus f cannot be a solution of (1) if Φ is bounded below. If $\limsup_{x\to\infty} f(x) = \infty$, $\liminf_{x\to\infty} f(x) = -\infty$, then $\liminf_{x\to\infty} f^{(n)}(x) = -\infty$ for all values of n, which is again impossible.

We have the following analogue to Theorem 4 in [1].

THEOREM 2. *If f is a solution of* (1) *for some value of n, x_0 is a point such that $f(x_0)<x_0$, and $c>x_0$ is a fixed point of f, then $f(x_0)\geqq\Phi(x_0)$.*

Proof. Since $f(x_0)<x_0$, $f(c)=c>x_0$, there exists a point x_1 between x_0 and c such that $f(x_1)=x_0$. Since $f(x_1)<x_1$, there exists a point x_2 between x_1 and c such that $f(x_2)=x_1$ or $f^{(2)}(x_2)=x_0$. By an inductive procedure we obtain an increasing sequence $\{x_n\}$, lying between x_0 and c, such that $f^{n-1}(x_{n-1})=x_0$ and thus $f^{(n)}(x_{n-1})=f(x_0)$. If n is a number such that f is a solution of (1), then, since $x_{n-1}>x_0$, $f(x_0)=f^n(x_{n-1})\geqq\Phi(x_{n-1})>\Phi(x_0)$.

THEOREM 3. *If f is a solution of* (1) *with fixed point at c and $\Phi(d)=c$, $\Phi(x)>c$ for $x>d$, then either* (i) *$f(x)>c$ for $x>d$, or* (ii) *$f(x)<c$ for $x>d$, and $f(x)>c$ for $m\leqq x\leqq M$, where $m=\inf_{x>d}f(x)$, $M=\sup_{x>d}f(x)$. If $\lim_{x\to\infty}\Phi(x)=\infty$ and $c<d$, then if f is a solution for an odd value of n, $f(x)\geqq\Phi(x)$ for $d\leqq x\leqq d_1$; if f is a solution for an even value of n, $f^{(2)}(x)\geqq\Phi(x)$ for $d\leqq x\leqq d_2$, where $d_1=\inf\{x\mid x>d, f(x)\geqq x\}$, $d_2=\inf\{x\mid x>d, f^{(2)}(x)\geqq x\}$.*

We note that $c\leqq d$.

Proof. There can be no point x_1 to the right of d such that $f(x_1)=c$, for at such a point we would have $f^{(n)}(x_1)=c=\Phi(d)<\Phi(x_1)$. Thus either $f(x)>c$ for $x>d$ or $f(x)<c$ for $x>d$. If the latter holds we see that there can be no point x_2 between m and M such that $f(x_2)=c$, otherwise there would be a point x_3 to the right of d such that $f(x_3)=x_2$ and $f^{(n)}(x_3)=c=\Phi(d)<\Phi(x_2)$, $n>1$. Hence either $f(x)<c$ on the interval $[m, M]$ or $f(x)>c$ on this interval. If the first of these possibilities holds, we would have $f^{(n)}(x)<c$ when $x>d$ for all n. Since this is impossible, the first part of the theorem is established.

If $\lim_{x\to\infty}\Phi(x)=\infty$, it follows from Theorems 2 and 3 of [1] that if f is a solution for an odd value of n, then $\lim_{x\to\infty}f(x)=\infty$ while if f is a solution for an even value of n $\lim_{x\to\infty}f^{(2)}(x)=\infty$. The final part of the theorem now follows from Theorem 4 of [1].

For the inequality

$$f^{(n)}(x)\geqq 0 \tag{1a}$$

we have more definite information.

LEMMA. *If f is a solution of* (1a) *for some value of n, then*

(i) $\liminf_{x\to-\infty}f(x)>-\infty$,

(ii) $f(x)>x$, $f^{(2)}(x)>x$ *for* $x<0$,

(iii) *there exists a positive number δ_0 such that $f(x)\geqq 0$, $-\delta_0<x<0$.*

Proof. Theorem 1 implies (i). To establish (ii) we note that f cannot have negative fixed points if it is a solution, hence, on $(-\infty, 0)$ we have either $f(x)>x$ or $f(x)<x$. But if $f(x)<x$ on $(-\infty, 0)$, then $f^{(n)}(x)<x<0$ on this interval, for all values of n, hence $f(x)>x$ on $(-\infty, 0)$. Similar reasoning shows that $f^2(x)>x$ for $x<0$. To establish (iii) we note that if f takes negative values, in

every interval $(-\delta, 0)$, $\delta>0$, then each such interval is mapped on an interval with negative points by all iterates of f. This is impossible if f is a solution. Hence, for some positive number δ_0, $f(x)\geqq 0$ on $(-\delta_0, 0)$.

We let $\alpha=\inf\{x \mid x>0, f(x)<0\}$. If $f(x)\geqq 0$ for $x>0$, $\alpha=\infty$.

LEMMA. *If f is a solution of* (1a), *then there exists a positive number δ_0 such that $0\leqq f(x)\leqq\alpha$ for $-\delta_0\leqq x\leqq\alpha$.*

Proof. If there are points x in $[0, \alpha]$ such that $f(x)>\alpha$, then f maps $[0, \alpha]$ on a set containing $[0, \alpha]$ as well as negative points; the iterates of f also map $[0, \alpha]$ on such a set. This situation is impossible if f is a solution. If, for each positive number δ, the interval $[-\delta, 0]$ contains points x for which $f(x)>\alpha$, then, since $f^2(0)\geqq 0$, each interval $[-\delta, 0]$ is mapped by $f^{(2)}$ on a set containing $[0, \alpha]$ as well as negative points; moreover, if n is even then $f^{(n)}(x)$ takes negative values at some points in $-\delta<x<0$. Under these conditions f cannot be a solution of (1a) for an even value of n. On the other hand, if n is odd, then $n-1$ is even. If $\alpha<\infty$, then there exists a positive number γ such that $-\gamma$ is in the range of f. If f takes values greater than α on each interval $[-\delta, 0]$, then $f^{(n)}(x) =f[f^{(n-1)}(x)]$ takes negative values on the set $x<f(x)<0$. This is again impossible if f is a solution.

The two preceding lemmas yield two numbers $\alpha\geqq 0$, $\delta_0>0$ such that

$$0 < f(x) \leqq \alpha, \qquad -\delta_0 \leqq x \leqq \alpha, \quad \text{and} \quad \alpha = \inf\{x \mid x > 0, f(x) < 0\}.$$

LEMMA. *If f is a solution of* (1a) *and $\alpha<\infty$, then $f(x)<x$ for $x>\alpha$ and* $\limsup_{x\to\infty} f(x)<\infty$.

This result is proved by a method similar to that used in the preceding lemmas.

LEMMA: *If there exist numbers $\alpha\geqq 0$, $\delta>\lambda>0$ such that*

$$\text{(i)} \qquad 0 < f(x) \leqq \alpha, \qquad -\delta \leqq x \leqq \alpha,$$

$$\text{(ii)} \qquad \alpha = \inf\{x \mid x > 0, f(x) < 0\},$$

$$\text{(iii)} \qquad f(x) \geqq x + \lambda, \qquad f^2(x) \geqq x + 2\lambda, \qquad x < -\delta,$$

and f satisfies a Lipschitz condition with Lipschitz constant $L<1$ on the set $(-\infty, \delta)\cup(\alpha, \infty)$ then $f^k(x)\geqq\min(-\delta, x+k\lambda)$ for $x=-\delta$, for $k=1, 2, \cdots$.

We note that $f(x)<x$ for $x>\alpha$.

Proof. The proof is by induction. The lemma is clearly true for $k=1, 2$. Let us suppose that it is true for $k\leqq k_0$ where $k_0\geqq 2$. Let x' be a point to the left of $-\delta$. If $-\delta\leqq f^{(k)}(x')\leqq\alpha$ for a value of k less than or equal to k_0, then, by (i) $0\leqq f^{(k_0+1)}(x')\leqq\alpha$. If there exists a value of k less than or equal to k_0 such that $f^{(k)}(x')<-\delta$, then by our inductive hypothesis

$$\begin{aligned} f^{(k_0+1)}(x') &\geqq \min(-\delta, f^k(x') + (k_0 - k + 1)\lambda) \\ &= \min(-\delta, x' + (k_0 + 1)\lambda). \end{aligned}$$

It remains to deal with points x' such that $x'<-\delta_0$ and $f^{(k)}(x')\geqq\alpha$ for $k<k_0$. If $f^{(k_0+1)}(x')<\min(\delta,\ x+(k+1)\lambda)$ then, since $f(\alpha)=0$, there exists a point $\mathcal{E}$ between α and $f^{k_0}(x)$ such that $f(\mathcal{E})=x'+(k+1)\lambda$ (note that $\alpha<\infty$ in the case we are treating). Again, since $f^{(k_0-1)}(x')>f^{(k_0)}(x')$, by (ii), and $f^{(k_0-1)}(-\delta)\leqq\alpha$, there exists a point η between x'' and $-\delta_0$ such that $f^{(k_0-1)}(\eta)=\mathcal{E}$ and $f^{(k_0)}(\eta)=x'+(k_0+1)\lambda$. Also by our inductive hypothesis, $\eta<x'+\lambda$. Since $f^{(k_0)}(x')-f^{(k_0)}(\eta)>\alpha+\delta>\lambda$, however, we have a contradiction to the assumption that f satisfies a Lipschitz condition with Lipschitz constant $L<1$. The induction is now complete.

THEOREM 4. *If a continuous function $f(x)$ is a solution of* (1a), *then*:
(a) *f is bounded below as x tends to minus infinity*;
(b) *there exist numbers $\alpha>0$, $\delta_0>0$, such that $0\leqq f(x)\leqq\alpha$ for $-\delta_0\leqq x\alpha$*;
(c) *there exists a positive number λ_0 such that $f(x)\geqq x+\lambda_0$, $f^{(2)}(x)\geqq x+2\lambda_0$ for $x<-\delta_0$*;
(d) *if f takes negative values for $x>0$, then f is bounded above as x tends to infinity and there exists a number λ_1 such that $f(x)<x-\lambda_1$ for $x>\alpha$.*

If f satisfies a Lipschitz condition with Lipschitz constant $L<1$ on the set $(-\infty,\ -\delta)\cup(\alpha,\ \infty)$, and (a), (b), (c), *and* (d) *are fulfilled, then f is a solution for all sufficiently large values of n.*

Proof. The first part of this theorem is contained in the lemmas we proved. To prove the second part we first note that if $\lambda_0\geqq\delta_0$ then $\lambda_0'=\delta_0/2$ has all properties stated for λ_0 and $\delta_0>\lambda_0'$. Certainly if $-\delta_0\leqq x\leqq\alpha$, then $f^n(x)\geqq 0$ for $n>1$. If $x<-\delta_0$, then, by the preceding lemma $f^{(k)}(x)\geqq-\delta_0$ if $k>(-m-\delta_0)/\lambda_0$, where $m=\min[-\delta_0,\ \inf_{x<\delta_0}f(x)]$. If $x>\alpha$, then, for some value of k less than or equal to $(M-\alpha)/\lambda_1+1$, $f^k(x)\leqq\alpha$, by (d); consequently either $0\leqq f^{(k)}(x)\leqq\alpha$ for $k>(M-\alpha)/\lambda_1+1$, or, if $f^{(k)}(x)<-\delta_0$ for some k less than or equal to $(M-\alpha)/\lambda_1+1$, then $f^{(j)}(x)\geqq-\delta_0$ for $j>(M+\lambda_1-\alpha)/\lambda_1-(m+\delta_0)/\lambda_0$, where $M=\max(\alpha,\ \sup_{x>\alpha}f(x))$. Hence for each value of x, $f^{(k)}(x)\geqq-\delta_0$ for $k\geqq(M-\alpha)/\lambda_1-(m+\delta_0)/\lambda_0'+2$. If x is such that $f^{(k)}(x)\leqq\alpha$ when $k\geqq[(M-\alpha)/\lambda_1-(m+\delta_1)/\lambda_0'+2]$ then $f^n(x)$ lies between 0 and α for $n>[(M-\alpha)/\lambda_1-(m+\delta_0)/\lambda_0'+2]$. (Here [ ] denotes integral part.) If, on the other hand, $f^k(x)>\alpha$ for the above value of k, then $f^j(x)<\alpha$ for some value of j between k and $k+[(M-\alpha)/\lambda_1]$. For this value of j, say j_0, $-\delta_0\leqq f^{j_0}(x)\leqq\alpha$, for $j>j_0$, $0\leqq f^{j_0}(x)\leqq\alpha$. Hence, if the hypotheses are fulfilled, f satisfies (1a) for all $n>2+2[(M-\alpha)/\lambda_1]-[(m+\delta_0)/\lambda_0']$.

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References

1. G. Brauer, A functional inequality, this MONTHLY, 68 (1961) 638–642.

2. G. M. Ewing and W. R. Utz, The functional equation $f^n(x)=f(x)$, Can. J. Math., V (1953) 101–103.

ON THE BOUNDS FOR GEGENBAUER POLYNOMIALS

P. R. KHANDEKAR, Motilal Vigyan Mahavidyalaya, Bhopal, India

1. The main results. Gegenbauer polynomials are defined by

$$(1.1)\qquad (1-2t\cos\theta+t^2)^{-\lambda}=\sum_{n=0}^{\infty}C_n^{\lambda}(\cos\theta)t^n.$$

The aim of this paper is to prove the following bounds, which do not seem to have been noticed before.

$$(1.2)\qquad \left|C_n^{\lambda}(\cos\theta)-C_{n+2}^{\lambda}(\cos\theta)\right|<\frac{2}{\Gamma(2\lambda)n^{(2-2\lambda)}}+\frac{4}{\Gamma(\lambda)(n+2)^{(1-\lambda)}},$$

$$(1.3)\qquad \left|\int_{-1}^{x}C_n^{\lambda}(\xi)d\xi\right|<\frac{2}{n+\lambda}\left[\frac{1}{\Gamma(\lambda)(n+1)^{(1-\lambda)}}+\frac{1}{\Gamma(2\lambda)(n-1)^{(2-2\lambda)}}\right],$$

(1.4)

$$\left|C_{n+1}^{\lambda}(x)-xC_n^{\lambda}(x)\right|<\frac{1}{\Gamma(2\lambda)n^{(2-2\lambda)}}+\frac{2}{\Gamma(2\lambda)(n-1)^{(2-2\lambda)}}+\frac{2}{\Gamma(\lambda)(n+1)^{(1-\lambda)}},$$

$$(1.5)\qquad \begin{aligned}\left|C_{n+1}^{\lambda}(x)+C_n^{\lambda}(x)\right|&<\frac{1}{\Gamma(2\lambda)n^{(2-2\lambda)}}+\frac{2}{\Gamma(2\lambda)(n-1)^{(2-2\lambda)}}\\&\quad+\frac{2}{\Gamma(\lambda)(n+1)^{(1-\lambda)}}+\frac{2^{(1-\lambda)}(1+x)}{\Gamma(\lambda)n^{(1-\lambda)}(1-x^2)^{1/2\lambda}},\end{aligned}$$

$$(1.6)\qquad \begin{aligned}\left|\frac{d}{dx}C_n^{\lambda}(x)\right|&=2\lambda\left|C_{n-1}^{\lambda+1}(x)\right|\\&<\frac{n+2\lambda}{1-x^2}\left[\frac{1}{\Gamma(2\lambda)n^{(2-2\lambda)}}+\frac{2}{\Gamma(2\lambda)(n-1)^{(2-2\lambda)}}+\frac{2}{\Gamma(\lambda)(n+1)^{(1-\lambda)}}\right]\\&\quad+\frac{(2\lambda-1)2^{1-\lambda}}{\Gamma(\lambda)(n+1)^{1-\lambda}(1-x^2)^{1/2\lambda+1}}\end{aligned}$$

for $0<\lambda\leqq 1$.

2. Preliminaries. From (1.1) we get

$$(2.1)\qquad C_n^{\lambda}(\cos\theta)=\sum_{k=0}^{n}\alpha_k\alpha_{n-k}e^{-i(n-2k)\theta},$$

where

$$(2.2)\qquad \alpha_k=(-1)^k\binom{-\lambda}{k}=\frac{(\lambda)_k}{k!},\qquad k=1,2,3,\cdots,\quad \alpha_0=1.$$

Let

$$a_n = \frac{(n-1)!n^\nu}{\Gamma(\nu+n)} \qquad n = 1, 2, 3, \cdots;$$

then $a_{n+1}/a_n \leqq 1$ if $0 \leqq \nu \leqq 1$. Also from [1], p. 23, we have $\lim_{n\to\infty} a_n = 1$; hence $a_n \geqq 1$. Now

$$\frac{(\nu)_n}{n!} = \frac{\Gamma(\nu+n)}{\Gamma(\nu)n!} = \frac{n^{\nu-1}}{\Gamma(\nu)a_n}.$$

Hence

$$\frac{(\nu)_n}{n!} \leqq \frac{1}{\Gamma(\nu)n^{(1-\nu)}}. \tag{2.3}$$

Also from [2], pp. 77–81, we have $\Gamma(n+\nu)/\Gamma(n+1) \geqq (n+1)^{(\nu-1)}$, $0 \leqq \nu \leqq 1$, $n = 1, 2, 3, \cdots$. Hence

$$\frac{(\nu)_n}{n!} = \frac{\Gamma(\nu+n)}{\Gamma(\nu)n!} \geqq \frac{1}{\Gamma(\nu)(n+1)^{(1-\nu)}}. \tag{2.4}$$

From (2.3) and (2.4) for $0 < \nu \leqq 1$, $n = 1, 2, 3, \cdots$ we get

$$\frac{1}{\Gamma(\nu)n^{(1-\nu)}} \geqq \frac{(\nu)_n}{n!} \geqq \frac{1}{\Gamma(\nu)(n+1)^{(1-\nu)}}. \tag{2.5}$$

Similarly, for $2 > \nu \geqq 1$ and $n = 1, 2, 3, \cdots$, we have

$$\frac{(n+1)^{(\nu-1)}}{\Gamma(\nu)} \geqq \frac{(\nu)_n}{n!} \geqq \frac{n^{(\nu-1)}}{\Gamma(\nu)}. \tag{2.6}$$

From (2.2) we easily get

$$\alpha_k \geqq \alpha_{k+1}. \tag{2.7}$$

From [3] p. 169, we have

$$| C_n^\lambda(x) | < \frac{2^{(1-\lambda)}}{\Gamma(\lambda)n^{(1-\lambda)}(1-x^2)^{1/2\lambda}}, \qquad -1 < x < 1, \quad 0 < \lambda \leqq 1, \tag{2.8}$$

and we have also the recurrence relations of the Gegenbauer polynomials given by

$$2(n+\lambda)C_n^\lambda(x) = \frac{d}{dx}[C_{n+1}^\lambda(x) - C_{n-1}^\lambda(x)], \tag{2.9}$$

$$(n+2\lambda)C_n^\lambda(x) = \frac{d}{dx}C_{n+1}^\lambda(x) - x\frac{d}{dx}C_n^\lambda(x), \tag{2.10}$$

$$(2.11)\qquad (1-x^2)\frac{d}{dx}C_n^\lambda(x) = (n+2\lambda)xC_n^\lambda(x) - (n+1)C_{n+1}^\lambda(x),$$

$$(2.12)\qquad \frac{d}{dx}C_n^\lambda(x) = 2\lambda C_{n-1}^{\lambda+1}(x).$$

We have also for $|z|<1$

$$(2.13)\qquad (1-z)^{-\lambda} = \alpha_0 + \alpha_1 z + \cdots + \alpha_k z^k + \cdots,$$

where α_k is given by (2.2). With the help of (2.5) we also obtain

$$(2.14)\qquad \alpha_k < \frac{1}{\Gamma(\lambda)k^{(1-\lambda)}}\cdot$$

3. From (2.1) we obtain $C_{2m-1}^\lambda(\cos\theta)=2\sum_{k=0}^{(m-1)}\alpha_k\alpha_{n-k}\cos(n-2k)\theta$, $n=2m-1$, and $C_{2m}^\lambda(\cos\theta)=2\sum_{k=0}^{(m-1)}\alpha_k\alpha_{n-k}\cos(n-2k)\theta+\alpha_m^2$, $n=2m$. Therefore, if $n=2m-1$ then

$$(3.1)\qquad \begin{aligned} C_{2m-1}^\lambda(\cos\theta) - C_{2m+1}^\lambda(\cos\theta) &= -2\alpha_{n+2}\cos(n+2)\theta \\ &\quad + 2\sum_{k=0}^{(m-1)}(\alpha_k\alpha_{n-k} - \alpha_{k+1}\alpha_{n-k+1})\cos(n-2k)\theta, \end{aligned}$$

and if $n=2m$ then

$$(3.2)\qquad \begin{aligned} C_{2m}^\lambda(\cos\theta) - C_{2m+2}^\lambda(\cos\theta) &= \alpha_m^2 - \alpha_{m+1}^2 - 2\alpha_{n+2}\cos(n+2)\theta \\ &\quad + 2\sum_{k=0}^{(m-1)}(\alpha_k\alpha_{n-k} - \alpha_{k+1}\alpha_{n-k+1})\cos(n-2k)\theta. \end{aligned}$$

From (2.7) we have $\alpha_k \geqq \alpha_{k+1}$ and $\alpha_{n-k}\geqq\alpha_{n-k+1}$. Hence

$$(3.3)\qquad \begin{aligned} &2\sum_{k=0}^{(m-1)}(\alpha_k\alpha_{n-k} - \alpha_{k+1}\alpha_{n-k+1})\cos(n-2k)\theta \\ &\qquad < 2\sum_{k=0}^{(m-1)}(\alpha_k\alpha_{n-k} - \alpha_{k+1}\alpha_{n-k+1}) \\ &\qquad < \frac{(2\lambda)_n}{n!} - \frac{(2\lambda)_{n+2}}{(n+2)!} + 2\alpha_{n+2}, \qquad n = 2m-1, \end{aligned}$$

or

$$(3.4)\qquad < \frac{(2\lambda)_n}{n!} - \frac{(2\lambda)_{n+2}}{(n+2)!} - \alpha_m^2 + \alpha_{m+1}^2 + 2\alpha_{n+2}, \quad n = 2m.$$

The results (3.3) and (3.4) we obtain with the help of (2.13). Hence, with the help of (3.3) and (3.4), for all values of n, (3.1) and (3.2) give

$$(3.5) \qquad \left| C_n^\lambda(\cos\theta) - C_{n+2}^\lambda(\cos\theta) \right| < \left| \frac{(2\lambda)_n}{n!} - \frac{(2\lambda)_{n+2}}{(n+2)!} \right| + 4\alpha_{n+2}.$$

Now for $0<\lambda\leqq\frac{1}{2}$ we get

$$(3.6) \qquad \frac{(2\lambda)_n}{n!} - \frac{(2\lambda)_{n+2}}{(n+2)!} < \frac{2}{(n+2)\Gamma(2\lambda)n^{(1-2\lambda)}} < \frac{2}{\Gamma(2\lambda)n^{(2-2\lambda)}},$$

and for $\frac{1}{2}\leqq\lambda<1$ we get

$$(3.7) \qquad \frac{(2\lambda)_{n+2}}{(n+2)!} - \frac{(2\lambda)_n}{n!} < \frac{2}{\Gamma(2\lambda)(n+1)^{(2-2\lambda)}} < \frac{2}{\Gamma(2\lambda)n^{(2-2\lambda)}}.$$

The results (3.6) and (3.7) we obtain with the help of (2.5) and (2.6). Hence (3.5) reduces to (1.2) with the help of (2.14).

4. Integrating both sides of (2.9) with respect to x between the limits $(-1, x)$ we obtain

$$\int_{-1}^x C_n^\lambda(\xi)d\xi = \frac{1}{2(n+\lambda)}\left[C_{n+1}^\lambda(x) - C_{n-1}^\lambda(x) + (-1)^n \left\{ \frac{(2\lambda)_{n+1}}{(n+1)!} - \frac{(2\lambda)_{n-1}}{(n-1)!} \right\} \right].$$

Therefore, because of (3.6), (3.7), and (1.2),

$$(4.1) \qquad \left| \int_{-1}^x C_n^\lambda(\xi)d\xi \right| < \frac{2}{n+\lambda}\left[\frac{1}{\Gamma(\lambda)(n+1)^{(1-\lambda)}} + \frac{1}{\Gamma(2\lambda)(n-1)^{(2-2\lambda)}} \right].$$

5. We also easily get

$$(5.1) \qquad \left| \frac{(2\lambda)_n}{n!} - \frac{(2\lambda)_{n+1}}{(n+1)!} \right| < \frac{(2\lambda)_n}{(n+1)!} < \frac{1}{\Gamma(2\lambda)n^{(2-2\lambda)}},$$

and from (4.1) we get

$$(5.2) \qquad \left| (n+2\lambda-1)\int_{-1}^x C_n^\lambda(\xi)d\xi \right| < \frac{2}{\Gamma(\lambda)(n+1)^{(1-\lambda)}} + \frac{2}{\Gamma(2\lambda)(n-1)^{(2-2\lambda)}}.$$

Integrating both sides of (2.10) with respect to x between the limits $(-1, x)$ and using the results (5.1) and (5.2) we obtain the results (1.4) and (1.5). For getting (1.5) we also use the result (2.8). Now (2.11) can be put in the form

$$(1-x^2)\frac{d}{dx}C_n^\lambda(x) = (n+2\lambda)\{xC_n^\lambda(x) - C_{n+1}^\lambda(x)\} + (2\lambda-1)C_{n+1}^\lambda(x).$$

Using (1.4), (2.8) and (2.12) we obtain the result (1.6).

References

1. E. D. Rainville, Special Functions, Macmillan, New York, 1960.
2. Walter Gautschi, J. Math. and Physics, 38 (1959–60) 77–81.
3. G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ., 23 (1959).

NOTE ON THE PARTITION FUNCTION

MORRIS NEWMAN, National Bureau of Standards

Let $p(n)$ denote the number of unrestricted partitions of n. The well-known Ramanujan congruences

$$p(5n-1) \equiv 0 \pmod{5} \tag{1}$$

$$p(7n-2) \equiv 0 \pmod{7} \tag{2}$$

$$p(11n-5) \equiv 0 \pmod{11} \tag{3}$$

show that there are arithmetic progressions on which $p(n)$ vanishes modulo 5, 7, 11. In some recent articles [1], [2], [3] the author has proved certain congruences for $p(n)$ modulo 13, special instances of which are

$$p\left(84n^2 - \frac{1}{24}(n^2-1)\right) \equiv 0 \pmod{13}, \qquad (n, 6) = 1 \tag{4}$$

$$p(11\Delta_n + 6) \equiv 11 \cdot 6^n \pmod{13}, \qquad \Delta_n = \frac{13}{24}(13^{2n}-1). \tag{5}$$

Congruence (4) shows that there are quadratic progressions on which $p(n)$ vanishes. The purpose of this note is to point out a consequence of (5).

THEOREM. *Let r be a fixed integer. Then neither of the congruences*

$$p(13k-7) \equiv r \pmod{13} \tag{6}$$

$$p(13^2k-7) \equiv r \pmod{13} \tag{7}$$

can hold for all sufficiently large integers k.

Proof. Set $a_k = 13^c k - 7$, $c = 1, 2$. We determine k so that

$$a_k = 11\Delta_n + 6,$$

$$k = 13^{2-c}\frac{11 \cdot 13^{2n-1} + 1}{24}.$$

For these values of k we see that $p(a_k) \equiv 11 \cdot 6^n \pmod{13}$ and the truth of the theorem is clear.

Whether $p(n)$ can vanish modulo 13 on other arithmetic progressions is still unsettled.

This work was supported (in part) by the Office of Naval Research.

References

1. M. Newman, Congruences for the coefficients of modular forms and some new congruences for the partition function, Canad. J. of Math., 9 (1957) 549–552.

2. ———, Further identities and congruences for the coefficients of modular forms, *Ibid.*, 10 (1958), 577–586.

3. ———, Periodicity modulo m and divisibility properties of the partition function, Transactions of the American Mathematical Society, 97 (1960), 225–236.

and

$$(21)\qquad 24c_{4p} \equiv \frac{c_1^4}{p^4} + \frac{12c_1^2c_2}{p^3} + \frac{24c_1c_3 + 12c_2^2}{p^2} + \frac{24c_4}{p} \pmod p \qquad (p > 4).$$

Again, if we take $n = p^2$ in (13), we find that

$$(22)\qquad \frac{c_1}{p} \equiv \sum_{r=1}^{p-1} \frac{rc_{p-r}c_{rp}}{p} \pmod p$$

which presumably is a consequence of (19), (20), etc.

As a simple application of the above it follows that if

$$c_n = 1 \quad (p \mid n), \qquad c_n = p \quad (p \nmid n),$$

then $\sum_0^\infty c_nx^n$ is not a pth power for $p > 2$. Also it is easily verified that this is the case for $p = 2$.

Supported in part by NSF grant G 16485.

References

1. J. Arkin, A congruence for the coefficients of the kth power of a power series, this MONTHLY, 71 (1964) 899–900.

2. I. Schur, Arithmetische Eigenschaften der Potenzsummen einer algebraischen Gleichung, Compositio Math., 4 (1937) 432–444.

GEGENBAUER POLYNOMIALS AND FRACTIONAL DERIVATIVES

G. G. BILODEAU, Boston College

1. The purpose of this note is to establish a new formula for Gegenbauer polynomials which will in general involve the use of fractional derivatives. Specifically we will show that

$$(1.1)\qquad C_n^\lambda(x^{1/2}) = \frac{(-2)^{n+1}x^{(n+3)/2}\pi^{1/2}}{(n+1)!\Gamma(\lambda)} D_x^{n+\lambda+1/2}[(1 - x^{1/2})^{n+1}x^{(n/2)+\lambda-1}]$$

for $\lambda > -\frac{1}{2}$, $\lambda \neq 0$ and $n = 1, 2, 3, \cdots$. $C_n^\lambda(x)$ is the Gegenbauer polynomial of degree n which for our purposes will be defined (cf. [1] p. 175) by

$$(1.2)\qquad C_n^\lambda(x) = \frac{1}{\Gamma(\lambda)} \sum_{k=0}^{[n/2]} (-1)^k \frac{\Gamma(n - k + \lambda)}{k!(n - 2k)!} (2x)^{n-2k}.$$

To prove (1.1) we will restrict ourselves to $x > 0$.

2. When $\lambda + \frac{1}{2}$ is nonintegral, the expression on the right hand side of (1.1) needs to be interpreted as a fractional derivative. Since the expression in the brackets of (1.1) is of the form $x^{\lambda-1/2}P(x^{1/2})$ for some polynomial $P(x)$, we shall define the operator D_x^α only for such cases.

$$P_n(u) = \frac{(-1)^{n+1}u^{n+3}}{(n+1)!}(u^{-1}D_u)^{n+1}[(1-u)^{n+1}u^{n-1}]$$

valid for $n=0, 1, 2, \cdots, u\neq 0$.

The Chebyshev polynomial $T_n(u)$ defined by

$$T_n(u) = \cos n\theta, \qquad u = \cos\theta, \qquad n = 0, 1, 2, \cdots$$

does not fit under this work inasmuch as it is a limiting case of Gegenbauer polynomials as $\lambda\to 0$. We can show, however, by proceeding in a similar way, that

$$T_n(u) = (-1)^{n+1}n\left(\frac{\pi}{2}\right)^{1/2}\frac{u^{n+3}}{(n+1)!}(u^{-1}D_u)^{n+1/2}[(1-u)^{n+1}u^{n-2}]$$

for $n = 1, 2, \cdots$.

Reference

1. A. Erdélyi, *et al.*, Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York, 1953

AN ITERATION PROCEDURE FOR QUASI-INVERSES

DAVID M. BURTON, University of New Hampshire

A rapidly converging iteration procedure for finding inverses in a Banach algebra with identity has been indicated by R. A. DeMarr (A.M.S. Notices, Vol. 10 No. 4). With suitable modifications, we adapt this procedure below to the case of an algebra without identity, where the concept of quasi-inverse replaces that of the inverse.

If x and y are elements of a Banach algebra A, the circle operation $x\circ y$ is defined by the equation $x\circ y=x+y-xy$. An element x of A is said to have a quasi-inverse if there exists an $x^0\in A$ such that $x\circ x^0=x^0\circ x=0$; such an x^0 is called the quasi-inverse of x. It is well known that x has a quasi-inverse if $\|x\|=\delta<1$.

The quasi-inverse x^0 can be obtained by considering the sequence of iterates $\{y_n\}$ defined by:

$$y_0 = 0, \quad \text{and} \quad y_n = -x + 2xy_{n-1} + y_{n-1}^2 - xy_{n-1}^2 \qquad \text{for } n = 1, 2, \cdots.$$

It is easily shown by an inductive argument that each y_n commutes with x. Thus, we have

$$\|x\circ y_n\| = \|(x\circ y_{n-1})^2\| \leqq \|x\circ y_{n-1}\|^2$$

and consequently, by induction,

$$\|x\circ y_n\| \leqq \|x\circ y_0\|^{2^n} = \|x\|^{2^n} = \delta^{2^n}.$$

The inequality

$$\|(x \circ y_n) - (x \circ x^0)\| = \|x \circ y_n\| \leqq \delta^{2^n}$$

yields $(1-\|x\|)\|y_n - x^0\| \leqq \delta^{2^n}$, which implies that $\|y_n - x^0\| \leqq \delta^{2^n}/1-\delta$.

Thus the sequence of iterates $\{y_n\}$ converges in norm quite rapidly to the quasi-inverse of x. It should be noted that if the algebra does have an identity e, then $e-x^0=(e-x)^{-1}$.

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Md. 20740.

THE EXTERIOR OPERATOR AND BOUNDARY OPERATOR

HYMAN GABAI, University of Illinois, UICSM

In [1], Frank Reese Harvey noted that a topology for a set X can be characterized by specifying the open sets, closed sets, a closure operator or an interior operator, and he presented a list of derived set operator axioms. In this paper, exterior operator axioms and boundary operator axioms which characterize a topological space are presented. In a topological space the exterior of a set is the complement of its closure, and the boundary of a set is the intersection of its closure with the closure of the complement of the set. We denote subsets of X by A and B.

1. The exterior operator. Let X be a nonempty set. An exterior operator on X is a function e which maps the subsets of X into the subsets of X and which satisfies the following four statements:

(E1) For each A and B: $e(A)\cap e(B)=e(A\cup B)$.

(E2) For each A: $A\cap e(A)=\varnothing$.

(E3) For each A: $e(A)\subseteq e(\sim e(A))$.

(E4) $e(\varnothing)=X$.

We first show that these statements are satisfied by the exterior operator in any topological space.

(E1) $e(A)\cap e(B)=(\sim\overline{A})\cap(\sim\overline{B})=\sim(\overline{A}\cup\overline{B})=\sim(\overline{A\cup B})=e(A\cup B)$.

(E2) $A\cap e(A)=A\cap(\sim\overline{A})=\varnothing$.

(E3) $e(\sim e(A))=e(\sim[\sim\overline{A}])=e(\overline{A})=\sim\overline{\overline{A}}=e(A)$.

(E4) $e(\varnothing)=\sim(\overline{\varnothing})=\sim\varnothing=X$.

The inequality

$$\|(x \circ y_n) - (x \circ x^0)\| = \|x \circ y_n\| \leqq \delta^{2^n}$$

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(E2) $A \cap e(A) = A \cap (\sim\overline{A}) = \emptyset$.

(E3) $e(\sim e(A)) = e(\sim[\sim\overline{A}]) = e(\overline{A}) = \sim\overline{\overline{A}} = e(A)$.

(E4) $e(\emptyset) = \sim(\overline{\emptyset}) = \sim\emptyset = X$.

(K4) $\overline{\varnothing}=\varnothing\cup b(\varnothing)=\varnothing$.

Then $\tau=\{U\subseteq X: \overline{X\sim U}=X\sim U\}$ is the unique topology on X such that for each $A\subseteq X$ the closure of A is $\overline{A}$. Also, for each $A\subseteq X$ the boundary of A is $\overline{A}\cap(\overline{\sim A})=(A\cup b(A))\cap((\sim A)\cup b(\sim A))=(A\cup b(A))\cap((\sim A)\cup b(A))=b(A)$.

In any topological space the closure of a set A is $A\cup b(A)$ so τ is the only topology satisfying the conditions of the theorem.

References

1. F. R. Harvey, The derived set operator, this Monthly, 70 (1963) 1085–1086.
2. D. Bushaw, Elements of general topology, Wiley, New York, 1963.

A NOTE ON MATRIX NOTATION

Ignace I. Kolodner, University of New Mexico

The purpose of this note is to introduce a simplified and, in many ways, fool-proof matrix notation. In section 1 we discuss matrices, in section 2, the representation of vectors, and in section 3, the representation of operators on a finite dimensional vector space.

1. Matrices. Let A be an $m\times n$ matrix. The entry of A at the intersection of row i with column j is denoted by ${}_iA^j$; the ith row of A is denoted by ${}_iA$ and the jth column of A by A^j. Besides the advantages to be discussed below, this sort of indexing has a mnemonic value: if one tends to push an index into the conventional position, the row index will move horizontally to the right, and the column index vertically downwards.

Columns and rows are usually denoted by l.c. letters unless they are extracted from larger matrices. A column needs no column index but has a row index; thus the ith component of the column x is denoted by ${}_ix$. If several columns are denoted by the same letter, they should be indexed in the column index position; for example x^1, x^2, or A^1, A^2 etc., for various columns originating from an $m\times n$ matrix A with $n>1$. Similarly, rows need no row index, but require a column index to indicate components. If several rows are denoted by the same letter, they should be indexed in the row index position.

The identity matrix (of any order) will be denoted by I. Thus ${}_iI^j=\delta_{ij}$, where δ_{ij} is the Kronecker symbol.

With the above notation the binary compositions of matrices (of suitable order) may be defined as follows. Let x be an n-column, and y an n-row. The compositions yx, $B+A$, cA (c a scalar), BA are given by the formulae:

$$yx = \sum_{1\leq\alpha\leq n} y^\alpha{}_\alpha x,$$

$${}_i(B+A)^j = {}_iB^j + {}_iA^j,$$

$${}_i(cA)^j = c\,{}_iA^j,$$

$${}_i(BA)^j = {}_iBA^j.$$

(K4) $\overline{\varnothing}=\varnothing\cup b(\varnothing)=\varnothing$.

Then $\tau=\{U\subseteq X: \overline{X\sim U}=X\sim U\}$ is the unique topology on X such that for each $A\subseteq X$ the closure of A is $\overline{A}$. Also, for each $A\subseteq X$ the boundary of A is $\overline{A}\cap(\overline{\sim A})=(A\cup b(A))\cap((\sim A)\cup b(\sim A))=(A\cup b(A))\cap((\sim A)\cup b(A))=b(A)$.

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$${}_i(BA)^j = {}_iBA^j.$$

THE QUADRATIC CHARACTER OF -3 IN FINITE PRIME FIELDS

GERALD STOLLER, Harvard University

It is an old result that -3 is a quadratic residue of primes of the form $6n+1$ and a quadratic nonresidue of primes of the form $6n+5$. For a proof of this fact using Gauss's Lemma (see [1] p. 75). A simple proof of this result can be given by using the existence of primitive elements. See [2] p. 106 for the necessary information on primitive elements.

Let $p=6n+1$ be a prime and let b be a primitive element modulo p. Then $1\equiv b^m \pmod{p}$ if and only if $(p-1)\mid m$. Let $u\equiv b^{2n} \pmod{p}$, then $u\not\equiv 1 \pmod{p}$ and $u^3\equiv 1 \pmod{p}$. Thus we get

$$0 \equiv u^3 - 1 \equiv (u-1)(u^2+u+1) \pmod{p}$$

whence $0\equiv u^2+u+1 \pmod{p}$. Multiplying both sides of this congruence by 4, we get $0\equiv 4u^2+4u+4 \pmod{p}$ which yields

$$-3 \equiv 4u^2 + 4u + 1 \equiv (2u+1)^2 \pmod{p}.$$

This gives us an explicit form of the square roots of -3 modulo p. (Note that $-3=(2\omega+1)^2$ where ω is a complex cube root of 1 other than 1.)

Now let $p=6n+5$ be a prime and let b be a primitive element modulo p. Suppose that -3 is a quadratic residue modulo p, then there is an element c such that $c^2\equiv -3 \pmod{p}$. Define u by the congruence $c\equiv 2u+1 \pmod{p}$, then we get that $u^2+u+1\equiv 0 \pmod{p}$ whence $u^3\equiv 1 \pmod{p}$. Let k be an integer such that $b^k\equiv u \pmod{p}$, then $b^{3k}\equiv u^3\equiv 1 \pmod{p}$ so $(p-1)\mid(3k)$. Since $p-1$ $(=6n+4)$ is relatively prime to 3, it follows that $(p-1)\mid k$ whence $1\equiv b^k\equiv u \pmod{p}$. Thus $c\equiv 2u+1\equiv 3 \pmod{p}$, consequently $12\equiv c^2+3\equiv 0 \pmod{p}$, which is a contradiction. Therefore -3 is a quadratic nonresidue modulo p.

(A contradiction can also be obtained by observing that $u\equiv 1 \pmod{p}$ and $u^2+u+1\equiv 0 \pmod{p}$ imply $3\equiv 0 \pmod{p}$. This was suggested by Professor George Whaples.)

The same method also serves to establish the quadratic character of -1 in finite prime fields.

References

1. G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, 3rd ed., Oxford University Press, London, 1954.

2. I. M. Vinogradov, Elements of number theory, Dover, New York, 1954.

THE MACLAURIN SERIES FOR e^x

J. R. ISBELL, University of Washington

The following interpretation of the terms of the series $e^x=1+x+\cdots+x^n/n!+\cdots$ would, I think, have been regarded as a proof in the eighteenth century. It can motivate a proof now; and Professor J. P. Ballantine informs me that Professor H. E. Slaught sometimes used it in that way at the University

of Chicago. The use I propose for it is in showing that

$$e = \limsup \left(1 + \frac{1}{n}\right)^n$$

is finite and approximately $1+1+\frac{1}{2}+ \cdots +1/m!$. Proofs can come later, after e^x and convergence of series have been defined.

$(1+1/n)^n$ is of course the value of 1 dollar drawing 100% interest compounded n times a year. "Passing to the limit," we compound interest continuously, getting value $v(t)$ at time $t \geqq 0$. The value consists of (i) the principal, 1, the first term of the series; plus (ii) simple interest t; plus further terms, the $(n+1)$-th being the simple interest earned by the nth. If the nth term is t^n/n at each time t, it earns $(t^n/n!)\ \Delta t$ between t and $t+\Delta t$, making $\int_0^t (s^n/n!)\ ds$ in all.

$D_x f[g(x)] = g(x)$

DONALD W. HIGHT, Kansas State College of Pittsburg
A. GLEN HADDOCK, Arkansas College, Batesville

A routine problem in calculus [1], "Differentiate $y=\cosh^{-1}(\sec x)$," motivated the following question: what are the functions f and g such that $D_x f[g(x)] = g(x)$. A partial solution of the problem was found that required only that g be invertible. $D_x f[g(x)] = g(x)$ if $f[g(x)] = \int g(x)dx$. Let $\int g(x)dx = v(x)$ and $g(x)=t$ then the equation is satisfied provided $f(t) = v[g^{-1}(t)]$. Example: given $g(x) = \ln x$ then $v(x) = x \ln x - x + C$ and $g^{-1}(t) = e^t$ so $f(t) = te^t - e^t + C$.

If f is given, the problem of finding g is as straightforward if certain conditions are met: $D_x f[g(x)] = g(x)$ if $f'[g(x)]g'(x) = g(x)$ or if

$$\frac{f'[g(x)]}{g(x)} g'(x) = 1.$$

The problem then essentially becomes one of solving

$$\int \frac{f'(t)}{t} dt = x \quad \text{where} \quad \int \frac{f'(t)}{t} dt = g^{-1}(t).$$

Again, by this method, g needs to be invertible but $f'(t)/t$ must also be integrable. Example: let $f(t) = ct^n$ ($c \neq 0$, $n \neq 1$), then

$$\int \frac{nct^{n-1}}{t} dt = \int nct^{n-2} dt = \frac{nct^{n-1}}{n-1} \cdot$$

Therefore,

$$t = \sqrt[n-1]{\frac{(n-1)x}{nc}} = g(x).$$

The example of the previous paragraph also works nicely by this method.

This problem seemed to be very interesting and simple but one in which both students and teachers become confused. The authors now ask if there are other solutions, or solutions dependent on different conditions.

Reference

1. A. E. Taylor, Calculus with analytic geometry, Prentice-Hall, Englewood Cliffs, N. J., 1959, p. 327.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

A METHODS COURSE FOR MATHEMATICS TEACHERS

DONOVAN A. JOHNSON, University of Minnesota

Mathematics teaching is a field in which knowledge of the subject matter is the first necessity. For the teacher of the secondary school, it is necessary that this background in mathematics be of the type and depth recommended by the recent publications of the Committee on the Undergraduate Program in Mathematics (CUPM). The high school mathematics teacher needs a background in calculus, number theory, foundations of mathematics, abstract algebra, non-euclidean geometry, set theory, symbolic logic, probability and statistics. But a required sequence of mathematics courses is not sufficient. The mathematics teacher must be able to read mathematical literature so that he will be able to learn more mathematics independently. He will need to be up-to-date in his mathematical language, symbolism, assumptions and proofs in his classroom presentations. Besides he should find pleasure in reading mathematics books and in presenting new mathematical ideas to his students.

Teaching mathematics, however, involves more than knowing and enjoying the subject. The mathematics teacher must be able to motivate his students, he must be able to guide them to discover ideas and he must be able to evaluate the achievement of his students. The methods course should serve as a bridge between the knowledge of subject matter, principles of psychology, and actual practice in the classroom. Hence, the methods course and student teaching should be concurrent and usually after a background in mathematics and education have been attained.

Mathematics with its abstract symbolism, its sequential organization, its logical structure, its wide application, has unique learning problems. At one

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Teaching mathematics, however, involves more than knowing and enjoying the subject. The mathematics teacher must be able to motivate his students, he must be able to guide them to discover ideas and he must be able to evaluate the achievement of his students. The methods course should serve as a bridge between the knowledge of subject matter, principles of psychology, and actual practice in the classroom. Hence, the methods course and student teaching should be concurrent and usually after a background in mathematics and education have been attained.

Mathematics with its abstract symbolism, its sequential organization, its logical structure, its wide application, has unique learning problems. At one

extreme it involves memorizing facts and practicing skills. At the other extreme, the solving of a problem, the proof of a theorem, the application of a generalization, the building of a mathematical structure requires a high level of creative thinking. Thus, the teacher of mathematics will need to know how to teach concepts, skills, proof, and productive thinking. The current emphasis on discovery, problem solving, and attitudes poses problems of adaptability and flexibility in the classroom that require far greater skill than a lecture-recitation presentation.

The well-informed high school mathematics teacher today must be up-to-date regarding the many recommendations being made for curriculum revision. The methods course will need to study these and other curriculum issues so that evaluations, professional judgments and adaptations can be made. These recommendations involve new topics, new vocabulary, new procedures and new methods. Besides, a variety of research projects in mathematics education have implications for the teacher of mathematics. Thus, the mathematics teacher should be able to implement these findings in the classes and to participate intelligently in current research projects.

The variety of instructional aids currently available for the mathematics teacher has multiplied tremendously in recent years. Some of this increase is due to the emphasis on the discovery method, some of it is due to funds from NDEA, and some of it is due to the normal invention of new devices. In any case the mathematics teacher will need to render professional judgment in the selection and use of these teaching aids. These aids include programmed texts and teaching machines, TV lessons, supplementary books and pamphlets, charts, films, filmstrips, models, overhead projectuals, games, exhibits, and chalkboard devices. It should be an objective of the methods course to show what materials are available, the choice of appropriate materials, and how the materials can contribute to the learning of mathematics.

The discussion above suggests that the methods course should show the prospective mathematics teacher how to:

Guide the student to discover mathematical concepts;
Stimulate the learning of mathematics;
Develop ability to solve mathematical problems;
Build understanding, accuracy and efficiency in computational skills;
Develop desirable attitudes and appreciations of mathematics;
Teach the student how to study mathematics;
Evaluate the learning of concepts, skills, and problem solving;
Provide a program of enrichment and acceleration for gifted students;
Plan an effective program for the slow learner;
Evaluate new curriculum proposals;
Procure, use, and evaluate new instructional aids;
Select appropriate goals for mathematics instruction;
Plan a variety of lessons and units;
Find new applications, new ideas, and new materials.

Reference Texts:

Jerome S. Bruner, *The Process of Education*, Harvard University Press, Cambridge, 1960.

H. M. Martyn Cundy, and A. P. Rollett, *Mathematical Models*, Oxford Press, London, 1952.

L. A. Kenna, *Understanding Mathematics with Visual Aids*, Littlefield Adams and Co., Patterson, New Jersey, 1962.

Morris Kline, *Mathematics in Western Culture*, Oxford University Press, New York, 1953.

NCTM, Twenty-seventh Yearbook, *Enrichment Activities for the Grades, 1963.*

NCTM, Twenty-eighth Yearbook, *Enrichment Activities for High School, 1963.*

H. Steinhaus, *Mathematical Snapshots*, Oxford University Press, New York, 1950.

Pamphlets:

NCTM, *The Revolution in School Mathematics.*

NCTM, *An Analysis of New Mathematics Programs.*

NCTM, *A Guide to the Use and Procurement of Teaching Aids for Mathematics.*

NCTM, *Program Provisions for the Mathematically Gifted Student in the Secondary School.*

NCTM, *Education in Mathematics for the Slow Learner.*

Edwina Deans, *Elementary School Mathematics, New Directions*, U. S. Office of Education, 1963.

Adrien L. Hess, *Mathematics Projects Handbook*, D. C. Heath, Boston, 1962.

It is assumed that the methods teacher will also have available a library of mathematics texts, films, tapes, overhead projectuals, pamphlets, books, charts, models, demonstration equipment and supplies.

Preparing to teach mathematics is not a matter of learning mathematics *or* learning how to teach mathematics. It must be a union of mathematics and methods. It would seem that the background in mathematics precedes instruction in how to teach it. Ideally the courses in mathematics should be taught in such a way that the student learns from these experiences how to teach. However, the variety of current methods, materials, curricula, research, and learning theories require presentation in a special methods course. Thus, all major groups establishing ideal programs of preparation for mathematics teaching have included student teaching and a methods course in their recommendations.

AN OBSERVATION OF TEACHING METHODS

SISTER HELEN CLARE, S. L., Loretto Heights College, Loretto, Colorado

My observations have been, for the most part, from the pupil's side of the teacher's desk. For the past twenty years, at least one period of six weeks each year has been spent in the student chair. At times, it has been my good fortune to have been the recipient of good teaching; at other times, less good.

What factors contribute to good teaching? Let me attempt to answer this question by analyzing the teaching methods of a particular professor. This approach may offer one or two suggestions for inspiring effective teaching. I have tried to discover "why" upon leaving some classes there was such a sense of satisfaction and profit; and why from other classes, little profit seemed to be derived.

The one outstanding in my memory of college professors is Dr. Emil Artin, who was formerly associated with the University of Hamburg, and under whom I studied at the University of Notre Dame. He began his course with a concept

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The one outstanding in my memory of college professors is Dr. Emil Artin, who was formerly associated with the University of Hamburg, and under whom I studied at the University of Notre Dame. He began his course with a concept

so simple that even a child could understand. From the simple idea he developed slowly and with perfect planning his entire course. I cannot recall that he ever attempted a presentation for which he had to apologize because he had failed to pave the way. The class could see, and by degrees could anticipate the trend that the entire course was taking. The effect upon the student was a feeling of security and independence regarding the matter at hand. Perfect preparation produced the integrated effect that any teacher could well be proud to produce.

As the members of the class were teachers themselves, they could not fail to appreciate the rich background of "this" teacher—rich, not only in the field of mathematics but in the arts as well. Although he was a genius in his subject, he was well informed outside his field. His background enabled him to correlate and to point out relationships of ideas. Each new development served as a stepping stone to something higher and beyond. The student, as a result, felt that he was making discoveries, as Van Doren ably expresses it in [1]: "The art of teaching is the art of assisting discovery to take place." The beauty of this teaching was further enhanced by the fact that this teacher was what Mortimer Adler in [2] calls a "primary teacher." "Let us call 'those' living teachers who perform the function of original communication the primary teachers." Here is a living instructor who knows something which cannot be found in books. He has something which he has himself discovered and has not yet made available through books to others. He has the ability to teach without a book. Never in the six weeks in either of the two classes of his that I attended, do I remember his bringing a book to class, or referring to a note. When he was asked, "What text do you use?" he replied, "Text! I teach. 'You' make the text."

His class was so different from those classes in which note taking was a waste of time and energy. I say a waste of energy, because all the matter being voiced could be found in some particular chapter of the text. His was different from the general run of lecture courses of which it has been remarked: "The notes of the teacher become the notes of the pupil without entering the minds of either." Obviously, a member of this class was not merely taking notes; his jottings had to be accompanied with intellectual activity if the notes were to be interpreted later.

Such a teacher is bound to inspire his class, and inspiration is the spark of learning. To teach effectively one who enters a class must be well prepared both remotely and proximately—remotely, by a rich background of varied knowledge, and proximately by a logically developed plan of approach to his subject. He must enable his pupils to discover truths for themselves whereby they acquire a feeling of success and independence.

This paper, published in the Catholic School Journal, January, 1946, was sent by the author as a tribute to a great teacher—the late Emil Artin, after she heard the papers given in his honor at the Boulder Association Meeting, August, 1963.

References

1. Mark Van Doren, Liberal education, Holt, New York, 1943.
2. Mortimer J. Adler, How to read a book, Simon and Schuster, New York, 1940.

The Commission on Engineering Education is an independent organization with headquarters in Washington, D. C., established to provide direct action for the improvement and appreciation of engineering education.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

COLLABORATING EDITORS: J. BARLAZ, Rutgers-The State University; L. CARLITZ, Duke University; H. S. M. COXETER, University of Toronto; H. EVES, University of Maine; A. E. LIVINGSTON, University of Alberta; and A. WILANSKY, Lehigh University.

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. Solutions to problems appearing in previous issues of the Monthly should not be sent to Professor Starke.

ELEMENTARY PROBLEMS

All solutions of Elementary Problems should be sent to A. E. Livingston, University of Alberta, Edmonton, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before February 28, 1965.

E 1731. *Proposed by Gus Mavrigian, Youngstown University*

Let $x>0$, and let $x_1=[x(x)^b]^a$, $x_2=[x(x\cdot x_1)^b]^a$, $\cdots$, $x_n=[x(x\cdot x_{n-1})^b]^a$. Determine the function $f(x)=x_n$, and find $\lim_{n\to\infty}x_n$ if $ab<1$.

E 1732. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College*

If $x_i>0$ $(i=1, 2, \cdots, n)$, show that

$$\sum_{j=1}^{n}\left\{\prod_{i=1}^{n}x_i\left[\left(\sum_{k=1}^{n}x_k\right)-x_j\right]\Big/x_j\right\}\geqq n(n-1)\prod_{i=1}^{n}x_i.$$

E 1733. *Proposed by E. F. Assmus, Jr., Wesleyan University*

Let G be a group, H a finite normal subgroup of G which is not trivial. Suppose H has the property that whenever an element of G commutes with some nontrivial element of H it is, in fact, in H. Prove that G is finite.

E 1734. *Proposed by Omar Khayyam, Jr., University of California, Berkeley*

The functions f and g are defined by the series:

$$f(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}m^nx^n}{m!n!},\qquad g(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{m^nx^n}{m!n!}.$$

Prove that $(f(e))^{g(1)}(g(e))^{f(-1)}=1$.

E 1735. *Proposed by William Emerson, University of California, Berkeley*

Determine all real valued functions f defined on the positive integers N which satisfy (1) $(m, n)=1$ implies $f(mn)=f(m)f(n)$, and (2) $m<n$ implies $f(m) \leqq f(n)$ for all $m, n \in N$.

E 1736. *Proposed by E. R. Barnes, Student, Morgan State College*

Evaluate

$$S = \sum_{p=2}^{\infty}\left(\frac{1}{p}\sum_{q=2}^{\infty}\left(\frac{1}{q}\right)^{p}\right).$$

E 1737. *Proposed by F. L. Bookstein, University of Michigan*

Define a telescoping product for a natural number t as a sequence of n integers $a_i>1$, $1\leqq i\leqq n$, such that a_i divides a_{i-1} for $2\leqq i\leqq n$ and such that $\prod_{i=1}^{n} a_i=t$. Define a telescoping sum for a natural number t as a sequence of n positive integers a_i, $1\leqq i\leqq n$, such that $a_i\leqq a_{i+1}$, $1\leqq i\leqq n-1$, and such that $\sum_{i=1}^{n} a_i=t$.

Let q have prime factorization $p_1^{b_1}p_2^{b_2} \cdots p_k^{b_k}$. Let c_i be the number of telescoping sums for b_i. Prove that the number of telescoping products for q is $\prod_{i=1}^{k} c_i$.

E 1738. *Proposed by Michael Fried, Bell Aerosystems*

Given $T_1=2$, $T_2=3$, $T_{2n}=T_{2n-1}+2T_{2n-2}$, $T_{2n+1}=T_{2n}+T_{2n-1}$. Find an explicit expression for T_i.

SOLUTIONS OF ELEMENTARY PROBLEMS

Sum of Squares of Digits

E1651 [1964, 90]. *Proposed by Azriel Rosenfeld, Budd Electronics, Long Island City, New York.*

Prove that no multidigit integer is equal to the sum of the squares of its digits.

Solution by N. J. Fine, Pennsylvania State University. We consider the problem for an arbitrary base $b\geqq 2$. Let $N=a_0+a_1b+a_2b^2+ \cdots +a_nb^n$, with $0\leqq a_i<b$, $0<a_n$. If $n\geqq 2$, then N cannot satisfy the stated condition, for that would imply

$$\begin{aligned} b^2 - b + 1 \leqq a_n(b^n - a_n) \leqq a_1(b - a_1) + a_2(b - a_2) + \cdots + a_n(b^n - a_n) \\ = a_0(a_0 - 1) \leqq (b - 1)(b - 2), \end{aligned}$$

which is false. For $n=1$, we have $a_1(b-a_1)=a_0(a_0-1)$, which is equivalent to

$$b^2 + 1 = (2a_0 - 1)^2 + (b - 2a_1)^2.$$

Now if b^2+1 is a prime (as in the case $b=10$), its representation as a sum of two

squares is unique, except for order and signs, so $b-2a_1=\pm b$, and $a_1=0$ or b. Both being ruled out, there is no solution. If b^2+1 is not a prime, there is a solution. For if b is odd, $a_0=a_1=(b+1)/2$ is a solution. If b is even, then $b^2+1=x^2+y^2$, where x is even, y odd, and $1<x, y<b$; then $a_1=(b-x)/2$, $a_0=(y+1)/2$ provide a solution. The case of a one-digit number N is trivial.

Also solved by Shair Ahmad, R. G. Albert, Joseph Arkin, C. R. Atherton, J. W. Baldwin, W. E. Bodden, W. H. Bonney, Joel Brawley, Jr., Maxey Brooke, J. L. Brown, Jr., John Burslem, Robert Burton, Leonard Carlitz, J. C. Caughran, Allen Chuck and Peter Goldstein (jointly), M. J. Cohen, L. E. DeNoya, R. B. Eggleton, Harold Finkelstein, C. N. Frye, Michael Goldberg, Emil Grosswald, J. D. Haggard, H. S. Hahn, Ned Harrell, M. H. Hayamizu, Ralph Herbert and Duane Kefel and Paul Welsh (jointly), K. S. Hirschel, Stephen Hoffman, R. F. Jackson, R. A. Jacobson, J. E. Jean, Jr., Erwin Just and Norman Schaumberger (jointly), P. L. Kingston, Frank Kocher, Kenneth Kramer, E. S. Langford, H. R. Lewis, G. K. Liebschner, Robert Maas, J. J. Malone, Jr., D. C. B. Marsh, D. E. Moxness, P. N. Muller, Sam Newman, E. T. Ordman, C. B. A. Peck, Dean Phelps, Stanton Philipp, L. J. Pratte, George Purdy, S. W. Reyner, Henry Ricardo, P. A. Scheinok, D. L. Silverman, H. D. Snyder, Jr., Jerome Solheim, David Sookne, E. L. Spitznagel, Jr., and K. P. Yanosko (jointly), Harlan Stevens, E. E. Strock, P. K. Subramanian, R. L. Syverson, G. C. Thompson, A. M. Vaidya, Simon Vatriquant, Gary Venter, Andy Vince, L. J. Warren, W. C. Waterhouse, R. E. Wilder, K. S. Williams, D. G. Wilson, Kari Ylinen, K. L. Yocom, and the proposer.

A Weak Version of Stirling's Formula

E1652 [1964, 90]. *Proposed by Erwin Just, Bronx Community College.*

Prove that

$$\prod_{k=0}^{m-1} k! > (m/e)^m$$

for all positive integral m.

Solution by W. C. Waterhouse, Harvard University. The usual proof of Stirling's Formula [See, for example, D. V. Widder, *Advanced Calculus*, pp. 384–387] gives $m!>(m/e)^m(2\pi m)^{1/2}$, so we have the sharper inequality $(m/2\pi)^{1/2}(m-1)!>(m/e)^m$.

Also solved by A. N. Aheart, R. G. Albert, J. W. Baldwin, M. Barnebey, S. S. Blakney, R. J. Bridgman, John Burslem, Robert Burton, Jim Campbell, J. C. Caughran, D. I. A. Cohen, F. J. Dickey, Ragnar Dybvik, J. W. Ellis, P. G. Engstrom, N. J. Fine, M. L. Faulkner and M. G. Murdeshwar (jointly), D. M. Good, Emil Grosswald, H. S. Hahn, M. H. Hayamizu, C. V. Heuer, Stephen Hoffman, J. E. Humphreys, R. W. Hurd, R. F. Jackson, S. F. Kapoor, P. G. Kirmser, Robert Kopp, Kenneth Kramer, E. S. Langford, Nicholas Macri, D. C. B. Marsh, Gus Mavrigian, P. N. Muller, Dave Nixon, Paul Pang, Stanton Philipp, L. J. Pratte, George Purdy, S. W. Reyner, B. E. Rhoades, V. K. Rohatgi, Bernard Rosner, M. S. R. K. Sastry, Norman Schaumberger, Perry Scheinok, R. A. Smith, Al Somayajulu, David Sookne, Harlan Stevens, Eric Sturley, A. M. Vaidya, Simon Vatriquant, Julius Vogel, Raymond Whitney, K. S. Williams, L. B. Winrich, Itaru Yamaguchi, and the proposer.

Most solvers used mathematical induction (and the fact that $(1+n^{-1})^n \uparrow e$) to verify the inequality of the problem, while some observed that it follows easily from the geometric-arithmetic mean inequality.

A Combinatorial Problem

E1653 [1964, 90]. *Proposed by Arthur Engel, Stuttgart, Germany.*

There are given $p_n = [en!]+1$ points in space. Each pair of these points is connected by a line, and each line is colored with one of n different colors. Show that there is at least one triangle all of whose sides are of the same color.

Solution by J. W. Ellis, Louisiana State University in New Orleans. Define a sequence $\{b_n\}$ inductively by $b_1=2$ and $b_{n+1}=(n+1)b_n+1$. We will prove: (1) When the segments connecting a set of b_n+1 points are colored with n colors, at least one single-color triangle results; (2) $p_n \geqq b_n+1$ for all n.

Statement (1) is clear for $n=1$. Suppose it is true for $n=k$, and let a set of $b_{k+1}+1$ points be given. Starting at any point A in this set, there are b_{k+1} segments joining A to the remaining points; since $b_{k+1}>(k+1)b_k$, one of the $k+1$ colors (call it "green") must be used at least b_k+1 times in coloring those segments. Thus we have a subset B consisting of b_k+1 points, each joined to A by a green segment. If any segment joining two points of B is green, they will form with A an all-green triangle; otherwise the segments of B are all colored with the k remaining colors, and the induction hypothesis assures us that a monochromatic triangle exists in this case also.

Now if $a_n = b_n/n!$ for each n, we see that $a_1=2$ and $a_{n+1}=a_n+1/(n+1)!$ Thus $\{a_n\}$ is the (increasing) sequence of partial sums of the usual series for e. Therefore, for all n, $a_n<e$ and consequently $b_n<en!$, $b_n \leqq [en!]$, and finally $b_n+1 \leqq p_n$.

Comment by R. E. Greenwood, University of Texas. This result was obtained in R. E. Greenwood and A. M. Gleason "Combinatorial Relations and Chromatic Graphs," Canadian Journal of Mathematics, 7 (1955) 1–7, especially page 5.

It is also interesting to note that this relation is an upper bound for a certain class of Ramsey numbers, which were defined by F. P. Ramsey, "On a Problem in Formal Logic," Proceedings of the London Mathematical Society, series 2, 30, (1929), 264–286. A summary of the known results on the Ramsey numbers appears in Herbert J. Ryser, Combinatorial Mathematics (Carus Mathematical Monographs, No. 14, published by the Association in 1963) in Chapter 4, pp. 38–46, especially page 43.

Of course, Problem E 1653 is a generalization of the Putnam Mathematical Competition for 1953, problem #2, morning session, which appeared in the Monthly for Oct. 1953, page 541 and which was rephrased as Problem E 1321 by Bostwick in the June–July 1958 issue of the Monthly.

Also solved by W. H. Bonney, John Burslem, Robert Burton, Gary Chartrand, D. I. A. Cohen, M. L. Faulkner and M. G. Murdeshwar (jointly), S. F. Kapoor, D. C. B. Marsh, J. W. Moon, Stanton Philipp, Harlan Stevens, George Purdy, W. C. Waterhouse, and the proposer.

Other references: R. K. Grey, *Another combinatorial problem* (perhaps not yet published); R. Sprague, *Unterhaltsame Mathematik.*

Subsets of the *r*th Roots of Unity

E1654 [1964, 90]. *Proposed by Ralph Greenberg, University of Pennsylvania.*

A set of numbers is said to be *special* if the sum of the numbers is zero. Let $N(r)$ denote the number of special proper subsets of the set of rth roots of unity. Show that $N(r)=0$ if and only if r is a prime.

Solution by A. M. Vaidya, Pennsylvania State University. Suppose r is composite, and let p be a prime dividing r (so that $p<r$). If ω is a primitive pth root of unity, then $1, \omega, \omega^2, \omega^3, \cdots, \omega^{p-1}$ is clearly a special proper subset of the set of the rth roots of unity.

Next suppose r is a prime p, and again let ω be as above. ω satisfies $f(x) = 1+x+x^2+\cdots+x^{p-1}=0$. Suppose there is a special proper subset

$$\omega^{k_1}, \omega^{k_2}, \cdots, \omega^{k_n}; \qquad 0 \leqq k_i \leqq p-1,\ n < p.$$

Then ω also satisfies $g(x)=x^{k_1}+x^{k_2}+\cdots+x^{k_n}=0$. Thus ω satisfies $g(x)=0$ and $f(x)-g(x)=0$, and one of these equations is of degree less than $p-1$. This is a contradiction, for ω, being a primitive pth root of unity, cannot satisfy an equation of degree less than $\phi(p)=p-1$.

Also solved by R. G. Albert, J. W. Baldwin, John Burslem, Robert Burton, Leonard Carlitz, S. R. Cavior, H. S. Hahn, S. F. Kapoor, C. G. Linder, D. C. B. Marsh, Stanton Philipp, E. L. Spitznagel, Jr., and K. P. Yanosko (jointly), Harlan Stevens, Rory Thompson, L. Y. L. Tong, W. C. Waterhouse, and the proposer.

An Algorithm for the L.C.M. of n Integers

E1655 [1964, 90]. *Proposed by A. J. Goldman, National Bureau of Standards.*

Determine the validity of the following asserted algorithm for finding the least common multiple L of a finite sequence $X=(x_1, x_2, \cdots, x_n)$ of positive integers. Beginning with $X^{(1)}=X$, at the mth step one has a finite sequence

$$X^{(m)} = (x_1^{(m)}, x_2^{(m)}, \cdots, x_n^{(m)}).$$

If all components of $X^{(m)}$ are equal, their common value is L and the algorithm terminates. If not, choose a minimum component $x_k^{(m)}$ of $X^{(m)}$ and form $X^{(m+1)}$ by $x_k^{(m+1)}=x_k^{(m)}+x_k$, $x_i^{(m+1)}=x_i^{(m)}$ for $i\neq k$.

Solution by N. J. Fine, Pennsylvania State University. Let $M_k=\max x_i^{(k)}$. There are only finitely many k such that $M_k \leqq L$. Let m be the largest such index. Suppose that $X^{(m+1)}$ exists. By a suitable re-numbering, we will then have

$$X^{(m)} = (rx_1, x_2^{(m)}, \cdots, x_n^{(m)}),$$

$$X^{(m+1)} = ((r+1)x_1, x_2^{(m)}, \cdots, x_n^{(m)}),$$

with r a positive integer and

$$L \neq rx_1 \leqq x_2^{(m)} \leqq \cdots \leqq x_n^{(m)} = M_m \leqq L.$$

Now $(r+1)x_1 \leqq L$ implies that $M_{m+1} \leqq L$, which is false, by the maximality of m. Hence

$$rx_1 < L < (r+1)x_1.$$

This contradicts the fact that L is a multiple of x_1. Hence $X^{(m+1)}$ does not exist, and

$$X^{(m)} = (M_m, \cdots, M_m).$$

But then M_m is a common multiple of all the x_i, so that $L \leqq M_m \leqq L$. Hence

$$X^{(m)} = (L, \cdots, L),$$

as required.

Also solved by R. G. Albert, John Burslem, Robert Burton, Jim Campbell, D. I. A. Cohen, David Cohoon, Michael Goldberg, W. E. Gould, H. S. Hahn, Leroy Junker, E. S. Langford, Richard Laver, D. C. B. Marsh, E. J. Ordman, David Sookne, Simon Vatriquant, Andy Vince, W. C. Waterhouse, and the proposer.

Langford and Marsh point out that the algorithm contains precisely $\sum_{j=1}^{n} L/x_j - n$ steps, L being the l.c.m. of $x_1, x_2, \cdots, x_n$.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08900. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before May 31, 1965.

5207 [1964, 562]. *Correction.* Change "degree not exceeding n" to "degree not exceeding $n-1$."

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Solve in closed form the integral equations

I. $$\phi(x) + \int_0^x t\phi(t)F'(z)dt = g(x),$$

II. $$\int_0^x \phi(t)F(z)dt + f(0)F(x^2) = f(x),$$

where $z = x(x-t)$ and the prime indicates differentiation with respect to z. $F(z)$ is the modified Bessel function $I_0(2z^{1/2})$. The free functions $f(x)$, $g(x)$ are assumed to be continuously differentiable.

5237. *Proposed by I. I. Hirschman, Jr. and Guido Weiss, Washington University, St. Louis*

Let $F(x)$ be a complex measurable function on $0 < x < 1$ such that $F(x) \notin L^2(0, 1)$ and let P be the subset of $L^2(0, 1)$ consisting of those functions $\phi(x)$ for which $F(x)\overline{\phi(x)} \in L^1(0, 1)$ and $\int_0^1 F(x)\overline{\phi(x)}dx = 0$. Show that P is dense in $L^2(0, 1)$.

5238. *Proposed by Z. Govindarajulu, Case Institute of Technology*

Show that for $\alpha > -1$,

$$1 - \sum_{s=1}^{\infty} \frac{\alpha(1-\alpha)\cdots(s-1-\alpha)}{s!(2s+1)} = 2^{2\alpha} \frac{\Gamma^2(1+\alpha)}{\Gamma(2+2\alpha)}.$$

$$X^{(m)} = (M_m, \cdots, M_m).$$

But then M_m is a common multiple of all the x_i, so that $L \leqq M_m \leqq L$. Hence

$$X^{(m)} = (L, \cdots, L),$$

as required.

Also solved by R. G. Albert, John Burslem, Robert Burton, Jim Campbell, D. I. A. Cohen, David Cohoon, Michael Goldberg, W. E. Gould, H. S. Hahn, Leroy Junker, E. S. Langford, Richard Laver, D. C. B. Marsh, E. J. Ordman, David Sookne, Simon Vatriquant, Andy Vince, W. C. Waterhouse, and the proposer.

Langford and Marsh point out that the algorithm contains precisely $\sum_{j=1}^{n} L/x_j - n$ steps, L being the l.c.m. of $x_1, x_2, \cdots, x_n$.

ADVANCED PROBLEMS

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SOLUTIONS OF ADVANCED PROBLEMS

Ring with Idempotents

5082 [1963, 335]. *Proposed by the Junior Research Seminar for High School Students of Summer 1962, Lehigh University.*

Let R be a ring in which, if either $x+x=0$ or $x+x+x=0$, it follows that $x=0$. Suppose that a, b, c and $a+b+c$ are all idempotents in R. Does it follow that $ab=0$?

Editorial Note. A negative answer to the proposed question is one of the results in J. G. Mauldon, Nonorthogonal idempotents whose sum is idempotent, p. 963–973, in this issue of the MONTHLY.

Also solved by George Bergman.

Words in the Commutator Subgroup of a Free Group

5138 [1963, 899]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Show that in a free group, if a word of length $2n$ lies in the commutator subgroup, then it can be expressed as a product of at most $n-1$ commutators.

Solution by Stephen Montague, University of Illinois. The proof will be by induction on n. If $w \in F$, where F is the free group in question, then to say that w is in the commutator subgroup F' is to say that the letters in w pair off into pairs consisting of an element of the generating set and its inverse. Write commutators in the form $(a, b)=aba^{-1}b^{-1}$. Now any nonidentity element of the commutator subgroup must have length at least 4, and the only elements of this length are individual commutators of two free generators, so the theorem holds for $n=2$.

Assume the theorem true for $n \leqq k$, and consider $w=x_1x_2 \cdots x_{2k+2}$. By the preceding, if $w \in F'$, then for some j, $x_j=x_1^{-1}$. Hence w may be written $w=x_1Ax_1^{-1}B$, where B may be the empty word. (If A was the empty word then w would have reduced length $2k$ or less and by induction could be written as a product of $k-1$ commutators.) Then if the length of A is m, the length of B is $2k-m$, so $w=x_1Ax_1^{-1}A^{-1}AB=(x_1, A)AB$, where AB has length at most $2k$. Since AB is clearly in F', by induction it is the product of at most $k-1$ commutators, so w is the product of at most $k=(k+1)-1$ commutators, proving the theorem.

Also solved by T. R. Berger, C. C. Lindner, M. D. Mavinkurve (India), and the proposer.

Variations of the Basis for a Normed Vector Space

5139 [1963, 899]. *Proposed by Dennis Travis, Columbia University*

Let V be a finite dimensional normed vector space over R. Let $x_1, \cdots, x_n$ be a basis for V. Prove that there exists a $k>0$ such that if $y_1, \cdots, y_n$ is a set of vectors with the property that $\|x_i-y_i\|<k$ for $i=1, \cdots, n$, then the y_i are independent.

I. *Solution by James Duemmel, Wright-Patterson Air Force Base, Ohio.* If such a k does not exist, then there exist sequences $\{(y_1^m, \cdots, y_n^m)\} \subseteq V^n$ and

$\{(\alpha_1^m, \cdots, \alpha_n^m)\} \subseteq R^n$ such that $\|y_i^m - x_i\| < 1/m$ for $i=1, 2, \cdots, n$, $\sum_{i=1}^n \alpha_i^m y_i^m = 0$ and $\sum_{i=1}^n |\alpha_i^m| \neq 0$. We may assume $\sum_{i=1}^n |\alpha_i^m| = 1$. But this makes $\{(\alpha_1^m, \cdots, \alpha_n^m)\}$ a sequence in a compact subset of R^n. Since such a sequence has a convergent subsequence, we assume the sequence itself is convergent to $(\alpha_1, \cdots, \alpha_n)$. Then $\sum_{i=1}^n |\alpha_i^m| = 1$, $\lim_{m\to\infty} \alpha_i^m = \alpha_i$ and $\lim_{m\to\infty} y_i^m = x_i$ for $i=1, \cdots, n$. Hence, $0 = \lim_{m\to\infty} \sum_{i=1}^n \alpha_i^m y_i^m = \sum_{i=1}^n \alpha_i x_i$. But this cannot be true since $\sum_{i=1}^n |\alpha_i| = 1$ and $x_i, \cdots, x_n$ form a basis.

II. *Solution by G. A. Heuer, Concordia College.* Without loss of generality, V may be represented as R^n and an element x of V as an n-tuple $(x^1, \cdots, x^n)$ of real numbers. The norm property $\|kx\| = |k| \cdot \|x\|$ guarantees that a polynomial in $(x^1, \cdots, x^n)$ is a continuous function on V to R. Therefore the determinant function d defined on the topological product V^n by $d(y_1, \cdots, y_n) = \det(y_i^j)$ is continuous. Since $d(x_1, \cdots, x_n) \neq 0$, there is a neighborhood of $(x_1, \cdots, x_n)$ in V^n throughout which $d \neq 0$, and the result follows.

Also solved by D. F. Dawson, C. F. Evans, D. P. Giesy, Roy H. Hines, Jr., K. O. Leland, J. Levy and P. Meyers, A. E. Livingston and M. G. Murdeshwar, Maurice Machover, M. D. Mavinkurve (India), F. T. Metcalf, Veselin Perić (Yugoslavia), J. R. Retherford, E. F. Steiner, F. R. Swenson, W. C. Waterhouse, R. J. Whitley, A. B. Willcox, and the proposer.

The result can be found in the literature where it appears as a property of a basis in a Banach space. It is also valid in a Hilbert space—implying that finite dimensionality of the vector space is not a necessary condition.

On Goldbach's Conjecture

5140 [1963, 899]. *Proposed by A. A. Mullin, University of Illinois*

Consider the algebraic system determined by the additive and multiplicative monoids of nonnegative integers N. Put $E_n = \{p_{j_1} \cdot p_{j_2} \cdots p_{j_{2n+1}} : p_j \in A\}$ where A is the set of odd primes and $n \in N$. Put $S = \bigcup_{n=0}^\infty E_n$. Let $T \subseteqq N$ and consider the following two conditions:

$$\text{(i)} \qquad T + T \subseteqq \overline{T}, \qquad\qquad \text{(ii)} \qquad T \cdot T \subseteqq \overline{T},$$

where $\overline{T}$ is the set-theoretical complement of T relative to N. Prove that S satisfies conditions (i) and (ii). Prove that if S is not maximal over N relative to both conditions (i) and (ii) holding simultaneously, then Goldbach's conjecture is not a theorem. (Incidentally, if one considers whether or not every even integer greater than 4 can be represented as the sum of two elements of S, then one has a weakened version of Goldbach's conjecture. Possibly the weaker version will yield more readily to solution.)

Solution by Ralph Greenberg, University of Pennsylvania. S satisfies (i) and (ii) because $S+S$ consists entirely of even integers and $S \cdot S$ consists of integers with an even number of prime factors. Suppose that S is not maximal in N, i.e., for some subset R of N, $S \subset R$, $S \neq R$, and conditions (i) and (ii) hold for R. Then some element of N is contained in R and is not the product of an odd number of odd primes. This element cannot be the product of an even number of odd primes since $R \cdot R$ would then contain an element of S in violation of (ii). Clearly 2 and

4 are not contained in R (since $2+3=5$ and $4+3=7$ are contained in R). Therefore, by (i), this even number has no representation as the sum of two elements of R, and a fortiori cannot be represented as the sum of two odd primes.

Also solved by Robert Bowen, and by E. S. Langford.

Absolutely Convex Ideals

5141 [1963, 1013]. *Proposed by C. W. Kohls, Syracuse University*

An ideal I in a commutative ring is said to be primary if, whenever $ab\in I$, either $a\in I$ or $b^n\in I$ for some positive integer n. Evidently a prime ideal is primary. It is known that in any ring $C(X)$ of continuous real-valued functions, every prime ideal is absolutely convex (see Gillman and Jerison, *Rings of Continuous Functions*, p. 69). Show that, in fact, every primary ideal in $C(X)$ is absolutely convex.

Solution by the proposer. The notation of Gillman and Jerison [*GJ*] will be used. Let P be a primary ideal in $C(X)$, and let $|f|\leqq|g|$, with $g\in P$. We may assume that $|g|\leqq 1$ [*GJ*, p. 30]. Put $h=\sum_{n\in N}2^{-n}|g|^{1/n}$; then $h\in C(X)$, because the series converges uniformly. Define k by $k(x)=g(x)/h(x)$ for $x\notin \mathbf{Z}(h)$. $k(x)=0$ for $x\in\mathbf{Z}(h)$. Now for each $n\in\mathbf{N}$,

$$2^{-4n^2}|g| \leqq h^{2n}. \tag{1}$$

It follows that $2^{-4}|g|\leqq|h|^2$, $|g(x)/h(x)|\leqq 16|h(x)|$ for $x\notin\mathbf{Z}(h)$, and $k\in C(X)$. Inequality (1) also implies that $2^{-4n^2}|f|\leqq|h^n|^2$ for each $n\in\mathbf{N}$; thus, f is a multiple of h^n [*GJ*, p. 21]. Next, for $x\notin\mathbf{Z}(k)=\mathbf{Z}(g)=\mathbf{Z}(h)$, we have $|f(x)/k(x)|$ $=|h(x)|\,|f(x)/k(x)h(x)|=|h(x)|\,|f(x)/g(x)|\leqq|h(x)|$; so f is a multiple of k.

Since $kh=g\in P$, either $k\in P$ or $h^n\in P$ for some $n\in\mathbf{N}$. In either case, f is a multiple of an element of P; so $f\in P$.

An Orthogonal Sequence

5143 [1963, 1013; 1964, 439]. *Proposed by J. B. Roberts, Reed College*

Let $n_1, n_2, \cdots$ be a sequence of integers each greater than unity. Put $p_0=1$, $p_j=n_1\cdots n_j$ for $j\geqq 1$. Let ϕ be a function belonging to L^2 of period 1 and satisfying

$$\sum_{j=1}^{n_k}\phi\left(x+\frac{j-1}{n_k}\right)=0 \qquad k=1,2,\cdots.$$

Then the sequence $\{\phi_n\}$ is orthogonal on (0, 1) when $\phi_n(x)=\phi(p_n(x)$. (This generalizes a problem, p. 43 of Kaczmarz and Steinhaus, *Theorie der Orthogonal Reihen.*)

Solution by W. C. Waterhouse, Harvard University. Say $s<k$, and let $N=p_k/p_s$. Then we have

$$\int_0^1 \phi(p_s x)\phi(p_k x)dx = (1/p_k)\int_0^{p_k}\phi(y/N)\phi(y)dy$$
$$= (p_s/p_k)\int_0^N \phi(y/N)\phi(y)dy \qquad (\phi(y/N) \text{ has period } N)$$
$$= (1/N)\sum_{m=1}^{N}\int_0^1 \phi(y)\phi\left(\frac{y+m-1}{N}\right)dy$$
$$= (1/N)\sum_{r=1}^{N/n_k}\int_0^1 \phi(y)\sum_{j=1}^{n_k}\phi\left(\frac{y+r-1}{N}+\frac{j-1}{n_k}\right)dx = 0.$$

Also solved by L. Carlitz, P. G. Engstrom, N. J. Fine, and the proposer.

From the fact that $\sum_{j=1}^{T}\phi(x+(j-1)/n_k))=0$ when T is a multiple of n_k, Carlitz obtains a corresponding summation result

$$\sum_{j=1}^{p_k}\phi_r\left(x+\frac{j-1}{p_k}\right)\phi_s\left(x+\frac{j-1}{p_k}\right)=0, \qquad r<s\leqq k,$$

which is comparable to formulae appearing in *Some Finite Summation Formulas of Arithmetic Character*, Publicationes Mathematicae, 6 (1959) 262–268 and Acta Math. Acad. Scient. Hung., 11 (1960) 15–22.

Matrix Differential Equation

5144 [1963, 1013]. *Proposed by Reuben Hersh, Fairleigh Dickinson University, Teaneck, N. J.*

If F_0, A_i, B_i, $i=1, \cdots, k$ are given constant $n\times n$ matrices, solve the matrix differential equation

$$\frac{dF}{dt}=\sum_{i=1}^{k} A_iFB_i, \qquad F(0)=F_0.$$

Solution by P. G. Kirmser, Kansas State University. Let R be an operator which rearranges a rectangular matrix of dimension $m\times n$ (taken as partitioned into a row matrix of n column matrices each of dimension $m\times 1$) into a single column matrix of dimension $nm\times 1$, partitioned into n column matrices each $m\times 1$. This operation is equivalent to a transposition of a row matrix of matrices in which the individual matrices which form the elements are not themselves transposed.

Let R^{-1} be an operator which rearranges the column matrix of column matrices back into the original row matrix of column matrices. Then

$$\sum_{i=1}^{k} A_iFB_i = R^{-1}\left\|\sum_{i=1}^{k} A_i\cdot xB_i^T\right\| RF,$$

where $\left\|\sum_{i=1}^{k} A_i\cdot xB_i^T\right\|$ is the nivellateur of Sylvester (see C. C. MacDuffee, *The Theory of Matrices*, p. 89); and the given equation becomes the n^2 system:

$$R\frac{dF}{dt}=\left\|\sum_{i=1}^{k} A_i\cdot xB_i^T\right\| RF,$$

which has the unique solution

$$RF = \exp\left(\left\| \sum_{i=1}^{k} A_i \cdot x B_i^T \right\| t\right) RF_0.$$

Thus

$$F = R^{-1} \exp\left(\left\| \sum_{i=1}^{k} A_i \cdot x B_i^T \right\| t\right) RF_0.$$

Also solved by N. P. Bhatia, L. Carlitz, P. G. Engstrom, J. H. Halton, Imanuel Marx, R. F. Rinehart, H. Schwerdtfeger, and the proposer.

Iterative Nesting of Triangles

5146 [1963, 1014]. *Proposed by R. J. Cormier, University of Missouri*

In the Euclidean plane, denote the distance between two points p and q by pq, and let k_1, k_2, k_3 be any three positive real numbers less than 1. Let p_1, q_1, r_1 be three noncollinear points and let p_n, q_n, r_n be defined inductively as follows for $n>1$:

$$\begin{aligned} p_{n-1}r_n &= k_1 \cdot p_{n-1}q_{n-1}, & p_{n-1}r_n + r_nq_{n-1} &= p_{n-1}q_{n-1}, \\ q_{n-1}p_n &= k_2 \cdot q_{n-1}r_{n-1}, & q_{n-1}p_n + p_nr_{n-1} &= q_{n-1}r_{n-1}, \\ r_{n-1}q_n &= k_3 \cdot r_{n-1}p_{n-1}, & r_{n-1}q_n + q_np_{n-1} &= r_{n-1}p_{n-1}. \end{aligned}$$

Let t_n be any point interior to the triangle with vertices p_n, q_n, r_n. Does the sequence $\{t_n\}$ converge? If so, what is the position of the limit point relative to the points p_1, q_1, and r_1?

Solution by Michael Goldberg, Washington, D. C. The conditions require that, for all n, the vertices of the triangle $p_nq_nr_n$ lie on the sides of the triangle $p_{n-1}q_{n-1}r_{n-1}$. If p_1 and q_1 are considered as vectors with respect to r_1 as origin, then

$$\begin{aligned} r_2 &= p_1 + k_1(q_1 - p_1) = (1 - k_1)p_1 + k_1q_1, \\ p_2 &= (1 - k_2)q_1, \qquad q_2 = k_3p_1; \\ r_3 &= p_2 - k_1(p_2 - q_2) = (1 - k_1)p_2 + k_1q_2 \\ &\qquad\qquad = k_1k_3p_1 + (1 - k_1)(1 - k_2)q_1, \\ q_3 &= (1 - k_3)r_2 + k_3p_2, \qquad p_3 = (1 - k_2)q_2 + k_2r_2; \\ r_4 &= (1 - k_1)p_3 + k_1q_3. \end{aligned}$$

This reduces to

$$\begin{aligned} r_4 &= [k_1(1 - k_3) + k_2(1 - k_1) + k_3(1 - k_2)](1 - k_1)p_1 \\ &\quad + [k_1(1 - k_3) + k_2(1 - k_1) + k_3(1 - k_2)]k_1q_1 \\ &= c[(1 - k_1)p_1 + k_1q_1] = cr_2, \end{aligned}$$

which has the unique solution

$$RF = \exp\left(\left\| \sum_{i=1}^{k} A_i \cdot x B_i^T \right\| t\right) RF_0.$$

Thus

$$F = R^{-1} \exp\left(\left\| \sum_{i=1}^{k} A_i \cdot x B_i^T \right\| t\right) RF_0.$$

Also solved by N. P. Bhatia, L. Carlitz, P. G. Engstrom, J. H. Halton, Imanuel Marx, R. F. Rinehart, H. Schwerdtfeger, and the proposer.

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Let t_n be any point interior to the triangle with vertices p_n, q_n, r_n. Does the sequence $\{t_n\}$ converge? If so, what is the position of the limit point relative to the points p_1, q_1, and r_1?

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$$\begin{aligned} r_2 &= p_1 + k_1(q_1 - p_1) = (1-k_1)p_1 + k_1q_1, \\ p_2 &= (1-k_2)q_1, \qquad q_2 = k_3p_1; \\ r_3 &= p_2 - k_1(p_2 - q_2) = (1-k_1)p_2 + k_1q_2 \\ &\qquad\qquad = k_1k_3p_1 + (1-k_1)(1-k_2)q_1, \\ q_3 &= (1-k_3)r_2 + k_3p_2, \qquad p_3 = (1-k_2)q_2 + k_2r_2; \\ r_4 &= (1-k_1)p_3 + k_1q_3. \end{aligned}$$

This reduces to

$$\begin{aligned} r_4 &= [k_1(1-k_3) + k_2(1-k_1) + k_3(1-k_2)](1-k_1)p_1 \\ &\quad + [k_1(1-k_3) + k_2(1-k_1) + k_3(1-k_2)]k_1q_1 \\ &= c[(1-k_1)p_1 + k_1q_1] = cr_2, \end{aligned}$$

The location of L in terms of the origin r_1 and the vectors p_1 and q_1 is given by

$$L = \{[(1 - k_1)p_1 + k_1q_1] + [k_1k_3p_1 + (1 - k_2)(1 - k_1)q_1]\}/(3 - c)$$
$$= [(1 - k_1 + k_1k_3)p_1 + (1 - k_2 + k_2k_1)q_1]/(3 - c).$$

Unbounded Metrics on Noncompact Spaces

5147 [1963, 1014]. *Proposed by Eric D. Nix, New York City*

Prove or disprove the proposition: Every noncompact metrizable space admits an unbounded metric.

Solution by W. J. Pervin, University of Heidelberg and Pennsylvania State University. This proposition is the content of a note by N. Levine, this MONTHLY, 68 (1961) 657–658. It also appears as a problem on page 249, R. Vaidynathaswamy, *Treatise on Set Topology*, (Madras, 1948).

Also solved by M. K. Fort, Jr., Charles Himmelberg, Meyer Jerison, L. R. King, K. O. Leland, M. D. Mavihkurve, S. M. Robinson, Herman Rubin, and W. C. Waterhouse.

Equivalent Ultrafilters

5148 [1963, 1014]. *Proposed by Hewitt Kenyon, George Washington University*

Let us agree that F is eventually in a set A if and only if F is a filter and $B \subset A$ for some member B of F. Let us agree further that F is equivalent to G if and only if F and G are filters such that each is eventually in every member of the other.

Suppose that h is a function mapping the set X onto the set Y; suppose that F is an ultrafilter eventually in X, and let G be the ultrafilter consisting of maps by h of subsets of X belonging to F. Show that F is equivalent to G if and only if F is eventually in the set of fixed points of h.

In the terminology of Kenyon and Morse, *Runs*, Pacific Journal of Math., 8 (4), 1958, the problem may be rephrased: Suppose that f is a function, x is a full run, and x is eventually in the domain of f. Show that $f{:}x$ runs the same as x if and only if x is eventually in the set of fixed points of f.

Solution by S. M. Robinson, Smith College. Let h be a mapping from X onto Y, F an ultrafilter on X and G the ultrafilter on Y which consists of the h-images of members of F. ($G = h[f]{:}f \in F$.) Let $Z = \{x \in X{:}\ h(x) = x\}$. We will establish the proposition that F is equivalent to G if and only if F is eventually in Z.

First assume that F is eventually in Z, i.e., there is an $f_1 \in F$ such that $f_1 \subseteqq Z$. Since $Z \subseteqq X$, this implies $Z \in F$. If f is any member of F, $f' = f \cap Z$ is also a member of F. Since $h[f'] = f'$, f' is also a member of G, and since $f' \subseteqq f$, G is eventually in f. On the other hand, if g is any member of G, then $g' = g \cap Z$ is also in G, for $z \in F$ and $h[Z] = Z$. Now $h^{-1}[g'] = g' \cup \{h^{-1}[g'] - Z\}$, and since $\{h^{-1}[g'] - Z\}$ is not a member of F, the fact that $h^{-1}[g']$ is in F implies that $g' \in F$. Thus, we have shown that F is eventually in every member of G and that G is eventually in every member of F, i.e., F and G are equivalent.

If F is not eventually in Z, Z is not in F, and therefore $X-Z\in F$. Let Φ denote the restriction to $X-Z$ of the function h. Since Φ has no fixed points we may partition $X-Z$ into three disjoint classes A, B and C such that $A\cap\Phi[A]=B\cap\Phi[B]=C\cap\Phi[C]=\varnothing$, the empty set. (See problem 5077 [1964, 219].) F being an ultrafilter, it follows that exactly one of the sets A, B, C is in F; say $A\in F$. Thus $\Phi[A]=h[A]=A'\in G$, and $A\cap A'=\varnothing$. Since A must meet every member of F, no member of F is contained in A'. Similarly no member of G is contained in A. Consequently F is not equivalent to G.

Covering a Sphere with Two Pairs of Small Circles

5149 [1963, 1014]. *Proposed by Michael Goldberg, Washington, D. C.*

Find r, the radius of the largest sphere on the surface of which two circles of given radius a, and two circles of given radius b, can be placed so that every point on the surface is within at least one of the regions (less than hemispheres) bounded by the four given circles.

Solution by W. J. Blundon, Memorial University of Newfoundland, and the proposer. Let the two circles of radius a intersect at points A and B. Let the points on the two circles farthest from AB be C and D, respectively. Then b is the circumradius of the triangles ACD and BCD. Let $AC=BC=AD=BD=e$. Let $AB=2c$, $CD=2d$. The situation is, thus, that the four circles intersect in the four vertices of a tetrahedron and, by symmetry, four of its edges are equal. Let r be the radius of the sphere. Then $a=\frac{1}{2}e^2(e^2-c^2)^{-1/2}$, $b=\frac{1}{2}e^2(e^2-d^2)^{-1/2}$. From this we have $4a^2c^2=e^2(4a^2-e^2)$, $4b^2d^2=e^2(4a^2-e^2)$. Let h be the height of the tetrahedron between the edges $2c$ and $2d$. Then, $h^2=e^2-c^2-d^2$.

If f is the distance from the center of the sphere to the midpoint of the edge $2d$, and g is the distance to the midpoint of the edge $2c$, then $h=f+g$, and

$$
\begin{aligned}
r^2 &= f^2+d^2 = c^2+g^2 = c^2+(h-f)^2,\\
f^2 &= c^2-d^2+h^2-2hf+f^2,\\
f &= (e^2-2d^2)/2h, \qquad f^2 = (e^2-2d^2)^2/4(e^2-c^2-d^2),\\
r^2 &= f^2+d^2 = (e^4-4c^2d^2)/4(e^2-c^2-d^2)\\
&= e^2(4e^2a^2+4e^2b^2-12a^2b^2-e^4)/4(e^2a^2+e^2b^2-4a^2b^2).
\end{aligned}
$$

If the derivative $dr^2/de=0$, then

$$(a^2+b^2)e^6-[2(a^2+b^2)^2+6a^2b^2]e^4+16a^2b^2(a^2+b^2)e^2-24a^4b^4=0.$$

This is a cubic in e^2. Solve for e^2 in terms of a^2 and b^2 and substitute in the equation for r^2.

Editorial Note. This is a composite of two independent solutions, one of which presented slightly more motivation for the process, and the other slightly more algebraic detail.

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This is a cubic in e^2. Solve for e^2 in terms of a^2 and b^2 and substitute in the equation for r^2.

Editorial Note. This is a composite of two independent solutions, one of which presented slightly more motivation for the process, and the other slightly more algebraic detail.

Real Analysis. By H. L. Royden. Macmillan, New York, 1963. xvi+284 pp. $9.00.

This well-planned and well-written book will inevitably be compared with the book of the same title by E. J. McShane and the reviewer (D. Van Nostrand Co., 1959). There are several similarities. The two books are about the same in length, and both are pitched at about the beginning graduate level. Both aim to present central and currently relevant selections from real function theory, general topology, and functional analysis. Both contain, in more or less general settings, such important tool-theorems of modern analysis as the Stone-Weierstrass theorem, the Ascoli theorem, the Fubini theorem, the Tychonoff product-space theorem, the Tietze extension theorem, the Hahn-Banach theorem, the Radon-Nikodym theorem, and the Riesz representation theorem. Both introduce, and make occasional use of, the Hausdorff maximality principle.

A notable difference between the two books lies in the presentation of integration. McShane-Botts introduces integration, using the Daniell technique from the outset, at the level of the Lebesgue-Stieltjes integral in R^n, obtaining as by-products the basic measure-theoretic facts as well as the Riemann and Riemann-Stieltjes integrals, and later outlining briefly the alternative measure-theoretic route to integration. Royden prefers pedagogically to present first the classical Lebesgue theory of measure and integration on the real line, later extending this to integration on a general abstract measure space, and finally including a good discussion of the Daniell-Stone integral and its equivalence to the measure-theoretic one.

Except for a brief notice of Kelley nets, Royden attempts no unified general treatment of convergence such as is found in the McShane-Botts book, nor does he treat linear operator theory, which the latter book carries as far as the spectral resolution for bounded hermitian operators on Hilbert space. On the other hand, Royden introduces a number of important and interesting topics not found in McShane-Botts, including convexity and the Krein-Milman theorem (with the neat Kelley proof), weak topologies of normed vector spaces and the Alaoglu theorem, and Borel equivalences of measurable spaces and a characterization of the isometries of $L^p[0, 1]$ into itself.

The exercises appear to be well selected, and quite a few are multi-part exercises extending the theory or containing interesting alternative proofs of theorems in the text.

TRUMAN BOTTS, University of Virginia

Differential Geometry. By Heinrich Guggenheimer. McGraw-Hill, New York, 1963. 378 pp. $12.50.

This book is a textbook, not a treatise, on local differential geometry, with some references to global theory. It is intended for a year course at the advanced undergraduate or beginning graduate level and presupposes only a knowledge of advanced calculus and linear algebra. There are illustrative examples in the text and each chapter ends with a set of exercises and a bibliography, the latter

including not only the classical texts but references to current research literature in the field.

The first few chapters, which deal with euclidean geometry in the plane, are intended for the student who needs additional geometrical background. Then material on transformation groups and on Lie group and Lie algebra theory which will be needed in the remainder of the text is presented. An introduction to tensor algebra is given but tensors are not used when matrix algebra will suffice. The classical material on curves and surfaces in differential geometry is presented from a modern viewpoint. The field used is always the real field. Topological methods are not used but the author often points out where algebraic topology is needed to solve a given problem. At the end of the book there are short chapters on Riemannian geometry and connections.

This book meets the need for a text in modern differential geometry using the theory of transformation groups. Aside from the first portions of Helgason's *Differential Geometry and Symmetric Spaces* and the recent *Foundations of Differential Geometry*, Vol. I, by Kobayashi and Nomizu, there seems to be no other.

ALICE T. SCHAFER, Wellesley College

A Second Course in Number Theory. By Harvey Cohn. Wiley, New York, 1962. 276 pp. $8.00.

We shall begin by briefly reviewing the contents of the 13 chapters. Chapter 1 is a review of elementary number and group theory (including quadratic congruences and the Jacobi symbol). Chapter 2 on "Characters" is obviously a preparation for Dirichlet's theorem that every arithmetic progression $ax+b$, with $(a, b)=1$, contains an infinity of primes. Chapter 3 discusses representations by quadratic forms and introduces examples of algebraic number fields. Chapter 4: Basis theorems. Chapter 5 contains Kronecker's Basis Theorem for Abelian groups, discusses minima of quadratic forms. The concluding section is "Korkine's and Zolatareff's Example" and mentions the work of the British school in the geometry of numbers (initiated by Mordell, Davenport, Mahler). Chapter 6 (the first of Part 2) is entitled "Unique factorization and units." The fundamental unit of a real quadratic field is discussed, also the euclidean algorithm. Chapter 7 "Unique factorization into ideals." Chapter 8 "Norms and ideal classes" contains in particular "Minkowski's theorem" and "norm estimate." Chapter 9 "Class structure in quadratic fields" discusses the splitting of rational primes in a quadratic field. Chapter 10 (the first of Part 3) is a beautifully written account of "Class number formulas and primes in arithmetic progressions." Chapter 11: "Quadratic Reciprocity." Chapter 12: "Quadratic forms and ideals." Chapter 13: "Compositions, orders and genera."

This is a most attractive book written by an expert. The author has succeeded with his informal style in writing a *very stimulating* book. The reviewer feels that Harvey Cohn's book is in a class by itself.

S. CHOWLA, Pennsylvania State University

Introduction to Differentiable Manifolds. By L. Auslander and R. E. MacKenzie. McGraw-Hill, New York, 1963. 219 pp.

This is a very valuable book; it has been much needed. The notion of differentiable manifold is relevant to a number of different mathematical disciplines besides differential geometry—algebraic geometry, Lie groups, and differential topology among them—but until now a student has been hard-pressed to find a treatment of the subject which was not aimed well above his head. This book will fill his needs; it should prove accessible to one who is familiar with linear algebra, point-set topology, and what is usually called advanced calculus. It is not a book on differential geometry in the usual sense of the term; geodesics and curvature do not appear, and tensors are not defined until the final chapter. Instead, it is an introduction to a number of important topics in modern mathematics in which the concept of differentiable manifolds plays a role.

The first six chapters develop, in careful and leisurely fashion, the basic ideas and facts of the subject, culminating in a proof of the theorem of Whitney which states that a differentiable manifold of dimension n may be differentiably imbedded in euclidean space of dimension $2n+1$. Along the way the student learns about the tangent and cotangent spaces, submanifolds, and the existence of a Riemannian metric. He also sees a number of examples, among them the projective spaces and projective analytic varieties.

The later chapters of the book deal with more advanced topics. Necessary and sufficient conditions for the existence of integral manifolds for a "p-dimensional vector field" are obtained, though not all the details of the proof are given. Lie groups and their Lie algebras are discussed in some detail, with particular reference to the general linear group; differentiable fiber bundles are treated, with special emphasis on the case of a principal bundle obtained from an action of a Lie group on a differentiable manifold.

The book is well-written and I have only one minor criticism in this regard. Too often the important definition of a section is submerged in the middle of an exposition which includes motivation, a geometric construction, and/or some subsidiary remarks. This is satisfactory for the reader who is plodding carefully in the authors' footsteps, but not so comfortable for others. Even such a reader may have trouble finding the crucial definition if he needs to refer to it a few pages later on.

J. R. MUNKRES, Massachusetts Institute of Technology

Flows in Networks. By L. R. Ford, Jr. and D. R. Fulkerson. Princeton University Press, 1962. 194 pp. $6.00.

The authors have expanded their elegant and important "Max-flow min-cut theorem" into the present book. This theorem appears in Chapter I with a shorter and clearer proof than that in their original paper. Because of the multiplicity of applications of this result and its subsequent extensions to programming problems, the book is both welcome and useful to workers in this field.

The first chapter implements the theorem with a workable algorithm for finding the maximum flow and establishes the link between flow and programming problems.

The second chapter develops feasibility theorems, i.e., criteria for the existence of network flows that satisfy additional linear inequalities. Applications are made to mixed graphs (in which some arcs are directed and some are not), partially ordered sets, and systems of distinct representatives.

Chapter III deals with transportation problems including the optimal assignment problems, a shortest chain algorithm, the well-known warehousing and caterer problems, and perhaps of greatest current applied interest, the study of project cost curves.

The concluding chapter generalizes the results to multi-terminal maximal flows from one set of nodes to another set. Unfortunately the recent related theorem of G. J. Minty (Reference [59] in Chapter III) was not presented in the text. The book is clearly written and should become a standard reference in programming literature.

FRANK HARARY, The University of Michigan

Diophantine Geometry. By Serge Lang. Interscience, New York, 1962. x+170 pp. $7.45.

"Diophantine geometry," by abuse of Langage, is what an algebraic geometer sees when he looks at Diophantine equations: points on algebraic varieties whose coordinates are rational numbers or integers. The two big theorems of the subject—Siegel's proof that an algebraic curve of positive genus has only finitely many integral points, and the Mordell-Weil theorem that the rational points on an abelian variety form a finitely generated group— were proved thirty-odd years ago. The big intervening event has been Roth's "$2+\epsilon$" theorem, which simplifies Siegel's proof. The author presents these results, dressed up and stylishly generalized, to an audience composed primarily of algebraic geometers. He gives these people in two introductory chapters a rapid account of what they need to know in algebraic number theory (and not one word more). The theorems alluded to above are then proved, and the book concludes on a simpler level with the Hilbert irreducibility theorem.

Granting the nondiscursive style (the sort that can "do" algebraic number theory without ever mentioning cyclotomic fields), it's extremely well done. The author's taste is impeccable, and the cognoscenti—those who have gone to Andre Weil's school of algebraic geometry—will find this book deeply rewarding. On the other hand, it will probably be alien corn for the number theorists. They will certainly find no numbers here, and the two central proofs are so deeply imbedded in geometry that even to extract a proof of the relatively elementary theorem of Mordell would require heavy surgery. Unless they are willing to bone up on the author's two previous books on algebraic geometry, "Diophantine geometry" is likely to remain for them a closed book.

ARTHUR MATTUCK, Massachusetts Institute of Technology

Stochastic Processes. By Emanuel Parzen. Holden-Day, San Francisco, 1962. xi+324 pp.

This is an introductory text, but does require a knowledge of elementary probability theory, since the two initial chapters which review this topic serve mainly to set the terminology. Succeeding chapters are devoted to introductory accounts of some of the more standard types of stochastic processes, such as normal and second-order stationary processes, Poisson and renewal processes, and Markov processes with both discrete and continuous parameter. A welcome feature of the presentation is the large number of worked out examples.

The book should be of use as a general background course for nonspecialists, and for specialized disciplines such as communication theory, control theory, etc., as a supplementary text at the upper division level.

A. V. BALAKRISHNAN, UCLA

The Structure of the Real Number System. By L. W. Cohen and G. Ehrlich. Van Nostrand, Princeton, 1963. 116 pp. $4.25.

After a short but careful introduction to set theory, the complex number system is constructed from a modification of the Peano axioms and proceeding, in order, through the construction of the natural numbers, integers, rational numbers and real numbers via Cauchy sequences.

On the way many algebraic structures are defined, recursive definitions are discussed, finite and denumerable sets and the axiom of choice are introduced, and an interesting chapter is devoted to showing that for ordered fields the additional requirements that the field be Archimedean and complete can be replaced by any one of six popular alternatives.

There are a number of exercises scattered through the book. Most of these extend in a routine way the results in the text, and a number of them are designated as being necessary parts of the text itself.

In this reviewer's opinion the book is a bit heavy in spots and there is an excess of definitions and notations not necessary to the main development, and not investigated in their own right. These things tend to remove much of the feeling of wonder the subject should evoke and we would guess that most readers' questions would be concerned with details rather than with ideas.

But even with this criticism it must be said that the book gives a good and careful development of the number system and discusses many of the related questions that arise in such a development. It should be useful as collateral reading for any mathematics student and as a text in courses dealing with the real numbers. It is perhaps most suitable for use as a text at the junior-senior undergraduate level.

The book seems quite free of misprints and errors. Only two will be mentioned here: in the last paragraph on p. 84 the $C(y)$, $C(x)$ should be $C_{(y)}$, $C_{(x)}$; and, in Exercise 5.16 (b) p. 101 the order of terms in the square brackets should be reversed.

J. B. ROBERTS, Reed College

Auto-Primer in Computer Programming. By Doris R. Entwisle. Blaisdell, New York, 1963. 345 pp. $6.50.

In spite of its broad title, this book is actually a text on the IBM 1620 Fortran II language. Besides ample treatment of all types of statements in the language (that given the FORMAT statement is quite extensive), it includes short chapters on flow-charting, checking and debugging, and internal operations. The student will get a thorough grounding in the Fortran language and some adeptness at programming in it. Although the material is not "programmed" in Skinner linear style, the designation "auto-primer" is fully justified by the pattern of brief exposition and exercises followed immediately by answers.

This reviewer prefers that the emphasis in computer programming courses be placed on the reformulation of problems for a computer and the flow-charting of their solutions rather than on the formal language used to implement these solutions. This book sorely lacks problems of a nontrivial nature and teaches little more than the Fortran language.

DONALD TARANTO, Carleton College

Algebraic Number Theory. By Edwin Weiss. McGraw-Hill, New York, 1963. 275 pp. $9.95.

This book is an introduction to algebraic number theory, using the modern approach of valuations, as distinct from the classical method of ideal theory. As the author states in his preface, the development is similar to that in Artin's *Algebraic Numbers and Algebraic Functions* (New York University, 1951) insofar as the common topics are concerned.

The author has been quite successful in achieving his purpose. The student who reads the book carefully should gain good insight into the results and techniques of the subject. There are many excellent examples, problems abound, and the author avoids too much generality when doing so seems advisable for clarity. But the book has a few rough edges; some additional editing would have removed its occasional errors and obscurities. They should, however, cause no great difficulty to an alert student. Also the reviewer felt that, at times, the author's style and the typography combined to make reading somewhat dull.

The first two chapters are devoted to the theory of (rank one) valuations. The third deals with the theory of extension fields of complete fields. Chapter 4 is on Dedekind rings, and is entitled "ordinary arithmetic fields." The next chapter specializes the assumptions to the classical case of the global field, i.e., algebraic number fields and algebraic function fields over finite constant fields. The method of idèles is used to prove finiteness of class number and the unit theorem. The last two chapters give standard facts on quadratic and cyclotomic fields.

Thus the book is a fairly complete and modern exposition of the classical results of algebraic number theory, up to but not including class field theory and results of an analytical nature. As such, it is the best source available, and it therefore lends itself very well to introductory graduate courses.

C. R. RIEHM, University of Notre Dame

Solved Problems: Gamma and Beta Functions, Legendre Polynomials, Bessel Functions. By O. J. Farrell and B. Ross. Macmillan, New York, 1963. vi+410 pp. $12.50.

Chapters 1, 3 and 5 of this book are concerned with the functions listed in the title. The three even numbered chapters contain applications of the pertinent functions. Each chapter contains a short introduction (longest, five pages) followed by a sequence of problems, each with its solution and many with appended remarks. The informal style used rescues the book from the monotony inherent in the presentation of more than two hundred problems and solutions. The problems vary in complexity from those fully suitable to a graduate course to others which require only the insertion of numerical values in a previously obtained result.

The index seems good and there is a useful bibliography of just under thirty items. The few printing errors noted by the reviewer should not bother a reader.

The authors have deliberately left themselves open to numerous complaints that certain problems could have been solved in much simpler ways but, within the framework stipulated in their preface, the authors have done a good job.

The book should be useful to a student who is learning to use these functions, particularly if no formal college course in such material is available to him.

E. D. RAINVILLE, The University of Michigan

Asymptotic Behavior and Stability Problems in Ordinary Differential Equations. By L. Cesari. Academic Press, New York, 2nd ed. 1963. 271 pp. $9.00.

One has only to glance at the sixty-nine page bibliography to determine that this book constitutes a comprehensive survey of an active field of research. I would class the book as a useful reference work. If you wish to refresh yourself on the contents of some paper you read a number of years ago, there is a good chance that you will find it discussed here, but probably, and necessarily, in a very brief form.

The four chapters are headed: The Concept of Stability and Systems with Constant Coefficients; General Linear Systems; Nonlinear Systems; Asymptotic Developments. A detailed and quite lengthy review of the first edition was made by J. A. Nohel, MR 22 #9673, and since there have been few changes in the present edition except for a few new references, and in one section to some more recent work of the author and J. K. Hale, it seemed unnecessary to go into any further details concerning the contents of the book.

My principal peeve against the book is the small sized type used. Each page could easily be spread out to two, and then, perhaps I would not have felt the need for a magnifying glass.

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An Introduction to Numerical Mathematics. By Eduard Stiefel. Translated by Werner C. Rheinboldt. Academic Press, New York, 1963. x+286 pp. $6.75.

This is an excellent translation of the second German edition of "Einführung in die numerische Mathematik." The author has selected a relatively small number of computational procedures of broad applicability and has discussed them in considerable detail. For example, the first three chapters which discuss the solution of linear equations, linear programming, and least-squares approximation are largely based on the exchange-method. By using this elementary form of presentation the use of matrix calculus is completely avoided. This is an advantage for the reader who has not studied linear algebra, but the reader familiar with linear algebra will be somewhat less satisfied with the presentation, losing as it does the power and beauty of the matrix representation.

The next two chapters are devoted to nonlinear algebra and eigenvalue problems. Here again the emphasis is on methods which lead to a general understanding of the problems with only passing reference to the latest techniques currently employed on automatic computers. The chapter on differential equations begins with an heuristic approach to numerical differentiation. In addition to the trapezoidal rule and Simpson's rule for numerical integration, the recently proposed method of repeated interval halving (Romberg method) is described. The major portion of this chapter is devoted to the basic ideas involved in the numerical solution of ordinary and partial differential equations, the emphasis being on an understanding of the ideas. The last chapter deals with approximations and includes interpolation and the use of Chebyshev expansions for obtaining best approximations.

Throughout the book the presentation is distinguished by its clarity. At many points there are discussions which give the reader a clear insight into some of the difficulties of numerical mathematics. Numerical examples of small size are used frequently to illustrate points in the body of the text and some examples of larger size are given in Appendix I. No problem sets have been included.

J. G. HERRIOT, Stanford University

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Associate Secretary, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

George Washington University: Professor H. F. Bright has been appointed Associate Dean of Faculties; Professor Solomon Kullback has been appointed Chairman of the Department of Statistics.

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PERSONAL ITEMS

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Editorial Note. Attention is directed to the splendid September 1964 issue of the *Scientific American*, in which all of the principal articles are devoted to mathematics. While the supply lasts, copies may be secured from the *Scientific American*, Mr. Jerome Feldman, 415 Madison Avenue, New York, New York 10017, for 60 cents each, postpaid. There is no discount for quantity.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-FIFTH SUMMER MEETING OF THE ASSOCIATION

The Forty-fifth Summer Meeting of the Mathematical Association of America was held at the University of Massachusetts, Amherst, Massachusetts, from Monday, August 24, through Wednesday, August 26, 1964, in conjunction with summer meetings of the American Mathematical Society, the Institute of Mathematical Statistics, the Society for Industrial and Applied Mathematics, the Pi Mu Epsilon Fraternity, and Mu Alpha Theta. The session of the Association on Wednesday at 2:15 P.M. was a joint session with the Society for Industrial and Applied Mathematics, and the session on Wednesday at 4:00 P.M. was a joint session with the Institute of Mathematical Statistics. There were registered 1495 persons, including 958 members of the Association.

Sessions of the Association were held on Monday morning and afternoon, on Tuesday morning and on Wednesday afternoon. All sessions were held in the Bowker Auditorium of Stockbridge Hall at the University of Massachusetts. Presiding offlcers at the three Earle Raymond Hedrick Lectures were President R. H. Bing, Professor F. A. Ficken, and President-Elect R. L. Wilder; at the remainder of the session on Monday morning Professor Leonard Gillman; at the remainder of the session on Monday afternoon Professor G. B. Price; at the session on a Mathematics Workshop in Africa Professor F. R. Olson; at the joint session with the Society for Industrial and Applied Mathematics Dr. A. S. Householder; and at the joint session with the Institute of Mathematical Statistics Professor Z. W. Birnbaum. The thirteenth series of Earle Raymond Hedrick Lectures was delivered by Professor E. E. Floyd of the University of Virginia. The Program Committee for the meeting consisted of Leonard Gillman, Chairman; Max Beberman, H. G. Jacob, Jr., L. H. Loomis, F. R. Olson and D. E. Richmond.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: *Periodic Maps*, Lecture I, by Professor E. E. Floyd, University of Virginia.

These lectures will be published in a forthcoming volume on topology in *MAA Studies in Mathematics*.

Mathematics for Liberal Arts Students, by Professor H. L. Alder, University of California, Davis.

The speaker outlined a course for students whose major does not require a mathematics course and who have only minimum mathematical preparation. Objectives should be: to give the students an appreciation of mathematics; to present *interesting* mathematics (this can be achieved by maximizing the ratio of ideas and theorems to definitions), to let the student solve some real mathematical problems (specifically to let him make discoveries, formulate conjectures, and prove theorems), and to make the student aware that mathematics is a fruitful field for research (unsolved problems should be stated wherever possible).

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Recommendations of CUPM's Panel on Mathematics for the Biological, Management and Social Sciences, by Professor Samuel Goldberg, Oberlin College.

The Panel tentatively recommends a three-year college course of study in mathematics for prospective graduate students in the biological, management and social sciences. Full details of the curriculum, including outlines of recommended courses in analysis, probability, statistics, linear algebra, and computing, are in a brochure available upon request to the CUPM Central Office, P. O. Box 1024, Berkeley 1, California.

Discussion of the Recommendations of the CUPM Panel.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Floyd.

Mathematical Training in the Soviet Union—Comments on a Visit to Novosibirsk, by Professor I. M. Singer, Massachusetts Institute of Technology.

The speaker reported on sidelights of the Symposium on Partial Differential Equations held in Novosibirsk in August, 1963.

The Cambridge Conference and After, by Professor P. J. Hilton, Cornell University.

The Cambridge Conference on School Mathematics met for a month in the summer of 1963. It reported the main burden and tenor of the discussions in *Goals for School Mathematics* which was published in the fall. This report has caused considerable controversy and much dismay in the teaching profession. It is clearly necessary to re-emphasize the tentative, probing, experimental nature of the recommendations, and to discuss follow-ups of the conference, both actual and potential. This does not exclude the possible conclusion that, at least temporarily, a halt should be called to the projection of innovations.

Logic for Undergraduates, by Professor Anil Nerode, Cornell University.

There are several types of logic courses suitable for undergraduate mathematics department offerings. The possibilities are outlined in terms of their relation to the evolution of the subject and the role of the subject for various classes of students.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Floyd.

Business Meeting of the Association.

A Mathematics Workshop in Africa, by Professor Walter Prenowitz, Brooklyn College.

This paper is a report on the Entebbe Mathematics Workshop, which has been held for the past three summers at Entebbe, the former capital of Uganda, on the shore of Lake Victoria. The workshop has been held under the auspices of Educational Services, Incorporated, and supported by the Agency for International Development of the State Department. It has included participants from eleven African countries, Great Britain and the United States. The workshop is devoted to curricular experimentation and the production of text materials in pre-college mathematics for use in English speaking, tropical African countries.

Discussion on the Mathematics Workshop in Africa.

FOURTH SESSION OF THE ASSOCIATION

Joint Session with the Society for Industrial and Applied Mathematics

Interweaving the Applications of Mathematics into the Undergraduate Curriculum, by Professor Bernard Friedman, University of California, Berkeley, and Dr. H. O. Pollak, Bell Telephone Laboratories.

Professor Friedman believes that the interweaving of applications in the curriculum is not achieved by introducing long digressions, but by trying to maintain some contact with reality and

by avoiding excessive compartmentalization of subject matter. It should also be emphasized that mathematics is the study of problems, not of theorems. Finally, the main point is not the subject matter but the attitude with which the subject is taught.

Dr. Pollak pointed out that applications of mathematics have a claim for inclusion in undergraduate courses, partly because of the variety of motivations among mathematics students. It is good applied mathematics, good pure mathematics, and good pedagogy, occasionally to present mathematics to the student in the form "Here's a situation, understand it," rather than "Here's a problem, solve it." A variety of such situations, drawn from engineering and relating to several undergraduate mathematics courses, will soon be available through the Association. They will emphasize the abstraction of the mathematical model, treatment of the resulting mathematics, and its engineering interpretation.

Joint Session with the Institute of Mathematical Statistics

Statistical Inference and Stochastic Processes, by Professor J. R. Blum, University of New Mexico.

The purpose of this talk is to show that relatively advanced notions of statistical decision theory can be handled by elementary mathematical techniques in a course in the sophomore or junior year. The first semester of such a course would be devoted to a classical introduction to probability theory on the level of Feller or Parzen. The second semester would then start with a discussion of convex sets and the supporting hyperplane theorem in finite dimensional spaces. This would be most appropriately followed by the linear programming problem and the simplex method. If time permits, a discussion of Game Theory could follow.

Next we turn to the statistical decision problem. At this level this should probably be kept to the case of a finite number of decisions. The notions of loss and risk now follow easily, as well as the idea of admissibility. Next in order are Bayes rules and the relation between admissible and Bayes rules. Most of these ideas are intuitively motivated by simple geometric notions.

Finally, minimax procedures are defined, and it is easily shown that the problem of finding minimax rules in the finite case reduces to the linear programming problem which has already been discussed.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in Bowker Auditorium of Stockbridge Hall as follows: On Monday at 5:00 P.M. the following color animations by Bruce Cornwell were shown: "Seven Bridges of Königsberg," "How Do We Count?," "Sets, Crows, and Infinity," and "Possibly So, Pythagoras." These were followed at 5:41 P.M. by a showing of "The Mathematician and the River" (color), an ETS film in the Horizons of Science series. On Monday at 7:30 P.M. there was a showing of "Four Line Conics" (color, silent), by Fletcher, followed at 7:41 P.M. by "Random Events" (black and white), a PSSC physics film, and at 8:20 P.M. by "Mathematical Induction" (color), an MAA produced film, by Professor Leon Henkin. On Tuesday at 7:00 P.M. there was the first general showing of "Mathematics for Tomorrow," (color), produced jointly by the MAA and NCTM during the past year; this was followed at 7:30 P.M. by "Complex Numbers via Matrices" (black and white), a Madison Project film, by Professor R. B. Davis, and at 8:10 P.M. by "What is an Integral?," an MAA produced kinescope, by Professor Edwin Hewitt.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Sunday at 10:00 A.M. in the Council Chambers of the Student Union at the University of Massachusetts with thirty members present. Among the items of business transacted were the following:

The Board reelected Professor H. L. Alder of the University of California, Davis, as Secretary of the Association for the five year term 1965–1969.

The Board elected Professors A. E. Livingston of the University of Alberta and Joshua Barlaz of Rutgers—The State University as additional Associate Editors of the MONTHLY. They will collaborate with Professor Starke and have special responsibility for dealing with solutions submitted for Elementary Problems and Advanced Problems, respectively.

The Board elected Professor D. E. Thoro of San Jose State College as an additional Editor of the MATHEMATICS MAGAZINE to be in charge of book reviews.

The Board voted to invite Professor J. W. Milnor of Princeton University to deliver the fourteenth series of Earle Raymond Hedrick Lectures at the 1965 Summer Meeting.

The Board voted to "express its appreciation to the International Business Machines Corporation, the Research Corporation and the Alfred P. Sloan Foundation for their generosity in providing funds to conduct a Cooperative Summer Seminar in 1965. This Seminar should contribute materially to the improvement of mathematical education. The interest in mathematics shown by these corporations is most gratifying." The Seminar will be held at Bowdoin College with Professor I. N. Herstein of the University of Chicago and Professor L. H. Loomis of Harvard University lecturing on algebra and analysis, respectively.

The Finance Committee announced that the price of back numbers of the MONTHLY (including Slaught papers and other supplements) will be increased from $1.25 to $1.50, effective January 1, 1965, and that the non-member subscription rate for the MONTHLY will be increased to $10 per year, effective August-September 1965.

The Board approved the following schedule of future meetings: Denver, Colorado, January 28–30, 1965; Cornell University, August 30–September 2, 1965; Chicago, Illinois, January 26–28, 1966; Rutgers—The State University, New Brunswick, New Jersey, August 29–31, 1966; Houston, Texas, January 26–28, 1967; University of Toronto, August 28–30, 1967; University of Wisconsin, Madison, August 26–28, 1968.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Tuesday morning with President Bing presiding. The Secretary reported that the membership was 14,780, an increase of 10% since the corresponding date last year.

The Secretary then reported on some of the actions taken by the Board of Governors on Sunday. He expressed the deepest gratitude of the MAA to Professor A. B. Willcox as Executive Director of CUPM, and announced the appointment of his successor, Professor B. E. Rhoades, previously Associate Director of CUPM. He introduced the new Associate Director of CUPM, Professor R. H. McDowell of Washington University, and the new Associate Director of the Committee on Educational Media, Professor P. E. Miles of the University of Wisconsin.

He announced that the appointment of MAA representatives at all university and college departments of mathematics was nearing completion. To date, 620 representatives have been appointed in 21 of the 27 Sections.

The Secretary announced certain actions taken by the Editor of the MATHEMATICS MAGAZINE and the Board of Governors concerning the MATHEMATICS MAGAZINE, including the reinstatement of book reviews beginning with the November-December 1964 issue. A policy of division of book reviews between the MATHEMATICS MAGAZINE and the MONTHLY has been agreed upon: reviews in the MATHEMATICS MAGAZINE will be confined to books of interest to students and teachers in the first two years of college mathematics. This will include not only calculus and precalculus textbooks but also other books at this general level. Some overlapping with the books reviewed in the MONTHLY and the MATHEMATICS TEACHER is to be expected and, indeed, may be desirable, but it is hoped that the overlap will be minimal.

The Board approved certain measures designed to make widely known these and other new features of the MATHEMATICS MAGAZINE in order that all those who believe

The Institute of Mathematical Statistics met Wednesday through Saturday. The Wald Lectures were delivered by Professor E. L. Lehman of the University of California, Berkeley, who spoke on "Topics in Non-Parametric Statistics."

The Society for Industrial and Applied Mathematics met on Wednesday. The von Neumann Lecture was given by Professor Solomon Lefschetz of RIAS on "Recent Advances in the Stability of Nonlinear Controls" at 1:00 P.M. in Bowker Auditorium of Stockbridge Hall.

The Pi Mu Epsilon Fraternity held sessions for contributed papers on Tuesday at 3:30 P.M. and 8:30 P.M. and on Wednesday at 2:00 P.M. in Room 151–152, Goessman Laboratory. A dinner meeting was held Tuesday at 6:30 P.M. in recognition of Pi Mu Epsilon's fiftieth anniversary in the Commonwealth Room of the Student Union. At the dinner, Professor J. S. Frame, Director General of Pi Mu Epsilon, spoke on the "History and Future of Pi Mu Epsilon." President Bing presented a certificate to Pi Mu Epsilon congratulating it on the extent to which it has achieved its goal of promoting scholarship in mathematics. A Dutch treat breakfast meeting for Pi Mu Epsilon members was held on Wednesday at 8:00 A.M.

The Governing Council of Mu Alpha Theta, the National High School and Junior College Mathematics Club, sponsored by the MAA, met on Wednesday at 9:00 A.M. in Room E 21 of Machmer Hall.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of A. E. Andersen, Chairman; H. L. Alder, R. H. Breusch, H. G. Jacob, Jr., Lorraine D. Lavallee, Torsten Norvig, Everett Pitcher, R. W. Wagner, and G. L. Walker.

Registration headquarters were located in the Ballroom on the first floor of the Student Union. Dormitory accommodations and cafeteria facilities were provided by the University of Massachusetts. The Mathematical Sciences Employment Register was maintained in Rooms E 13 and E 15 of Machmer Hall, and book exhibits were in the Student Union Ballroom.

An all day excursion by chartered bus to Old Sturbridge Village left the Student Union at 9:00 A.M. on Wednesday. SIAM conducted a beer party on Wednesday at 8:00 P.M. in the Gymnasium on the campus of Amherst College.

HENRY L. ALDER, *Secretary*

ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendments adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting at the University of Massachusetts in Amherst on August 23, 1964, elected to membership in the Association the sixth set of applicants for academic membership (for election of the other five sets, see pages 337–38 of the April 1962, page 953 of the November 1962, pages 479–80 of the April 1963, page 1044 of the November 1963, and page 469 of the April 1964 issues of this MONTHLY). Approval for election to membership was given to the following 15 applicants for academic membership.

University of Arizona
Arkansas Polytechnic College
University of Idaho
University of Massachusetts
Mercy College
Mississippi State University
New York City Community College
University of North Carolina
North Central College
Oklahoma State University
Rice University
Sweet Briar College
University of Toledo
Wayne State University
West Chester State College

HENRY L. ALDER, *Secretary*

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Annual Meeting at the Denver Hilton Hotel, Denver, Colorado on January 27–29, 1965. The Register will be conducted from 9:00 A.M. to 5:00 P.M. on each of these three days.

There is no charge for registration, either to job applicants or to employers, except when the late registration fee for employers is applicable. Provision will be made for anonymity of applicants upon request and upon payment of $3.00 to defray the cost involved in handling anonymous listings.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence, Rhode Island 02906 for application forms or for position description forms. These forms must be completed and returned to Providence not later than December 15, 1964 in order to be included in the listings at the Annual Meeting in Denver. Position Description forms which arrive after this closing date, but before December 31, 1964 will be included in the Register at the meeting for a late registration fee of $3.00. The printed listings will be available for distribution both during and after the meeting.

It is important that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Chancellor's Hall, State Education Department of the State of New York, at Albany, New York, on Saturday, May 16, 1964. A. H. Fox, Chairman of the section, presided at the morning session, and C. S. Ogilvy, Vice-Chairman, presided at the afternoon session. There were 130 persons in attendance, including 110 members of the Association.

At the business meeting the following officers were elected: Chairman, C. S. Ogilvy, Hamilton College; Vice-Chairman, M. W. Pownall, Colgate University; Secretary-Treasurer, N. G. Gunderson, University of Rochester. H. M. Gehman reported on MAA publications, MAA representatives, projects in cooperation with NSF, and the proposed increase in dues. Miss Nura Turner reported another successful year for the Contest Committee, but recommended that the Ontario-Quebec Section of Region II be considered separately from the Upstate New York Section of Region II for National Award purposes. A motion urging this was passed.

The program was as follows:

1. *Recent developments in N. Y. State secondary education*, by Frank Hawthorne, N. Y. State Education Department.

2. *Some aspects of the Madison Project*, by R. M. Exner, Syracuse University.

3. *The N. Y. State College Proficiency Examination Program*, by R. A. Beaver, S.U.N.Y. at Albany.

The New York State Department of Education offers proficiency examinations in seventeen fields for individuals who have prepared in these fields outside of regular college classes. College credit so gained may be granted by 134 colleges and universities of the State. There are two examinations in mathematics, Mathematics A, which is the C.E.E.B.'s Advanced Placement examination in calculus and allows credit for one year of calculus, and Mathematics B, which includes all of elementary calculus and analytics and allows up to two years of college credit.

4. *The algebra of reflexive relations*, by F. D. Parker, S.U.N.Y. at Buffalo.

A reflexive relation in a finite set can be characterized by a Boolean matrix, 1 in position (ij)

indicating that the ith element is related to the jth element. Under addition, the matrices form a commutative semigroup with identity in which every matrix is idempotent; under multiplication, they form a semigroup with identity. The same identity (the equality relation) serves as the identity for *both* operations, and multiplication is distributive over addition. A few elementary theorems are presented and an equivalence relation, as well as applications to logic.

5. *An inclusion theorem for Nörlund means*, by P. T. Schaefer, S.U.N.Y. at Albany.

Hardy (*Divergent Series*, p. 67) gives a sufficient condition that a regular Nörlund method include $(C, 1)$. The following theorem sharpens this result. *Theorem:* A sufficient condition that the regular Nörlund method (N, q_n) include (C, m) for integral $m \geqq 1$ is that $(-1)^m \Delta^m q_n \geqq 0$ for all $n \geqq -m$, where $q_{-1} = q_{-2} = \cdots = 0$, $\Delta q_n = q_n - q_{n+1}$ and $\Delta^{p+1} q_n = \Delta(\Delta^p q_n)$. It was indicated how this theorem can be used to obtain an inclusion theorem for M. Fekete's Taylor-Nörlund method ("New Methods of Summability," J. London Math. Soc. 33 (1958), 466–470).

6. *Moments and cumulants—and consequences*, N. P. Salz, Cornell Aeronautical Laboratory.

Given the arbitrary power of any infinite series with nonzero constant term, the author shows that the coefficients of the resulting series can easily be expressed, independently of those of lower order, in terms of the coefficients of the original series. The result, depending on the partitions of integers, follows from an equation relating the moments to the cumulants of a statistical distribution.

7. *Accumulated truncation errors in Simpson's rule*, by C. E. Rhodes, Alfred University.

In approximating a definite integral by Simpson's Rule on a computer, keeping a fixed number of digits, the error due to truncating the decimal values increases with n, the number of subintervals used, in a cyclic manner. A simple method is developed for estimating this truncation error as a function of n. This can easily be applied if a graph of the function to be integrated is available. There is extremely close agreement between the error so determined and the actual error in test integrals evaluated on a computer.

8. *Utterly integer valued entire functions*, by Daihachiro Sato, University of Saskatchewan at Regina.

9. *Neighbourly vertices in a polyhedron*, by Viktors Linis, University of Ottawa.

Gale has shown that for every integer $k \geqq m$ there exists a convex polyhedron in E^{2m} having k vertices such that every m of these are neighbourly. Gale's proof is nonconstructive; a new constructive proof is given based on the properties of systems of polynomials over $(0, 1)$.

N. G. GUNDERSON, *Secretary*

JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The annual meeting of the Pacific Northwest Section of the MAA was held at Washington State University, Pullman, Washington, on June 19, 1964. Approximately 120 people attended the meeting of which 62 were members of the section. K. S. Ghent presided over the business meeting which followed the noon luncheon. The following officers were elected for the coming year: Chairman, H. E. Chrestenson of Reed College; Vice-Chairman, S. A. Jennings of Victoria University, and Secretary-Treasurer, L. H. McFarlan of the University of Washington.

Following invitations from Professor Moursund and Jennings, it was decided that the annual meeting of the section for 1965 would be held at the University of Oregon at Eugene, Oregon and that of 1966 at the University of Victoria, Victoria, British Columbia. Professor Ghent announced that A. T. Lonseth of Oregon State University would be the new section governor. The retiring governor, D. C. Murdoch reported on the activities of the Board of Governors for the past three years.

The program meeting started with an hour address by Roy Dubisch of the University of Washington. He reported on the work of Educational Services Incorporated in developing a modern mathematics program in Africa.

The morning session of the meeting was completed by a Symposium on CUPM Recommendations for the Training of Elementary School Teachers. This was moderated by Wilfred Barnes of Washington State University. Speakers were: Mrs. Nancy Corrick,

elementary school teacher from Spokane; S. A. Jennings, formerly of the University of British Columbia, and Robert A. Willson, Supervisor of Mathematics Programs for Washington State Department of Education.

The afternoon program consisted of the following three papers. The first two were part of the SIAM program which was held in conjunction with the meeting of the section.

1. *Methods and extensions of Liapunov theory*, by R. J. Roth, Boeing Company.

2. *Symmetry and significance in mathematics*, D. S. Carter, Oregon State University.

The paper emphasized the relationship between the symmetry groups of mathematical structures and the logically significant relations within the structure. Under reasonable definitions of logically significant, every logically significant relation is invariant under the symmetry group of the structure. Conditions are discussed under which the converse holds. These ideas serve to explain Klein's Erlangen program. These are especially important in applications, where one must know which relations of the mathematical model are significant to the application.

3. *Complete residue systems in the gaussian integers*, J. H. Jordan and C. J. Potratz, Washington State University.

The congruence $a+bi$ is an equivalence relation. The following are representations for the set of equivalence classes: (1) the set of lattice points $x+yi$, $0 \leq y < d$, $0 \leq x < a^2+b^2/d$, where $d=(a, b)$; (2) two adjoining squares of lattice points with a^2 and b^2 points respectively; (3) a best representation which in some manner has least absolute value, is composed of a square centered at the origin and four triangles attached to the side at appropriate places; (4) a generated representation for $\gamma\delta$ from given representations for γ and δ. The division algorithm is a corollary to (3).

L. H. McFarlan, *Secretary*

CUPM PUBLICATIONS

The following publications may be obtained free of charge by writing to: CUPM Central Office, Post Office Box 1024, Berkeley, California 94701.

Report No. 1—Five Conferences on the Training of Mathematics Teachers.

Report No. 3—The Production of Mathematics Ph.D.'s in the U. S.—Revised.

Report No. 7—Ten Conferences on the Training of Teachers of Elementary School Mathematics, Fall 1962.

Report No. 8—Annual Report, August 1962–August 1963.

Report No. 9—Ten Conferences on the Training of Teachers of Elementary School Mathematics, Fall 1963.

Report No. 10—Annual Report, August 1963–August 1964.

Reports No. 2, 4, 5, and 6—Discontinued.

Recommendations for the Training of Teachers of Mathematics (A Summary).

Course Guides for the Training of Teachers of Junior High and High School Mathematics.

Course Guides for the Training of Teachers of Elementary School Mathematics (Preliminary Report).

Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists.

Recommendations on the Undergraduate Mathematics Program for Work in Computing.

Preliminary Recommendations for Pregraduate Preparation or Research Mathematicians.

Tentative Recommendations for the Undergraduate Mathematics Program for Students in the Biological, Management and Social Sciences.

Study No. 1—A Catalogue Survey of College Mathematics Courses.

Study No. 2—Mathematics Text Materials for the Undergraduate Preparation of Elementary School Teachers.

CUPM Basic Library List.

CALENDAR OF FUTURE MEETINGS

Forty-eighth Annual Meeting, Denver-Hilton, Denver, Colorado, January 28–30, 1965.

Forty-sixth Summer Meeting (Fiftieth Anniversary Celebration), Cornell University, Ithaca, New York, August 30–September 2, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, May 1, 1965

ILLINOIS, Southern Illinois University, Carbondale, May 14–15, 1965.

INDIANA

IOWA, University of Dubuque, Dubuque, April 23, 1965.

KANSAS, Washburn University, Topeka, April 10, 1965.

KENTUCKY, Eastern Kentucky State College, Richmond, Spring, 1965.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 12–13, 1965.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1965.

MINNESOTA

MISSOURI, University of Missouri, Columbia, Spring, 1965.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 30–May 1, 1965.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, College of San Mateo, February 6, 1965.

OHIO

OKLAHOMA, University of Arkansas, Fayetteville, Spring, 1965.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 18, 1965.

PHILADELPHIA

ROCKY MOUNTAIN, The Colorado School of Mines, Golden, Colorado, Spring, 1965.

SOUTHEASTERN, Wake Forest College, Winston Salem, North Carolina, April 9–10, 1965.

SOUTHERN CALIFORNIA, Claremont Men's College, March 13, 1965.

SOUTHWESTERN, Arizona State University, Tempe, Spring, 1965.

TEXAS, Texas Christian University, Fort Worth, April 9–10, 1965.

UPPER NEW YORK STATE, Colgate University, Hamilton, May 15, 1965.

WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Montreal, December 26–31, 1964.

AMERICAN MATHEMATICAL SOCIETY, Denver, Colorado, January 26–29, 1965.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Illinois Institute of Technology, Chicago, June 21–25, 1965.

ASSOCIATION FOR COMPUTING MACHINERY, Cleveland, August 24–26, 1965.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 25–27, 1965.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Sheraton-Mount Royal Hotel, Montreal, December 30, 1964.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Boston, Mass., May 6–7, 1965.

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Forty-eighth Annual Meeting, Denver-Hilton, Denver, Colorado, January 28–30, 1965.

Forty-sixth Summer Meeting (Fiftieth Anniversary Celebration), Cornell University, Ithaca, New York, August 30–September 2, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, May 1, 1965

ILLINOIS, Southern Illinois University, Carbondale, May 14–15, 1965.

INDIANA

IOWA, University of Dubuque, Dubuque, April 23, 1965.

KANSAS, Washburn University, Topeka, April 10, 1965.

KENTUCKY, Eastern Kentucky State College, Richmond, Spring, 1965.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 12–13, 1965.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1965.

MINNESOTA

MISSOURI, University of Missouri, Columbia, Spring, 1965.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 30–May 1, 1965.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, College of San Mateo, February 6, 1965.

OHIO

OKLAHOMA, University of Arkansas, Fayetteville, Spring, 1965.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 18, 1965.

PHILADELPHIA

ROCKY MOUNTAIN, The Colorado School of Mines, Golden, Colorado, Spring, 1965.

SOUTHEASTERN, Wake Forest College, Winston Salem, North Carolina, April 9–10, 1965.

SOUTHERN CALIFORNIA, Claremont Men's College, March 13, 1965.

SOUTHWESTERN, Arizona State University, Tempe, Spring, 1965.

TEXAS, Texas Christian University, Fort Worth, April 9–10, 1965.

UPPER NEW YORK STATE, Colgate University, Hamilton, May 15, 1965.

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FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Montreal, December 26–31, 1964.

AMERICAN MATHEMATICAL SOCIETY, Denver, Colorado, January 26–29, 1965.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Illinois Institute of Technology, Chicago, June 21–25, 1965.

ASSOCIATION FOR COMPUTING MACHINERY, Cleveland, August 24–26, 1965.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 25–27, 1965.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Sheraton-Mount Royal Hotel, Montreal, December 30, 1964.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Boston, Mass., May 6–7, 1965.

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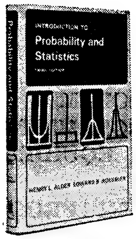

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* See also pp. 1133, 802, 803, 804.

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DECEMBER 1964

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A DISSECTION PROBLEM FOR SETS OF POLYGONS

MICHAEL GOLDBERG, Bureau of Naval Weapons
AND B. M. STEWART, Michigan State University

Introduction. If there are given n congruent simple planar polygons, which we describe as being of polygonal shape P, we seek to dissect each of these polygons in exactly the same way by straight line cuts into $K=K(n, P)$ pieces such that the total set of nK pieces may be fitted together, without reflection or overlapping, exactly to fill a large polygon directly similar to the original smaller polygons. Then a variety of problems may be studied.

For a given n does there exist a number $K(n, P)$ for which the dissection problem proposed above is possible? Well-known theorems about the dissection of polygons allow us to conclude that finite values of $K(n, P)$ are obtainable for every n and every shape P. There exists a finite dissection D' carrying any polygon P' into any other polygon P'' of the same area. (For example, see W. W. R. Ball, Mathematical Recreations and Essays, 11th Edition, London, 1939, pp. 87–93.) After perhaps some preliminary dissection D'', let the given n polygons of shape P be adjoined without overlapping to form a polygon P' and let the above theory be used to carry P' into P'' where P'' is of shape P. The dissections D'' and D' may have divided each of the n smaller polygons in various ways. But by superimposing these dissections we can arrive at a dissection D, still finite, which is the same for each of the n small polygons.

Since the answer to the previous question is affirmative, the following questions are reasonable.

For a given n what is the minimal value $k(n, P)$ of $K(n, P)$? In a few cases we are able to give best possible answers of the type $k(n, P)=1$ or $k(n, P)=2$, but in general, after producing $K(n, P)$, we can express our answers only in the form $k(n, P) \leqq K(n, P)$.

Does there exist an integer $u(P)$, dependent on P but independent of n, so that the dissection problem is solvable for all n, using $u(P)$ for $K(n, P)$? We show for a triangle T that $u(T)=5$, hence it follows for a polygon P_m of m sides that $u(P_m)=5(m-2)$.

Knowing the existence of $u(P)$ we ask a further question. What is the minimal value $g(P)$ of $u(P)$? We express our answers in the form $g(P) \leqq u(P)$ and the principal theorems of this paper are the following:

THEOREM 1. *For any triangle T, $g(T) \leqq 5$.*

THEOREM 2. *For any parallelogram M, $g(M) \leqq 4$.*

This result for a parallelogram improves an earlier result for a square S showing that $g(S) \leqq 5$. (See E 1010, this MONTHLY, 59 (1952) 699–700.)

We study in special detail the case of the square and for many values of n we are able to show $k(n, S) \leqq 3$. Furthermore, in a surprising number of cases we arrive at $k(n, S)=2$, which is a best possible result when n is not a perfect square. (Solutions for $k(n, S)=2$ when $n=5, 10, 13$ were given by T. Sundara

Row, Geometric Exercises in Paper Folding, 4th Edition, Open Court, 1958, pp. 25–27.)

One way to visualize these problems is to take n congruent paper polygons and stack them neatly so that each scissor cut made will cut all n polygons in the same way. If the paper used has sides of different color, then the accidental turning over of a piece, which is not intended in the problem, can be avoided. Even when a suitable dissection for a given n is known, the problem still has real recreational possibilities, for it is a challenging puzzle to give some uninitiated person the handful of pieces and ask him to assemble either the small polygons or the large polygon.

Preliminary considerations. In describing our various results we shall draw a figure to show how each small polygon should be dissected, labeling such a figure with an I and a series of letters or numbers corresponding to the appropriate discussion in the text. We will fix attention on some one edge of the small polygon, assuming it to be of unit length and referring to it as the "chosen edge." Where other lengths need to be described they will be marked on the figure by one, two, . . . , dashes and referred to in the text by $s = s_1, s_2, \cdots$, respectively. The regions of the small polygon will be designated by capital letters $A, B \ldots$.

The large polygon has its chosen edge of length $\sqrt{n}$ and appears in a figure labeled with a II and a series of letters or numbers to correspond to the text. Triangular, rectangular, square, parallelogram and trapezoidal regions of these large polygons will be designated by letters T, R, S, M and Z, respectively, with necessary subscripts.

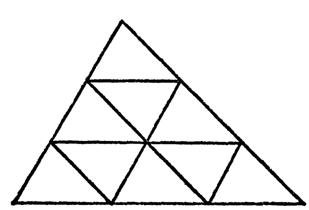
The letters n, a, b, and r which appear in the discussion and proof denote nonnegative integers.

Part 1. Triangles

1.1. The case $n=a^2$. When $n=a^2$ is the square of the positive integer a, it is easy to obtain the result $k(n, T)=1$. For from the well-known relation

$$a^2 = 1 + 3 + \cdots + (2a - 1),$$

it is easy to see how the $n=a^2$ uncut triangles T may be arranged in trapezoidal rows, each containing an odd number of copies of T, and then placed so as to fill exactly the large triangle T_1. The figure $II(T.1)$ illustrates the case $a=3$, $n=9$.

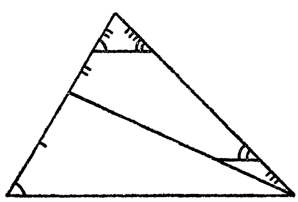


$II(T.1)$

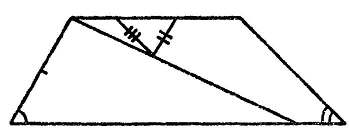
In the descriptions which follow we will refer to $II(T.1)$ as the "standard" filling of T_1 with a^2 copies of T, even in those cases where the individual T have been dissected into several parts.

1.2. Conversion of a triangle to a specified trapezoid or parallelogram. In $I(T.4)$ we show a four-part dissection of the triangle T and in $II(T.4)$ we show the reorganization of the parts into a trapezoid Z with the specified edge $s=s_1$ and the same base angles $\theta=\theta_1$ and θ_2 as in T. The left side of T, on which are indicated the section s_1 (with one mark) and the two sections of equal length s_2 (with two marks), will be called the "chosen side" of T and considered to be of unit length. Two of the cuts in $I(T.4)$ run parallel to the base of T. The location of the second of these cuts is determined by the equal segments s_3 (with three marks) which are on the "other side" of T. It is readily checked that the following condition is sufficient to guarantee that the dissection shown in $I(T.4)$ is indeed four-part:

$$1/3 \leqq s < 1. \tag{1}$$



$I(T.4)$

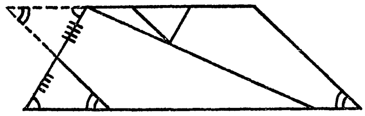


$II(T.4)$

Note 1. Two of the trapezoids Z, with one rotated through 180°, may be put together to form a parallelogram D with an angle θ and side s. If u is the base of T and if q is the length of the other side of D, we may compare the areas $qs \sin \theta = 2(u/2) \sin \theta$ to find $q=u/s$.

Note 2. An odd number of the trapezoids Z may be combined into a long trapezoid with base angles θ and θ_2 and with side s adjacent to θ.

Note 3. A five-part dissection will convert T either into Z or into a parallelogram M with an angle θ_2 and the same altitude as Z. For if to the dissection in $I(T.4)$ we add one cut, parallel to the "other side" and bisecting the segment s, then the cutting and rearrangement suggested by the dotted lines in $I(T.5)$ is easily justified.



$I(T.5)$

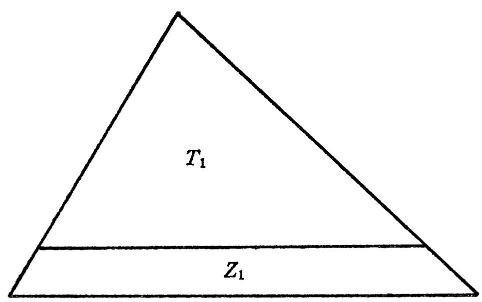
Note 4. An even number, say $2t$, of the triangles T may be combined by the five-part dissection in $I(T.5)$ into a long trapezoid with base angles θ and θ_2 and with side s adjacent to θ. For we may combine $2t-1$ of the trapezoids Z as in Note 2 and then adjoin one parallelogram M.

1.3. The case $n=a^2+r$, $2a/3<r<2a+1$. For the case described in this heading when $r=2t+1$ is odd, we will show that $k(n, T)\leqq 4$; and when $r=2t$ is even, we will show that $k(n, T)\leqq 5$. We employ the dissections described in $I(T.4)$ and $I(T.5)$ with $s=\sqrt{n}-a$.

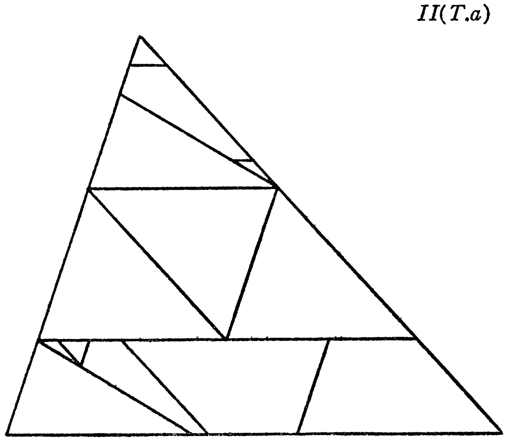
The first concern is to check condition (1) which is equivalent to

$$(a+1/3)^2 < n = a^2 + r < (a+1)^2 \quad \text{or} \quad (6a+1)/9 < r < 2a+1.$$

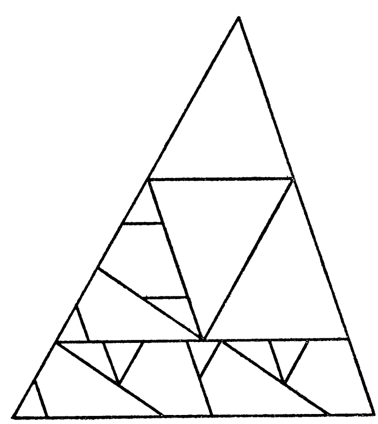
It follows that condition (1) is satisfied because of the covering hypotheses that r is an integer and $2a/3<r<2a+1$.



$II(T.a)$



$II(T.4.a)$



$II(T.5.a)$

The method of filling the large triangle is indicated in figure $II(T.a)$. For both subcases the upper triangle T_1 with "chosen edge" a is filled in the "standard" way with a^2 copies of T as explained in $II(T.1)$. The remaining trapezoidal strip Z_1 has base angles θ and θ_2 with side s adjacent to θ.

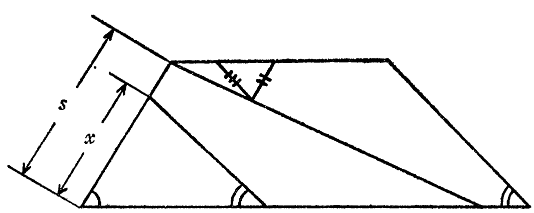
If $r=2t+1$, we apply Note 2 and use $2t+1$ of the trapezoids Z to fill Z_1. Hence we have $k(n, T)\leqq 4$. The figure $II(T.4.a)$ illustrates the case $n=7$.

If $r=2t$, we apply Note 4 and use $2t-1$ of the trapezoids Z and one parallelogram M to fill Z_1. Hence we have $k(n, T)\leqq 5$. The figure $II(T.5.a)$ illustrates the case $n=6$.

In these figures we have adopted the economy of indicating only as many of the dissections as are essential to understanding the principles involved.

1.4. The case $n=a^2+r$, $0<r\leqq 2a/3$. For the case described in this heading we begin by making the change of variables $a=b+1$, $r=R-(2b+1)$, so that b and R are positive integers in terms of which the new description of this case is $n=b^2+R$, $2b+1<R\leqq(8b+5)/3$. We will show $k(n, T)\leqq 5$, employing an initial dissection into four parts like that in $I(T.4)$—so that Z can be formed, followed by one more cut parallel to the "other side" as shown in $I(T.5.b)$, where the length of the segment x will be specified later. The following condition is sufficient, in addition to (1), to guarantee that the dissection shown in $I(T.5.b)$ is indeed a five-part dissection:

$$0 < x \leqq s. \tag{2}$$



$I(T.5.b)$

Depending on whether R is even or odd, we use $R=2t$, or $R=2t+1$, to define the integer t. We take $s=(\sqrt{n}+b)/R$ and $x=ts-b$.

To check condition (1) we must study whether $(b-R/3)^2\leqq n=b^2+R<(R-b)^2$. These inequalities are equivalent to $2b+1<R\leqq 6b+9$ which are guaranteed by the covering hypotheses $2b+1<R\leqq(8b+5)/3$.

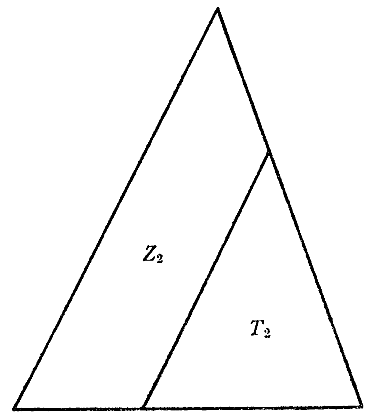
To check condition (2) we must study whether $0<ts-b\leqq s$. Respectively, these inequalities are equivalent to $(R-2t)b^2<t^2$ and $(t-1)^2\leqq b^2(R+2-2t)$.

If $R=2t$, we must check that $0<t\leqq\sqrt{2}\,b+1$, or that $0<R\leqq 2\sqrt{2}\,b+2$. But this follows since we have assumed for this case $2b+1<R\leqq(8b+5)/3$ and note that $(8b+5)/3<2\sqrt{2}\,b+2$ inasmuch as $4<3\sqrt{2}$ and $0<b$.

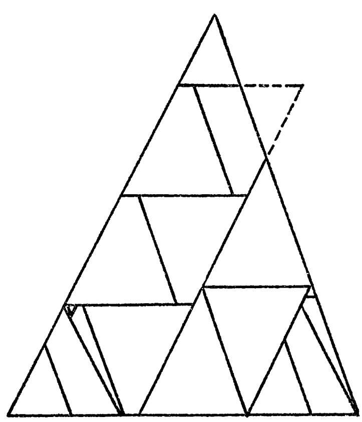
Similarly, if $R=2t+1$, we must check that $b<t\leqq\sqrt{3}\,b+1$, or that $2b+1<R\leqq 2\sqrt{3}\,b+3$. But this follows in the same manner as above from the hypotheses covering this case.

The method of filling the large triangle is indicated in figure $II(T.b)$. For both subcases the triangle T_2 with "chosen edge" b is filled in the "standard"

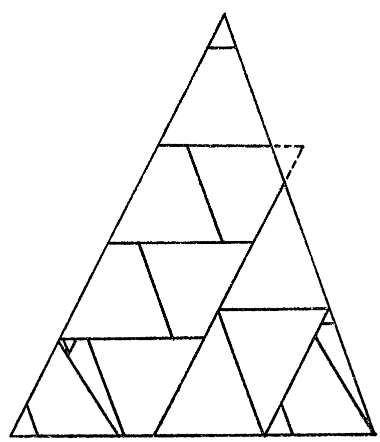
way with b^2 copies of T as explained in $II(T.1)$. The remaining trapezoidal strip Z_2 has parallel bases of length $\sqrt{n}$ and b and a "base" edge of length $(\sqrt{n}-b)u$.



$II(T.b)$



$II(T.5.b.0)$



$II(T.5.b.1)$

In case $R=2t$, we see that $s=(\sqrt{n}+b)/R$ implies

$$x = ts - b = \sqrt{n} - ts. \tag{3}$$

We apply Note 1 to see that $2t$ trapezoids Z, combined in pairs to form t parallelograms D, will almost exactly fill Z_2, for D has an angle θ, one edge s, and the other edge $q=u/s=uR/(\sqrt{n}+b)=(\sqrt{n}-b)u$. In fact, as indicated in $II(T.5.b.0)$ which illustrates the case $n=10$ and as checked by (3), the projecting triangle of "chosen edge" x just matches the deficiency in filling Z_2. So the five-part dissection in $I(T.5.b)$ is effective and $k(n, T)\leqq 5$.

In case $R=2t+1$, we see that $s=(\sqrt{n}+b)/R$ implies

$$x = ts - b = \sqrt{n} - (t + 1)s. \tag{4}$$

Again we apply Note 1 to see that $2t$ trapezoids Z may be combined to form t parallelograms D with angle θ and edges s and $q=(\sqrt{n}-b)u$. These t parallelograms and one more trapezoid Z almost exactly fill Z_2. In fact, as indicated in $II(T.5.b.1)$ which illustrates the case $n=11$ and as checked by (4), the projecting triangle of "chosen edge" x just matches the deficiency in filling Z_2. So the five-part dissection in $I(T.5.b)$, with appropriate x, is effective and $k(n, T)\leqq 5$.

1.5. Theorem 1 and Corollary 1. It is easy to check that the various cases treated in Sections 1.1, 1.3 and 1.4 provide an exhaustive and mutually exclusive classification of the positive integers. For the separation provided by $n=a^2+r$, $0\leqq r<2a+1$ is easily checked since $(a+1)^2=a^2+(2a+1)$. After the classes $r=0$ and $2a/3<r<2a+1$ have been removed, there remains only the class $0<r\leqq 2a/3$. Since five is the largest value of $K(n, T)$ required in any of the cases, our principal result follows:

THEOREM 1. *For any triangle T, $g(T)\leqq 5$.*

We turn our attention to the existence of $u(P)$. It is known that every simple closed plane polygon P_m of m sides can be decomposed into $m-2$ triangles by means of $m-3$ straight line cuts lying in the interior of P_m. (For example, see N. J. Lennes, Amer. J. Math., 33 (1911) 45–47. Also see Howard Eves, A Survey of Geometry, Allyn and Bacon, 1963, pp. 237–239.) To each of these $m-2$ triangles, T_i, we may apply Theorem 1 to claim $g(T_i)\leqq 5$. Then the $m-2$ large triangles of respective shapes T_i may be recomposed to form the large polygon of shape P_m. This establishes the existence of $u(P_m)=5(m-2)$, and proves the following:

COROLLARY 1. *For any polygon P_m of m sides, $g(P_m)\leqq 5(m-2)$.*

It is possible that polygons with special symmetries may admit much smaller values for $u(P_m)$ than provided in the corollary. Thus when $m=4$, the corollary suggests $g(P_4)\leqq 10$. But in the next part of this paper for any parallelogram M we show that $g(M)\leqq 4$. If a polygon P_m can be decomposed into C_M parallelograms and C_T triangles, then the result $g(P_m)\leqq 4C_M+5C_T$ may be a considerable improvement over the bound provided by Corollary 1.

Part 2. Parallelograms

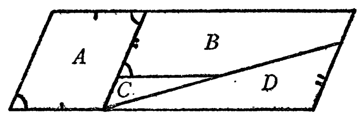
2.1. The case $n=a^2$. For a given parallelogram M we assume the "base" to be of length x, the "left" edge to be the "chosen edge" of unit length, and the included angle to be θ. It is easy to see when $n=a^2$ that $k(n, M)=1$. For the uncut parallelograms may be arranged in a rows each containing a copies of M to fill exactly the large parallelogram M_1 of base ax, chosen edge a, and included angle θ.

2.2. The case $n \neq a^2$, $n > 4$. When $a \geqq 2$ we note that the cases not covered in the preceding section can be classified by using

$$\text{(a) } n = a^2 + r, \quad 0 < r \leqq a; \qquad \text{(b) } n = a(a+1) + r, \quad 0 < r \leqq a.$$

Consider the four-part dissection shown in $I(M.4)$, where we take

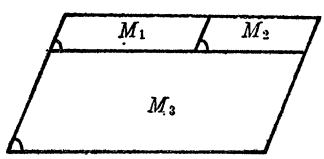
$$s_1 = (\sqrt{n} - r)x/(n - r) \quad \text{and} \quad s_2 = (\sqrt{n} - 1)/a.$$



$I(M.4)$

The dissection is valid if $0 < s_1 < x$ and if $1/2 \leqq s_2 < 1$. In both cases (a) and (b) we have $r \leqq a < \sqrt{n}$, so it follows that $0 < s_1$. Since $n > 4$ implies $\sqrt{n} < n$, it follows that $s_1 < x$. Since the conditions (a) and (b) imply $\sqrt{n} - 1 > a - 1$, it will follow that $s_2 \geqq 1/2$ providing that $(a-1)/a \geqq 1/2$; but this condition holds since it has been assumed in this section that $a \geqq 2$. Finally, since (a) and (b) imply $\sqrt{n} < a + 1$, it follows that $s_2 < 1$.

To fill the large copy of M of chosen edge $\sqrt{n}$ and base $\sqrt{n}\,x$ we proceed as indicated in $II(M.4)$ to consider separately the parallelograms M_1, M_2 and M_3. We indicate in the following chart the dimensions of these regions and the number of pieces of types A, B, C, D which we will use in filling them.

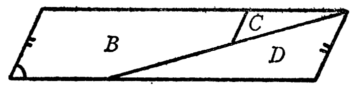


$II(M.4)$

Region	Edge	Base	# of A	# of B	# of C	# of D
M_1	1	rx	r	r	r	r
M_2	1	$(\sqrt{n} - r)x$	$n - r$	0	0	0
M_3	$\sqrt{n} - 1$	$\sqrt{n}x$	0	$n - r$	$n - r$	$n - r$

The filling of M_1 and M_2 is quite obvious: for r complete copies of M will fill M_1; and the base length s_1 of A is deliberately chosen so that $n - r$ copies of A will fill M_2.

As shown in figure (M') the parts B, C, D may be assembled to form a parallelogram M' with the same base angle θ as in M, but with the chosen edge of length s_2.



(M')

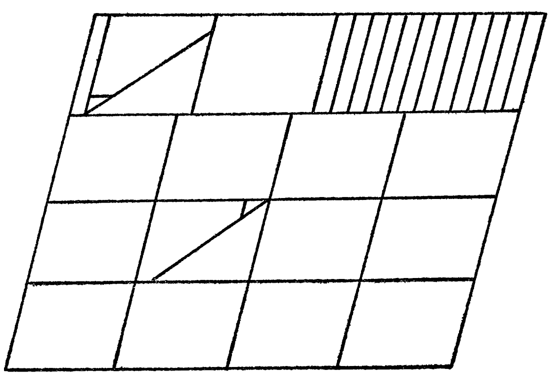
Then $n-r$ copies of M' may be assembled in a rows, each of which contains either a copies or $a+1$ copies, in cases (a) or (b), respectively, and these will exactly fill M_3. For from the definition of s_2, we obtain for the chosen edge the dimensional check $as_2=\sqrt{n}-1$.

As a confirmation we compute the base length v of M' by comparing areas. From $1\cdot x=1\cdot s_1+s_2\cdot v$, we find

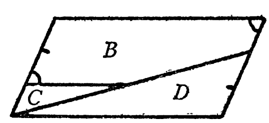
$$v=\frac{x-s_1}{s_2}=\frac{n-\sqrt{n}}{n-r}\cdot\frac{ax}{\sqrt{n}-1}=\frac{\sqrt{n}\,ax}{n-r}.$$

In case (a), $n-r=a^2$, so $av=\sqrt{n}x$; in case (b), $n-r=a(a+1)$, so $(a+1)v=\sqrt{n}x$. Since the base length of M_3 is $\sqrt{n}x$, the check is complete.

In figure $II(M.4.b)$ we illustrate the case $n=14=3\cdot 4+2$.



$II(M.4.b)$



$I(M.3)$

2.3. The cases $n=2$ and $n=3$. The cases $n=2$ and $n=3$ may be handled uniformly with the three-part dissection indicated in $I(M.3)$ in which we set $s=1/\sqrt{n}$. Since $1/2<1/\sqrt{n}<1$ holds for both $n=2$ and $n=3$, the dissection is indeed of the form shown.

The parts B, C, D may be assembled as in figure (M') to form a parallelogram M' of base angle θ and chosen edge s. Since $sn=\sqrt{n}$, it follows that n copies of M' will fill the large copy of M of chosen edge $\sqrt{n}$ and base $\sqrt{n}x$.

2.4. Theorem 2 and corollaries. Combining the cases in Sections 2.1, 2.2 and 2.3, we find that we have established the following:

THEOREM 2. *For any parallelogram M, $g(M) \leqq 4$.*

It is noteworthy that the reassembly of pieces called for in the proof of Theorem 2 demands only translation and no rotation!

COROLLARY 2. *For any parallelogram M, $k(ac^2, M) \leqq k(a, M)$.*

Proof. This corollary follows readily from the observation in Section 2.1 that $k(c^2, M) = 1$ and the idea of expanding the figure for the case $n = a$ by a factor of c.

COROLLARY 3. *For any parallelogram M, if $b < a < 4b$, then $k(ab, M) \leqq 3$.*

Proof. Since $n = ab$ and $b < a < 4b$, we find $1/2 < \sqrt{n}/a < 1$. We return to the three-part dissection in $I(M.3)$ and set $s = \sqrt{n}/a$. The new parallelogram M' has its base given by $t = x/s = ax/\sqrt{n} = x\sqrt{n}/b$. Therefore a rows each containing b of the M' will fill exactly a parallelogram similar to M of chosen edge $as = \sqrt{n}$ and base $bt = x\sqrt{n}$.

Part 3. Squares

3.1. Three-part dissections of the square. Obviously every result for the parallelogram is valid for the square, but because of the special symmetries of the latter figure for many of the numbers n we can obtain values for $k(n, S)$ smaller than 4. However, note that in addition to translation of pieces, we now employ some rotation, but, as we agreed, no reflection.

First we recall that Corollaries 2 and 3 imply

$$(S.3.1) \qquad \textit{if } b < a < 4b, \qquad \textit{then } k(abc^2, S) \leqq 3.$$

Further special results are as follows:

$$(S.3.2) \qquad k(4c, S) \leqq 3;$$

$$(S.3.3) \qquad k(4c + 1, S) \leqq 3.$$

To establish $(S.3.2)$ we determine the integer x so that $(x+1)^2 \leqq c < (x+2)^2$ and we write $c = x^2 + y$ where we know that $2x+1 \leqq y < 4x+4$. Then it follows that $(2x+2)^2 \leqq 4c < (2x+4)^2$, so that if we set $n = 4c$ we have $1 \leqq (\sqrt{n} - 2x)/2 < 2$. Therefore, if we set $s = 2/(\sqrt{n} - 2x)$, we may use the three-part dissection shown in $I(M.3)$.

In the figure $II(S.3.2)$ the large square is of edge $\sqrt{n}$ and the various regions have the dimensions and accommodate the number of pieces shown in the following table:

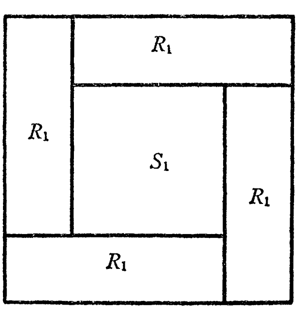
Region	Base	Altitude	# of B, C, D
S_1	$2x$	$2x$	$(2x)^2$
R_1	$(\sqrt{n}+2x)/2$	$(\sqrt{n}-2x)/2$	y

The square S_1 is filled with sets of B, C, D in the form of unit squares. Since B, C, D may be rearranged to form a rectangle which has the base $s = 2/(\sqrt{n} - 2x)$

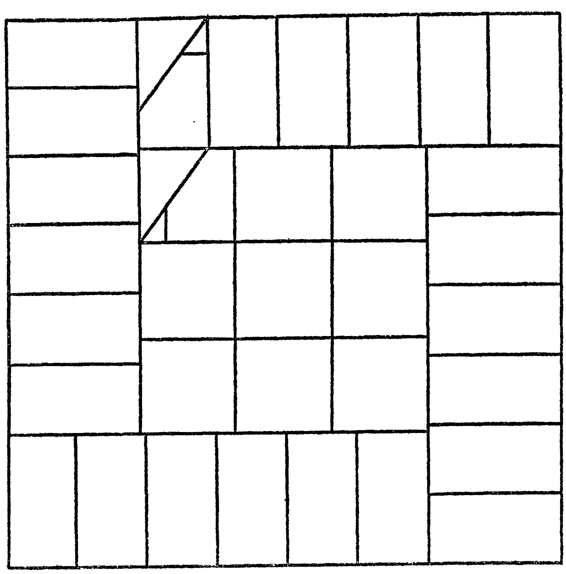
$=(\sqrt{n}+2x)/2y$ and the altitude $1/s$, it follows that y of these sets will exactly fill R_1. Since there are four of the rectangles R_1, it follows that the total count for each of B, C, D used in filling $II(S.3.2)$ is given by $4x^2+4y=4c=n$, which completes the proof of $(S.3.2)$.

Note that the method in $II(S.3.2)$ depends upon the congruence of the four rectangles R_1—so it will not apply to a parallelogram which is not a square, not even to a rhombus or rectangle. For an example of the use of $(S.3.2)$, where we cannot use $(S.3.1)$ or the following $(S.3.4)$, we may take $n=76=4\cdot 19$. We shall not include this figure, however, for the figure given to illustrate $(S.3.3)$ uses the same ideas.

To establish $(S.3.3)$ we first suppose $c\geqq 2$ and then we determine $x\geqq 0$ so that $(x+1)(x+2)\leqq c<(x+2)(x+3)$. If we write $c=x(x+1)+y$, then we have $2x+2\leqq y<4x+6$. But we have $n=4c+1=(2x+1)^2+4y$, so that we obtain $(2x+3)^2\leqq n<(2x+5)^2$. We may rearrange this pair of inequalities to see that $1\leqq(\sqrt{n}-(2x+1))/2<2$. Therefore, when $c\geqq 2$, the dissection and composition used for $(S.3.2)$ will apply to prove $(S.3.3)$ if we merely replace $2x$ by $2x+1$ at every step.



$II(S.3.2)$



$II(S.3.3)$

When $c=0$, we have $n=1$ and we know $k(1, S)=1$. When $c=1$, we have $n=5$, and in the following $(S.3.4)$ we show $k(5, S)=2$. This completes the proof of $(S.3.3)$.

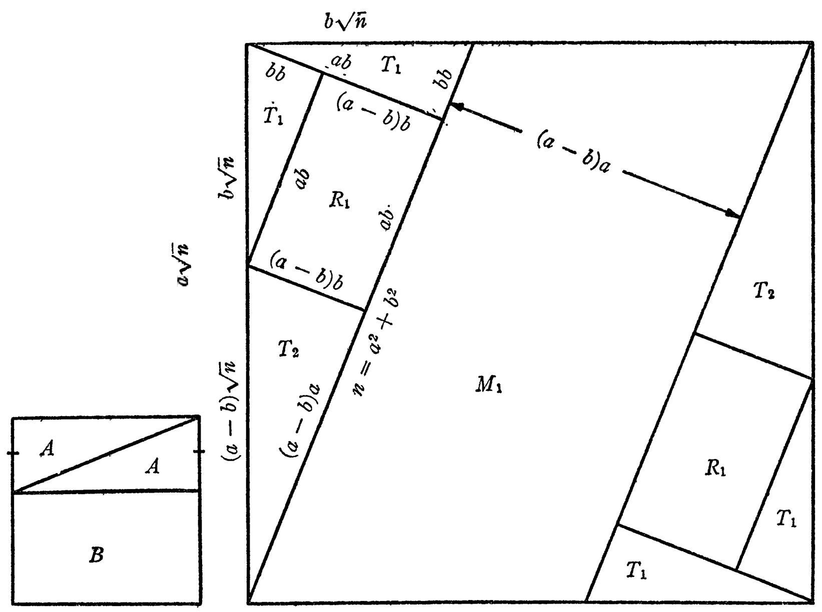
As an example we illustrate in figure $II(S.3.3)$ the case $n=33$ which is chosen because it cannot be reached by our other two-part or three-part dissections.

Next we shall show a three-part solution when n is the sum of two squares:

$$(S.3.4) \qquad k(a^2 + b^2, S) \leqq 3.$$

We should immediately point out, however, that for "one-fourth" of these cases we are able to reach two-part dissections in the next section of this paper.

Of course we may assume $a \geqq b > 0$. Because of Corollary 2 we might assume a and b to be relatively prime, but our proof makes no use of this simplification.



$I(S.3.4)$ $\qquad$ $II(S.3.4)$

In figure $I(S.3.4)$ we find it convenient to take the side of the given square to be a and to take the marked length $s = s_1 = b$. The large square in $II(S.3.4)$ has the edge $a\sqrt{n}$. In the following table we describe the dimensions of the various regions in $II(S.3.4)$ and the ways in which they may be filled.

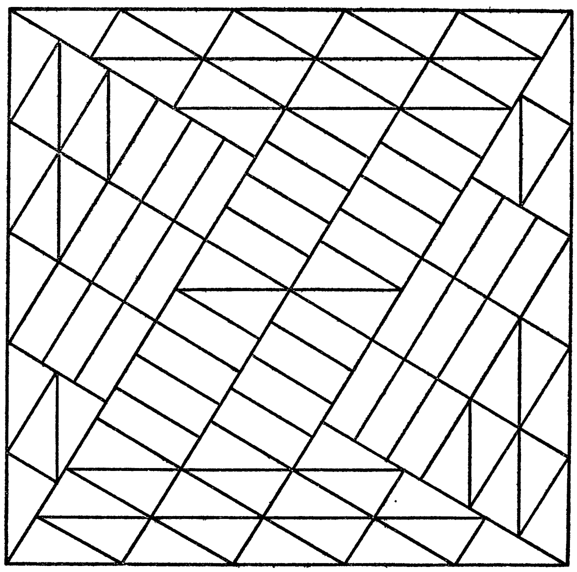
Region	Base	Altitude	# of A	# of B
T_1	ba	bb	b^2	0
T_2	$(a-b)a$	$(a-b)b$	$(a-b)^2$	0
R_1	ba	$(a-b)b$	0	b^2
M_1	n	$(a-b)a$	$(a-b)4b$	$(a-b)(a+b)$

The method of filling the triangles T_1 and T_2 has already been explained in $II(T.1)$ and the way to fill R_1 is obvious since B is a rectangle of dimensions a and $a-b$. To explain the filling of M_1 we may divide M_1 into $a-b$ parallelogram strips of base n and altitude a. Since $n = a^2 + b^2 = (2b)b + (a+b)(a-b)$, each of these strips may be filled with $a+b$ of the rectangles B and with $2(2b) = 4b$ of the triangles A, for these triangles in pairs form a rectangle of dimensions a and b.

If we assume $b > 0$, there are sure to be two triangles A to use to make the strip into a parallelogram.

Taking into account the frequencies of T_1, T_2 and R_1, we readily find the grand total of pieces used to be $2n$ for A and n for B. This completes the proof of $(S.3.4)$.

As an illustration we show the figure for $n = 34$, since this is the smallest n for which we cannot use the previous three-part dissections, nor the later two-part dissections.



$k(34, S) \leqq 3$

3.2. Two-part dissections of the square. In the results which follow we shall take it as obvious that when we write $k(n, S) \leqq 2$ for certain n, then it is understood that $k(n, S) = 1$ if and only if n is a perfect square. For when n is a square we may use the method in Section 2.1; and when n is not a square, at least one cut must be made to introduce edges of irrational length so that the large square of edge $\sqrt{n}$ can be realized. Furthermore, from Corollary 2, a result $k(n, S) = 2$ implies $k(nc^2, S) = 2$ for all $c \geqq 1$.

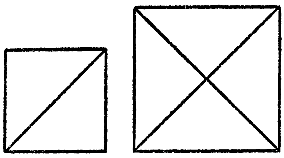
We know only three cases which do not fall into our general result $(S.2.4)$ so we describe these separately as follows:

$(S.2.1)$ $$k(2, S) = 2;$$

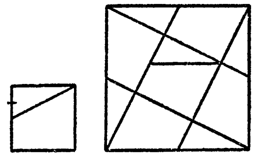
$(S.2.2)$ $$k(5, S) = 2;$$

$(S.2.3)$ $$k(10, S) = 2.$$

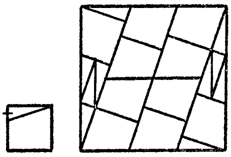
As proof we offer the following figures. In $(S.2.1)$ the unit square is bisected along one of its diagonals. In $(S.2.2)$ we take $s = 1/2$. In $(S.2.3)$ we take $s = 1/3$.



(S.2.1)



(S.2.2)

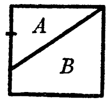


(S.2.3)

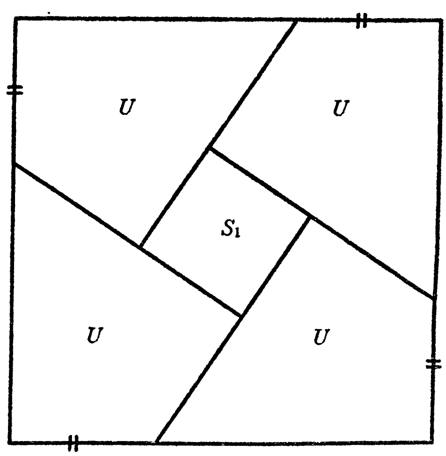
In the following paragraphs we prove:

(S.2.4) *If* $t \geqq 1$ *and* $x \geqq 4t - 1$, *then* $k(x^2 + (x - t)^2, S) \leqq 2$.

When $t=1$, our proof will include all $x=3$; the case $x=2$, however, may also be allowed since it corresponds to $n=5$ and the special result (S.2.2). Again when $t=2$, our proof requires $x \geqq 7$; the cases $x=3$, 4, 6, however, correspond to $n=10$, 20, 52, respectively, and these are covered by (S.2.3), or by Corollary 2 and (S.2.2), or by Corollary 2 and the previous case $t=1$. Thus when $t=2$, only the case $x=5$ with $n=34$ remains in doubt, our best result being the three-part dissection given in (S.3.4). Similarly, when $t=3$ one method or another yields a two-part solution, except for the cases $n=17$, 29, 89, 149 for which our best result is the three-part dissection given by (S.3.3) or by (S.3.4).



I(S.2.4)



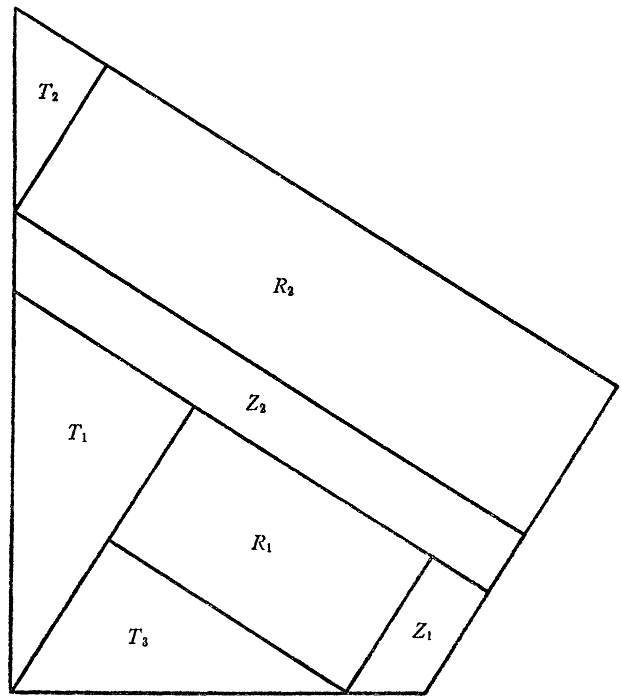
II(S.2.4)′

To establish (S.2.4) we use the dissection shown in I(S.2.4) in which the side of the given square is x and in which $s=x-t$. In describing how to fill the large square of edge $x\sqrt{n}$ we distinguish the four cases: $x=4a$, $4a+1$, $4a+2$, $4a+3$. Although much the same general plan is used in the four cases, there is one major difference according as x is even or odd, and some further slight differences according to the least nonnegative remainder when x is divided by 4.

First we subdivide the large square as shown in II(S.2.4)′ where the central square S_1 is of edge tx. The other four pieces U have the edge marked with two

dashes of length $t\sqrt{n}$ and the smallest angle of U is equal to the larger acute angle of A.

When x is even, all four of these regions U are to be filled in the same way. When x is odd, two of these regions are filled in one way (*i*) and two in another way (*ii*). In $II(S.2.4)$ we show the region U subdivided in a manner described in detail in the accompanying tables.



$II(S.2.4)$

First we show the tables when $x=2q$ is even.

Region	Base	Altitude	# of A	# of B
T_1	$(q-1)x$	$(q-1)(x-t)$	$(q-1)^2$	0
T_2	$(q-t)x$	$(q-t)(x-t)$	$(q-t)^2$	0
T_3	$(t-1)x$	$(t-1)(x-t)$	$(t-1)^2$	0
Z_1	$(q-1)x-t(x-t)$	x	$q-2t$	$q-1$
R_1	$(q-1)x-(t-1)(x-t)$	$(t-1)x$	$(t-1)(q-2t+1)$	$(t-1)(q-1)$
Z_2	$(q-1)(x-t)+tx$	x	$2q-1+t$	t
R_2	$q(x-t)+tx$	$(q-t)x$	0	$2q(q-t)$

We pass over the method of filling T_1, T_2, T_3 by reference to $II(T.1)$. To justify the counts given for Z_1 we rewrite the base of Z_1, separating the two cases. When $x=4a$, we write

$$(2a - 1)4a - t(4a - t) = (a - 1)(4a + t) + (a - t)(4a - t) + t.$$

Since two B's will form a rectangle that is x by $x+t$, it follows that we can fill Z_1 with $2(a-1)+1=q-1$ of the B and with $2(a-t)=q-2t$ of the A. But when $x=4a+2$, the base of Z_1 appears in the form

$$2a(4a + 2) - t(4a + 2 - t) = a(4a + 2 + t) + (a - t)(4a + 2 - t).$$

Hence Z_1 may be filled with $2a=q-1$ of the B and with $2(a-t)+1=q-2t$ of the A. The counting results are the same for the two subcases, but the actual arrangement of the pieces is different: when $x=4a$, there is a B at the pointed end of Z_1; when $x=4a+2$, there is an A.

The region R_1 may be divided into $t-1$ strips of altitude x and each can be filled exactly like Z_1, except that each strip requires an extra A at the end to change it from a trapezoid to a rectangle.

Whether $x=4a$ or $x=4a+2$, the base of Z_2 may be written $(q-1)(x-t)+tx$ so we may use $2(q-1)+t+1$ of the A and t of the B to fill Z_2. The base of R_2 may be written in the form $q(x-t)+tx=q(x+t)$, so if R_2 is divided into $q-t$ strips of altitude x, each of these can be filled with $2q$ copies of B.

In both subcases the above arguments involve nonnegative dimensions and nonnegative counting numbers if $a \geqq t$, but the covering assumption $x \geqq 4t-1$ justifies this condition.

Finally, we total the A's and B's, recalling that there are four of the regions U and the central S_1, the latter to be filled in the standard way with t^2 of the given squares. For both A and B we find the total count given by $8q^2-4tq+t^2 = 2x^2-2tx+t^2=n$. This completes the proof of $(S.2.4)$ when x is even.

Next we show tables for $II(S.2.4)$ when $x=2q-1$ is odd.

Region	Base	Altitude	# of A	# of B
T_1	$(q-1)x$	$(q-1)(x-t)$	$(q-1)^2$	0
T_2	$(q-t-1)x$	$(q-t-1)(x-t)$	$(q-t-1)^2$	0
T_3	$(t-1)x$	$(t-1)(x-t)$	$(t-1)^2$	0
Z_1	$(q-1)x-t(x-t)$	x	$q-2t$	$q-1$
R_1	$(q-1)x-(t-1)(x-t)$	$(t-1)x$	$(q-2t+1)(t-1)$	$(q-1)(t-1)$
R_2	$q(x-t)+tx$	$(q-t-1)x$	$q-t-1$	$(q-t-1)(2q-1)$
$Z_2(i)$	$(q-1)(x-t)+tx$	x	$2q-1+t$	t
$Z_2(ii)$	$(q-1)(x-t)+tx$	x	0	$2q-1$

The method of filling T_1, T_2, T_3 is the usual one of $II(T.1)$. To justify the counts for Z_1, we rewrite the base of Z_1, separating the two cases. For $x=4a+1$, we note that

$$2a(4a + 1) - t(4a + 1 - t) = a(4a + 1 + t) + (a - t)(4a + 1 - t);$$

hence it follows that we can fill Z_1 with $2(a-t)+1=q-2t$ of the A and $2a=q-1$ of the B. But when $x=4a+3$, the base of Z_1 is given by

$$(2a + 1)(4a + 3) - t(4a + 3 - t)$$
$$= a(4a + 3 + t) + (a - t + 1)(4a + 3 - t) + t;$$

hence we can fill Z_1 with $2(a-t+1)=q-2t$ of the A and $2a+1=q-1$ of the B. The difference between the two cases is that when $x=4a+1$, there is an A at the pointed end of Z_1; but when $x=4a+3$, there is a B.

In both cases R_1 is divided into $t-1$ strips of altitude x, each of which may be filled in the same way as Z_1, except for the addition of an extra A in each strip to square off the end.

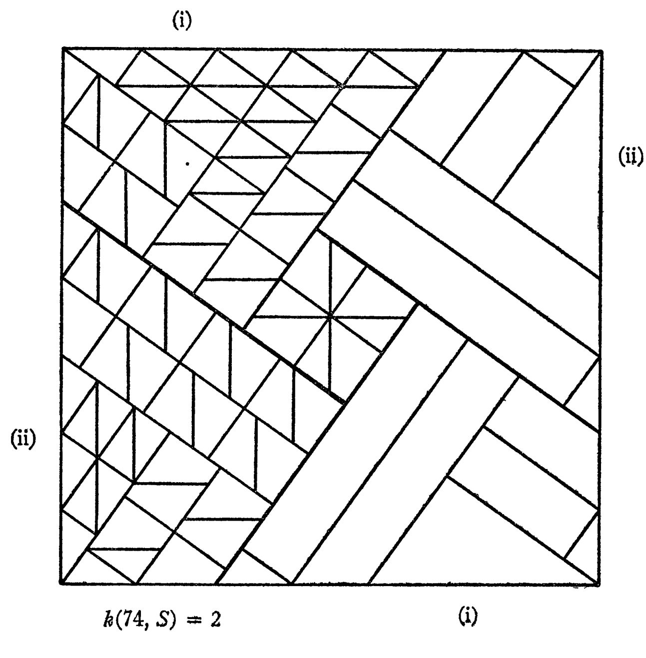
The base of Z_2 may be written in two ways:

$$(q-1)(x-t)+tx=(q-1)(x+t)+t;$$

hence we may fill Z_2 in two different ways:

(i) using $2(q-1)+t+1=2q-1+t$ of the A and t of the B;
(ii) using none of the A and $2(q-1)+1=2q-1$ of the B.

In filling R_2 we divide R_2 into $q-t-1$ strips of altitude x and we use the method (ii) on each strip, remembering to add an extra A in each strip to square off the end.



$k(74, S)=2$

The dimensions and counting numbers used in the above arguments are non-negative if $q\geqq 2t$, and this condition is satisfied because of the covering assumption that $x\geqq 4t-1$.

Finally, we total the A's and B's with instructions to use method (i) in filling Z_2 for two of the regions U, but to use method (ii) in filling Z_2 for the other two U. The four regions U together with the central S_1 use parts A and B

Of a related nature are problems which do not specify that the unit polygons need be cut in the same way and these have already received some attention in the literature. (For example, see E. Fourrey, Curiosités géometriques, Paris, 4th ed., 1938, pp. 109–125.)

It can be seen that a relaxation of our requirement that there should be no reflection of pieces may change the results. This is particularly clear in a class of problems considered by S. Golomb and reported upon by M. Gardner (Scientific American, May, 1963). In our terminology, Golomb seeks for each n all possible P such that $k(n, P)=1$. Since he allows reflection, he obtains some answers which we would reject. Using reflection, we find that any parallelogram M whose edges are of lengths 1 and $\sqrt{n}$ has $k(n, M)=1$; without reflection, we must restrict the parallelogram to be a rectangle.

Dissection theorems, both two- and three-dimensional, in fascinating variety are presented by V. G. Boltyanskii, *Equivalent and Equidecomposable Figures*, D. C. Heath, 1963.

FIXED POINT THEOREMS

A. GLEN HADDOCK, Oklahoma State University and Arkansas College

1. Introduction. In recent years many new notions have been introduced into mathematics. Among these is the notion of isometry. An isometry of E^n onto E^n is a distance preserving transformation of E^n onto E^n. It has been shown [1] that any isometry of E^n onto E^n can be represented by $n+1$ or fewer reflections, and furthermore that if I is an isometry of a subset A of E^n onto a subset B of E^n, then I can be extended to an isometry of E^n onto E^n. In the case of E^2 it is shown [2] that an isometry I must be one of the following transformations:

(1) Identity (every point fixed)
(2) Rotation (one fixed point)
(3) Translation (no fixed point)
(4) Reflection (line of fixed points)
(5) Glide reflection (no fixed point)

We shall use the above results from geometry along with the following results from topology to obtain some theorems that are new and to extend known results. If M is a continuum which does not separate the plane, then M is the intersection of a monotonic descending sequence of topological 2-cells. If $\{C_i\}$ is a finite collection of topological 2-cells in the plane whose intersection is nondegenerate, there is a topological 2-cell C such that the boundary of C is contained in the union of the boundaries of the C_i's and C contains the union of the C_i's. Both results may be found in [3].

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A well-known unsolved problem in topology is the following: *If M is a compact continuum in the plane and does not separate the plane, and T is a continuous*

transformation such that $T(M)=M$, *does* T *necessarily allow a fixed point?* The answer is not known even if T is restricted to be a periodic transformation. The question has been answered in the affirmative in certain cases. The main result along this line, obtained by P. A. Smith [4], is stated as follows: "Let K be a point set in Euclidean m-space and T a topological transformation of K into itself of finite prime period p. If every continuous single-valued image in K of every sphere of dimension less than or equal $pm-m-1$ is deformable to a point, then T leaves fixed at least one point of K."

It has also been shown that if T is a one-to-one continuous and orientation preserving transformation of the Euclidean plane E^2 onto itself which leaves a bounded continuum M invariant, and if M does not separate E^2, then some point of M is left fixed by T. This result was first obtained by M. L. Cartwright and J. E. Littlewood [5]; later O. H. Hamilton [6] obtained the same result using a much shorter method.

2. Special results. We show that if T is a periodic transformation of the plane into itself which leaves a plane continuum M invariant, and if M does not separate the plane, then some point of M is left fixed by T. We also show that if T is an isometric transformation then some point of M is fixed under T. These results are not contained in either of the previously mentioned results.

LEMMA 1. *Let* C *be a circle and* T *be a transformation of* C *onto itself such that* T *is a rotation which is not periodic on any point of* C (*that is, a rotation through some angle* A *such that* $kA\neq 0 \pmod{2\pi}$ *for any integer* $k\neq 0$). *Then the closure of the union of the points* $T^i(x)$ *is* C *for any* x *in* C.

THEOREM 1. *Let* T *be an isometry of a compact continuum* M *of the plane onto itself. If* T *is periodic on no point of* M, *then* $\overline{\bigcup_i T^i(x)}$ *is a continuum for any point* x *in* M.

Proof. Consider the different types of isometries in the plane. The hypothesis that T is not periodic on any point of M excludes types (1) and (4). The hypothesis that M is compact excludes types (3) and (5). Hence it follows that T must be a rotation about some point P, so that $T^i(x)$ lies on some circle with center P. Since T is not periodic on some point x, $\overline{\bigcup_i T^i(x)}$ is a circle for every point $x\neq P$.

THEOREM 2. *Let* T *be a periodic transformation of the plane into itself which leaves the compact continuum* M *invariant. If* M *does not separate the plane, then* T *leaves a point of* M *fixed.*

Proof. Let p be the period of T, and M be the intersection of the monotonic descending sequence $\{C_i\}$ of topological 2-cells. Let $S_i=\bigcup_{j=0}^{p-1} T^j(C_i)$; it is now shown that $\bigcap_{i=1}^{\infty} S_i=M$. It is obvious that $\bigcap_{i=1}^{\infty} S_i\supset M$. To show that $\bigcap_{i=1}^{\infty} S_i\subset M$, assume that there exists a point x in $\bigcap_{i=1}^{\infty} S_i$ which is not in M. There exists an open set U about M such that x is not in U. Since T is continuous, there exists an open set V about M such that $T^k(V)\subset U$, for $k=0, 1, 2, \cdots, p-1$. Since $\bigcap_{i=1}^{\infty} C_i=M$ it follows that there exists a C_j for some j such that $C_j\subset V$. Therefore

x is not in S_j. This is a contradiction and hence $\bigcap_{i=1}^{\infty} S_i = M$.

Let Q_i be defined as the topological 2-cell which contains $\bigcup_{j=0}^{p-1} T^j(C_i)$ and whose boundary $F(Q_i)$ is contained in the set $\bigcup_{j=0}^{p-1} T^j(F(C_i))$. It is now shown that $\bigcap_{i=1}^{\infty} Q_i = M$. It is obvious that $\bigcap_{i=1}^{\infty} Q_i \supset M$. To show that $\bigcap_{i=1}^{\infty} Q_i \subset M$, suppose that there is a point x in $\bigcap_{i=1}^{\infty} Q_i$ such that x is not in M. It follows that x must be in a bounded component of $E^2 - S_i$ for each $i > k$ for some integer k. Let y be in the unbounded component of $E^2 - S_i$ for all i. It follows that S_i separates x from y for each $i > k$, and from a known theorem x is separated from y by $\bigcap_{i=1}^{\infty} S_i = M$, [3]. This is a contradiction and hence it follows that $\bigcap_{i=1}^{\infty} Q_i = M$.

We now show that $T(Q_i) \subset Q_i$. From the definition of Q_i it is known that $T(F(Q_i)) \subset Q_i$. Let x be an interior point of Q_i such that $T(x)$ is in Q_i. Assume that $T(Q_i)$ is not contained in Q_i. Then there is a point y in the interior of Q_i such that $T(y)$ is not in the bounded component of $T(F(Q_i))$. Since Q_i is connected and x and y are interior points of Q_i, there is an arc $\widehat{xy}$ from x to y contained in the interior of Q_i. It follows that $T(\widehat{xy})$ does not intersect $T(F(Q_i))$. Therefore, $T(x)$ and $T(y)$ are not separated by $T(F(Q_i))$. This contradicts the assumption that $T(x)$ was in the bounded component of the complement of $T(F(Q_i))$ and that $T(y)$ was in the unbounded component. It follows now by the Brouwer Fixed Point Theorem that each Q_i contains a point which is fixed under T, and hence M must contain a point which is fixed under T.

THEOREM 3. *If T is an isometry of a compact continuum M of the plane into itself and M does not separate the plane, then T leaves a point of M fixed.*

Proof. Again consider the different types of isometries in the plane. As noted above, any isometry of a subset of the plane into itself may be extended to an isometry of the whole plane into itself. Observe that neither of the types (3) and (5) transforms a compact set into itself and that (1) leaves every point fixed. Therefore, it is necessary to consider only types (2) and (4). First consider type (4), and let L represent the line about which the reflection occurs. If M intersects the line then the theorem is true. If M does not intersect L, then $T(M)$ is separated from M by L. This contradicts either the hypothesis that T transforms M into M or the hypothesis that M is a continuum. Now consider type (2). If T is a rotation about a point in M then the theorem is true. Suppose that T is a rotation about a point p not in M. It is easy to see that if T is periodic at some point of the plane other than p, then T is periodic at every point of the plane and the period would be the same for each point other than p. It would follow from Theorem 2 that M contains a point which is fixed under T. If T is not periodic on any point of the plane other than p, then it follows that $\overline{\bigcup_i T^i(x)}$ is a circle for any point of M and p is the center. This contradicts the fact that p is not in M, and M does not separate the plane.

3. A related problem. We show (Theorem 7) that the answer to the question concerning fixed points in nonseparating plane continua under periodic transformations is contained in the answer to the following question. Let $M \subset E^n$ be

References

1. L. M. Blumenthal, Theory and Applications of Distance Geometry, Oxford University Press, 1953.
2. F. A. Sherk, An Integrated Introduction to Geometry, Oklahoma State University Press, 1958.
3. R. L. Moore, Foundations of Point Set Theory, Amer. Math. Soc. Colloq. Publ., 13 (1932).
4. P. A. Smith, A Theorem on Fixed Points for Periodic Transformations, Ann. Math., 35 (1934) 572–578.
5. M. L. Cartwright and J. E. Littlewood, Some Fixed Point Theorems, Ann. Math., 54 (1951) 1–37.
6. O. H. Hamilton, A Short Proof of the Cartwright-Littlewood Fixed Point Theorem, Canad. J. Math., 6 (1954) 522–524.

ON THE DEFINITION OF A TANGENT-LINE

H. A. THURSTON, University of British Columbia

1. Introduction. The purpose of this note is to point out that some of the definitions of *tangent-line* commonly found in text-books and lecture-notes are invalid (or, at any rate, incomplete) and to indicate how they may be made good. We work in two dimensions: the generalization to three or more dimensions is obvious.

2. A common definition of tangent is as follows: the tangent at the point $(a, F(a))$ to the graph of $y=F(x)$ is the line through that point with slope $F'(a)$. Thus to find, say, the tangent to a parabola at the end of the latus rectum, we set up a system of cartesian coordinates, and apply the definition. If, for instance, we set up a system in which the equation of the parabola is $y=x^2/2a$ and the point in question $(a, \frac{1}{2}a)$, then the tangent will have slope 1, and so we find the tangent to be the line making an angle of $\pi/4$ with the axis of the parabola. The result is satisfactory, but the logic behind it is not: there is no *a priori* guarantee that if we had used different axes of coordinates we should have obtained the same result. Until the line determined by the definition has been proved invariant under change of axes, the definition cannot be accepted as valid. (The same criticism applies to certain careless definitions of area as an integral.) The proof of invariance is, of course, quite easy. We do, however, have to be careful about axes with respect to which the tangent is "vertical" (i.e., parallel to the y-axis).

A more serious defect is that not every plane curve can have an equation of the form $y=F(x)$: a circle cannot, though a semi-circle can. One way to overcome this defect is to define tangents for curves which are *locally* of the form $y=F(x)$.

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A more serious defect is that not every plane curve can have an equation of the form $y=F(x)$: a circle cannot, though a semi-circle can. One way to overcome this defect is to define tangents for curves which are *locally* of the form $y=F(x)$.

3. Another possible approach is to take parametric equations $x=X(t)$, $y=Y(t)$ for the curve and to define the tangent at $(X(c), Y(c))$ to be the line through that point with direction-ratio $X'(c):Y'(c)$, provided that this direction-ratio exists. It is easy to prove that this line is invariant under changes of axes —even vertical tangents now give no trouble—but we must also prove invariance under changes of parametrization.

4. A third approach is to define the tangent purely geometrically.

5. We shall consider all three definitions of tangent, defining only tangents to an *arc*, in order to avoid complications due to multiple points. We shall call the three types of tangent *explicit*, *parametric*, and *geometrical* respectively. It is clear that every explicit tangent is a parametric tangent, and we shall prove (in Section 13) that every parametric tangent is a geometrical tangent. Conversely, it is known that not every geometric tangent is a parametric tangent (A. J. Ward, On Jordan curves possessing a tangent everywhere, Fundamenta Mathematica, 28 (1937) pp. 280–288); and we shall show (in Section 18) that not every parametric tangent is an explicit tangent.

Preliminary details.

6. DEFINITIONS. *An arc is a homeomorph of a closed bounded interval of the real-number line; any homeomorphism which gives it is a parametrization of it.*

7. *Notation.* Throughout this paper the parametrization

$$x = X(t), \qquad y = Y(t), \qquad t \in I$$

will be abbreviated by the symbol $(\mathbf{X})$. $[\mathbf{X}]$ will denote the arc of which $(\mathbf{X})$ is a parametrization. c will denote a number in I, and C the point whose parameter is c. (It will in fact be the point at which we define the tangent.)

8. DEFINITION. *Two sets of points are locally equivalent at C if there is a circle with centre C inside which they coincide.*

The geometrical approach.

9. DEFINITION. *The line L through a point C of an arc is a geometrical tangent to the arc at C if, given any positive number, there is a circle with centre C such that the angle between the line PC and the line L is less than the given number for every point P (other than C) which is both on the arc and in the circle.*

10. Clearly an arc cannot have more than one geometrical tangent at each point.

11. THEOREM. *If the arc $[\mathbf{X}]$ has a geometrical tangent at C with direction-cosines u, v; then*

$$\text{(i)} \qquad \frac{X(t) - X(c)}{TC} \to u \quad \textit{and} \quad \frac{Y(t) - Y(c)}{TC} \to v \quad \textit{as} \quad t \to c.$$

Conversely, if (i) *holds and* $(u, v) \neq (0, 0)$, *then the line through* C *with direction-cosines* u, v *is a geometrical tangent to the arc at* C.

Proof. It is clear that the angle between two lines tends to zero if and only if each direction-cosine of one tends to the corresponding direction-cosine of the other. Moreover, because a homeomorphism is bicontinuous, one point tends to another if and only if the parameter of the first point tends to that of the second.

Note. If C is an end-point, the limits are one-sided. This remark applies also to later results.

The parametric approach.

12. DEFINITION. *If a given arc has a parametrization* (**X**) *and if the ratio* $X'(c):Y'(c)$ *exists, then the line through* C *with this direction-ratio is the tangent at* C *to the arc with respect to the parametrization* (**X**). (*It can easily be proved invariant under change of axes.*) *A tangent with respect to a parametrization is called a parametric tangent.*

13. THEOREM. *Any parametric tangent is also a geometrical tangent.*

Proof. Let L be the tangent at C with respect to a parametrization (**X**). Then $X'(c)$ and $Y'(c)$ exist and are not both zero. Therefore

$$\frac{X(t) - X(c)}{t - c} \to X'(c) \quad \text{as} \quad t \to c,$$

and similarly for Y. Then

$$\frac{X(t) - X(c)}{((X(t) - X(c))^2 + (Y(t) - Y(c))^2)^{1/2}} \to \frac{X'(c)}{(X'(c)^2 + Y'(c)^2)^{1/2}}$$

as $t \to c$, and similarly for Y. Thus the direction-cosines of TC (where T is the point whose parameter is t) tend to those of L.

14. COROLLARY. *An arc cannot have more than one parametric tangent at a given point.*

The explicit approach.

15. DEFINITION. *If an arc is locally equivalent at* C *to the graph of* $y = F(x)$ *in some system of cartesian coordinates, if* C *has coordinates* $(a, F(a))$, *and if* $F'(a)$ *exists; then the line through* C *with slope* $F'(a)$ *is an explicit tangent at* C *to the arc with respect to this system of coordinates.*

16. Any explicit tangent is a parametric tangent, because $y = F(x)$ is simply the parametrization $x = E(t)$, $y = F(t)$, where E is the identity-function. (It follows that an arc cannot have more than one explicit tangent at a given point.)

We shall prove next a partial converse of this fact: provided an arc is (locally) the graph of a function F, a nonvertical parametric tangent is an explicit tangent. Finally, in Section 18, we shall show that the full converse is false.

Thus $(u+v\cdot t)/(v-u\cdot t)$ would be a function of x^*, and so t would be a function of x^*. That is, the function which x^* is of t would have a (local) inverse.

Now $x^*=v\cdot(t^2\cdot\sin t^{-1}+\frac{1}{2}t)-u\cdot(t^3\cdot\sin t^{-1}+\frac{1}{2}t^2)$ if $t\neq 0$, and so

$$\begin{aligned} D_t x^* &= v\cdot(2t\cdot\sin t^{-1} - \cos t^{-1} + \tfrac{1}{2}) \\ &\quad - u\cdot(3t^2\cdot\sin t^{-1} - t\cdot\cos t^{-1} + t) \quad \text{if} \quad t\neq 0. \end{aligned}$$

If $v\neq 0$, then in any neighbourhood of zero the term $v\cdot(-\cos t^{-1}+\frac{1}{2})$ varies from $\frac{3}{2}v$ to $-\frac{1}{2}v$ and so changes sign. In such a neighbourhood we can take t arbitrarily small, and in particular we can take the neighbourhood of t so small that $D_t x^*$ has the sign of $v(-\cos t^{-1}+\frac{1}{2})$ when the latter expression equals $3v/2$ or $-v/2$.

If, however, $v=0$, then we have $D_t x^*=-u\cdot t\cdot(3t\cdot\sin t^{-1}-\cos t^{-1}+1)$, where u is either 1 or -1. Again, $D_t x^*$ changes sign in every neighbourhood of zero.

Hence, the function which x^* is of t is continuous but nonmonotonic, and so has no inverse. It is clear then that the arc is not of the form $y^*=F(x^*)$ in *any* Cartesian system.

A CORRECTION FOR "EXTENSION OF GROUPOIDS WITH OPERATORS"

T. Tamura and D. G. Burnell, University of California, Davis

In the paper by T. Tamura and D. G. Burnell entitled "Extension of groupoids with operators," this Monthly, 71 (1964) 385–391, line 21 on page 389 should read "a homomorphic image of" instead of "isomorphic to." Throughout the paper, the following condition is implicitly assumed: $\alpha x=\beta x$ for all x implies $\alpha=\beta$.

The authors express their thanks to Dr. H. J. Hoehnke for his remark on this paper.

Mathematical Swifties

"F leaves one point invariant", Tom said fixedly.
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R. T. SMYTHE

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146

THE DOMINATORS OF A SEMIGROUP

JACK LATIMER, University of California, Davis

DEFINITION 1. *Let S be a semigroup. An element $x \in S$ will be called a dominator of S if, and only if, $xyx = x$ for all $y \in S$. The set D of all dominators of S will be called the dominator of S.*

The dominator of a semigroup may be empty. We will show that any group of order greater than 1 has no dominators. On the other hand, right-zero semigroups ($xy = y$ for all x, y), left-zero semigroups ($xy = x$ for all x, y), and rectangular semigroups ($xyx = x$ for all x, y) each have the property that every element is a dominator, hence $D = S$.

In Theorem 1 we determine the nature of semigroups having nonempty dominators and the structure of the dominator. In Theorem 2 we show, by means of extension theory, that it is always possible to construct semigroups with a given set of dominators and with a given Rees factor semigroup. We need a few definitions before stating the theorems. These definitions and notations conform to those found in [3].

DEFINITION 2. *A semigroup is called a band if every element of S is idempotent ($x^2 = x$).*

Right-zero, left-zero, and rectangular semigroups are all examples of bands.

DEFINITION 3. *By an ideal of a semigroup S we mean a subset I of S such that $SI \subseteq I$ and $IS \subseteq I$.*

THEOREM 1. *A semigroup S contains a dominator if, and only if, it contains an ideal I which is a rectangular band. Then I is the dominator of S.*

Proof. Let S contain a nonempty dominator D. If $x \in D$, $y \in S$, and if z is any element of S, then

$$(xy)z(xy) = [x(yz)x]y = xy$$

hence $xy \in D$. Similarly $yx \in D$. Hence D is an ideal of S. This, of course, means that D is a subsemigroup of S. Now, since D is the dominator of S, $xyx = x$ for all $y \in S$ and, in particular, for all $y \in D$. Thus D is a rectangular band.

Now, suppose that S contains a nonempty ideal I, which is a rectangular band. We show that I is the dominator of S. Clearly if $x, y \in I$, then $xyx = x$ since I is rectangular. Suppose that $x \in I$ and $y \notin I$, then xy and $(xy)x \in I$ since I is an ideal. It is not immediately clear, however, that $xyx = x$, since $y \notin I$. Since I

is a band, however, and $xyx \in I$, it follows that

$$xyx = (xyx)^2 = x[(yx)(xy)]x = x$$

since yx, xy, and $(yx)(xy) \in I$ (I is an ideal). This proves that S has a dominator since we have shown that every element of I dominates S. Thus if D is the dominator of S we have $I \subseteq D$. Now if $x \in D$ then $xyx = x$ for all $y \in S$. In particular if $y \in I$ then $xyx = x$, but since I is an ideal $x = xyx \in I$. Hence $D \subseteq I$, and it follows that $I = D$.

This completes the proof of Theorem 1.

DEFINITION 4. *An ideal of a semigroup S is called minimal if it does not properly contain any ideal of S.*

A minimal ideal, if it exists, is unique, and it is called the *kernel* of S.

We add that if an ideal of a semigroup is a rectangular band, it is the kernel, since a rectangular band contains no proper ideal.

COROLLARY 1. *A semigroup S has a unique dominator if, and only if, it has a zero element.*

Proof. If S has a unique dominator a, then $\{a\}$ is an ideal of S by Theorem 1, hence a is a zero. The converse is clear since the zero is the kernel of S.

COROLLARY 2. *The only group with a dominator is the group of order* 1.

Proof. A dominating element must be idempotent, and the only idempotent in a group is the identity element. But the identity element cannot dominate unless it is the only element in the group.

DEFINITION 5. *An element x of a semigroup S is called a commutative element of S if it commutes with every element of S.*

COROLLARY 3. *An element x of a semigroup S is a commutative dominator of S if, and only if, it is a zero element of S (hence, if, and only if, it is a unique dominator of S).*

Proof. If x is a zero, then clearly x commutes and dominates. Conversely, if x is a commutative dominator, then for any $y \in S$

$$x = xyx = x^2y = xy$$

and

$$x = xyx = yx^2 = yx$$

since x is commutative and idempotent. Thus $x = xy = yx$, and x is a zero element of S, and by Corollary 1, the only dominator of S.

COROLLARY 4. *If x is a dominator of a semigroup S, the set C_x of all elements of S which commute with x is a subsemigroup with a zero element.*

The proof will be left to the reader.

This lemma is proved in [1].

Now, let $(D, \cdot)$ be any rectangular semigroup, and $(Z, \circ)$ be any semigroup with a zero element 0. Let $Z^0 = Z - \{0\}$, and let $E = D \cup Z^0$. We now define an operation $(*)$ in E as follows: Let a and b be two arbitrary, but fixed, elements of D (possibly $a = b$). Then if $x, y \in E$ we define $x * y$ as follows:

$$x * y = \begin{cases} x \cdot y \text{ if } x, y \in D \\ x \cdot a \text{ if } x \in D, y \in Z^0 \\ b \cdot y \text{ if } x \in Z^0, y \in D \\ x \circ y \text{ if } x, y \in Z^0 \text{ and } x \circ y \neq 0 \text{ in } Z^0 \\ b \cdot a \text{ if } x, y \in Z^0 \text{ and } x \circ y = 0 \text{ in } Z^0. \end{cases} \tag{1}$$

This is an example of an ideal extension E of D by Z based on [2] or [3]. We can prove directly, however, that (1) is associative and that D is the dominator of E without using the general theory in [2] or [3]. This proof will be left to the reader.

Acknowledgments. I wish to thank Dr. Takayuki Tamura of the University of California at Davis for the help and advice he gave to me in the writing of this paper, and I wish to thank the referee who reviewed this paper for his suggested revisions which enhanced the clarity of the paper.

References

1. N. Kimura, The structure of idempotent semigroups, Pacific J. Math., 8 (1958) no. 2.
2. A. H. Clifford, Extensions of semigroups, Trans. Amer. Math. Soc., 68 (1950) 165–173.
3. A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, vol. 1, Math. Surveys no. 7, AMS, 1961.
4. D. Rees, On semigroups, Proc. Camb. Phil. Soc., 36 (1940) pt. 4.

A PROBLEM IN GRAPH THEORY

P. ERDÖS, A. HAJNAL AND J. W. MOON, University College, London and
Math. Inst. of the University of Budapest

A *graph* consists of a finite set of vertices some pairs of which are *adjacent*, i.e., joined by an edge. No edge joins a vertex to itself and at most one edge joins any two vertices. The *degree* of a vertex is the number of vertices adjacent to it. The *complete k-graph* has k vertices and $\binom{k}{2}$ edges.

We shall say that a graph G has property (n, k), where n and k are integers with $2 \leqq k \leqq n$, if G has n vertices and the addition of any new edge increases the number of complete k-graphs contained in G. For example, let $A_k(n)$ denote a graph with n vertices and $n(k-2) - \binom{k-1}{2}$ edges which consist of a complete $(k-2)$-graph each vertex of which is also joined to each of the $n-(k-2)$ remaining vertices. $A_k(n)$ contains no complete k-graphs but it is easily seen that with the addition of any new edge a complete k-graph is formed. Hence, $A_k(n)$ has property (n, k).

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Now, let $(D, \cdot)$ be any rectangular semigroup, and $(Z, \circ)$ be any semigroup with a zero element 0. Let $Z^0 = Z - \{0\}$, and let $E = D \cup Z^0$. We now define an operation $(*)$ in E as follows: Let a and b be two arbitrary, but fixed, elements of D (possibly $a = b$). Then if $x, y \in E$ we define $x * y$ as follows:

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We shall say that a graph G has property (n, k), where n and k are integers with $2 \leqq k \leqq n$, if G has n vertices and the addition of any new edge increases the number of complete k-graphs contained in G. For example, let $A_k(n)$ denote a graph with n vertices and $n(k-2) - \binom{k-1}{2}$ edges which consist of a complete $(k-2)$-graph each vertex of which is also joined to each of the $n-(k-2)$ remaining vertices. $A_k(n)$ contains no complete k-graphs but it is easily seen that with the addition of any new edge a complete k-graph is formed. Hence, $A_k(n)$ has property (n, k).

We wish to determine the "minimal (n, k) graphs," i.e., those graphs with property (n, k) and with the minimal number of edges. We prove the following result.

THEOREM 1. *For every pair of integers n and k, with $2 \leqq k \leqq n$, the only minimal (n, k) graph is $A_k(n)$.*

We will apply Theorem 1 to prove a conjecture of Erdös and Gallai (see [1]). A set of vertices is said to represent the edges of a graph if each edge contains at least one of these vertices. A graph G is said to be *edge p-critical* if the maximal number of vertices necessary to represent all the edges of G is p, but if any edge is omitted the remaining edges can be represented by $p-1$ vertices. For example the complete $(p+1)$-graph is edge p-critical. In [1] it is conjectured that an edge p-critical graph can have at most $\binom{p+1}{2}$ edges. Theorem 1 immediately implies this conjecture. In fact we prove

THEOREM 2. *Every edge p-critical graph has at most $\binom{p+1}{2}$ edges and the only edge p-critical graph with $\binom{p+1}{2}$ edges is the complete $(p+1)$-graph.*

Finally we would like to state a conjecture. A bipartite graph (k, l) is a bipartite graph having k green and l blue vertices. A complete bipartite graph (k, k) is a graph where all green and blue vertices are adjacent. We now say that a bipartite graph (n, m) has property (n, m, k, k) if any new edge increases the number of complete bipartite (k, k) graphs in our graph (we assume $k \leqq n$, $k \leqq m$).

Problem. Is it true that every (n, m) graph with property (n, m, k, k) has at least $(k-1)(n+m-k+1)$ edges?

A weaker conjecture would be that every bipartite graph (n, m) which contains no complete bipartite (k, k) but which loses this property when any new edge is added has at least $(k-1)(n+m-k+1)$ edges.

One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult. One can easily formulate the analogous conjecture for property (n, m, k, l), but we leave this to the reader.

Proof of Theorem 1. We first show that $A_k(n)$ is a minimal (n, k) graph and then we show that it is the only one. We begin by establishing the inequality

$$f_k(n) \geqq f_k(n-1) + (k-2), \qquad \text{for } n = k+1,\ k+2, \cdots, \tag{1}$$

where $f_k(n)$ denotes the number of edges in a minimal (n, k) graph.

Let G be any minimal (n, k) graph where $n \geqq k+1$. There exist nonadjacent vertices in G, say p and q, as the complete n-graph is clearly not a minimal (n, k) graph. Since $G+(p, q)$, the graph obtained from G by adding an edge joining p and q, contains at least one more complete k-graph than G, it must be that p and q are both adjacent to all the vertices of some complete $(k-2)$-graph. Hence, if we let G^* denote the graph obtained from G by removing q and then joining

p by an edge to every vertex which originally was adjacent to q but not to p, it follows that G^* has at least $k-2$ fewer edges than G. We may assert that G^* has property $(n-1, k)$. For if a and b are nonadjacent vertices in G^*, both different from p, then the addition of the edge (a, b) still forms at least one new complete k-graph since none of the complete k-graphs formed by adding (a, b) to G could have contained both p and q and in G^* the vertex p can serve wherever q was required before; in the remaining cases the addition of a new edge to G^* forms the same new complete k-graphs as were formed by the addition of the same edge to G. Since G^* contains at least $f_k(n-1)$ edges, inequality (1) now follows.

It is obvious that $f_k(k)=\binom{k}{2}-1$. This combined with (1) implies that

$$f_k(n) \geqq \binom{k}{2} - 1 + (n-k)(k-2) = n(k-2) - \binom{k-1}{2}, \tag{2}$$

$$\text{for } n = k+1, k+2, \cdots.$$

But $A_k(n)$ is an example of a graph having property (n, k) and with only $n(k-2)-\binom{k-1}{2}$ edges. Therefore, it must be that $A_k(n)$ is a minimal (n, k) graph and that equality holds throughout in (1) and (2).

We now use induction to show that $A_k(n)$ is the only minimal (n, k) graph. For any fixed admissible value of k this is certainly the case when $n=k$. Assume that the assertion is valid whenever $k \leqq n < m$, for some integer m, and consider any minimal (m, k) graph G. From the fact that equality holds in (1) it is not difficult to see that G^*, constructed as before, must be a minimal $(m-1, k)$ graph. Hence, we may suppose that G^* is the same as $A_k(m-1)$.

If in G^* the vertex p, using the same notation as before, is one of the $k-2$ vertices adjacent to every other vertex in G^*, then in G it must be that q is adjacent to all the other $k-3$ such vertices and to one of the remaining vertices. This is so that the addition of the edge (p, q) to G will form at least one new complete k-graph. Each of the other $m-k$ vertices is adjacent to either p or q but not both for otherwise p and q would be mutually adjacent to more than $k-2$ vertices and G would contain more than $f_k(m)$ edges. We may suppose that one such vertex h is not adjacent to p. But it is now easily seen that the addition of the edge (p, h) would not form a new complete k-graph in G, contradicting the definition of G. The only alternative is that p is one of the vertices of degree $k-2$ in G^*. From the definition of G^* it now follows that G differs from G^* only by the presence of the vertex q of degree $k-2$ which is adjacent to the same $k-2$ vertices as in p. This implies that G is the same as $A_k(m)$ which completes the proof of the theorem.

We may restate the above theorem in the following slightly weaker form: Of all graphs with n vertices which contain no complete k-graphs, where $2 \leqq k \leqq n$, but which lose this property when any new edge is added, the graph $A_k(n)$ and only that graph has the minimal number of edges. This statement

References

1. J. Mikusinski, Operational Calculus, Pergamon Press, 1959.
2. D. V. Widder, The inversion of a convolution transform whose kernel is a Laguerre polynomial, this MONTHLY, 70 (1963) 291–293.

RETRACTION OF "N-TH POWERS IN THE FIBONACCI SERIES"

FLOYD BUCHANAN, Buffalo, New York

In the June-July issue of this MONTHLY, in my article, I stated that the rank of the first term divisible by q^k, but by no higher power than k of the prime q is equal to $q^{k-1}j(q)$. This statement would be true, as it has been pointed out to me, if the first term divisible by q is divisible by no higher power than the first in q. The correct statement of Lucas' theorem is: If p is an odd prime, and U_n is the first term divisible by p^λ but not $p^{\lambda+1}$, then U_{pn} is the first term divisible by $p^{\lambda+1}$ but not $p^{\lambda+2}$. My error was in assuming that the theorem implied $j(p^2) \neq j(p)$, which it does not. I have been informed that $j(p^2) \neq j(p)$ for primes less than 10,000 but it is not known whether there are any higher primes for which this may not be true. Although this is an unsolved problem, I feel that my article is not correct and not complete and would like to retract it.

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland 20740.

MODULES OVER COMMUTATIVE RINGS

W. G. LEAVITT, University of Nebraska

The following is another short proof of the fact that for a commutative ring with unit R, any finitely based R-module is "dimensional" in the sense that all of its bases have the same number of elements.

THEOREM. *Let R be a commutative ring with unit. If M is a unitary R-module with a basis of n elements, then all bases of M contain exactly n elements.*

Proof. (The method is that of [1], p. 115.) Let $\{\alpha_i\}$ $(i=1, \cdots, n)$ be a basis for M. It is easy to see that M cannot have an infinite basis. (See [2], p. 241–2. Applied to modules, the method shows that for a module with an infinite basis all bases have the same cardinality.) Thus let $\{\beta_j\}$ $(j=1, \cdots, m)$ be another

basis of M. Write $\alpha_i = \sum_{j=1}^{m} a_{ij}\beta_j$ $(i = 1, \cdots, n)$ and $\beta_j = \sum_{k=1}^{n} b_{jk}\alpha_k$ $(j=1, \cdots, m)$. If $A=[a_{ij}]$ and $B=[b_{ij}]$, it follows from the independence of the α_i's and the β_j's that

$$(1)\ AB = I_n \quad \text{and} \quad (2)\ BA = I_m,$$

where I_n and I_m are unit matrices. Conversely, the existence of relations (1) and (2) in a ring R implies the existence of an R-module with bases of lengths m and n, namely the module of all m-tuples. This module has, of course, the rows of I_m as a basis, but also has as an alternative basis the rows of A. This is clear, since from (2) each row of I_m is a linear combination of the rows of A, while from (1), $XA=0$ implies $XI_n=X=0$, so the rows of A are independent.

Now any homomorphism of R preserves the relations (1) and (2), and so any nonzero homomorphic image of R also admits a module with bases of lengths m and n. But if we apply Zorn's lemma in the usual way (relative to ideals not containing the unit, partially ordered by set inclusion) we obtain a maximal ideal I of R. Since R/I is a field, its modules are vector spaces all of whose bases are of the same length. Thus since R/I is a homomorphic image of R, we must conclude that $m=n$.

References

1. W. G. Leavitt, The module type of a ring, Trans. Am. Math. Soc., 103 (1962) 113–130.
2. N. Jacobson, Lectures in abstract algebra, vol. II, Van Nostrand, New York, 1953.

ON SOLUTIONS OF CERTAIN RICCATI DIFFERENTIAL EQUATIONS

JAMES L. ALLEN, Ball State Teachers College, AND F. MAX STEIN, Colorado State University

1. Rao's transformation. In [1] Rao presented the transformation

$$y = uv - g/h \tag{1}$$

that reduced the Riccati differential equation

$$y' = f + gy + hy^2 \tag{2}$$

to one in which the variables were separable,

$$u' = \left(\frac{W}{h}\right)^{1/2}(1 - Ku + u^2), \tag{3}$$

if

$$\frac{hW' - (3h' - 2gh)W}{2h^{1/2}W^{3/2}} \equiv K, \tag{4}$$

where K is constant and where

$$W = fh^2 + g'h - gh'. \tag{5}$$

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that is defined over $(-\pi/\sqrt{3}, \pi/\sqrt{3})$ and which is $-1/2$ at the origin, we see from (8) that $C=1$ and $\sqrt{fh}=1$. Thus the transformation (6) reduces (10) to

$$u' = 1 + u + u^2. \tag{11}$$

The solution of (11) is $u=(\sqrt{3}/2)\tan(\sqrt{3}x/2+a)-1/2$, and hence the solution of (10) that is $-1/2$ at $x=0$ is

$$y = ([\exp(x^{4/3})]/2)[\sqrt{3}\tan(\sqrt{3}x/2) - 1].$$

Observe that Rao's method does not apply to (10) since W in (5) is not defined at the origin. Even on an interval not containing the origin, the labor involved in this case in obtaining $W^{3/2}$ in (4) is quite formidable.

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Reference

1. P. R. P. Rao, The Riccati differential equation, this MONTHLY, 69 (1962) 995.

ON A PROOF OF HERMITE'S IDENTITY

YOSHIO MATSUOKA, Kagoshima University, Japan

The following identity, due to Hermite, is well known:

$$[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx],$$

where x and n denote any real number and any natural number, respectively, and $[x]$ denotes, as usual, the greatest integer not exceeding x; (see, for example, [1] p. 118 and p. 324).

In this note we give another proof of it. Let

$$f(x) = [nx] - [x] - \left[x + \frac{1}{n}\right] - \cdots - \left[x + \frac{n-1}{n}\right].$$

Then

$$\begin{aligned} f\left(x+\frac{1}{n}\right) &= [nx+1] - \left[x+\frac{1}{n}\right] - \left[x+\frac{2}{n}\right] - \cdots - \left[x+\frac{n-1}{n}\right] - [x+1] \\ &= [nx] - [x] - \left[x+\frac{1}{n}\right] - \cdots - \left[x+\frac{n-1}{n}\right] = f(x). \end{aligned}$$

On the other hand, if $0 \leqq x < 1/n$, we have $f(x)=0$. Therefore $f(x)\equiv 0$, which was to be proved.

Reference

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$$= [nx] - [x] - \left[x+\frac{1}{n}\right] - \cdots - \left[x+\frac{n-1}{n}\right] = f(x).$$

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A PROOF OF THE FUNDAMENTAL THEOREM OF ARITHMETIC

E. A. MAIER, University of Oregon

The standard proofs of the fundamental theorem of arithmetic are based upon results derived from properties of the greatest common divisor of two integers. These properties in turn are obtained from the well-ordering principle and the division algorithm. The following proof depends upon these latter two propositions but does not involve the concept of greatest common divisor.

LEMMA 1. *Let α be rational and let b be the least positive integer such that $b\alpha$ is an integer. If c and $c\alpha$ are integers, then $b \mid c$.*

Proof. By the division algorithm there exist integers q and r such that $c=bq+r$, $0 \leqq r < b$. Then $r\alpha$ is an integer since

$$r\alpha = (c - bq)\alpha = c\alpha - (b\alpha)q.$$

Hence $r=0$ by the definition of b.

LEMMA 2. *Let p be a prime and a an integer such that a/p is not an integer. If b is the least positive integer such that $b(a/p)$ is an integer, then $b=p$.*

Proof. Since $p(a/p)$ is an integer, from Lemma 1 we have $b \mid p$. Hence $b=1$ or $b=p$. But $b \neq 1$ since a/p is not an integer. Hence $b=p$.

THEOREM. (The Fundamental Theorem of Arithmetic.) *Every positive integer greater than 1 is factorable into primes uniquely, apart from the order in which the factors occur.*

Proof. It is readily established that every integer greater than 1 may be factored into a product of primes.

To establish uniqueness, suppose that there exist positive integers which may be factored into primes in more than one way.

Let n be the least such integer, say $n=p_1p_2 \cdots p_r = q_1q_2 \cdots q_s$. Since the factorization of a prime is unique, r and s are greater than 1.

If $p_1 = q_j$ for some j, then $n/p_1 = n/q_j$ is an integer less than n which lacks unique factorization. This contradicts the manner in which n was defined.

If $p_1 \neq q_j$ for all j, then, since p_1 and q_1 are primes, q_1/p_1 is not an integer. Thus, by Lemma 2, the least positive integer b such that $b(q_1/p_1)$ is an integer is p_1. Also

$$q_2q_3 \cdots q_s(q_1/p_1) = n/p_1 = p_2p_3 \cdots p_r$$

is an integer and hence, by Lemma 1, $p_1 \mid q_2q_3 \cdots q_s$. However, p_1 is different from all the q's, and it follows that $q_2q_3 \cdots q_s$ lacks unique factorization. This again contradicts the manner in which n was defined and the proof is complete.

The following is an immediate consequence of the theorem.

COROLLARY. *If a prime p divides the product ac of two integers, then $p \mid a$ or $p \mid c$.*

Alternatively, one may establish the corollary directly from Lemmas 1 and 2 and then proceed with the proof of the fundamental theorem in the usual fashion. To establish the corollary directly, suppose that $p|ac$ but $p\nmid a$. Then $c(a/p)$ is an integer, but a/p is not. By Lemma 2, p is the least positive integer b such that $b(a/p)$ is an integer and hence, by Lemma 1, $p|c$.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

PREGRADUATE TRAINING IN MATHEMATICS—A REPORT OF A CUPM PANEL

A. B. WILLCOX, Executive Director of CUPM

1. The mathematical theory of curriculum construction.[1] Every mathematician and increasingly many college sophomores know of the intimate connections between differential equations $y'=f(x, y)$ and direction fields. In this section we introduce an application of a differential equation (of sorts), describing, first, the space in which the associated direction field and integral curves are embedded. To our knowledge, this particular application of the notion of a differential equation cannot be found in print at present. We are confident, moreover, that it will not be found in print in the future, outside of this article, the entire purpose of the theory having been accomplished when the reader reaches section 2.

Let us denote by A a "space"[2] each of whose points represents a *mathematical activity appropriate for a college student.* For example, one point might denote "studying the calculus integrated with material from linear algebra"; others, "studying the calculus with some differential equations but with no use of concepts from linear algebra," or, "studying linear algebra and calculus as separate parallel courses," or, "beginning homological algebra, but requiring remedial work in long division."

Using the time-honored device of "thinking away" complexity by the use of simple diagrams, we picture A as shown in Fig. 1.

Denoting the real number system by R, we picture $A\times R$ as follows. R is treated as a "time axis," following common usage.

$A\times R$ will be called a (mathematics) *curriculum space* and a function F defined on the interval $[0, 4]$ in R and having values in A will be called a *pregradu-*

THE RUTGERS PROGRAM FOR RETRAINING IN MATHEMATICS OF COLLEGE GRADUATE WOMEN

HELEN M. MARSTON, Rutgers—The State University

This program was started at Rutgers three and one half years ago, primarily because of the shortage of mathematics teachers. It was sponsored by the Ford Foundation in the belief that college graduate women who have raised families would be both able and eager to retrain for some of the many jobs requiring mathematical ability and up-to-date knowledge. It was preceded by a year-long survey in which school principals and teachers, college teachers, industrial personnel managers, and more than 21,000 college graduate women in the area of northern New Jersey were questioned as to their interests and needs.

Now, at the end of the first three and one half years, there are 188 women and 2 men who have successfully completed one or more semesters in the program. To some of these, retraining has meant a single course, to some it has meant several courses, and to 30 it has meant enrolling (either subsequently or concurrently) in graduate school. Some women enroll with teaching as a very definite goal, some wish to do anything but teach, and some have no idea what they would like to do. Some study in the hope of an immediate job, and others study so they will be prepared to work when their children are older. Mathematical backgrounds vary all the way from one college course to an M.A., and from as little as five years ago to thirty years ago.

The 6 courses which now constitute the program have been designed, insofar as possible, to help all of these women. Statistics and/or Computer Programming are the courses taken by most who plan to get jobs in industry, and a professionalized subject-matter course called "Background for Teaching the New Mathematics" is taken by most who plan to teach. "Background for Teaching Mathematics in the Elementary School" will be offered for the first time in 1964–65. The noncredit course called "Review, College Freshman and Sophomore Mathematics" has been the most popular, and this is frequently followed by the course in Calculus. All courses are offered at both the Newark and New Brunswick campuses, scheduled at the times most convenient for women with families. All courses are staffed with top-notch professors. (The bright and busy 40-year-old is hardly a captive audience!) Scholarships have been given where needed.

In spite of the wide range in ages (25–68) and the diversity in backgrounds and goals, these women have a sort of self-selective homogeneity: they are well above average in intelligence, they are highly motivated, and they share a common courage and desire to do something constructive with their talents. Among those who consider their retraining complete, 60 are already holding jobs as a fairly direct result of their courses and the counseling and placement help which they received in the program. Thirty-five are teaching in junior or senior high schools, 3 are teaching in college, and most of the other 22 have jobs with industrial research. Nine are already on second jobs since completing their retraining, and others have received promotions and raises.

Without exception, these women seem to be successful and happy in their work, and skillful in their management of home and family adjustments. One mother of 5, now doing full-time teaching, ends her letter to us with, "Now that you've introduced me to studying again, I do want to continue. Say thank you to 'Mr. Ford' when you see him. I'm delighted with the change in my life." And from the letter of one school superintendent: "It is undeniable that the regular teacher training programs have been unable to adequately supply the number of teachers of mathematics needed to fill all positions in public and private schools with completely qualified personnel. . . . I regard your training program as an invaluable service in a field for which there is currently a critical shortage of teachers."

"The mature woman" is becoming more popular with employers, and there seems to be a continuing demand for women with recent courses in mathematics. Since the shortage of teachers in science is also critical, it is our hope that we may be able to expand this program to "The Rutgers Program for Retraining in Mathematics and Science."

SCHOOL AND COLLEGE ENROLLMENTS 1973

Statisticians at the U. S. Office of Education, Department of Health, Education, and Welfare, have taken a look into the future and foresee ten years hence:

Fifty-four million students enrolled in public and private elementary and secondary schools in the fall of 1973. This is 7.1 million more than in 1963.

Eight million students seeking degrees in colleges and universities, nearly double the 4.5 million enrolled in 1963.

About 2.2 million teachers in public and private elementary and secondary schools, an increase of 375,000.

More than three million high school graduates, an increase of approximately 800,000, and

Students getting 788,000 bachelor degrees—300,000 more than in 1963.

The figures are highlights of projections of the major items of educational statistics being prepared for publication later this year by the Office of Education. Statisticians emphasized that all projections are based on the assumption that the 1954–55 to 1963–64 trend will continue through 1973–74.

The predicted total of 62 million students in schools and colleges in 1973 indicates a nearly 80 per cent increase in enrollments since 1953.

Projections indicate that in 1973 the number of high school students will have more than doubled, and the number of degree-seeking college students will have more than tripled the 1953–54 totals. Ten years ago there were 6.8 million in high schools and 2.2 million in colleges. Ten years from now it is expected that 16 million youths will be in high schools and eight million in degree courses at colleges.

News Release from Office of Education
U. S. Dept. of Health, Education, and Welfare

1st row: 1, 2, $\cdots$, 10 of clubs
2nd row: 10, 1, 2, $\cdots$, 9 of diamonds
3rd row: 9, 10, 1, $\cdots$, 8 of hearts
4th row: 8, 9, 10, 1, $\cdots$, 7 of spades.

Pick them up, face down, from right to left, a column at a time, always starting with the first row. The bottom card will be the ten of clubs, and the top, the eight of spades. Now distribute them, face down, in four equal rows from left to right. Find a formula which identifies the card in each of the forty positions.

E 1743. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove that $(\sinh x-x)\left\{\operatorname{arc}\sec(2+\operatorname{sech} x)-\pi/3\right\}$ is a steadily increasing function of x.

E 1744. *Proposed by Yasser Dakkah, S. S. Boys' School, Qalgilya, Jordan*

In the plane of a given triangle, locate a point whose distances from the three vertices have the smallest possible sum of squares.

E 1745. *Proposed by E. S. Langford, North American Aviation, Anaheim, California*

(1) Consider a wineglass in the shape of half a prolate spheroid. A marble is introduced into the glass. What is the radius of the smallest marble which will not touch the bottom?

(2) The similar problem for a paraboloid of revolution.

E 1746. *Proposed by Michael Fried, University of Michigan*

Given positive integers k and n, find all integers x such that $n|x$, $n+1|x+1$, $\cdots$, $n+k|x+k$.

E 1747. *Proposed by Joseph Arkin, Spring Valley, N. Y.*

If $\sigma(n)$ is the sum of the divisors of n, show that $\sigma(6q+5)\equiv 0 \pmod 6$ for all positive q. Is this an instance of a more general rule?

SOLUTIONS OF ELEMENTARY PROBLEMS

A Condition for a Semigroup to be an Abelian Group

E 1629 [1963, 891; 1964, 687]. *Proposed by F. M. Sioson, University of Hawaii*

Show that any associative system S satisfying the identity $x^2y=y=yx^2$ is a commutative group.

Comment by D. F. Dawson, North Texas State University. Solution II as given is incorrect and, in particular, the statement regarding a weakened hypothesis does not hold. Consider the following counterexample:

S:

	a	b
a	a	a
b	b	b

.

Clearly S is associative since each element of S is a right identity. Also $ab^2=a$, $ba^2=b$, $aa^2=a$, $bb^2=b$. But S is not a group nor is it commutative.

The Additive Identity in a Simple Algebraic System

E1656 [1964, 90]. *Proposed by G. A. Heuer and D. B. Erickson, Concordia College.*

Let $(R;\ +,\ \cdot)$ be a system such that $(R;\ +)$ is a cancellation semigroup, $(R;\ \cdot)$ is a semigroup, and "$\cdot$" is right and left distributive over "$+$". Let $z\in R$ be such that $zx=xz=z$ for all $x\in R$. Is z an additive identity?

Solution by Jack Nebb, University of North Carolina. First, $z+z=zz+zz=z(z+z)=z$. If $x\in R$, $x+z=x+(z+z)=(x+z)+z$. Hence, $x=x+z$ by cancellation. Similarly, $x=z+x$, and z is indeed an additive identity.

Remark. The following hypotheses were not needed: "$\cdot$" is right distributive over "$+$", "$\cdot$" is associative, and z is a right identity for "$\cdot$".

Also solved by Shair Ahmad, R. G. Albert, D. J. Allen, Charles Atherton, E. R. Barnes, W. E. Bodden, G. A. Bogar, Joel Brawley, Jr., L. P. Bush, Jim Campbell, D. I. A. Cohen, David Cohoon, I. E. DeNoya, J. W. Ellis, Ann Endsley, Francis Florey, Michael Gemignani, W. E. Gould, H. S. Hahn, D. J. Hansen, C. V. Heuer, K. S. Hirschel, Stephen Hoffman, J. E. Humphreys, S. F. Kapoor, Max Klicker, E. S. Langford, J. R. C. Leitzel, C. C. Linder, Nicholas Macri, D. C. B. Marsh, J. R. Merriman, T. M. O'Leary, Stanton Philipp, J. R. Porter, David Reel, Dorothy S. Rutledge, P. A. Scheinok, David Sookne, E. L. Spitznagel, Jr., and K. P. Yanosko (jointly), C. F. Stephens, W. C. Waterhouse, W. L. Werner, H. E. Wickes, K. L. Yocom, and the proposers.

As in the solution above, most solvers showed that $z+z=z$. This can of course be proved under considerably less restrictive hypotheses: All that is required of $(R;\ \cdot)$ and z is that (i) $z, zz, z(z+z)\in R$, (ii) $zz=z$, and (iii) $z(z+z)=zz+zz$.

Subgroup A for which $A\cup\{x, x^{-1}\}$ is a Subgroup

E1657 [1964, 91]. *Proposed by Michael Gemignani, University of Notre Dame.*

Let G be any group and A a subgroup of G. Let $x\in G$, $x\notin A$. We say x *augments* A if $A_x=A\cup(x, x^{-1})$ is also a subgroup of G. Suppose x augments A. Show that A_x is cyclic of order 2, 3, or 4.

Solution by Richard Sinkhorn, University of Houston. Denote the identity in G by i. Pick any $y\in A$. If A_x is a subgroup, $xy\in A_x$. But $xy\notin A$, for otherwise, since $y^{-1}\in A$, we would have $xy\cdot y^{-1}=x\in A$. Thus either $xy=x$ or $xy=x^{-1}$, i.e., either $y=i$ or $y=x^{-2}$. Thus A_x contains no more than the elements i, x, x^{-1}, x^{-2} (which need not be distinct!).

Since $i \in A$ and $x \notin A$, $x \neq i$ and thus $x^{-1} \neq i$. Suppose $x^{-2} \neq i$. Then since A_x as a subgroup is closed under multiplication, either $x^{-1} \cdot x^{-2} = i$ or $x^{-1} \cdot x^{-2} = x$. In the former case $x^{-2} = x$ and $x^{-1} = x^2$, and A_x is the cyclic group $\{i, x, x^2\}$. In the latter case $x^{-2} = x^2$ and $x^{-1} = x^3$, and A_x is the cyclic group $\{i, x, x^2, x^3\}$.

The remaining alternative $x^{-2} = i$ gives $x^{-1} = x$. In this case A_x is the cyclic group $\{i, x\}$.

Also solved by R. G. Albert, D. J. Allen, C. R. Atherton, E. R. Barnes, W. E. Bodden, W. H. Bonney, F. L. Bookstein, Joel Browley, Jr., Robert Burton, J. N. S. Cardoso, D. I. A. Cohen, David Cohoon, Henry Davis and Victor Keiser (jointly), F. J. Dickey, W. E. Gould, L. A. Guillou, H. S. Hahn, D. J. Hansen, C. V. Heuer, Stephen Hoffman, J. E. Humphreys, R. A. Jacobson, Colonel Johnson, Jr., S. F. Kapoor, Robert Kopp, E. S. Langford, Richard Laver, C. C. Linder, Robert Maas, J. W. Mades, D. C. B. Marsh, J. R. Merriman, P. N. Muller, M. G. Murdeshwar, Stanton Philipp, D. J. Samuelson, P. A. Scheinok, David Sookne, E. L. Spitznagel, Jr., and K. P. Yanosko (jointly), Rory Thompson, A. M. Vaidya, Simon Vatriquant, W. C. Waterhouse, K. L. Yocom, and the proposer.

Random Polygons Inscribed in a Circle

E1658 [1964, 91]. *Proposed by D. L. Silverman, Beverly Hills, California.*

Points are selected at random on the circumference of a circle until they form the vertices of an inscribed polygon which encloses the center of the circle. Prove that the "expected polygon" is a pentagon.

I. *Solution by Robert Burton, Service Bureau Corporation, New York City.* Let M_n be a random variable equal to the angular measure of a minimal covering arc for n randomly selected points. Then the convex polygon determined by the points encloses the center if and only if $M_n > \pi$.

Since the probability that a given point is an end-point of a minimal covering arc is $2/n$, and since, given a fixed end-point, the conditional probability that the remaining $n-1$ points all lie on the same semi-circular arc is $2 \cdot (1/2)^{n-1}$, it follows that $p_n = P\{M_n < \pi\} = 2 \cdot (1/2)^{n-1}/(2/n) = n/2^{n-1}$, for $n \geqq 1$. Then $U_n = p_{n-1} - p_n = (n-2)/2^{n-1} = P\{X = n\}$, where X is a random variable equal to the number of points required, and $E(X-1) = \sum_{n=2}^{\infty}(n-1)(n-2)/2^{n-1} = 4$, so that $E(X) = 5$.

II. *Solution by F. G. Schmitt, Jr., Ann Arbor, Michigan.* One may as well solve Problem E1658 in n dimensions: Let $x_1, x_2, \cdots$ be a sequence of points scattered at random on an $(n-1)$-sphere in E^n. The random variable N_n is defined to be the smallest number m such that the convex hull of the points $x_1, \cdots, x_m$ contains the center of the sphere. Find the expected value $E(N_n)$.

Solution. $N_n > k$ iff the points $x_1, \cdots, x_k$ are contained in some hemisphere. J. G. Wendel [*A Problem in Geometric Probability*, Mathematica Scandinavica 2 (1962), 109–111] has observed that this event has the same probability as that of obtaining the nth head on or after the kth toss of an honest coin. Thus if M_n denotes the total number of tails before the nth head, then $\Pr(N_n > k) = \Pr(M_n + n \geqq k) = \Pr(M_n + n + 1 > k)$. Hence M_n and $N_n - n - 1$ have the same negative binomial distribution with $p = 1/2$, for which the mean is n. Thus

$E(N_n)=E(M_n+n+1)=2n+1$. In particular $E(N_2)=5$.

Remark. $\mathrm{Var}(N_n)=\mathrm{Var}(M_n)=2n$. Hence $\mathrm{Var}(N_2)=4$.

Also solved by Jack Abad, R. G. Albert, Maxey Brooke, Allan Church, D. I. A. Cohen, Michael Goldberg, H. S. Hahn, R. F. Jackson, E. S. Langford, Winton Laubach, D. C. B. Marsh, K. L. Yocom, and the proposer.

Not all these solvers interpreted "expected polygon" as in the solutions given above: Writing P_n for the probability that n but not $n-1$ points selected at random on the circumference, form a polygon "surrounding" the center of the circle, most of these solvers maximized P_n. Depending upon the probabilistic assumptions made, some found the "expected polygon" to be a quadrilateral, some a hexagon.

A Characterization of the Parabola

E1659 [1964, 91]. *Proposed by Jose Gallego-Diaz, Universidad del Zulia, Maracaibo, Venezuela.*

A parabola has the property that the circumcircle of the triangle formed by three tangents to the curve passes through a fixed point (the focus). Does this property characterize the parabola?

I. *Solution by Michael Goldberg, Washington, D. C.* Take as origin O, the intersection of two fixed tangents of slopes m and n. Let F be the fixed point. Then each circle through O and F intersects the two fixed tangent lines in two points. The line through these two points is to be a new tangent to the sought curve. The envelope of the family of such lines is the sought curve. This curve is a parabola as shown below.

The circle whose diameter is OF cuts the two fixed tangents in points whose join is taken as the tangent at the vertex of the parabola. The axis of the parabola is taken as the normal to this line through F. Then, each new tangent to this parabola forms the third side of a triangle whose other two sides are the original two tangents. The circumscribing circles of these triangles pass through F. The tangents, therefore, correspond to the family of lines as constructed in the foregoing paragraph.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* Assuming only differentiable functions are considered, the answer is "Yes." Consider two non-perpendicular tangents to a curve and establish coordinate axes so that these tangents have equations $y=0$, $y=mx$. All other tangents not through their intersection nor parallel to either will be of the form $Ax+By+1=0$. The vertices of the triangle formed are $(0, 0)$, $(-1/A, 0)$, $(-1/(A+mB), -m/(A+mB))$. Assuming that the circumcircle of such triangles passes through a fixed point, (x_0, y_0), we use the 4×4 determinantal form of the circle

$$\left| x^2+y^2\ x\ y\ 1 \right| = 0$$

to obtain $(x_0^2+y_0^2)(A^2+mAB)+(y_0-x_0)(A+mB)+my_0A=0$ which can be put in the form $B=(pA^2+qA)/(r-mpA)$ (p, q, r constants depending on x_0, y_0, m). For $B=f(A)$, we have the curve determined as the envelope of these tangents,

in parametric form $x=-B'/(AB'-B)$, $y=1/(AB'-B)$. Routine, but tedious, elimination of the parameter, A, yields:

$$qr(rx+qy-mp)^2-4pr^2(rx+qy-mp)+4pr^2(mq+r)x-4mp^2r^2=0$$

which is, indeed, the equation of a parabola.

Also solved by D. I. A. Cohen and the proposer.

An Incomplete Partially Ordered Set

E1660 [1964, 91]. *Proposed by Seymour Kass, Illinois Institute of Technology.*

Give an example of a strongly partially ordered set which has the property that every pair of unrelated elements has a sup and an inf while every pair of related elements has neither.

I. *Solution by Richmond G. Albert, West Newton, Massachusetts.* Let S consist of all finite open intervals of real numbers each containing zero and the relation R be that of proper containment (as sets of numbers).

II. *Solution by D. L. Silverman, Beverly Hills, California.* An easily verified example is the set [0, 1] with the order relation $\prec$ defined as follows: $a\prec b$ if and only if $a<b$ and either
(1) $a=0$
(2) $b=1$
(3) a and b are both rational or
(4) a and b are both irrational.

Also solved by David Cahoon, S. P. Franklin, W. E. Gould, E. S. Langford, D. C. B. Marsh, David Sookne, and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before June 30, 1965.

5245. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College*

Let f be a real valued function defined on [0, 1]. If the set of zeros of f is uncountable and nowhere dense, can f be continuous?

5246. *Proposed by Solomon Marcus, University of Bucharest, Rumania*

Let f be continuous in $[a, b]$. For each $\epsilon>0$, let $\phi(\epsilon)$ be the greatest number η such that $|x'-x''|\leqq\eta$ implies $|f(x')-f(x'')|\leqq\epsilon$. Does there exist such an f that ϕ is continuous and nondifferentiable in some point $\epsilon_0\in(0,+\infty)$?

in parametric form $x=-B'/(AB'-B)$, $y=1/(AB'-B)$. Routine, but tedious, elimination of the parameter, A, yields:

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SOLUTIONS OF ADVANCED PROBLEMS

Distributive Binary Operations Modulo *n*

5150 [1963, 1015]. *Proposed by J. R. Clay, University of Washington.*

Let $Z_n = \{0, 1, 2, \cdots, n-1\}$. There are n^{n^2} distinct closed binary operations that can be defined on Z_n. One of these is $+$, addition modulo n. How many of the remaining are left distributive over $+$?

Solution by J. L. Pietenpol, Columbia University. If the operation $*$ is left distributive, then $a * b = a * (1 + \cdots + 1) = b \cdot (a * 1)$, where $\cdot$ is ordinary multiplication modulo n. The operation is thus determined by the n quantities $a * 1$, which can be chosen independently. The number of such operations is therefore n^n.

Also solved by R. G. Albert, R. L. Farrell, G. A. Heuer, Hewitt Kenyon, R. B. Killgrove, F. D. Parker, Necdet Ücoluk, W. C. Waterhouse, J. Ernest Wilkins, Jr., K. L. Yocom, and the proposer.

Editorial Note. It was observed by several solvers that only n operations are both left and right distributive.

Decomposition Space of a Solid Torus

5152 [1963, 1106]. *Proposed by J. D. Sondow, Princeton University*

Prove that the decomposition space whose points are the interior points of the solid torus in E_3 and the longitudinal circles of the boundary is topologically the 3-sphere. Intuitively this amounts to identifying all medians with one of them.

Solution by L. R. King, University of Virginia. A meridian cross section (two discs) of the solid torus becomes a 2-sphere S in the decomposition space X that separates X into two disjoint open 3-cells C_1 and C_2. Since $C_1 \cup S$ is a closed 3-cell, $i = 1, 2$, X is homeomorphic to the 3-sphere.

Also solved by Patrick Shanahan and the proposer.

Groups with Composition Series (p, p, q)

5153 [1963, 1107]. *Proposed by Seymour Kass, Illinois Institute of Technology.*

Let G be a group of order p^2q, where p and q are distinct primes. Prove that if G has a composition series with sequence of indices (p, p, q), then G is abelian.

I. *Solution by J. J. Zeltmacher, Jr., University of Illinois.* The assertion is false. A simple counterexample is given by $G = S_3 \times \sigma(2) \supset S_3 \supset A_3 \supset \{1\}$, where $\sigma(2)$ is the cyclic group of order 2. G is not abelian and has a composition series with sequence of indices (2, 2, 3).

II. *Solution by D. T. Sigley, Downey, California.* If $p < q$ there are many counterexamples to the stated proposition. However, if the order g of G is such

that $g=p^2q$ with $p>q$, then the statement is true by the following argument. The invariant subgroup H of index p under G is of order pq.

If H is Abelian, then its group of isomorphisms I is of order $(p-1)(q-1)$ and, hence, does not contain an element of order p. Any element a of G that is adjoined to H to produce G must transform each element of H into itself, for the conjugation by a on H would have its pth power 1 and hence must be the identity map. Hence the group G so formed is Abelian.

On the other hand, consider H to be non-Abelian. There is only one such group which exists if and only if $p\equiv 1 \pmod{q}$. It has no invariant subgroup of index p (order q). See, e.g., Marshall Hall, *Theory of Groups*, p. 49. In this case G would have factors of composition (p, q, p) but not (p, p, q). This completes the proof when $p>q$.

Also solved by Anders Bager, J. H. Biggs, C. V. Heuer, William Scott, Donna J. Seaman, and Hermann Simon.

Contraction Map by Stochastic Matrix

5154 [1963, 1107]. *Proposed by Allen S. Davis, University of Oklahoma*

Show that, with respect to a suitable metric, a square stochastic matrix M with positive entries defines a contraction map of the space X of probability vectors. (Hence, as is well known, $xM=x$ has a unique solution in X.)

Solution by the proposer. We assume that the rows of M are not identical. Let ξ be the n-dimensional column vector whose components are all 1. Define $|(x_1, \cdots, x_n)|=(|x_1|, \cdots, |x_n|)$. For all $p, q\in X$, let $\delta(p, q)=|p-q|\xi$. Then (X, δ) is a complete metric space with the usual topology. Let θ be the row vector whose components are the column minima of M, and note that $0<\theta\xi<1$. Define

$$Q=\frac{1}{1-\theta\xi}(M-(\xi\theta)').$$

Then $M=(\xi\theta)'+(1-\theta\xi)Q$ and Q is stochastic. For $p, q\in X$, we have $\delta(pM, qM)$ $=|(p-q)(\xi\theta)'+(1-\theta\xi)(p-q)Q|\xi\leqq(1-\theta\xi)\delta(p, q)$, since $(p-q)(\xi\theta)'=0$, $|xQ|$ $\leqq|x|Q$, and $Q\xi=\xi$. Since $\theta\xi>0$, we have found an $\alpha<1$ such that $\delta(pM, qM)$ $\leqq\alpha\delta(p, q)$, for all $p, q\in X$. The final fact that $xM=x$ has a solution in X now follows from the fixed point theorem. (See, e.g., Kolmogorof and Fomin, *Functional Analysis*, I, pp. 43–46.)

Area-Perimeter Relationships and Lattice Points

5155 [1963, 1107]. *Proposed by Joseph Hammer, University of Sydney, Australia*

Given a convex domain D, its area $A(D)$ and its half perimeter, $S(D)$. Prove that if $A(D)>rS(D)$, where r is any positive integer, then D contains r lattice points. (See Bender, *Area-Perimeter relations for two-dimensional lattices*, this MONTHLY, 69 (1962) 742–744, where the case $r=1$ is proved.)

Solution by Robert Bowen, Fairfield, California. Suppose D were a counterexample with respect to lattice L. Since at most $r-1$ points of L lie inside D, it is possible to choose $(a, 0) \in L$ such that $x \equiv a \pmod r$ implies that (x, y) is not inside D. Let L' be the lattice consisting of the points $\{u, v\} = (ur+a, vr)$. Then no points of L' lie inside D and, because of the increase in the unit of length, $A'(D) = (1/r^2)A(D) > (1/r)S(D) = S'(D)$. This contradicts Bender's theorem.

See also the proposer's paper, *On a general area-perimeter relation for two dimension lattices*, this MONTHLY 71 (1964) 534–5.

Series of Trigonometric Functions

5156 [1963, 1107]. *Proposed by Don Kirkham, Iowa State University*

For $-\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi$, prove that

$$\theta \cot \theta + \tfrac{1}{3} \cot^3 \theta(\tan \theta - \theta) - \tfrac{1}{5} \cot^5 \theta(-\tfrac{1}{3} \tan^3 \theta + \tan \theta - \theta)$$
$$+ \tfrac{1}{7} \cot^7 \theta(\tfrac{1}{5} \tan^5 \theta - \tfrac{1}{3} \tan^3 \theta + \tan \theta - \theta) - \cdots = \tfrac{1}{8}(\pi^2 - 4\theta^2).$$

Solution by P. J. de Doelder, Technological University, Eindhoven, Netherlands. We take $x = \tan \theta$, and there follows:

$$\begin{aligned}
\phi(x) &= \frac{\arctan x}{x} + \frac{1}{3x^3}(x - \arctan x) - \cdots \\
&= \left(1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \cdots\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{5}x^2 + \frac{1}{7}x^4 - \cdots\right) + \cdots \\
&= \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) - \left(1\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{5} + \cdots\right)x^2 \\
&\quad + \left(1\cdot\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{7} + \frac{1}{5}\cdot\frac{1}{9} + \cdots\right)x^4 + \cdots \\
&= \frac{\pi^2}{8} - \frac{1}{2}x^2 + \frac{1}{4}\left(1 + \frac{1}{3}\right)x^4 - \frac{1}{6}\left(1 + \frac{1}{3} + \frac{1}{5}\right)x^6 + \cdots.
\end{aligned}$$

We define $f(x) = \frac{1}{2}x^2 - \frac{1}{4}(1+\frac{1}{3})x^4 + \cdots$ and by differentiating we have

$$f'(x) = x - \left(1 + \frac{1}{3}\right)x^3 + \left(1 + \frac{1}{3} + \frac{1}{5}\right)x^5 - \cdots = \frac{\arctan x}{1 + x^2},$$

and from this, $f(x) = \frac{1}{2}(\arctan x)^2$. Therefore $\phi(x) = \pi^2/8 - \frac{1}{2}(\arctan x)^2 = \pi^2/8 - \frac{1}{2}\theta^2$.

Also solved by L. Carlitz, Stephen Fisk, P. R. Rider, and Hermann Simon.

Editorial Note. In the above proof we need the fact that $|x| \leqq 1$ in order to justify the rearrangement of terms; this restricts the proof to the case $0 < |\theta| \leqq \pi/4$. None of the proofs submitted consider the convergence situation if $|\theta| > \pi/4$, so that the identity is not yet proved for such values.

The proposer, however, provides numerical verification of the result when $\theta=7\pi/6$. This interesting identity arose in theoretical work in soil physics.

Rational Bases for Rational Numbers

5157 [1963, 1107; 1964, 325]. *Proposed by R. L. Graham, Bell Telephone Laboratories*

Suppose that $S=(s_1, s_2, \cdots)$ is a monotone sequence of positive rational numbers which has the property that every sufficiently large rational number is of the form $\sum_{k=1}^{\infty} \epsilon_k s_k$, where ϵ_k is 0 or 1 and all but a finite number of the ϵ_k are 0. Prove (or disprove) that all positive rationals are of the form $\sum_{k=1}^{\infty} \epsilon_k s_k$.

Solution by John R. Isbell, Institute for Advanced Study. This is not true. Enumerate all rationals greater than 1 in a list (u_i). Construct finite blocks $B_{i+1}=(s_{n_i+1}, \cdots, s_{n_{i+1}})$ (for $i\geqq 0$) of positive rationals, with the sum of B_i equal to u_i but no sum $\sum \epsilon_k s_k$ equal to 1, as follows. To start, write $n_0=0$ and $s_0=1$. Then s_1, in the block B_1, and all following s_{j+1} in B_{i+1} are chosen as follows. Finitely many numbers t_m are forbidden values for s_{j+1} because they, with preceding s_k's, could represent 1. We forbid also all values which would make $s_{n_i+1}+\cdots+s_{j+1}$ one of the numbers $u_{i+1}-t_m$, another finite set. Also $s_{j+1}<s_j$. We require $s_{j+1}>\frac{1}{2}s_{n_i}$ unless this requirement would make the sum of B_{i+1} already greater than u_{i+1}; in the latter case, s_{j+1} is to be exactly that number which makes the sum u_{i+1}, and n_{i+1} is defined to be $j+1$, ending the block. In the contrary case s_{j+1} must be less than $u_{i+1}-s_{n_i+1}-\cdots-s_j$. Beyond that, s_{j+1} is chosen freely.

The choice of s_{j+1} is always possible; it is either determined or free in an interval of rational values with finitely many values forbidden. It is not possible to represent 1 as $s_{k_1}+\cdots+s_{k_p}$ (all k's>0). This is clear if the last index k_p is not a block-end n_i. If $k_p=n_i$, suppose $k_{p-1}<n_i-1$; then s_{n_i-1} is a forbidden number, since s_{n_i} is a t_m and the sum of B_i is u_i. In case k_{p-1} is n_i-1, consider the largest r such that $k_{p-r}=n_i-r$. This index n_i-r is greater than $n_{i-1}+1$, because the whole block B_i has sum $u_i>1$ and cannot all appear in the supposed representation of 1. But the sum $s_{k_{p-r}}+\cdots+s_{k_p}$ is a t_m, and s_{n_i-r-1} is a forbidden number. We have contradictions in both cases; 1 is not representable. Finally, every u_i is representable, i.e. every block is completed, because all of its terms except the last are larger than $\frac{1}{2}s_{n_{i-1}}$.

(If arbitrary irrational values s_k are permitted as in the original statement of the problem, one can even make the set of representable positive rationals $\sum \epsilon_k s_k$ be any desired set. The question, exactly what sets can be represented with rational s_k, is hard but perhaps not impossible.)

Also disproved, in the original form, by Paul Erdös.

Product of All Elements in a Finite Abelian Group

5158 [1963, 1107]. *Proposed by S. D. Chatterji, University of New South Wales, Australia*

Given a finite commutative group $A = \{a_1, \cdots, a_n\}$. What is the product $a_1a_2 \cdots a_n$ equal to?

Solution by Richard Laatsch, Miami University, Oxford, Ohio. If A contains a unique element x of order 2, the product is x. Otherwise the product is the identity e. For, if $y \in A$ and $y \neq y^{-1}$, then the pair y, y^{-1} can be removed from consideration. (In particular, the answer is e for every group of odd order.) This leaves the subgroup $H = \{x \in A : x^2 = e\}$. H has order 2^m for some integer m. To show this, if $e, a, b \in H$, then e, a, b, ab are distinct; if c is another element of H, then c, ca, cb, cab are all new and distinct elements; if d is another element of H, the products with d give again as many new and distinct elements; etc. To show distinctness at any stage we have, for example, that if $acd = ab$, then $d = a^2bc = bc$, contrary to the choice of d. Also the product of all the elements of H is e, unless $m = 1$, since every generating element (a, b, c, etc.) occurs as a factor an even number of times.

Also solved by Anders Bager, K. F. Bailie, P. T. Bateman, J. H. Biggs, Robert Bowen, H. G. Bray, L. Carlitz, David Carlson, A. J. Chandy, D. I. A. Cohen, M. S. Demos, John de Pillis, Harvey Friedman, Harry Gonshor, Ralph Greenberg, A. P. Hallstrom, J. H. Halton, G. A. Heuer, R. A. Jacobson, C. Donald La Budde, S. Lajos, E. S. Langford, J. F. Leetch, Tung-Po Lin, M. D. Mavinkurve, Jong Kuen Park, F. D. Parker, P. R. Parthasarathy, C. B. A. Peck, Stanton Philipp, S. M. Robinson, Azriel Rosenfeld, S. W. Saunders, Sister Maris Stella Schrot, William Scott, D. T. Sigley, Helmut Simon, Richard van de Velde, S.J., Seth Warner, Robert L. Wilson, Jr., Robert Lee Wilson, K. L. Yocom, J. J. Zeltmacher, Jr., and the proposer.

Editorial Note. Several solvers called attention to occurrences of this result in the literature. The earliest reference is to Frobenius and Stickelberger, J. Reine Angew. Math., 86 (1879) 240. A recent appearance is in L. J. Paige, Bull. Amer. Math. Soc., 53 (1947) 590.

Semigroup Isomorphism for a Set of Number Pairs

5159 [1963, 1107]. *Proposed by S. D. Chatterji, University of New South Wales, Australia*

Show that a monoid M (semigroup with identity) of objects (k, l), k and l nonnegative integers, with the composition law

$$(k, l) \circ (k', l') = (k + k', l + l' + 2ll')$$

is isomorphic to the positive integers considered as a monoid under multiplication. If one orders M by the rules $(k, l) < (k'l')$ if $k < k'$, or if $k = k'$ and $l < l'$, then M is also order isomorphic to the positive integers (with natural ordering).

Solution by Dave Nixon and Jim Wahab, Charlotte College, N. C. If n is any positive integer, there exist unique nonnegative integers k and l such that $n = 2^k(2l+1)$. The function f such that $f(n) = (k, l)$ is an isomorphism. The statement concerning order is in error since it would imply the existence of infinitely many positive integers smaller than the correspondent of (k, l) for $k > 0$.

Also solved by E. S. Langford, Stanton Philipp, J. L. Pietenpol, A. M. Vaidya, W. C. Waterhouse, and K. L. Yocom.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

COLLABORATING EDITORS: K. O. MAY, University of California, Berkeley, and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457. Programmed Materials: K. O. May, University of California, Berkeley, Calif. 94704. Films: E. P. Vance, Oberlin College, Oberlin, Ohio 44074.

REVIEWS IN PUBLICATIONS OF THE MATHEMATICAL ASSOCIATION OF AMERICA *AMERICAN MATHEMATICAL MONTHLY* AND *MATHEMATICS MAGAZINE*

Under the editorship of Professor Dmitri Thoro of San Jose State College, a section of book reviews has been reinstated in *Mathematics Magazine*. In conformity with recommendations by the Committee on Publications and the Board of Governors, reviews in that journal will be confined to books of interest to students and teachers of the first two years of college mathematics. This will include not only calculus and pre-calculus textbooks but also other books at this general level.

The resumption of reviewing in the *Magazine* permits the *Monthly* to concentrate on material extending from the beginning of the junior year through the first year of graduate education. This "division of labor" should result in more adequate notice of new publications than would otherwise be feasible.

Foundations of General Topology. By Ákos Császár. Macmillan, New York, 1963. xix+380 pp. $15.00.

Grundlagen der allgemeinen Topologie. By Ákos Császár. Akadémiai Kiadó, Budapest, 1963. 367 pp. $9.00.

This book which is the (English) second edition of *Fondements de la Topologie Générale*, contains an additional four chapters, mainly devoted to the work of the late J. Czipszer, together with some revisions to earlier chapters. The German edition coincides with the English one, except for the revision of Chapter 16 and the fact that the format and typography are better.

This monograph is a detailed description of a category which contains (in the sense of having isomorphic subcategories) the categories of topological spaces, uniform spaces and proximity spaces. The objects of the category (called syntopogeneous spaces) are pairs $(E, \mathcal{S})$, where E is a set and $\mathcal{S}$ a family of binary relations on $\mathcal{P}(E)$ which satisfies certain axioms. The morphisms (continuous functions) from $(E, \mathcal{S})$ to $(E', \mathcal{S}')$ are those functions $f: E \to E'$, such that for each relation in $\mathcal{S}'$, the relation on $\mathcal{P}(E)$ induced (in the obvious way) by f^{-1} is a subset of a relation in $\mathcal{S}$.

If $(E, \mathcal{O})$ is a topological space, it can be identified with $(E, \{<\})$, where $A < B$ if $A \subseteq B^0$. If $(E, \mathcal{U})$ is a uniform space, it can be identified with $(E, (<_V)\ V \in \mathcal{U})$, where $A <_V B$ if $V[A] \subseteq B$. Similarly, a proximity space can be identified with a syntopogeneous space. These identifications are such that morphisms correspond to morphisms.

The first six chapters are devoted to a detailed discussion of relations on $\mathcal{P}(E)$ and operations that can be performed on them. In the next three chapters, syntopogeneous spaces and continuous functions are defined, and various opera-

tions on $\mathcal{S}$ discussed. Products are then defined (Chapter 10) and the analogues of the T_i separation axioms introduced (Chapter 14). Complete spaces and compact spaces are defined in terms of the convergence of suitable classes of filter bases, and completion and compactification theorems proved (Chapters 15 and 16). In the German edition, Chapter 16 is improved considerably by a discussion of completion and compactification from the point of view of extension spaces.

Let $\mathfrak{J}$ denote the family $(<_\epsilon)_{\epsilon>0}$ of relations $<_\epsilon$ on R, where $A <_\epsilon B$ if $\sup A + \epsilon \leqq \inf (\mathrm{R} - B)$. The theorem of Czipszer in Chapter 12 asserts that every $(E, \mathcal{S})$ is (in a certain sense) the inverse image of $\mathfrak{J}$. In order to explain this it is convenient to enlarge the category by adding more morphisms: let $\Phi = (\phi_\alpha)_{\alpha \in \Omega}$ be a family of sets ϕ_α of functions $f: E \to E'$; Φ is a morphism, from $(E, \mathcal{S})$ to $(E', \mathcal{S}')$, if each ϕ_α is such that for a relation in $\mathcal{S}'$, there is one relation in $\mathcal{S}$ which contains all the relations induced by $\{f^{-1} | f \in \phi_\alpha\}$ (i.e., Φ is a family of "equicontinuous" sets of functions). Czipszer's theorem essentially states that given $\mathcal{S}$, there exists a morphism $\Phi: (E, \mathcal{S}) \to (\mathrm{R}, \mathfrak{J})$ with $\Phi^{-1}\mathfrak{J} = \mathcal{S}$.

Czipszer's theorem has as a consequence the fact that every $(E, \mathcal{S})$ can be embedded in a product of fundamental cubes. A fundamental cube is the product E of a number of copies of $[0, 1]$ equipped with the coarsest $\mathcal{S}$ which makes the set of projections: $(E, \mathcal{S}) \to (\mathrm{R}, \mathfrak{J})$ an "equicontinuous" set. Chapters 17 and 18 (due almost entirely to Czipszer) are devoted to a discussion of the number of copies of $[0, 1]$ needed for such an embedding.

The remaining chapters discuss quasi-metrics, the weight of metrizable spaces and totally bounded spaces.

The basic interesting idea in this book is that a topological space, a uniform space and a proximity space are particular examples of a syntopogeneous space. Those who wish to read a unified account of the three separate theories will find it of considerable interest. Otherwise, the main novelty is the theorem of Czipszer.

The reviewer feels that the main value of this exposition, which lies in the fact that it is a unification of three theories, is decreased by its tendency, at several points, to become bogged down in a mass of details. This feeling is reinforced by the fact that not much new light is shed on the separate theories.

J. C. TAYLOR, McGill University

An Introduction to Computational Methods. By K. A. Redish. Wiley, New York, 1962. xii+211 pp. $5.75.

This is a carefully written and well produced book which is intended for the occasional computer and for engineering and science students. It includes the standard topics of an introductory numerical analysis course, e.g., simultaneous linear algebraic equations, interpolation, differentiation and integration. Unfortunately, the methods are basically oriented to desk calculation and consequently will find little application in this country where most colleges and some modern high schools have electronic computers.

G. H. GOLUB, Stanford University

Introduction to Knot Theory. By R. H. Crowell and R. H. Fox. Ginn, Boston, 1963. x+182 pp. $8.00.

It is a famous theorem that any two embeddings of a circle in the plane are topologically equivalent. In contrast, it is intuitively obvious that a circle embedded in three-dimensional space can be knotted in many different ways. Mathematical curiosity at once prompts the question: "How can two knots be *proved* to be really different?" This natural and apparently innocuous question poses the central problem of knot theory and leads to algebraic developments of surprising variety and richness.

The authors have succeeded admirably in their purpose of providing an introduction to this fascinating subject at a level accessible to graduate students and advanced undergraduates. The topological prerequisites are minimal, amounting to no more than the basic concepts of point-set theory which are now frequently presented in rigorous calculus courses. No specific knowledge of algebra beyond elementary concepts is presupposed, but a student who has not had some experience in thinking about abstract groups and rings is likely to have difficulty. A number of problems are given at the end of each chapter. Most of them require a good understanding of the text and a little thought; a few require substantial insight and ingenuity.

The notions of fundamental group and of representation of a group by generators and relations are basic in knot theory, but are important concepts in their own right. The careful and self-contained expositions of these topics are worth noting for reference purposes. Here, for example, one will find a complete elementary proof that the fundamental group of a circle is infinite cyclic. An extensive chronological bibliography begins with Gauss's 1833 paper on linking numbers and ends with a list of thirty-three papers for 1962, some of which had not yet appeared in print when the book went to press. The bibliography is accompanied by a guide to the literature which indicates the various developments of knot theory beyond the scope of the text, and is of great value for anyone wishing to go more deeply in the subject.

H. F. TROTTER, Princeton University

A Programming Language. By K. Iverson. Wiley, New York, 1962. 286 pp. $8.95.

In this book, the author presents his own version of what he considers to be "an adequate programming language." Unfortunately, the language and its presentation both suffer from the lack of a unifying principle. The general impression received by the reviewer was of an *ad hoc* collection of notations and techniques for describing problems for digital computation. Some of the ideas are interesting but one wonders at the overall collection. The following is a list of chapter headings: The Language, Microprogramming, Representation of Variables, Search Techniques, Metaprograms, Sorting, The Logical Calculus.

E. K. BLUM, Wesleyan University

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E. K. BLUM, Wesleyan University

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Leonard M. Blumenthal of the University of Missouri has received the Distinguished Faculty Award of that University, consisting of a bronze plaque and one thousand dollars.

Professor Lyle E. Mehlenbacher, University of Detroit, represented the Association at the inauguration of William T. Jerome, III, as President of Bowling Green State University on September 16.

Professor Eric Reissner, Massachusetts Institute of Technology, has received an honorary degree of Doctor of Engineering from the Hanover Institute of Technology in Hanover, Germany.

University of California, Berkeley: Dr. B. R. Kripke, University of Texas, has been appointed Assistant Professor; Professor Ivan Niven, University of Oregon, has been appointed Visiting Professor; Dr. F. W. Warner, III, Massachusetts Institute of Technology, has been appointed Acting Assistant Professor and Assistant Research Mathematician; Associate Professor W. G. Bade has been promoted to Professor; Professor E. A. Bishop, on leave for the academic year 1964–65, has been appointed Research Professor in the Miller Institute for Basic Research, Berkeley; Professor J. L. Kelley, on leave for the academic year 1964–65, will spend the year in Kanpur, India, taking part in the Kanpur Indo-American Project; Associate Professor R. S. Lehman, on sabbatical leave for 1964–65, will carry on research at the Mathematical Institute of the University of Göttingen, Germany; Professor P. E. Thomas, on leave for the spring semester 1965, will carry on research at the Institute for Advanced Study and at Oxford University, England.

University of California, Riverside: Professor V. L. Shapiro, University of Oregon, has been appointed Professor; Professor F. B. Jones has been appointed Chairman of the Mathematics Department; Associate Professor H. G. Tucker has returned after a sabbatical year at the Institute for Advanced Study; Assistant Professor Hajimu Ogawa has returned after a year's leave at the University of California, Berkeley.

University of Washington: Drs. M. H. McAndrew, International Business Machines, Yorktown Heights, New York, and W. E. Ritter, Dartmouth College, have been appointed Assistant Professors; Dr. J. V. Ryff, Harvard University, has been appointed Visiting Assistant Professor; Associate Professor J. P. Jans has been promoted to Professor.

Associate Professor G. E. Baxter, University of Minnesota, has been appointed Professor at the University of California, San Diego.

Dr. I. E. Block, UNIVAC Division of Sperry Rand Corporation, Blue Bell, Pennsylvania, has joined Auerbach Corporation, Philadelphia, Pennsylvania, as technical advisor to the director of the Information Sciences Division.

Associate Professor S. E. Bohn, Bowling Green State University, has been appointed Associate Professor at Miami University.

Dr. S. D. Chatterji, U. S. Army Mathematics Research Center, University of Wisconsin, will be lecturing at the Institut für Angewandte Mathematik of the University of Heidelberg, West Germany, during the academic year 1964–65.

The Department extends its invitation to all interested mathematicians. New Mexico State University cannot support travel, but there is a small amount of money available to help defray dormitory living expenses, particularly for mathematicians (including graduate students) from the Southwest. In any case, assistance in reserving motel or dormitory space in Las Cruces will be provided, as will auto transportation to and from the El Paso, Texas airport or railway station.

Inquiries should be directed to Professor Ralph Crouch, Chairman, Department of Mathematical Sciences, New Mexico State University, University Park, New Mexico.

TEACHER EDUCATION BOOKLETS RELEASED BY NCTM

A new series of eight booklets, TOPICS IN MATHEMATICS, written especially for elementary school teachers, represents the latest publishing endeavor of the NCTM. Lenore John of the University of Chicago served as coordinator of the writing team.

Each booklet deals with a separate subject; titles are as follows: *Sets*, *The Whole Numbers*, *Numeration Systems for the Whole Numbers*, *Algorithms for Operations with Whole Numbers*, *Numbers and Their Factors*, *The Rational Numbers*, *Numeration Systems for the Rational Numbers*, and *Number Sentences*.

The purpose of the booklets is to help elementary teachers prepare themselves to teach arithmetic in a manner consistent with the "modern" view. It is the hope of the NCTM that these booklets may be helpful to individual teachers and to groups of teachers in in-service programs.

The set of eight booklets, averaging fifty pages each, is available for $2.15 from the National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington, D. C. 20036.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE OKLAHOMA SECTION

The annual spring meeting of the Oklahoma-Arkansas Section of the MAA was held at the East Central State College, Ada, Oklahoma, on April 10–11, 1964. Dr. Gerald K. Goff, Chairman of the Section, presided. Two invited addresses were delivered: one on "Homogeneity" by Dr. R. H. Bing, President, Mathematical Association of America; the other on "Quadric Surfaces Associated with a Tetrahedron" by Dr. Nathan A. Court, Professor Emeritus, University of Oklahoma.

At the business meeting the following officers were elected: Chairman, James O. Danley, East Central State College, Ada, Oklahoma; Vice-Chairman, William R. Orton, University of Arkansas, Fayetteville, Arkansas; Secretary-Treasurer, Richard V. Andree University of Oklahoma, Norman, Oklahoma.

In addition to a panel discussion on "A Possible Cooperative Mathematics Program in Oklahoma Colleges," the following papers were presented:

1. *Proof of a generalization of Rorem's conjecture*, by Edgar Karst, University of Oklahoma.

2. *Reflections in some common categories, Part I*, by Montie Monzingo, University of Oklahoma.

3. *Reflections in some common categories, Part II*, by Joe Wimbish, University of Oklahoma.

4. *A normal equation in Hilbert space*, by Tetsundo Sekiguchi, University of Arkansas.

5. *Geometrizing dynamics*, by C. E. Springer, University of Oklahoma.

6. *Mathematical theory of optimal control in problems with time delay*, by M. Q. Jacobs, University of Oklahoma.

7. *On the incidence geometry of Deal*, by R. R. Kinkade, Oklahoma State University.

8. *Open functions and dimension*, by R. E. Hodel, University of Oklahoma.

9. *Some remarks on primitive roots*, by D. L. Wright, University of Tulsa.

10. *An Ascoli-type theorem*, by C. H. Cook, University of Oklahoma.

11. *Number theory functions in a semi-group ring which can be imbedded in a field*, by R. B. Deal, Oklahoma State University.

R. V. ANDREE, *Secretary*

ACKNOWLEDGEMENT

The Editorial Board acknowledges with thanks the services of the following mathematicians, not members of the Board, who have kindly assisted by evaluating papers submitted for publication in the MONTHLY.

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CALENDAR OF FUTURE MEETINGS

Forty-eighth Annual Meeting, Denver-Hilton Hotel, Denver, Colorado, January 28–30, 1965.

Forty-sixth Summer Meeting (Fiftieth Anniversary Celebration), Cornell University, Ithaca, New York, August 30–September 2, 1965.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May 1, 1965.

ILLINOIS, Southern Illinois University, Carbondale, May 14–15, 1965.

INDIANA

IOWA, University of Dubuque, Dubuque, April 23, 1965.

KANSAS, Washburn University, Topeka, April 10, 1965.

KENTUCKY, Eastern Kentucky State College, Richmond, Spring, 1965.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 12–13, 1965.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1965.

MINNESOTA

MISSOURI, University of Missouri, Columbia, Spring, 1965.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 30–May 1, 1965.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, College of San Mateo, February 6, 1965.

OHIO

OKLAHOMA, University of Arkansas, Fayetteville, Spring, 1965.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 18, 1965.

PHILADELPHIA

ROCKY MOUNTAIN, The Colorado School of Mines, Golden, Colorado, Spring, 1965.

SOUTHEASTERN, Wake Forest College, Winston Salem, North Carolina, April 9–10, 1965.

SOUTHERN CALIFORNIA, Claremont Men's College, March 13, 1965.

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TEXAS, Texas Christian University, Fort Worth, April 9–10, 1965.

UPPER NEW YORK STATE, Colgate University, Hamilton, May 15, 1965.

WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN MATHEMATICAL SOCIETY, Denver, Colorado, January 26–29, 1965.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Illinois Institute of Technology, Chicago, June 21–25, 1965.

ASSOCIATION FOR COMPUTING MACHINERY, Cleveland, August 24–26, 1965.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 25–27, 1965.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Boston, Massachusetts, May 6–7, 1965.

R. F. Rinehart, John Riordan, T. J. Rivlin, Herbert Robbins, J. B. Roberts, Louise J. Rosenbaum, Murray Rosenblatt, H. L. Royden, Z. Rubinstein, Mary E. Rudin, Walter Rudin, P. T. Rygg, H. J. Ryser, E. C. Schlesinger, I. J. Schoenberg, Jacob Schwartz, A. Selberg, J. L. Selfridge, Patrick Shanahan, D. Shanks, H. S. Shapiro, B. T. Sims, W. S. Snyder, J. R. Stallings, J. G. Stampfli, Sherman Stein, K. Stellmacher, Rothwell Stephens, B. M. Stewart, R. R. Stoll, A. H. Stone, J. D. Swift, Olga Taussky-Todd, H. P. Thielman, Emery Thomas, R. M. Thrall, P. A. Tulley, W. T. Tutte, A. M. Vaidya, H. H. Walbesser, Jr., R. J. Walker, A. D. Wallace, J. L. Walsh, Seth Warner, Roy Westwick, George Whaples, G. T. Whyburn, W. M. Whyburn, N. A. Wiegmann, H. S. Wilf, S. S. Wilks, K. S. Williams, T. A. Willke, T. T. Wu, Adil Yaqub, Bertram Yood, Burke Zane, E. M. Zaustinsky, M. Zedek.

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